Completing partially observed point patterns

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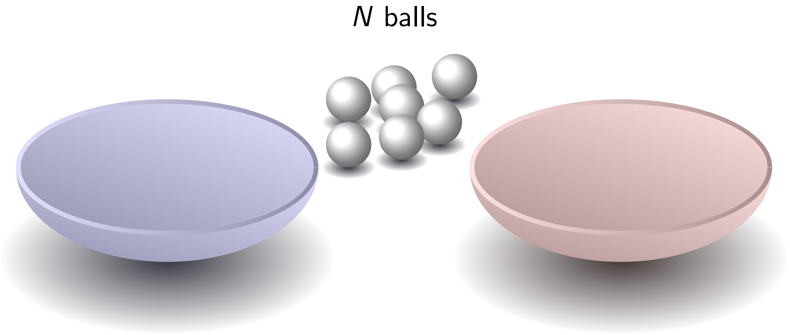
based on joint work with Hans Zessin and Benjamin Nehring

Potsdam, February 15, 2018

Warm-up for splitting
Direct problem



- compute $\mathcal{L}(N_q, N_q^*)$
- compute $\mathcal{L}(N_q^*|N_q) =: \Upsilon(N_q, \cdot)$

Warm-up for splitting

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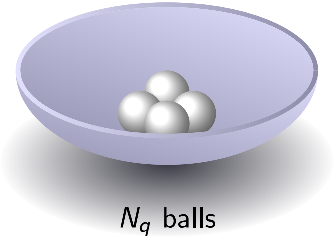

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Indirect problem

Now $\mathcal{L}(N)$ unknown

N_q balls

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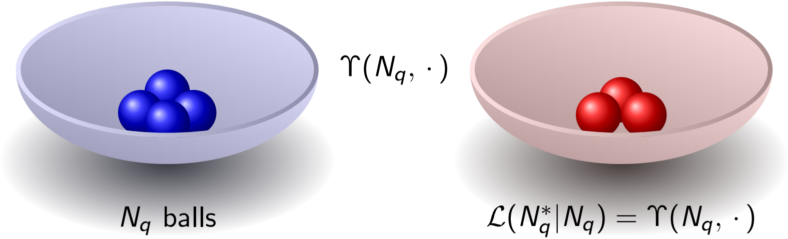

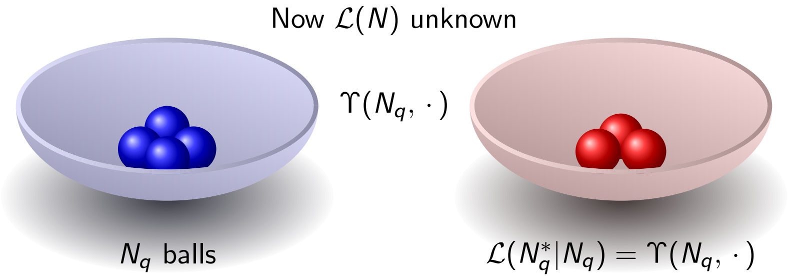

Which N satisfy the splitting equation

$$\mathbf{E} f(N_q, N^*_q) = \mathbf{E}\Big[\mathbf{E}\big[f(N_q, N^*_q)\big| N_q\big]\Big] = \iint f(k, l)\Upsilon(k, \mathrm{d}l)\mathbb{P}_q(\mathrm{d}l)$$

Warm-up for splitting

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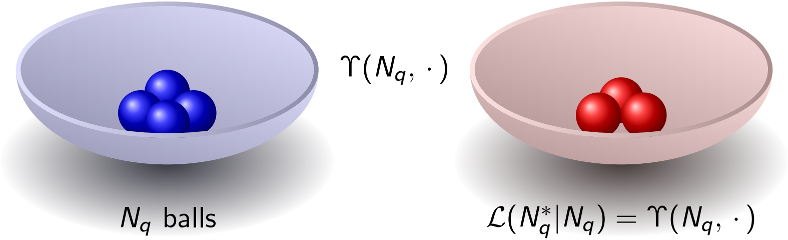

Which N satisfy the (dependent) convolution equation

$$\mathbf{E}g(N) = \mathbf{E}\Big[\mathbf{E}\big[g(N_q + N_q^*)\big|N_q\big]\Big] = \iint g(k+l)\Upsilon(k, \mathrm{d}l)\mathbb{P}_q(\mathrm{d}k)$$

N_q is observed, conditional law of N_q^* given $N_q = k$ is ...

