On the heterogeneous diffusion process

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Singular diffusions: analytic and stochastic approaches
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A Stratonovich SDE with irregular coefficients: Girsanov's example revisited

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2. Heterogeneous diffusion process

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Anomalous diffusion and ergodicity breaking in heterogeneous diffusion processes

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Space dependent diffusivity: diffusion in heterogeneous systems, e.g. Richardson diffusion in turbulence, transport in heterogeneous porous media, cytoplasmic diffusion in bacterial and eukaryotic cells...

$$\dot{X} = |X|^{\alpha}\dot{B}$$

$$\alpha \in \mathbb{R}$$

Mean square displacement, $X_0 = 0$:

$$\langle X_t^2 \rangle \sim t^{\gamma}, \quad t \to \infty$$

$\gamma = 1$: diffusion
$\gamma > 1$: superdiffusion
$\gamma < 1$: subdiffusion

3. Heterogeneous diffusion process

In the Itô interpretation: Girsanov (1960) — non-uniqueness for $\alpha \in (0, \frac{1}{2})$, also non-Markovian solutions. Zvonkin (1974) — for $\alpha \geqslant 1/2$ there is a unique strong solution, If $X_0 = 0$, $X_t \equiv 0$.

However: Cherstvy, Chechkin and Metzler considered the Stratonovich SDE:

$$X_t = X_0 + \int_0^t |X_s|^\alpha \circ \mathrm{d}B_s.$$

The Stratonovich integral is defined as a limit

$$\int_0^t |X_s|^\alpha \circ \mathrm{d}B_s := \lim \sum_k \frac{1}{2}\Big(|X_{t_{k+1}}|^\alpha + |X_{t_k}|^\alpha\Big)(B_{t_{k+1}} - B_{t_k})$$
$$= \int_0^t |X_s|^\alpha \,\mathrm{d}B_s + \frac{1}{2}[|X|^\alpha, B]_t$$

The definition of the integral contains the quadratic covariation process:

$$[|X|^\alpha, B]_t = \lim \sum_k \Big(|X_{t_{k+1}}|^\alpha - |X_{t_k}|^\alpha\Big)(B_{t_{k+1}} - B_{t_k})$$

4. Solution away from the origin

Assume $X_0 \neq 0$. Then for any $a > 0$ for $t < \tau_a \wedge \xi$, ξ — explosion time, $\tau_a = \inf\{t \geqslant 0\colon |X_t| \notin (a, \infty)\}$ the diffusion X solves

$$X_t = X_0 + \int_0^t |X_s|^\alpha \circ \mathrm{d}B_s = X_0 + \int_0^t |X_s|^\alpha \,\mathrm{d}B_s + \frac{\alpha}{2}\int_0^t |X_s|^{2\alpha-1} \operatorname{sign} X_s \,\mathrm{d}s$$

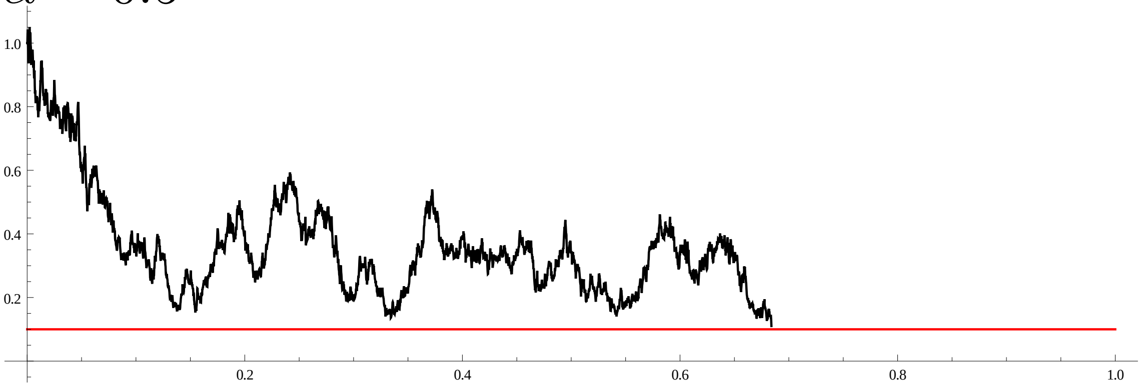
X can be found explicitly: denote $(x)^p = |x|^p \operatorname{sign} x$

$$\frac{\mathrm{d}X}{X^\alpha} = \circ\,\mathrm{d}B, \quad X_0 > 0,$$

$$X_t = \begin{cases} \left((1-\alpha)B_t + X_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}}, & \alpha < 1, \\ X_0 \mathrm{e}^{B_t}, \quad \alpha = 1, \\ \left(\frac{1}{X_0^{\frac{1}{\alpha-1}} - (\alpha-1)B_t}\right)^{\frac{1}{\alpha-1}}, & \alpha > 1 \end{cases}$$

