On exponential convergence of a distance between solutions of an SDE with discontinuous drift

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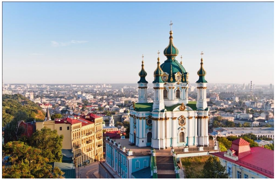


$$\begin{cases} d\varphi_t(x) = \left(-\lambda\varphi_t(x) + \alpha(\varphi_t(x))\right)dt, \ t > 0, \\ \varphi_0(x) = x, \end{cases}$$

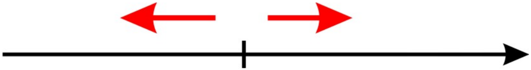
where $x \in \mathbb{R}$, $\lambda > 0$.

If α is Lip with constant L_α, then

$$|\varphi_t(x) - \varphi_t(y)| \le e^{(L_\alpha - \lambda)t}|x - y|.$$

$$\begin{cases} d\varphi_t(x) = \left(-\lambda\varphi_t(x) + \alpha(\varphi_t(x))\right)dt, \ t>0, \\ \varphi_0(x) = x, \end{cases}$$

Suppose $\alpha(x) = +1, \ x \geq 0, \ \alpha(x) = -1, \ x < 0.$



Then it is easily seen that there is no convergence, i.e.

$$|\varphi_t(x) - \varphi_t(y)| \nrightarrow 0, \ t \to \infty.$$

$$\begin{cases} d\varphi_t(x) = \big(-\lambda\varphi_t(x) + \alpha(\varphi_t(x)) + \beta(\varphi_t(x))\big)\,dt + \sigma(\varphi_t(x))dw_t, \ t > 0, \\ \varphi_0(x) = x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}$, $\lambda > 0$, $(w(t))_{t\geq 0}$ is a Wiener process.

(A1) (conditions on α) α is bounded, measurable and has compact support on $\mathbb{R}$;

(A2) (Lipschitz continuity of β) there exists $L_\beta > 0$ such that for all $x, y \in \mathbb{R}$,

$$|\beta(x) - \beta(y)| \leq L_\beta |x - y|;$$

(A3) (Lipschitz continuity of σ) there exists $L_\sigma > 0$ such that for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|;$$

(A4) (boundedness from below of σ) there exist $c_\sigma > 0$ such that for all $x \in \mathbb{R}$, $\sigma^2(x) \geq c_\sigma$.

Theorem

Let conditions (A1)-(A4) hold. Then there exists $\Lambda > 0$ *such that* $\forall\ \lambda > \Lambda\ \exists\ C_\lambda > 0, c_\lambda > 0\ \forall t \geq 0$:

$$\mathbb{E}(\varphi_t(x) - \varphi_t(y))^2 \leq C_\lambda (x-y)^2 e^{-c_\lambda t},$$

where $\varphi_t(x)$ *is a solution to equation (1) starting at the point* x.

Remark

Under the conditions (A1)-(A4) there exists a unique strong solution to equation (1). This result follows from [Zvonkin, 1974].

Proof

Let δ be such that

$$\frac{c_\sigma^2}{4\delta} > \|\alpha\|_\infty,$$

where $\|\alpha\|_\infty = \sup_{x\in\mathbb{R}} |\alpha(x)|$. Define a function γ such that $\gamma(x) = \|\alpha\|_\infty, x > \delta$; $\gamma(x) = -\|\alpha\|_\infty$, $x < -\delta$; $\gamma(x)$ is linear on $[-\delta, \delta]$.
Equation (1) can be rewritten as follows:

$$\begin{cases} d\varphi_t(x) = \left[-\lambda\varphi_t(x) + \big(\alpha(\varphi_t(x)) - \gamma(\varphi_t(x))\big) + \beta(\varphi_t(x)) + \gamma(\varphi_t(x))\right] dt + \\ \qquad\qquad\qquad\qquad\qquad\qquad \sigma(\varphi_t(x))dw_t, t \geq 0, \\ \varphi_0(x) = x. \end{cases}$$

