The modified Euler scheme for a weak approximation of solutions of stochastic differential equations driven by a Wiener process

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Let X be a diffusion process in $\mathbb{R}^d$ of the form

$$X_t = x + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad 0 \le t \le T, \tag{1}$$

where $W \in \mathbb{R}^m$ is a Wiener process, $a : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$.

Main goal

$$\mathbb{E}_x f(X_T) = ?$$

If X_T is known then one can simulate N independent copies of it, e.g. $X_T^{(i)}$, and approximate

$$\mathbb{E}_x f(X_T) \approx \frac{1}{N}\sum_{i=1}^{N} f(X_T^{(i)}).$$

Unfortunately, in most cases the law of X_T is not known. In this case one can use an approximation of it, e.g. $\hat{X}_T$. Then

$$\left|\mathbb{E}_x f(X_T) - \frac{1}{N}\sum_{i=1}^{N} f(\hat{X}_T^{(i)})\right| \le \left|\mathbb{E}_x f(X_T) - \mathbb{E}_x f(\hat{X}_T)\right| + \left|\mathbb{E}_x f(\hat{X}_T) - \frac{1}{N}\sum_{i=1}^{N} f(\hat{X}_T^{(i)})\right|.$$

Time discrete approximation

Consider a time discretization of the interval $[0;T]$ with step $h = \frac{T}{n}$:

$$t_k = kh, \quad k = 0, \ldots, n.$$

Let $\mathcal{F}_t = \sigma\{X_s : s \le t\}$.

Definition

We call cadlag process $Y^h = Y = \{Y_t, 0 \le t \le T\}$ a time discrete approximation if Y_{t_k} is $\mathcal{F}_{t_k}$-measurable and $Y_{t_{k+1}}$ can be expressed as a function of $Y_0, \ldots, Y_{t_k}, t_0, \ldots, t_k, t_{k+1}$ and a finite number of $\mathcal{F}_{t_{k+1}}$-measurable variables.

Example. (Maruyama'55) The simplest time discrete approximation of solution to (1) is the Euler approximation (or the Euler-Maruyama approximation)

$$Y_{t_{k+1}} = Y_{t_k} + a(Y_{t_k})h + \sigma(Y_{t_k})(W_{t_{k+1}} - W_{t_k}), \quad k = 0, \ldots, n-1 \tag{2}$$

with $Y_0 = x$.

Definition

We say that a time discrete approximation Y^h *converges weakly with order* $\beta > 0$ *to* X *at time* T *as* $h \to 0$ *if for all 'good enough' functions* f *there exist a positive constant* C*, which does not depend on* h*, such that*

$$|Ef(X_T) - Ef(Y_T^h)| \leq Ch^\beta. \quad (3)$$

Example. (Milshtein'78) The Euler approximation converges to X with the weak order $\beta = 1$.

How to improve the order of approximation?

1) The classical approach: by using the Ito-Taylor expansion.
P. E. Kloeden, E. Platen, *Numerical solution of stochastic differential equations*, Springer, Berlin, 1995.

2) Our approach.

It can be shown (Milshtein'78) that for proving (3) it's sufficient to prove

$$|Ef(X_h) - Ef(Y_h^h)| \le Ch^{\beta+1}. \quad (4)$$

On the interval $[0, h]$ the Euler scheme has the form

$$\tilde{X}_t = x + a(x)t + \sigma(x)W_t, \quad 0 \le t \le h.$$

Let us prove that Euler approximation has the weak order $\beta = 1$. We need to show that

$$\left|\mathbb{E}_x f\left(X_h\right) - \mathbb{E}_x f(\tilde{X}_h)\right| \le Ch^2.$$

If X is a process of form (1) and $g \in C^2(\mathbb{R})$ then

$$g(X_t) = g(x) + \int_0^t Ag(X_s)ds + \int_0^t Lg(X_s)dW_s, \tag{5}$$

where

$$Ag(y) = a(y)g'(y) + \frac{1}{2}b(y)g''(y), \quad Lg(y) = \sigma(y)g'(y).$$

Here $b(y) = \sigma^2(y)$.

