Gibbsian theory
as framework for the study of random dynamics

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An infinite-dimensional diffusion on finite time interval

$X = (X_i(t), i \in \mathbb{Z}^d, t \in [0,1])$ infinite-dimensional Brownian particle system, solution of the Stochastic Differential Equation

$$dX_i(t) = dW_i(t) + b_t(\theta_i X)\, dt,\ i \in \mathbb{Z}^d; \quad X(0) \sim \mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d}). \tag{1}$$

On the configuration space $\Omega := C([0,1], \mathbb{R}^{\mathbb{Z}^d}) \sim C([0,1], \mathbb{R})^{\mathbb{Z}^d}$

- $(W_i)_{i \in \mathbb{Z}^d}$ are independent Brownian motions
- θ_i denotes the space-shift by vector i
- the particle indexed by i is influenced by the other particles at time t through the adapted functional $b_t(\theta_i X)$.

Aims

- Find sufficient - but mild - conditions on the drift functional b and on the initial law μ which assure the **existence** of a shift-invariant weak solution to (1)
  $\Leftrightarrow$ construct a probability on Ω under which

$$\left(X_i(t) - \int_0^t b_s(\theta_i X)\, ds \right)_{i \in \mathbb{Z}^d, t \in [0,1]}$$

  are independent Brownian motions, and $X(0) \sim \mu$.
- Analyse the structure of the set of solutions
- Ergodicity or Gibbsianity of a solution, as probability measure on the infinite product space $C([0,1], \mathbb{R})^{\mathbb{Z}^d}$.
- Which conditions on the drift b can assure **uniqueness** of the solution?

Assumptions

- The drift functional $b : [0,1] \times C([0,1],\mathbb{R})^{\mathbb{Z}^d} \to \mathbb{R}$ is
  - adapted and local: $\exists \Delta \subset \mathbb{Z}^d$ finite, $\forall t \in [0,1]$,
$$b_t(\omega) = b_t(\omega_\Delta(s), s \in [0,t]).$$
  - uniformly sublinear:
$$\exists C > 0,\ \forall t, \omega \quad |b_t(\omega)| \leq C \left(1 + \sum_{i\in\Delta} \sup_{s\leq t} |\omega_i(s)|\right)$$
- The initial shift-invariant law $\mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ admits
  - finite-volume relative entropies wrt to a fixed $m \in \mathcal{P}(\mathbb{R})$:
$$\mathfrak{I}_R(\mu_\Lambda; m^{\otimes\Lambda}) := \begin{cases} \int_{\mathbb{R}^\Lambda} \ln\left(\frac{d\mu_\Lambda}{dm^{\otimes\Lambda}}\right) d\mu_\Lambda & \text{if it exists} \\ +\infty & \text{otherwise} \end{cases}$$

    and a finite specific entropy:
$$\mathfrak{I}(\mu) := \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \mathfrak{I}_R(\mu_\Lambda; m^{\otimes\Lambda}) < +\infty,$$
  - marginals with finite second moment: $\forall i \in \mathbb{Z}^d$, $\int x_i^2 \, \mu(dx) < +\infty$.

Examples

- A non-regular drift with delay.
  Define first β on $\mathbb{R}^\Delta$ by

$$\beta(x_\Delta) := \beta^+(x_\Delta)\,\mathbb{1}_{\left\{x_0 \geqslant \frac{1}{|\Delta|}\sum_{i\in\Delta} x_i\right\}} + \beta^-(x_\Delta)\,\mathbb{1}_{\left\{x_0 < \frac{1}{|\Delta|}\sum_{i\in\Delta} x_i\right\}}$$

  where the functions $\beta^+ \neq \beta^-$ have a sublinear growth.
  The function β depends on the relative value of x_0 wrt the barycentre of $x_\Delta = (x_i)_{i\in\Delta}$. It is discontinuous on the hyperplane $\{x_0 = \frac{1}{|\Delta|}\sum_{i\in\Delta} x_i\}$.
  Introducing a δ-delay, one now takes

$$b_t(\omega) := \beta(\omega_\Delta(0 \vee (t-\delta))).$$

- A long-term memory drift.

$$b_t(\omega) := \int_0^t f(s, \omega_\Delta(s))\, ds$$

  where $f(s,\cdot)$ has a sublinear growth.