Proof

Define

$$s(x) = \int_0^x \exp\left\{-2\int_0^y \frac{\alpha(z)-\gamma(z)}{\sigma^2(z)}dz\right\}dy,\ x \in \mathbb{R}.$$

Then for all $x \in \mathbb{R}$ there exists

$$s'(x) = \exp\left\{-2\int_0^x \frac{\alpha(z)-\gamma(z)}{\sigma^2(z)}dz\right\}.$$

Besides, the second derivative

$$s''(x) = 2\frac{\gamma(x)-\alpha(x)}{\sigma^2(x)}\exp\left\{-2\int_0^x \frac{\alpha(z)-\gamma(z)}{\sigma^2(z)}dz\right\}$$

is defined for almost all x w.r.t. Lebesgue measure.

Proof

For all $x, y \in \mathbb{R}$,

$$L_s^- |x-y| \le |s(x)-s(y)| \le L_s^+ |x-y|,$$

where

$$L_s^\pm = \exp\left\{ \pm 2 \int_{-\infty}^{\infty} \frac{|\alpha(z)-\gamma(z)|}{\sigma^2(z)} dz \right\},$$

$$\text{and} \quad L_s^- \le \|s'\|_\infty \le L_s^+.$$

Moreover, $|s'(x)-s'(y)| \le L_{s'} |x-y|$, where

$$L_{s'} = 2 \frac{\|\gamma - \alpha\|_\infty}{c_\sigma} L_s^+.$$

Since $s(\mathbb{R}) = \mathbb{R}$ and $s'(x) > 0$, $x \in \mathbb{R}$, $\Rightarrow$ $s \upuparrows$, $\exists\ s^{-1}(x)$, $x \in \mathbb{R}$, and for all $x, y \in \mathbb{R}$,

$$\frac{1}{L_s^+} |x-y| \le |s^{-1}(x) - s^{-1}(y)| \le \frac{1}{L_s^-} |x-y|.$$

Proof

Using Itô's formula we get

$$ds(\varphi_t(x)) = s'(\varphi_t(x))d\varphi_t(x) + \frac{1}{2}s''(\varphi_t(x))\sigma^2(\varphi_t(x))dt =$$
$$s'(\varphi_t(x))[-\lambda\varphi_t(x) + \beta(\varphi_t(x)) + \gamma(\varphi_t(x))]dt + s'(\varphi_t(x))\sigma(\varphi_t(x))dw_t. \quad (2)$$

Put $Y_t(x) = s(\varphi_t(x))$. Then equation (2) can be rewritten as follows

$$dY_t(x) = s'(s^{-1}(Y_t(x)))[-\lambda s^{-1}(Y_t(x)) + \beta(s^{-1}(Y_t(x)))+$$
$$\gamma(s^{-1}(Y_t(x)))]dt + s'(s^{-1}(Y_t(x)))\sigma(s^{-1}(Y_t(x)))dw_t.$$

If we prove the estimate for $\mathbb{E}(Y_t(x) - Y_t(y))^2$ then we get the statement of the theorem.

Proof

Applying Itŝ formula we obtain

$$d(Y_t(x)-Y_t(y))^2 =$$

$$2(Y_t(x)-Y_t(y))\Big(-\lambda\left[s'(s^{-1}(Y_t(x)))s^{-1}(Y_t(x))-s'(s^{-1}(Y_t(y)))s^{-1}(Y_t(y))\right]+$$
$$\left[s'(s^{-1}(Y_t(x)))\beta(s^{-1}(Y_t(x)))-s'(s^{-1}(Y_t(y)))\beta(s^{-1}(Y_t(y)))\right]+$$
$$\left[s'(s^{-1}(Y_t(x)))\gamma(s^{-1}(Y_t(x)))-s'(s^{-1}(Y_t(y)))\gamma(s^{-1}(Y_t(y)))\right]\Big)dt+$$
$$\big([s'(s^{-1}(Y_t(x)))\sigma(s^{-1}(Y_t(x)))-s'(s^{-1}(Y_t(y)))\sigma(s^{-1}(Y_t(y)))\big)^2dt+$$
$$2(Y_t(x)-Y_t(y))\big(s'(s^{-1}(Y_t(x)))\sigma(s^{-1}(Y_t(x)))-s'(s^{-1}(Y_t(y)))\sigma(s^{-1}(Y_t(y)))\big)dw_t.$$