$$\Rightarrow \quad \mathbb{E}_x g(X_t) = g(x) + \int_0^t \mathbb{E}_x Ag(X_s)ds. \tag{6}$$

Ito formula

For $\tilde{X}$ the Ito formula gives

$$\mathbb{E}_x g(\tilde{X}_t) = g(x) + \int_0^t \mathbb{E}_x \tilde{A} g(\tilde{X}_s) ds,$$

where

$$\tilde{A} g(y) = a(x) g'(y) + \frac{1}{2} b(x) g''(y).$$

Euler approximation

If $f \in C_b^4(\mathbb{R}), a, \sigma \in C_b^2(\mathbb{R})$, then

$$\mathbb{E}_x f(X_h) = f(x) + \int_0^h \mathbb{E}_x Af(X_s)ds = f(x) + \int_0^h \left(Af(x) + \int_0^s \mathbb{E}_x A(Af(X_r))dr \right) ds =$$

$$= f(x) + Af(x)h + \int_0^h \int_0^s \mathbb{E}_x A(Af(X_r))drds = f(x) + Af(x)h + O(h^2).$$

For $\tilde{X}$ we have

$$\mathbb{E}_x f(\tilde{X}_h) = f(x) + a(x)\int_0^h \mathbb{E}_x f'(\tilde{X}_s)ds + \frac{1}{2}b(x)\int_0^h \mathbb{E}_x f''(\tilde{X}_s)ds =$$

$$= f(x) + a(x)\int_0^h \left(f'(x) + \int_0^s \mathbb{E}_x \tilde{A} f'(\tilde{X}_r)dr \right) ds +$$

$$+ \frac{1}{2}b(x)\int_0^h \left(f''(x) + \int_0^s \mathbb{E}_x \tilde{A} f''(\tilde{X}_r) \right) ds =$$

$$= f(x) + Af(x)h + O(h^2).$$

$$\mathbb{E}_x f(X_h) = f(x) + Af(x)h + \frac{1}{2}A(Af(x))h^2 + O(h^3) \tag{7}$$

and

$$\mathbb{E}_x f(\tilde{X}_h) \stackrel{?}{=} f(x) + Af(x)h + \frac{1}{2}A(Af(x))h^2 + O(h^3), \tag{8}$$

where

$$\begin{aligned} A(Af(x)) = {} & \left(a(x)a'(x) + \frac{1}{2}a''(x)b(x)\right) f'(x) + \\ & + \left(a^2(x) + \frac{1}{2}a(x)b'(x) + a'(x)b(x) + \frac{1}{4}b(x)b''(x)\right) f''(x) + \\ & + \left(a(x)b(x) + \frac{1}{2}b(x)b'(x)\right) f'''(x) + \frac{1}{4}b^2(x)f^{(IV)}(x). \end{aligned}$$

$$\mathbb{E}_x f(\tilde{X}_h) = f(x) + Af(x)h + \\ + \frac{1}{2}\left(a^2(x)f''(x) + a(x)b(x)f'''(x) + \frac{1}{4}b^2(x)f^{(IV)}(x)\right)h^2 + O(h^3).$$

Instead of the Euler scheme we consider its modification of the form

$$\hat{X}_t = \tilde{X}_t + \Delta_t, \quad 0 \le t \le h,$$

where the corrector $\Delta = \Delta_t(\tilde{X}_t)$ has to be chosen in a such way that

$$\left| \mathbb{E}_x f\left(X_h\right) - \mathbb{E}_x f(\hat{X}_h) \right| \le Ch^3$$

for all 'good enough' functions f.