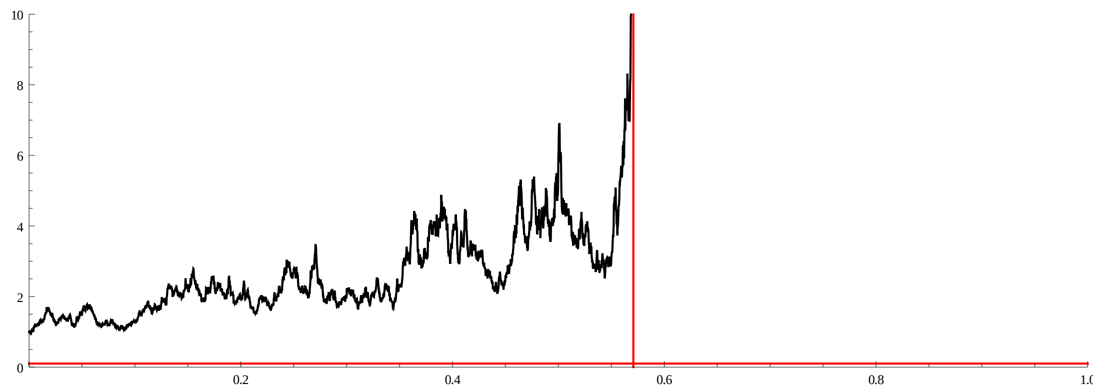
5. Solution away from the origin

$\alpha = 0.5$



$\alpha = 1.5$

$$\xi = \inf \left\{ t \geqslant 0 \colon B_t = \frac{X_0^{\frac{1}{1-\alpha}}}{\alpha - 1} \right\}$$



6. Solution by regularization: $\alpha \in (0,1)$

Consider smooth approximations $\sigma_\varepsilon(x)$ of $|x|^\alpha$ such that $\sigma_\varepsilon(x) > 0$ and approximations

$$X_t^\varepsilon = X_0 + \int_0^t \sigma_\varepsilon(X_s^\varepsilon) \circ \mathrm{d}B_s = X_0 + \int_0^t \sigma_\varepsilon(X_s^\varepsilon)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \sigma_\varepsilon(X_s^\varepsilon)\sigma_\varepsilon'(X_s^\varepsilon)\,\mathrm{d}s,$$

$$\frac{\mathrm{d}X^\varepsilon}{\sigma(X^\varepsilon)} = \circ\,\mathrm{d}B$$

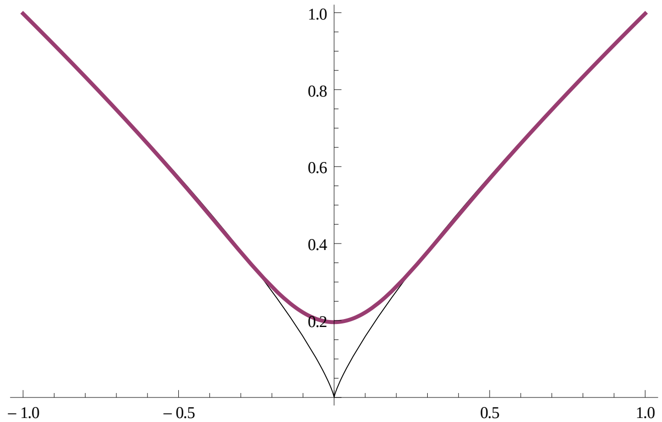
$$f_\varepsilon(x) = \int_0^x \frac{\mathrm{d}y}{\sigma_\varepsilon(y)}$$