Proof

Lemma

For all $x \in \mathbb{R}$,

$$(s'(s^{-1}(x))s^{-1}(x))' = \left(\frac{2(\gamma(u)-\alpha(u))}{\sigma^2(u)}u+1\right)\Bigg|_{u=s^{-1}(x)} > 1/2.$$

Proof. We have

$$(s'(s^{-1}(x))s^{-1}(x))' = \left(\frac{s''(u)}{s'(u)}u+1\right)\Bigg|_{u=s^{-1}(x)} =$$

$$\left(\frac{2(\gamma(u)-\alpha(u))}{\sigma^2(u)}u+1\right)\Bigg|_{u=s^{-1}(x)}$$

Proof

If $|u| = 0$,

$$\frac{2(\gamma(u) - \alpha(u))}{\sigma^2(u)} u + 1 = 1 > \frac{1}{2}.$$

If $|u| \geq \delta$,

$$\frac{2(\gamma(u) - \alpha(u))}{\sigma^2(u)} u + 1 = \frac{2(\operatorname{sgn} u \|\alpha\|_\infty - \alpha(u))}{\sigma^2(u)} u + 1 \geq 1 > \frac{1}{2}.$$

If $0 < u < \delta$, the condition

$$\frac{2(\gamma(u) - \alpha(u))}{\sigma^2(u)} u + 1 > \frac{1}{2}$$

is equivalent to $4(\gamma(u) - \alpha(u))u + \sigma^2(u) > 0$ or

$$\alpha(u) < \gamma(u) + \frac{\sigma^2(u)}{4u}.$$

Since $\|\alpha\|_\infty < \frac{c_\sigma^2}{4\delta}$ we get $\alpha(u) \leq \|\alpha\|_\infty < \frac{c_\sigma^2}{4\delta} \leq \gamma(u) + \frac{\sigma^2(u)}{4u}$.

Proof

Lemma

The function $s'(s^{-1}(x))\beta(s^{-1}(x)) : \mathbb{R} \to \mathbb{R}$ *satisfies a Lipschitz condition with constant* $(L_{s'}\|\beta\|_\infty + L_\beta L_s^+)/L_s^-$.

Lemma

The function $s'(s^{-1}(x))\gamma(s^{-1}(x)) : \mathbb{R} \to \mathbb{R}$ *satisfies a Lipschitz condition with constant* $(L_{s'}\|\gamma\|_\infty + L_\gamma L_s^+)/L_s^-$.

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Proof

Let us return to the proof of Theorem. By the mean value theorem, for each $t > 0$ there exists $\theta_t \in [Y_t(x) \wedge Y_t(y), Y_t(x) \vee Y_t(y)]$ such that

$$s'(s^{-1}(Y_t(x)))s^{-1}(Y_t(x)) - s'(s^{-1}(Y_t(y)))s^{-1}(Y_t(y)) = \left(\frac{2(\gamma(u) - \alpha(u))}{\sigma^2(u)} u \Big|_{u=s^{-1}(\theta_t)} + 1 \right) (Y_t(x) - Y_t(y)).$$