For constructing the corrector Δ_t we introduce the notion of Hermite polynomials. Let us remind their definition. Transition probability density $p_t(x,y)$ of the process $\tilde{X}_t$ has the form

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t b(x)}} \exp\left\{-\frac{(y-x-a(x)t)^2}{2tb(x)}\right\}.$$

Definition

The Hermite polinomials are the family of functions

$$\left\{H_t^{(m)}(x,y) : x,y \in \mathbb{R}, t>0, m \in \mathbb{N} \cup \{0\}\right\},$$

each of them satisfies an equality

$$\frac{\partial^m}{\partial y^m} p_t(x,y) = (-1)^m H_t^{(m)}(x,y) p_t(x,y).$$

$$H_t^{(0)}(x,y) = 1,$$

$$H_t^{(1)}(x,y) = \frac{(y - x - a(x)t)}{b(x)t},$$

$$H_t^{(2)}(x,y) = \frac{(y - x - a(x)t)^2}{b^2(x)t^2} - \frac{1}{b(x)t}$$

and so on.

In what follows we need the next relations

$$\left(H_t^{(1)}(x,y)\right)^2 = H_t^{(2)}(x,y) + \frac{1}{tb(x)}, \tag{9}$$

$$H_t^{(1)}(x,y)H_t^{(2)}(x,y) = H_t^{(3)}(x,y) + \frac{2}{tb(x)}H_t^{(1)}(x,y), \tag{10}$$

$$\left(H_t^{(2)}(x,y)\right)^2 = H_t^{(4)}(x,y) + \frac{4}{tb(x)}H_t^{(2)}(x,y) + \frac{2}{t^2b^2(x)}. \tag{11}$$

Lemma

Let $\tilde{X}_t = x + a(x)t + \sigma(x)W_t, t \geq 0$. *Then*
1) for $f \in C_b^m(\mathbb{R})$ *the following formula holds true*

$$\mathbb{E}_x f(\tilde{X}_t) H_t^{(m)}(x, \tilde{X}_t) = \mathbb{E}_x f^{(m)}(\tilde{X}_t). \tag{12}$$

2) for $f, a, \sigma \in C_b(\mathbb{R})$ *there exists constant* C, *which doesn't depend on* t *and* x, *such that*

$$\left|\mathbb{E}_x f(\tilde{X}_t) H_t^{(m)}(x, \tilde{X}_t)\right| \leq C t^{-\frac{m}{2}}. \tag{13}$$

We define

$$\Delta_t := t^2 \left(c_0(x) + c_1(x) H_t^{(1)}(x, \tilde{X}_t) + c_2(x) H_t^{(2)}(x, \tilde{X}_t) \right), \tag{14}$$

with

$$c_0(x) = \frac{1}{2} a(x) a'(x) + \frac{1}{4} a''(x) b(x),$$

$$c_1(x) = \frac{1}{4} a(x) b'(x) + \frac{1}{2} a'(x) b(x) + \frac{1}{8} b(x) b''(x) - \frac{1}{16} (b'(x))^2,$$

$$c_2(x) = \frac{1}{4} b(x) b'(x).$$

Theorem

If $f \in C_b^6(\mathbb{R})$, $a, \sigma \in C_b^4(\mathbb{R})$ *then the following bound holds true*

$$\left| \mathbb{E}_x f\left(X_h\right) - \mathbb{E}_x f(\hat{X}_h) \right| \leq Ch^3.$$

$$\mathbb{E}_x f(X_h) = f(x) + Af(x)h + \frac{1}{2}A(Af(x))h^2 + O(h^3) \tag{15}$$

and

$$\mathbb{E}_x f(\hat{X}_h) \stackrel{?}{=} f(x) + Af(x)h + \frac{1}{2}A(Af(x))h^2 + O(h^3). \tag{16}$$

Consider the scheme $\hat{X}$ with corrector Δ defined as in (14) with arbitrary coefficients $c_0(x), c_1(x)$ та $c_2(x)$. By the Taylor formula

$$\mathbb{E}_x f(\hat{X}_h) = \mathbb{E}_x f(\tilde{X}_h + \Delta_h) = \mathbb{E}_x f(\tilde{X}_h) + \mathbb{E}_x f'(\tilde{X}_h)\Delta_h + \frac{1}{2}\mathbb{E}_x f''(\tilde{X}_h)\Delta_h^2 +$$

$$+\frac{1}{6}\mathbb{E}_x f'''(\tilde{X}_h + \theta_h\Delta_h)\Delta_h^3, \quad \theta_h \in (0,1).$$