Results

Existence Theorem (Dereudre & R. '17)

Under the above assumptions,

- the SDE (1) admits at least one shift-invariant weak solution P with initial marginal law μ. Moreover the finiteness of the specific entropy of μ propagates at the path level:

$$\mathfrak{I}(\mu) < +\infty \quad \Rightarrow \quad \mathfrak{I}(P) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \mathfrak{I}_R(P_\Lambda; \mathbf{W}^{\otimes \Lambda}) < +\infty.$$

  where $\mathbf{W}$ is the Wiener measure with initial condition m.
  Each coordinate admits a uniform 2nd moment.

- The set of solutions with finite specific entropy is convex and its extremal elements are ergodic probability measure on Ω, i.e. trivial on the σ-field of shift-invariant sets. In particular, for any ergodic probability measure $\mu \in \mathcal{P}_s(\mathbb{R}^{\mathbb{Z}^d})$ with $\mathfrak{I}(\mu) < +\infty$ there exists an ergodic weak solution P of the SDE (1) which admits μ as marginal law at time 0.

Some related references in other frameworks

- **Markovian** drift:
  Doss-Royer ('78), Leha-Ritter ('85): weak solutions in l^2-**spaces**
- **Markovian** drift with unbounded linear operator
  Albeverio-Röckner ('91 +), Grothaus, Kondratiev, Kuna...:
  weak solutions via **Dirichlet forms**
- **Bounded** drift + unbounded linear operator
  Da Prato-Zabczyk ('92 +): mild/weak sol. in Hilbert/Banach spaces via **Girsanov theory**
- Non Markovian and non-regular drift, but bounded and **small**
  dai Pra-Redig-R.-Ruszel ('06, '10, '14) : weak existence and uniqueness of a Gibbsian solution via **cluster expansions**

Sketch of the proof

- Finite volume approximation on $(\Lambda_n)_n \nearrow \mathbb{Z}^d$.
  Define on $C([0,1],\mathbb{R})^{\Lambda_n}$

$$dP_n := \frac{d\mu_{\Lambda_n}}{dm^{\otimes\Lambda_n}}(\omega_{\Lambda_n}(0))\, e^{-H_{\Lambda_n}(\omega_{\Lambda_n}0_{\Lambda_n^c})}\, d\mathbf{W}^{\otimes\Lambda_n}$$

where

$$H_\Lambda(\omega) = -\sum_{i\in\Lambda}\Big(\int_0^1 b_t(\theta_i\omega)\, d\omega_i(t) - \frac{1}{2}\int_0^1 b_t^2(\theta_i\omega)\, dt\Big).$$

Define $P_n^{\mathrm{per}} \in \mathcal{P}(\Omega)$ a space-periodisation of P_n and

$$\bar{P}_n := \frac{1}{|\Lambda_n|}\sum_{i\in\Lambda_n} P_n^{\mathrm{per}} \circ \theta_i^{-1} \qquad \in \mathcal{P}_s(\Omega),$$

its shift-invariant version.

Tightness

Proposition: The sequence $(\bar{P}_n)_{n\in\mathbb{N}}$ is tight. Therefore there exists at least one limit point $\bar{P}$ which has a finite specific entropy.