Itô's formula: $$f_\varepsilon(X_t^\varepsilon) = f_\varepsilon(X_0) + B_t,$$

$$X_t^\varepsilon = f_\varepsilon^{-1}(B_t + f_\varepsilon(X_0))$$

$$X_t^\varepsilon \to X_t = f^{-1}(B_t + f(X_0)), \quad \varepsilon \to 0.$$

7. Solution by regularization: the “benchmark solution”

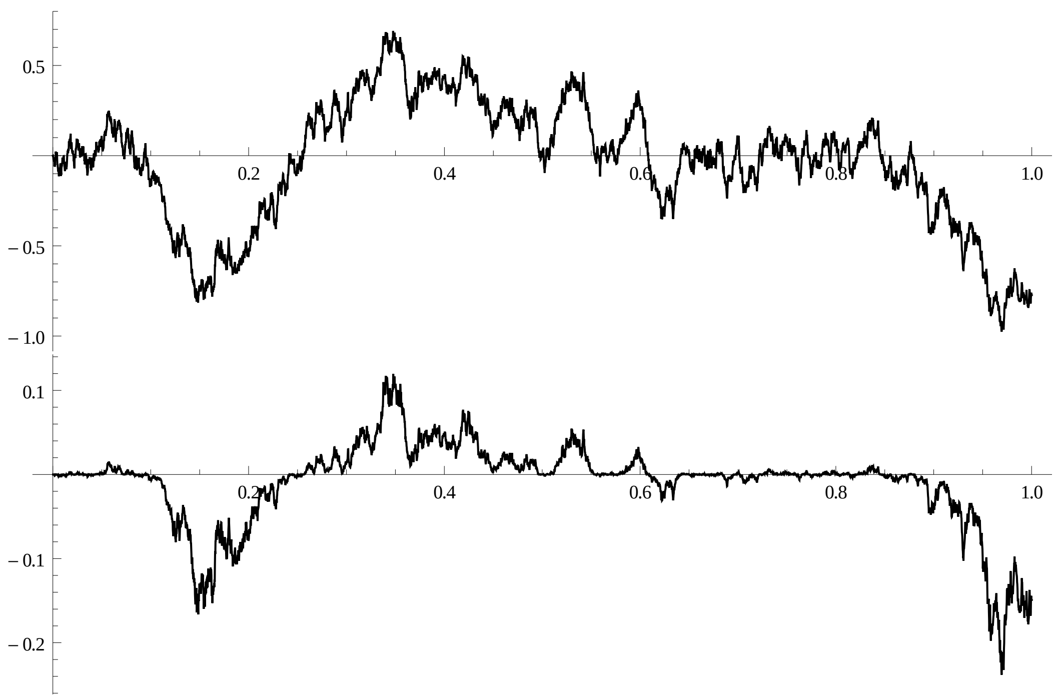


Recall notation: $(x)^p = |x|^p \operatorname{sign} x$

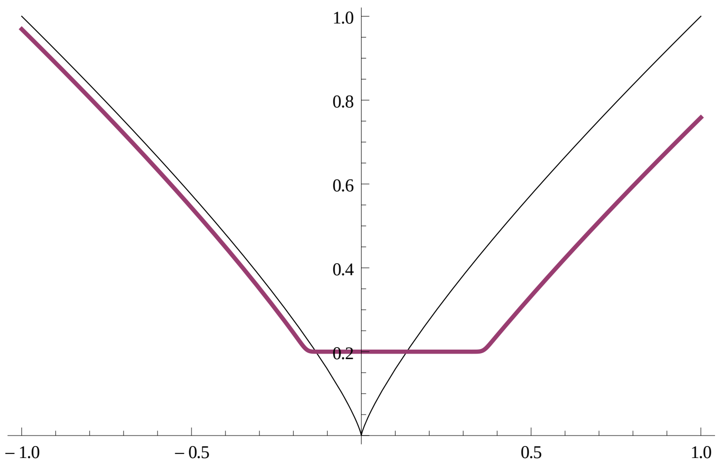
$$f_\varepsilon(x) \to f(x) = (x)^{1-\alpha} = |x|^{1-\alpha} \operatorname{sign} x$$

$$f_\varepsilon^{-1} \to f^{-1}(x) = ((1-\alpha)x)^{\frac{1}{1-\alpha}}$$

$$X_t^\varepsilon \to X_t^0 = F(B_t) = \left((1-\alpha)B_t + (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$