Proof

From Lemmas and Itô's formula we obtain

$$e^{\lambda t}(Y_t(x)-Y_t(y))^2 \le (s(x)-s(y))^2 +$$
$$\lambda\int_0^t e^{\lambda u}\left(1-2\left(\frac{2(\gamma(v)-\alpha(v))}{\sigma^2(v)}v\Big|_{v=s^{-1}(\theta_u)}+1\right)\right)(Y_u(x)-Y_u(y))^2du+$$
$$H\int_0^t e^{\lambda u}(Y_u(x)-Y_u(y))^2du+M_t,\ t\ge 0,$$

where

$$H=\frac{L_{s'}(\|\beta\|_\infty+\|\gamma\|_\infty)+L_s^+(L_\beta+L_\gamma)}{L_s^-}+\left(\frac{L_{s'}\|\sigma\|_\infty+L_\sigma L_s^+}{L_s^-}\right)^2,$$

$$M_t=2\int_0^t e^{\lambda u}[s'(s^{-1}(Y_u(x)))\sigma(s^{-1}(Y_u(x)))-s'(s^{-1}(Y_u(y)))\sigma(s^{-1}(Y_u(y))]\times$$
$$(Y_u(x)-Y_u(y))dw_u.$$

Proof

Taking the expectation we obtain

$$\mathbb{E}e^{\lambda t}(Y_t(x)-Y_t(y))^2 \le (s(x)-s(y))^2 + H\int_0^t \mathbb{E}e^{\lambda u}(Y_u(x)-Y_u(y))^2 du.$$

The Grönwall-Bellman inequality gives us

$$\mathbb{E}e^{\lambda t}(Y_t(x)-Y_t(y))^2 \le (s(x)-s(y))^2 e^{Ht}, t \ge 0.$$

Multiplying the both sides of this inequality by $e^{-\lambda t}$. We obtain

$$\mathbb{E}(Y_t(x)-Y_t(y))^2 \le (s(x)-s(y))^2 e^{(H-\lambda)t},\ t \ge 0.$$

Put $\Lambda = H$. Then for all $\lambda > \Lambda$, $H-\lambda < 0$. Since $Y_t(x) = s(\varphi_t(x))$,

$$\mathbb{E}(\varphi_t(x)-\varphi_t(y))^2 \le \frac{1}{(L_s^-)^2}(x-y)^2 e^{(H-\lambda)t},\ t \ge 0.$$

Setting $C_\lambda = \frac{1}{(L_s^-)^2}$ and $c_\lambda = \lambda - H$ we get the statement of Theorem.

Consider a d-dimensional SDE

$$\begin{cases} d\varphi_t(x) = \left(-\lambda\varphi_t(x) + \alpha(\varphi_t(x))\right)dt + \sum_{k=1}^{m} \sigma_k(\varphi_t(x))dw_k(t), t \geq 0, \\ \varphi_0(x) = x, \end{cases} \quad (3)$$

where $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, $\lambda > 0$, $(w(t))_{t\geq 0} = (w_1(t), \dots, w_m(t))_{t\geq 0}$ is a standard m-dimensional Wiener process, $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_1, \dots, \sigma_m) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are measurable functions.

Denote

$$S = \{x \in \mathbb{R}^d : x^d = 0\},$$

$$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^d > 0\}, \quad \mathbb{R}^d_- = \{x \in \mathbb{R}^d : x^d < 0\}.$$

Conditions on coefficient α

(A1) The function α is bounded.

(A2) *Lipschitz continuity on* $\mathbb{R}^d_\pm$: There exists $\tilde{K}_\alpha > 0$ such that for all $x, y \in \mathbb{R}^d_+$ or $x, y \in \mathbb{R}^d_-$,

$$|\alpha(x) - \alpha(y)| \le \tilde{K}_\alpha |x - y|.$$

It follows from (A2) that for all $\tilde{x} \in S$, there exist limits

$$\alpha_+(\tilde{x}) := \lim_{\substack{x \to \tilde{x}, \\ x \in \mathbb{R}^d_+}} \alpha(x), \alpha_-(\tilde{x}) := \lim_{\substack{x \to \tilde{x}, \\ x \in \mathbb{R}^d_-}} \alpha(x).$$

Conditions on coefficient σ

(B1) The function σ is bounded.