- Tightness criterion (Georgii '88)
  For any $\alpha > 0$, the level set of the specific entropy

$$\{P \in \mathcal{P}_s(\Omega), \mathfrak{I}(P) \leq \alpha\}$$

  is sequentially compact for the topology of the local convergence.
- In our framework

$$\begin{aligned}\mathfrak{I}(\bar{P}_n) &= \frac{1}{|\Lambda_n|}\mathfrak{I}_R(P_n; \mathbf{W}^{\otimes\Lambda_n}) \\ &\leq \frac{1}{|\Lambda_n|}\Bigg(\mathfrak{I}_R(\mu_{\Lambda_n}; m^{\otimes\Lambda_n}) + \sum_{i\in\Lambda_n}\frac{1}{2}E_{P_n}\Big(\int_0^1 b_t^2(\theta_i(\omega_{\Lambda_n}0_{\Lambda_n^c}))dt\Big)\Bigg).\end{aligned}$$

Characterization of the limit point $\bar{P}$

- $\bar{P}$ is the law of a Brownian semimartingale, weak solution of some SDE of type (1), for a certain L^2-drift $\bar{b}$, since it has a locally finite entropy. (Föllmer-Wakolbinger, '86)
- $\bar{P}$ minimizes the free energy functional $\mathfrak{I}^b$ defined by

$$\mathfrak{I}^b(Q) := \mathfrak{I}(Q) - \mathfrak{I}(Q \circ X(0)^{-1}) - \quad E_Q\Big(\int_0^1 b_t d\omega_0(t) - \frac{1}{2} \int_0^1 b_t^2 dt \Big).$$

- $\bar{P}$ is indeed a zero of the free energy $\mathfrak{I}^b$, which allows the identification of its drift $\bar{b}$.
- Its marginal law at time 0 is μ.

Gibbsian structure of the solution

$\bar{P}$ is a mixture of probability kernels: For any $\Lambda \subset \mathbb{Z}^d$,

$$\bar{P}(d\omega) = \int \Pi_\Lambda^{H,+}(\xi, d\omega)\, \bar{P}(d\xi)$$

for specific kernels where $\Pi_\Lambda^{H,+}$, that is

$\bar{P}$ **has a Gibbsian structure**

Uniqueness results

$$\begin{cases} dX_i(t) & = dW_i(t) - \frac{1}{2}\varphi'(X_i(t))\,dt + \beta\, b_t(\theta_i X)\,dt,\; i \in \mathbb{Z}^d \\ X(0) & \sim \mu \in \mathcal{G}_{\beta_0}(\psi) \end{cases} \tag{2}$$

(R. & Ruszel '14)

Under the following additional assumptions

- the dynamical self-potential φ is ultracontractive,
- the drift functional b is uniformly bounded,
- the initial potential ψ is strongly summable, and $\beta_0 < \frac{1}{\sup_i \sum_{\Lambda \ni i(|\Lambda|-1)} \|\psi_\Lambda\|}$ ($\Rightarrow$ initial state μ is in the *Dobrushin's uniqueness regime*)

there exists $\bar{\beta}$ such that, for $\beta < \bar{\beta}$, **uniqueness and Gibbsianness propagate**: at any time t, the law of the solution of (2) is a Gibbs measure uniquely determined by an absolutely summable interaction.

An infinite-dimensional diffusion on infinite time interval

$$dX_i(t) = dW_i(t) - \frac{1}{2}\varphi'(X_i(t))\,dt + \beta\, b(\theta_{i,t}X)\,dt,\ i \in \mathbb{Z}^d, t \in \mathbb{R} \qquad (3)$$

Uniqueness Theorem in perturbative regime (dai Pra & R. '04)
As before

- the dynamical self-potential φ is ultracontractive
- the drift functional b is uniformly bounded .

Then there exists an upper bound $\bar{\beta}$ for the dynamical inverse temperature, such that, for $\beta < \bar{\beta}$, the SDE (3) admits a **unique space-time shift-invariant** solution.

The unique law of the system (3) is itself a space-time Gibbs measure on $C((-\infty, +\infty), \mathbb{R})^{\mathbb{Z}^d}$, constructed via cluster expansion. At any time t, the law of the solution is a small perturbation of $\otimes_{i\in\mathbb{Z}^d} e^{-\varphi(x_i)}dx_i$, the stationary measure of the free dynamics ($\beta = 0$).