Consider each term separately.

$$\mathbb{E}_x f(\tilde{X}_h) = f(x) + Af(x)h + \\ + \frac{1}{2}\left(a^2(x)f''(x) + a(x)b(x)f'''(x) + \frac{1}{4}b^2(x)f^{(IV)}(x)\right)h^2 + O(h^3).$$

$$\mathbb{E}_x f'(\tilde{X}_h)\Delta_h =$$
$$= h^2 \mathbb{E}_x f'(\tilde{X}_h)\left(c_0(x) + c_1(x) H_h^{(1)}(x, \tilde{X}_h) + c_2(x) H_h^{(2)}(x, \tilde{X}_h)\right).$$

By (12) we have

$$\mathbb{E}_x f'(\tilde{X}_h) H_h^{(1)}(x, \tilde{X}_h) = \mathbb{E}_x f''(\tilde{X}_h),$$
$$\mathbb{E}_x f'(\tilde{X}_h) H_h^{(2)}(x, \tilde{X}_h) = \mathbb{E}_x f'''(\tilde{X}_h).$$

Then by the Ito formula

$$\mathbb{E}_x f'(\tilde{X}_h)\Delta_h =$$
$$= h^2 \left(c_0(x) f'(x) + c_1(x) f''(x) + c_2(x) f'''(x)\right) + O(h^3).$$

$$\frac{1}{2}\mathbb{E}_x f''(\tilde{X}_h)\Delta_h^2 =$$

$$= \frac{h^4}{2}\mathbb{E}_x f''(\tilde{X}_h)\left(c_0(x) + c_1(x)H_h^{(1)}(x,\tilde{X}_h) + c_2(x)H_h^{(2)}(x,\tilde{X}_h)\right)^2.$$

$\frac{1}{2}\mathbb{E}_x f''(\tilde{X}_h)\Delta_h^2$

$$\mathbb{E}_x f''(\tilde{X}_h) = O(1),$$

$$\mathbb{E}_x f''(\tilde{X}_h) H_h^{(1)}(x, \tilde{X}_h) = \mathbb{E}_x f'''(\tilde{X}_h) = O(1),$$

$$\mathbb{E}_x f''(\tilde{X}_h) H_h^{(2)}(x, \tilde{X}_h) = \mathbb{E}_x f^{(IV)}(\tilde{X}_h) = O(1),$$

$$\mathbb{E}_x f''(\tilde{X}_h) \left(H_h^{(1)}(x, \tilde{X}_h)\right)^2 = \mathbb{E}_x f''(\tilde{X}_h) \left(H_h^{(2)}(x, \tilde{X}_h) + \frac{1}{b(x)h}\right) =$$

$$= \mathbb{E}_x f^{(IV)}(\tilde{X}_h) + \frac{1}{b(x)h}\mathbb{E}_x f''(\tilde{X}_h) = O\left(\frac{1}{h}\right),$$

$\frac{1}{2}\mathbb{E}_x f''(\tilde{X}_h)\Delta_h^2$

$$\mathbb{E}_x f''(\tilde{X}_h) H_h^{(1)}(x,\tilde{X}_h) H_h^{(2)}(x,\tilde{X}_h) =$$

$$= \mathbb{E}_x f''(\tilde{X}_h)\left(H_h^{(3)}(x,\tilde{X}_h) + \frac{2}{b(x)h} H_h^{(1)}(x,\tilde{X}_h)\right) =$$

$$= \mathbb{E}_x f^{(V)}(\tilde{X}_h) + \frac{2}{b(x)h}\mathbb{E}_x f'''(\tilde{X}_h) = O\left(\frac{1}{h}\right),$$