(B2) *Lipschitz continuity on* $\mathbb{R}^d$: There exists $\tilde{K}_\sigma > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|\sigma(x) - \sigma(y)| \leq \tilde{K}_\sigma |x - y|.$$

(B3) *Uniform ellipticity*: There exists a constant $B_\sigma > 0$ such that for all $x \in \mathbb{R}^d$, $\theta \in \mathbb{R}^d$,

$$\theta^* \sigma(x) \sigma^*(x) \theta \geq B_\sigma |\theta|^2.$$

Under these assumptions there exists a unique strong solution to (3). This result follows from [Veretennikov, 1981].

Remark

Note that since σ is uniformly elliptic, the solution to equation (3) spends zero time on S. So we can redefine the function α on S in an arbitrary way.

Theorem 1

Let $\lambda > 0$ and conditions (A1), (A2), (B1), (B2), (B3) hold. Then $\exists \Lambda = \Lambda(\alpha, \sigma) > 0$ $\forall \lambda > \Lambda$ $\exists$ $C_1 = C_1(\lambda, \alpha, \sigma) > 0$ $\exists C_2 = C_2(\lambda, \alpha, \sigma) > 0 :$ $\forall x, y \in \mathbb{R}^d$, and

$$\mathbb{E}|\varphi_t(y) - \varphi_t(x)| \leq C_1 e^{-C_2 t}|y - x|. \tag{4}$$

Here $\varphi_t(x)$ is a solution to equation (3) starting at the point x.

Proof

It can be checked (see [Aryasova & Pilipenko, 2017]) that for all $p > 0$,

$$P\left\{\forall t \geq 0 : \varphi_t(\cdot) \in W^1_{p,loc}(\mathbb{R}^d, \mathbb{R}^d)\right\} = 1,$$

and for $x, y \in \mathbb{R}^d$,

$$\varphi_t(y) - \varphi_t(x) = \int_0^1 \big(\nabla\varphi_t(x + \xi(y - x)), y - x\big) d\xi,$$

where $\nabla\varphi_t(\cdot)$ is the distributional derivative.

$$\mathbb{E}|\varphi_t(y) - \varphi_t(x)| \leq \int_0^1 \mathbb{E}\Big|(\nabla\varphi_t(x + \xi(y - x)), y - x)\Big| d\xi \leq |y - x| \cdot \sup_{z \in \mathbb{R}^d} \mathbb{E}|\nabla\varphi_t(z)|.$$

Proof

If $\alpha_+(x) = \alpha_-(x), x \in S$, then $Y_t(x) := \nabla \varphi_t(x)$ is a solution to the SDE

$$\begin{cases} dY_t(x) = [-\lambda + \nabla \alpha(\varphi_t(x))] Y_t(x) dt + \sum_{k=1}^{m} \nabla \sigma_k(\varphi_t(x)) Y_t(x) dw_k(t), t \geq 0, \\ Y_0(x) = E, \end{cases}$$

where E is a $d \times d$-identity matrix. This formula is well known when $\alpha, \sigma \in C^1(\mathbb{R}^d)$. For Lipschitz continuous functions α and σ the result can be found in [Bouleau & Hirsch, 2010], Th. 3.3.1.

Proof

If $\alpha_+(x) \neq \alpha_-(x), x \in S$, then the distributional derivative of α is equal to

$$\nabla\alpha(x) + D(x)\delta_S, x \in \mathbb{R}^d,$$

where δ_S is the standard surface measure on S,

$$D(x) = \begin{pmatrix} 0 & \cdots & 0 & \alpha_+^1(x) - \alpha_-^1(x) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_+^d(x) - \alpha_-^d(x) \end{pmatrix}, \ x \in S.$$

$$Y_t(x) = E + \int_0^t [-\lambda + \nabla\alpha(\varphi_s(x))] Y_s(x) ds +$$
$$\int_0^t D(\varphi_s(x)) Y_s(x) \delta_S(\varphi_s(x)) ds + \sum_{k=1}^m \int_0^t \nabla\sigma_k(\varphi_s(x)) Y_s(x) dw_k(s).$$