Example 1 $\Upsilon(k, \cdot) = \mathrm{Poi}(1-q)$;

then $N \sim \mathrm{Poi}(1)$ and this is the only choice!

Example 2 $\Upsilon(k, \cdot) = \mathrm{Bin}\left(n-k, \frac{p(1-q)}{1-pq}\right)$;

then $N \sim \mathrm{Bin}(n, p)$

Example 3 $\Upsilon(k, \cdot) = \mathrm{NegBin}\big(n+k, p(1-q)\big)$;

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Integration by parts formula

N satisfies IBPF for some function $\pi : \mathbb{N}_0 \to \mathbb{R}_+$, if for bounded f,
$\mathbf{E}[Nf(N)] = \mathbf{E}[\pi(N)f(N+1)]$.

Problem

Given π, what is the distribution of N?

Examples

1. $\pi(k) = 1$ for all $k \in \mathbb{N}_0$, then $N \sim \text{Poi}(1)$
2. $\pi(k) = z(n-k)$ for $k = 0, 1, \dots, n$, then $N \sim \text{Bin}\left(n, \frac{z}{1+z}\right)$;
3. $\pi(k) = z(n+k)$ for $k \in \mathbb{N}_0$, then $N \sim \text{NegBin}(n, z)$.

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How to determine the law of N?

1. choose $f = 1_{\{k\}}$, then $k\mathbb{P}(N = k) = \pi(k)\mathbb{P}(N = k-1)$, $k = 1, 2, \ldots$
2. $\mathbb{P}(N = k) = \dfrac{\pi(k)\cdots\pi(1)}{k!}\mathbb{P}(N = 0)$
3. $\mathbb{P}(N = k) = \exp(-\pi)\dfrac{\pi[k]}{k!}$

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q-Splitting kernel

If N satisfies IBPF(π), then $\Upsilon(k, \cdot)$ satisfies IBPF($(1-q)\pi(k+\cdot)$).

N_q

N_q satisfies an IBPF. If N satisfies IBPF(π), then that function is the "average" $q\sum_j \pi(k+j)\Upsilon(k,j)$.

Equivalent statements

1. N satisfies IBPF(π)
2. N satisfies the splitting equation
3. N satisfies the (dependent) convolution equation

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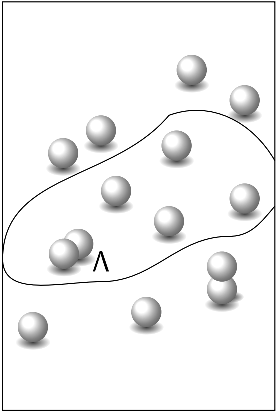
Point processes

A point process is a random point measure (r.v. N is now $\{N_\Lambda\}_\Lambda$).



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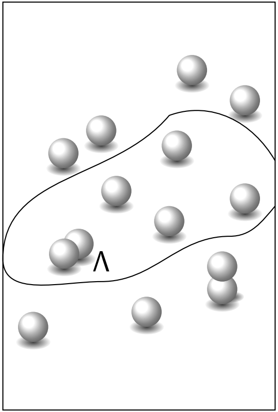
Poisson process

- $N_\Lambda \sim \mathrm{Poi}\big(m(\Lambda)\big)$
- given N_Λ, points are distributed iid
- $\Lambda \cap \Lambda' = \emptyset$, then N_Λ and $N_{\Lambda'}$ independent