8. Another example



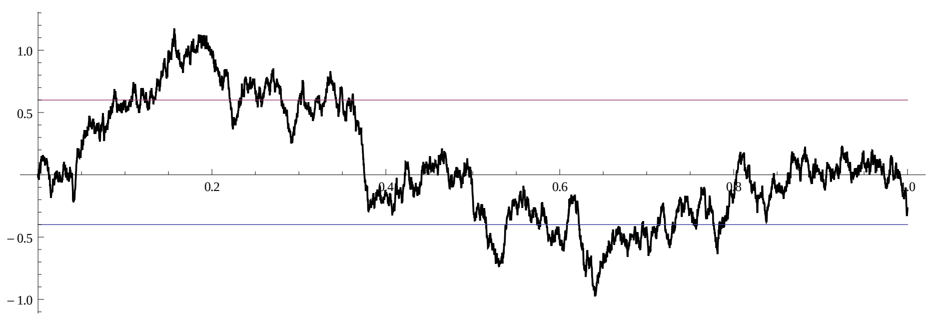
$$-A < 0 < B$$

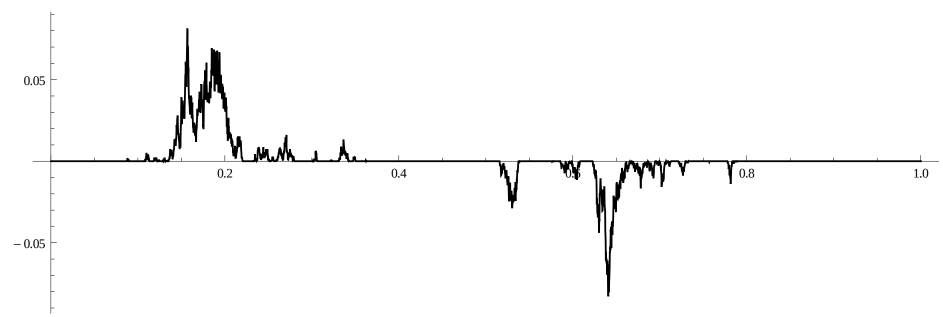
$$\sigma_\varepsilon(x) = \varepsilon, \quad -A\varepsilon \leqslant x \leqslant B\varepsilon,$$

$$f_\varepsilon(x) = \int_0^x \frac{\mathrm{d}y}{\sigma_\varepsilon(y)} = \frac{x}{\varepsilon}, \quad x \in [-A\varepsilon, B\varepsilon]$$

$$f_\varepsilon^{-1}(x) = \varepsilon x, \quad x \in [-A, B]$$

$$f_\varepsilon^{-1}(x) \to f(x) = \begin{cases} ((1-\alpha)(x-B))^{\frac{1}{1-\alpha}}, & x > B \\ 0, \quad x \in [-A, B] \\ -((1-\alpha)|x+A|)^{\frac{1}{1-\alpha}}, & x < -A \end{cases}$$





The limting process $X_t^{A,B} = f^{-1}(B_t) = F_{A,B}(B_t)$ equals to zero on random time intervals when $B_t \in [-\frac{A}{1-\alpha}, \frac{B}{1-\alpha}]$.

9. Itô's formula and existence of $[|X|^\alpha, B]$

We can prove that

$$X_t^0 = F(B_t) = ((1-\alpha)B_t + (X_0)^{1-\alpha})^{\frac{1}{1-\alpha}} \quad \text{or} \quad X_t^{A,B} = F_{A,B}(B_t)$$

is a solution with the help of the **generalized Itô formula**:

Föllmer–Protter–Shiryaev 1995: if F is absolutely continuous with locally square integrable derivative F' then $[F'(B), B]$ exists and

$$F(B_t) = F(0) + \int_0^t F'(B_s)\,\mathrm{d}B_s + \frac{1}{2}[F'(B), B]_t$$

In our case,

$$F(B) = ((1-\alpha)B + (X_0)^{1-\alpha})^{\frac{1}{1-\alpha}},$$

$$F'(B) = |(1-\alpha)B + (X_0)^{1-\alpha}|^{\frac{1}{1-\alpha}-1} = |F(B)|^\alpha$$

$$F' \in L^2_{\text{loc}}(\mathbb{R}) \quad \Leftrightarrow \quad 2\left(\frac{1}{1-\alpha} - 1\right) > -1 \quad \Leftrightarrow \quad \alpha > -1$$

Many solutions: the equation is underdetermined $\Rightarrow$ Impose more conditions!

10. Solutions spending zero time in zero

Let X be a (weak or strong) solution of

$$X_t = \int_0^t |X_s|^{1-\alpha} \circ \mathrm{d}B_s, \quad (X_0 = 0 \text{ for brevity})$$
$$\int_0^t \mathbb{I}(X_s = 0)\, \mathrm{d}s = 0 \text{ a.s.}, \quad t \geqslant 0.$$

The first guess: show that

$$\mathrm{Law}(|X|) = \mathrm{Law}\left(|(1-\alpha)B|^{\frac{1}{1-\alpha}}\right)$$

or in other words

$$\mathrm{Law}\left(\frac{1}{1-\alpha}|X|^{1-\alpha}\right) = \mathrm{Law}(|B|)$$

i.e. describe the law of the absolute value of X, and hence obtain **weak** solutions.

11. Reflected Brownian Motion

How to characterize the Reflected Brownian Motion?
Varadhan (lecture notes):

The RBM is the unique process $\mathbf{P}_x$ on the canonical probability space with the following properties:

1. $\mathbf{P}_x(Z_0 = x) = 1$

2. It behaves locally like Brownian motion on $(0, \infty)$, i.e. for any bounded smooth function $f\colon [0, \infty) \to \mathbb{R}$ that is a constant (w.l.o.g. $f = 0$) in some neighbourhood of 0 the process

$$f(Z_t) - f(x) - \frac{1}{2}\int_0^t f''(Z_s)\,\mathrm{d}s$$

is a martingale,

3. $$\mathbf{E}_x \int_0^\infty \mathbb{I}_{\{0\}}(Z_s)\,\mathrm{d}s = 0.$$

12. $\frac{1}{1-\alpha}|X|^{1-\alpha}$ is RMB, $\quad \alpha \in (-1,1)$

Denote $Z_t = \frac{1}{1-\alpha}|X_t|^{1-\alpha}$, Z spends zero time in zero.