$$\mathbb{E}_x f''(\tilde{X}_h)\left(H_h^{(2)}(x,\tilde{X}_h)\right)^2 =$$

$$= \mathbb{E}_x f''(\tilde{X}_h)\left(H_h^{(4)}(x,y) + \frac{4}{b(x)h} H_h^{(2)}(x,y) + \frac{2}{b^2(x)h^2}\right) =$$

$$= \mathbb{E}_x f^{(VI)}(\tilde{X}_h) + \frac{4}{b(x)h}\mathbb{E}_x f^{(IV)}(\tilde{X}_h) + \frac{2}{b^2(x)h^2}\mathbb{E}_x f''(\tilde{X}_h) =$$

$$= O\left(\frac{1}{h}\right) + \frac{2}{b^2(x)h^2}\mathbb{E}_x f''(\tilde{X}_h).$$

Last term

$$\frac{1}{6}\mathbb{E}_x f'''(\tilde{X}_h + \theta_h \Delta_h)\Delta_h^3 = O(h^3).$$

$$\mathbb{E}_x f(\hat{X}_h) = f(x) + Af(x)h + \frac{h^2}{2}\left[2c_0(x)f'(x) + \left(2c_1(x) + \frac{2c_2^2(x)}{b^2(x)} + a^2(x)\right)f''(x)\right.$$
$$\left. +(2c_2(x) + a(x)b(x))f'''(x) + \frac{1}{4}b^2(x)f^{(IV)}(x)\right] + O(h^3).$$

Recall that

$$A(Af(x)) = \left(a(x)a'(x) + \frac{1}{2}a''(x)b(x)\right)f'(x) +$$
$$+\left(a^2(x) + \frac{1}{2}a(x)b'(x) + a'(x)b(x) + \frac{1}{4}b(x)b''(x)\right)f''(x) +$$
$$+\left(a(x)b(x) + \frac{1}{2}b(x)b'(x)\right)f'''(x) + \frac{1}{4}b^2(x)f^{(IV)}(x).$$

$$c_0(x) = \frac{1}{2}a(x)a'(x) + \frac{1}{4}a''(x)b(x),$$
$$c_1(x) = \frac{1}{4}a(x)b'(x) + \frac{1}{2}a'(x)b(x) + \frac{1}{8}b(x)b''(x) - \frac{1}{16}(b'(x))^2,$$
$$c_2(x) = \frac{1}{4}b(x)b'(x).$$

If $a, \sigma \in C^6$ then

$$X_h = x + \int_0^h a(X_s)ds + \int_0^h \sigma(X_s)dW_s =$$

$$= x + \int_0^h \left(a(x) + \int_0^s Aa(X_r)dr + \int_0^s La(X_r)dW_r \right) ds +$$

$$+ \int_0^h \left(\sigma(x) + \int_0^s A\sigma(X_r)dr + \int_0^s L\sigma(X_r)dW_r \right) dW_s =$$

$$= x + a(x)\int_0^h ds + \sigma(x)\int_0^h dW_s + R_2 =$$

$$= x + a(x)h + \sigma(x)W_h +$$

$$+ Aa(x)\int_0^h\int_0^s dr ds + La(x)\int_0^h\int_0^s dW_r ds + A\sigma(x)\int_0^h\int_0^s dr dW_s + L\sigma(x)\int_0^h\int_0^s dW_r dW_s +$$

$$+ R_3.$$

Let Z_1 and Z_2 are two independent $N(0,1)$ random variables. Then

$$W_h = \sqrt{h} Z_1, \quad \int_0^h \int_0^s dW_r ds = \frac{1}{2} h^{\frac{3}{2}} \left(Z_1 + \frac{1}{\sqrt{3}} Z_2 \right).$$

$$\int_0^h \int_0^s dW_r dW_s = \frac{1}{2} \left(W_h^2 - h \right).$$

$$\int_0^h \int_0^s dW_r^i dW_s^k = ?$$

Thank you for attention!