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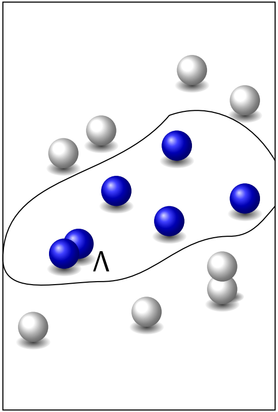
Gibbs process

- defined locally by
$$G(\,\cdot\,|\hat{\mathcal{F}}_\Lambda)(\mu) := \frac{e^{-V(\,\cdot\,|\mu_{\Lambda^c})}}{Z_{\Lambda,\mu}} P_\Lambda$$
- existence? uniqueness?



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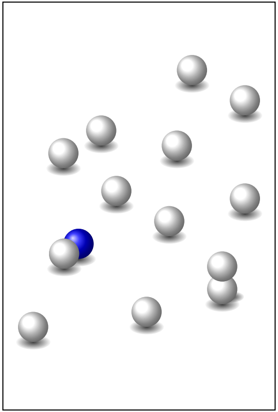
Gibbs process Nguyen, Zessin 79

DLR equations equivalent to IBPF

$$\iint h(x,\mu)\mu(\mathrm{d}x)G(\mathrm{d}\mu)$$
$$= \iint h(x,\mu+\delta_x)\, \mathrm{e}^{-V(x,\mu)}\, m(\mathrm{d}x)G(\mathrm{d}\mu)$$



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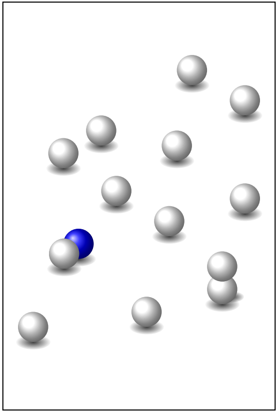
Papangelou process

replace $\mathrm{e}^{-V(\,\cdot\,,\mu)}\,\mathrm{d}m$ by $\pi(\mu,\,\cdot\,)$

$$\iint h(x,\mu)\mu(\mathrm{d}x)P(\mathrm{d}\mu)$$

$$= \iint h(x,\mu+\delta_x)\pi(\mu,\mathrm{d}x)P(\mathrm{d}\mu)$$



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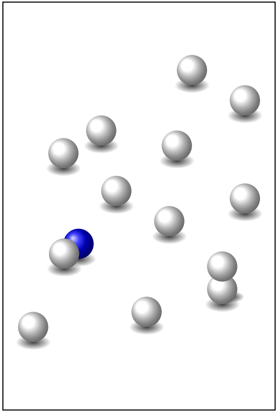
Papangelou process, examples

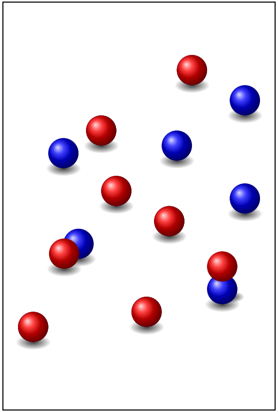
- $\pi(\mu, \cdot\,) = m$
- $\pi(\mu, \cdot\,) = z(m - \mu)$
- $\pi(\mu, \cdot\,) = z(m + \mu)$

Each N_Λ satisfies an IBPF.



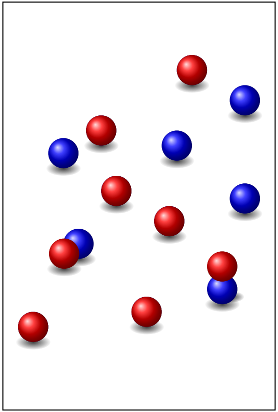
q-splittings and thinnings

- choose colour for each "ball" independently, e.g. blue with probability q
- joint law of red and blue point configurations is q-splitting $\mathcal{S}^q$
- marginals are thinnings
- conditional law of red point configuration given blue point configuration is splitting kernel