Let $f\colon [0,\infty) \to [0,\infty)$ be a smooth bounded function that is constant in a neighbourhood of zero. The function $g(x) = f(\frac{1}{1-\alpha}|x|^{1-\alpha}) = f(z)$ is also smooth and is constant in a neighbourhood of zero, and

$$g'(x) = f'(z)(x)^{-\alpha}, \qquad g''(x) \ = f''(z)|x|^{-2\alpha} - \alpha f'(z)|x|^{-\alpha-1}.$$

Applying the Itô formula (with a certain care!) yields

$$\begin{aligned} f(Z_t) &= \int_0^t f'(Z_s)(X_s)^{-\alpha}\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \Big(f''(Z_s)|X_s|^{-2\alpha} - \alpha f'(Z_s)|X_s|^{-\alpha-1}\Big)\,\mathrm{d}\langle X\rangle_s \\ &= \int_0^t f'(Z_s)(X_s)^{-\alpha}|X_s|^{\alpha}\,\mathrm{d}B_s + \frac{\alpha}{2}\int_0^t f'(Z_s)(X_s)^{-\alpha}(X_s)^{2\alpha-1}\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \Big(f''(Z_s)|X_s|^{-2\alpha} - \alpha f'(Z_s)|X_s|^{-\alpha-1}\Big)|X_s|^{2\alpha}\,\mathrm{d}s \\ &= \int_0^t f'(Z_s)\,\mathrm{sign}(X_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(Z_s)\,\mathrm{d}s \end{aligned}$$

13. Skew Brownian motion

Question: if $|Z| = |W|$ what is Z?

For example: $Z = W$ or $Z = |W|$ or $Z = -|W|$.

Let Z be a time-homogeneous Markov process.

$$|Z| \text{ is a reflected BM} \quad \Leftrightarrow \quad Z \text{ is a skew BM}$$

Markov process with the transition density

$$p_\theta(t,x,y) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(y-x)^2}{2t}} + \frac{\theta}{\sqrt{2\pi t}} \operatorname{sign} y \cdot \mathrm{e}^{-\frac{-(|x|+|y|)^2}{2t}}$$

Can be constructed by flipping of Brownian excursions with probabilities $\frac{1+\theta}{2} \uparrow$ and $\frac{1-\theta}{2} \downarrow$, for some $\theta \in [-1,1]$.

Or as a limit of symmetric random walks perturbed at zero:

$$\mathbf{P}(X_{n+1} - X_n = \pm 1 | X_n \neq 0) = \frac{1}{2},$$

$$\mathbf{P}(X_{n+1} - X_n = 1 | X_n = 0) = \frac{1+\theta}{2}, \qquad \mathbf{P}(X_{n+1} - X_n = -1 | X_n = 0) = \frac{1-\theta}{2}$$

14. Weak solutions

Theorem. Let $\alpha \in (-1, 1)$, and let X be a weak solution such that X is a strong Markov process spending zero time at 0. Then there is $\theta \in [-1, 1]$, such that

$$X \stackrel{\mathrm{d}}{=} \left((1-\alpha)B^\theta + (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

for a θ-skew Brownian motion B^θ.

15. Skew Brownian motion as a solution to an SDE

Harrison and Shepp, 1981: SMB B^θ, $\theta \in [-1, 1]$ is the unique strong solution of

$$B_t^\theta = B_t + \theta L_t^0(B^\theta),$$

$L_t^0(\cdot)$ is the symmetric local time at zero.

The SBM is a homogeneous strong Markov process however it is **not** the unique process whose absolute value is distributed like $|W|$.

Indeed consider variably skewed Brownian motion with a variable skewness parameter $\theta\colon \mathbb{R} \to (-1, 1)$ as a solution to the SDE

$$B_t^\Theta = B_t + \Theta(L_t(B^\Theta)), \quad t \geqslant 0,$$

where $\Theta(x) = \int_0^x \theta(y)\mathrm{d}y$. This is a Markov process with $|B^\Theta| \stackrel{\mathrm{d}}{=} |B|$ (Barlow et al., 2000); however, if θ is non-constant, B^Θ is not homogeneous Markov.

16. Strong solutions: the result

Theorem.

1. Let $\alpha \in (0, 1)$ and $\theta \in [-1, 1]$. Then

$$X_t^\theta = \left((1-\alpha)B_t^\theta + (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

is a strong solution which is a homogeneous strong Markov process spending zero time at 0.