Proof

By

$$\int_0^t D(\varphi_s(x))Y_s(x)\delta_S(\varphi_s(x))ds$$

we mean the integral

$$\int_0^t D(\varphi_s(x))Y_s(x)dL_s^S(\varphi(x)) = \int_0^t D(\varphi_s(x))Y_s(x)dL_s^0(\varphi^d(x)),$$

We obtain [Aryasova & Pilipenko, 2017]

$$Y_t(x) = E + \int_0^t [-\lambda + \nabla\alpha(\varphi_s(x))]Y_s(x)ds +$$
$$\int_0^t D(\varphi_s(x))Y_s(x)dL_s^0(\varphi^d(x)) + \sum_{k=1}^m \int_0^t \nabla\sigma_k(\varphi_s(x))Y_s(x)dw_k(s),$$

where

Proof

$$L_t^S(\varphi(x)) = L_t^0(\varphi^d(x)) = \underset{\varepsilon\downarrow 0}{\text{l.i.m.}} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|\langle \varphi_s(x), e_d\rangle|\le\varepsilon} ds =$$

$$\underset{\varepsilon\downarrow 0}{\text{l.i.m.}} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{|\varphi_s^d(x)|\le\varepsilon} ds,$$

where $e_d = (0,0,\dots,0,1)$. The existence and uniqueness of the solution follows from [Protter, 2004], Ch. V, Th. 7.

Proof

Define

$$h(t) = (2\lambda - 2K_\alpha - K_\sigma^2)t - 2|D|L_t^0(\varphi^d(x)),$$

where

$$K_\alpha := \operatorname{ess\ sup}_{x\in\mathbb{R}^d} |\nabla\alpha(x)|, K_\sigma := \operatorname{ess\ sup}_{x\in\mathbb{R}^d} |\nabla\sigma(x)|.$$

$$\|D\|_\infty = \sup_{x\in S} |D(x)|.$$

Lemma 1

For all $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E} e^{h(t)} |Y_t(x)|^2 \leq d.$$

Proof

By the Hölder inequality

$$\mathbb{E}|Y_t(x)| \le \left(\mathbb{E}e^{h(t)}|Y_t(x)|^2\right)^{1/2}\left(\mathbb{E}e^{-h(t)}\right)^{1/2} \le$$

$$d^{1/2}e^{(-\lambda+K_\alpha+\frac{1}{2}K_\sigma^2)t}\left(\mathbb{E}e^{2|D|L_t^0(\varphi^d(x))}\right)^{1/2}.$$

Proof

The process $(\varphi_t(x))_{t\geq 0}$ is a multidimensional semimartingale,

$$\widetilde{L}_t^0(\varphi^d(x)) = 2(\varphi_t^d(x))^+ - 2(x^d)^+ - 2\int_0^t \mathbb{1}_{\varphi_s^d(x)>0} d\varphi_s^d(x),\ t \geq 0.$$

It is also local time and (see [Revuz & Yor, 1999], Ch. VI, Corollary (1.9) and Th. (1.7))

$$\widetilde{L}_t^0(\varphi^d(x)) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\varphi_s^d(x)\leq\varepsilon} d\langle\varphi^d(x), \varphi^d(x)\rangle_s =$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\varphi_s^d(x)\leq\varepsilon} \sum_{k=1}^d \left(\sigma_k^d(\varphi_s(x))\right)^2 ds,$$

$$L_t^0(\varphi^d(x)) \leq \frac{1}{B_\sigma}\widetilde{L}_t^0(\varphi^d(x)).$$

We get

$$\sup_{x\in\mathbb{R}^d} \leq C_1 e^{-C_2 t}$$

holds. Hence,

$$\mathbb{E}|\varphi_t(y)-\varphi_t(x)| \leq C_1 e^{-C_2 t}|y-x|.$$

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Thank you