Moreover, X^θ is the unique strong solution which is a homogeneous strong Markov process spending zero time at 0 and such that

$$\mathbf{P}(X_t^\theta \geqslant 0 \mid X_0 = 0) = \frac{1+\theta}{2}, \quad t > 0.$$

2. Let $\alpha \in (-1, 0]$. Then $X_t^0 = \left((1-\alpha)B_t + (X_0)^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$ is the unique strong solution which is a homogeneous strong Markov process spending zero time at 0.

17. The main part of the proof

The crucial part of the proof is the **existence** of the quadratic variation $[|X^\theta|^\alpha, B]$. We show that

Theorem. Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ and let the θ-skew Brownian motion B^θ, $\theta \in (-1, 1)$, be the unique strong solution of $B^\theta_t = B_t + \theta L^0_t(B^\theta)$. Then the quadratic variation

$$[f(B^\theta, B), B]_t = \lim_{n\to\infty} \sum_{t_k \in D_n, t_k < t} \big(f(B^\theta_{t_k}, B_{t_k}) - f(B^\theta_{t_{k-1}}, B_{t_{k-1}})\big)(B_{t_k} - B_{t_{k-1}})$$

exists as a limit in u.c.p.

Moreover, let $\{f_n\}_{n\geqslant 1}$ be a sequence of continuous functions such that for each compact $K \subset \mathbb{R}^2$

$$\lim_{n\to\infty} \iint_K |f_n(x,y) - f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Then

$$[f_n(B^\theta, B), B]_t \xrightarrow{\text{u.c.p.}} [f(B^\theta, B), B]_t$$

18. Time reversion, $\theta \in (-1,1)$

We use the approach by Föllmer, Protter and Shiryaev, 1995:

$$[f(B^\theta,B),B]_t = \lim_{t_k\in D_n, t_k\leqslant t} \sum_{k=1}^{n} \Big(f(B^\theta_{t_k},B_{t_k}) - f(B^\theta_{t_{k-1}},B_{t_{k-1}})\Big)(B_{t_k}-B_{t_{k-1}})$$

$$= \int_0^t f(B^\theta_s,B_s)\,\mathrm{d}^*B_s - \int_0^t f(B^\theta_s,B_s)\,\mathrm{d}B_s$$

$$\lim_{t_k\in D_n, t_k\leqslant t} \sum_{k=1}^{n} f(B^\theta_{t_{k-1}},B_{t_{k-1}})(B_{t_k}-B_{t_{k-1}}) = \int_0^t f(B^\theta_s,B_s)\,\mathrm{d}B_s$$

$$\lim_{t_k\in D_n, t_k\leqslant t} \sum_{k=1}^{n} f(B^\theta_{t_k},B_{t_k})(B_{t_k}-B_{t_{k-1}}) = \int_0^t f(B^\theta_s,B_s)\,\mathrm{d}^*B_s$$

$$= \int_{T-t}^{T} f(\bar{B}^\theta_s,\bar{B}_s)\,\mathrm{d}\bar{B}_s$$

where $(\bar{B}^\theta_t,\bar{B}_t) = (B^\theta_{T-t},B_{T-t})$

Thus: show that $(\bar{B}^\theta_t,\bar{B}_t)$ is a semimartingale

19. Time reversion:

Time reversal technique by Haussmann and Pardoux, 1985: Let X be a Markovian diffusion in $\mathbb{R}^d$

$$\mathrm{d}X = b(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}W, \quad t \in [0,1]$$

$$X(t) \sim p(t,x), \quad \text{density with good properties,}$$

$$Lf(x) = \frac{1}{2}a^{ij}(x)f_{x_i x_j} + b^i(x)f_{x_i}, \quad a(x) = \sigma(x)\sigma^*(x)$$

Then $\bar{X} = (X_{1-t})_{t\in[0,1)}$ is a Markovian diffision with the generator

$$\bar{L}_t f(x) = \frac{1}{2}\bar{a}^{ij}(x)f_{x_i x_j} + \bar{b}^i(x)f_{x_i}$$

$$\bar{a}^{ij} = a^{ij}, \quad \bar{\sigma}^{ij} = \sigma^{ij}$$

$$\bar{b}^i(x) = -b^i(x) + \frac{\big(a^{ij}(x)p(1-t,x)\big)_{x_j}}{p(1-t,x)}$$

Hence, $\bar{X}$ has the same law as a solution of an SDE

$$\mathrm{d}\bar{X} = \bar{b}(\bar{X})\,\mathrm{d}t + \bar{\sigma}(\bar{X})\,\mathrm{d}\bar{W}, \quad t \in [0,1)$$

20. An SDE for (B^θ, B)

$$\begin{cases} B \\ B^\theta = B + \theta L^0(B^\theta) \end{cases} \Rightarrow \begin{cases} B \\ \mathrm{d}Y^\theta = \sigma(Y^\theta)\,\mathrm{d}B \end{cases} \qquad \sigma(y) = \begin{cases} \frac{2}{1-\theta}, & y < 0 \\ \frac{2}{1+\theta}, & y > 0 \end{cases}$$

$$r(Y^\theta) = B^\theta, \qquad r(y) = \frac{x}{\sigma(x)}$$

Theorem. Let for $\theta \in (-1,1)\backslash\{0\}$. Then $(\bar{Y}^\theta_t, \bar{B}_t) = (Y_{1-t}, B_{1-t})$ is a weak solution of

$$\bar{Y}^\theta_t = Y^\theta_T + \int_0^t \bar{b}^y(s, \bar{Y}^\theta_s, \bar{B}_s)\,\mathrm{d}s + \int_0^t \sigma(\bar{Y}^\theta_s)\,\mathrm{d}W_s,$$

$$\bar{B}_t = B_T + \int_0^t \bar{b}^z(s, \bar{Y}^\theta_s, \bar{B}_s)\,\mathrm{d}s + W_t, \quad t \in [0,1),$$

W being a standard Brownian motion.

Here: $\bar{b}^y(s,y,b)$ and $\bar{b}^y(s,y,b)$ are rather complicated functions, known explicitly.

21. Strong solutions: Proof $\alpha \in (0,1)$

For definiteness we set $X_0 = 0$.

1. For $\theta = 0$ (i.e. $B^\theta = B$) and $\alpha \in (-1,1)$: apply the generalized Itô formula by Föllmer–Protter-Shiryaev.

2. Let $\theta \in (-1,1)\backslash\{0\}$.

Take a sequence $\{h_n\}$ of C^1-functions such that, $h_n(0) = 0$, $h_n(x) = |(1-\alpha)x|^\alpha$ for $|x| \geqslant 1$ and $\sup_{x\in[0,1]} |h_n(x) - (1-\alpha)|x|^\alpha| \to 0$, $H_n(x) = \int_0^x h_n(y)\,\mathrm{d}y \in C^2$.

The conventional Itô formula for semimartingales

$$H_n(B_t^\theta) = \int_0^t h_n(B_s^\theta)\,\mathrm{d}B_s + \underbrace{\theta\int_0^t h_n(B_s^\theta)\,\mathrm{d}L_s(B^\theta)}_{=0} + \frac{1}{2}[h_n(B^\theta), B]_t + \underbrace{\frac{\theta}{2}[h_n(B^\theta), L(B^\theta)]_t}_{=0},$$

Hence, as $n \to \infty$,

$$H(B_t^\theta) = \int_0^t h(B_s^\theta)\,\mathrm{d}B_s + \frac{1}{2}[h(B^\theta), B]_t = \int_0^t |(1-\alpha)B_s^\theta|^\alpha \circ \mathrm{d}B_s.$$

22. Strong solutions: Proof $\alpha = 0$

Show that $X^\theta = B^\theta$ is not a solution for $\theta \neq 0$.

$$\int_0^t \mathbb{I}(B_s^\theta \neq 0)\,\mathrm{d}B_s = B_t \quad \text{a.s.}$$

Approximate $h(x) = \mathbb{I}(x \neq 0)$ by $h_n(x) \equiv 1$ in $L^2(\mathbb{R})$. Then

$$0 \equiv [1, B] = [h_n(B^\theta), B] \to [\mathbb{I}(B_\cdot^\theta \neq 0), B]$$

and

$$\begin{aligned}\int_0^t \mathbb{I}(B_s^\theta \neq 0) \circ \mathrm{d}B_s &= \int_0^t \mathbb{I}(B_s^\theta \neq 0)\,\mathrm{d}B_s + \frac{1}{2}[\mathbb{I}(B_\cdot^\theta \neq 0), B]_t \\ &= B_t \neq X_t^\theta = B_t + \theta L_t^0(B^\theta).\end{aligned}$$

23. Special case: explicit solution for $\theta = \pm 1$

Theorem. For $\alpha \in (0,1)$

$$X_t^1 = \Big((1-\alpha)\big(B_t - \min_{s\leqslant t} B_s\big)\Big)^{\frac{1}{1-\alpha}}$$

is a strong solution of $\mathrm{d}X_t = |X_t|^{1-\alpha} \circ \mathrm{d}B_t$, $X_0 = 0$.

Proof by substitution: consider a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and let $\tau_k = \min\{s \geqslant t_{k-1} \colon X_s = 0\} \wedge t_k$, $k = 1, \ldots, n$.

On $t \in [t_{k-1}, \tau_k)$ we have $m_{t_{k-1}} = m_t$ hence X is the unique solution on $[t_{k-1}, \tau_k)$:

$$\Big((1-\alpha)(B_t - B_{t_{k-1}}) + X_{t_{k-1}}^{1-\alpha}\Big)^{\frac{1}{1-\alpha}}$$

$$= \Big((1-\alpha)(B_t - B_{t_{k-1}}) + (1-\alpha)B_{t_{k-1}} - (1-\alpha)m_{t_{k-1}}\Big)^{\frac{1}{1-\alpha}} = X_t$$

Denote $I = \{k \colon B \text{ has a zero in } [t_{k-1}, t_k)\}$, then for $\frac{1-\alpha}{2} < \gamma < \frac{1}{2}$

$$\sum_{k\in I} |X_{t_k} - X_{\tau_k}| \leqslant C(\omega,\gamma) \sum_{k\in I} |t_k - t_{k-1}|^{\frac{\gamma}{1-\alpha}} \to \frac{\gamma}{1-\alpha}\text{-Hausdorff dim. of zeroes} = 0$$

On the relation between the Stratonovich and Itô equations I

$$X_t = \int_0^t |X_s|^\alpha \circ \mathrm{d}B_s \quad \overset{\text{formally?}}{\Longleftrightarrow} \quad X_t = \int_0^t |X_s|^\alpha \,\mathrm{d}B_s + \frac{\alpha}{2}\int_0^t (X_s)^{2\alpha-1}\,\mathrm{d}s.$$

Put X^θ into the Itô equation: for the existens of the Itô integral we need

$$\int_0^t |X_s^\theta|^{2\alpha}\,\mathrm{d}s \overset{\mathrm{d}}{=} \int_0^t |W_s|^{\frac{2\alpha}{1-\alpha}}\,\mathrm{d}s < \infty \quad \Leftrightarrow \quad \alpha > -1$$

and for the existence of the drift term we need (appy the Engelbert–Schmidt zero-one law)

$$\int_0^t |X_s^\theta|^{2\alpha-1}\,\mathrm{d}s \overset{\mathrm{d}}{=} \int_0^t |W_s|^{\frac{2\alpha-1}{1-\alpha}}\,\mathrm{d}s < \infty \quad \Leftrightarrow \quad \alpha > 0$$

Hence X^θ is a solution of the Itô equation for $\theta \in [-1, 1]$ and $\alpha \in (0, 1)$.

On the relation between the Stratonovich and Itô equations II

For $\alpha \in (-1, 0]$, consider the drift term in the *principal value sense*:

$$\text{v.p.} \int_0^t (W_s)^{\frac{2\alpha-1}{1-\alpha}} \, \mathrm{d}s := \lim_{\varepsilon \downarrow 0} \int_0^t (W_s)^{\frac{2\alpha-1}{1-\alpha}} \cdot \mathbb{I}(|W_s| > \varepsilon) \, \mathrm{d}s.$$

The principal value definition is intrinsically based on the symmetry of the Brownian motion and the asymmetry of the integrand and hence excludes the cases $\theta \neq 0$. Necessary and sufficient conditions for the existence of Brownian principal value integrals are given by Cherny, 2001.

$$\text{v.p.} \int_0^t (W_s)^{\frac{2\alpha-1}{1-\alpha}} \, \mathrm{d}s < \infty \quad \Leftrightarrow \quad \alpha > -1$$

Hence for $\alpha \in (-1, 0]$, X^0 is the solution of the Itô SDE

$$X_t = X_0 + \int_0^t |X_s|^\alpha \, \mathrm{d}B_s + \frac{\alpha}{2} \cdot \text{v.p.} \int_0^t (X_s)^{2\alpha-1} \, \mathrm{d}s.$$

26. Selection problem

Consider the perturbed equation: W be another indepenent BM,

$$X_t^\varepsilon = X_0 + \int_0^t |X_s^\varepsilon|^\alpha \circ \mathrm{d}B_s + \varepsilon W_t$$

Start with a simpler problem: Wong–Zakai approximation of B. For each $n \geqslant 1$, define

$$B_t^n = B_{\frac{k}{n}} + n\big(B_{\frac{k+1}{n}} - B_{\frac{k}{n}}\big)(t - \tfrac{k}{n}), \quad t \in [\tfrac{k}{n}, \tfrac{k+1}{n}], \quad k \geqslant 0$$

$$\sup_{t\in[0,1]} \big|B_t^n - B_t\big| \to 0 \text{ a.s.}, \quad n \to \infty$$

Thanks to Zvonkin (1974) there is a unique strong solution to

$$X_t^{n,\varepsilon} = X_0 + \int_0^t |X_s^{n,\varepsilon}|^\alpha \dot{B}_s^n \,\mathrm{d}s + \varepsilon W_t$$

Then: with probability 1,

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \sup_{t\in[0,1]} |X_t^{n,\varepsilon} - X_t^0| = 0, \qquad X_t^0 = \Big((1-\alpha)B_t + (X_0)^{1-\alpha}\Big)^{1/(1-\alpha)}$$