Examples

1. Poisson process P_m:
   $\mathsf{P}_m^q = \mathsf{P}_{qm}$, $\mathcal{S}^q = \mathsf{P}_{qm} \otimes \mathsf{P}_{(1-q)m}$
2. Difference process $\mathsf{D}_{z,m}$:
   $\mathsf{D}_{z,m}^q = \mathsf{D}_{\frac{qz}{1+(1-q)z},m}$,
   $\Upsilon(\nu, \cdot) = \mathsf{D}_{(1-q)z,m-\nu}$
3. Sum process $\mathsf{S}_{z,m}$:
   $\mathsf{S}_{z,m}^q = \mathsf{S}_{\frac{qz}{1-(1-q)z},m}$,
   $\Upsilon(\nu, \cdot) = \mathsf{S}_{(1-q)z,m+\nu}$



Splitting kernel ((1) Karr; (2) Nehring, R, Zessin)

1. If P is finite, then $\Upsilon(\nu, \cdot\,) \sim (1-q)^N P^!_\nu$.
2. If P satisfies IBPF for π, then $\Upsilon(\nu, \cdot\,)$ satisfies IBPF for $(1-q)\pi(\nu + \cdot\,, \cdot\,)$.

Thinnings (Nehring, R, Zessin)

If P satisfies IBPF for π, then also P^q does for
$q \int \pi(\mu+\nu, \cdot\,)\Upsilon(\mu, \mathrm{d}\nu)$.

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Characterization (Nehring, R, Zessin)

The following statements are equivalent

1. P solves IBPF for π;
2. P satisfies the splitting equation

$$\mathcal{S}_P(h) = \iint h(\mu,\nu)\Upsilon(\mu,\mathrm{d}\nu)P^q(\mathrm{d}\mu)$$

3. P satisfies the (dependent) convolution equation

$$P(\phi) = \iint \phi(\mu+\nu)\Upsilon(\mu,\mathrm{d}\nu)P^q(\mathrm{d}\mu)$$

Uniqueness of solutions of splitting and convolution equation

Uniqueness of solutions of IBPF implies uniqueness for splitting and convolution equation.

α-condensability (Ambartzumian)

P is α-condensable if there exists Q such that $Q^{1/\alpha} = P$.

- if P solves IBPF for σ, condensability "reduces" to solving $\sigma(\nu, \cdot) = q \int \pi(\nu + \mu, \cdot) \Upsilon(\nu, \mathrm{d}\mu)$

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Spatial birth processes

Let P solve IBPF for π, $(N_q)_q$ (point measure valued) process such that transition kernel

$$p_{q,q'}(\mu, \cdot\,) = \Upsilon_{q,q'}(\mu, \cdot\,)$$

solves an IBPF for $(q' - q) \int \pi(\mu + \kappa, \cdot\,)\Upsilon^{q'}(\mu, \mathrm{d}\kappa)$.

- law of N_q is P^q
- $q \mapsto N_q$ increasing

Cox processes and condensability

P is a Cox process iff $q \mapsto N^q$ extends to $\mathbb{R}_+$.

- (otherwise only on $[0, T]$ for some $T \geq 1$)
- exit space of pure birth process given by mixtures of Poisson pure birth

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Negative binomial process (Gregoire 84)

$P \sim \mathcal{BN}(r, \nu)$ if P has Laplace transform

$$\mathcal{L}(f) = \left[1 + \int 1 - e^{-f} \, \mathrm{d}\nu\right]^{-r}.$$

- shares only one-dimensional marginals with sum process

IBPF

If ν is finite, then $P \sim \mathcal{BN}(r, \nu)$ satisfies IBPF with kernel

$$\pi(\mu, \mathrm{d}x) = \frac{r + |\mu|}{1 + |\nu|} \nu(\mathrm{d}x).$$

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Splitting

If ν is finite, then then the q-splitting ernel of $P \sim \mathcal{BN}(r, \nu)$ is

$$\Upsilon(\mu, \cdot\,) = \mathcal{BN}\left(r + |\mu|, \frac{1-q}{1+q|\nu|}\nu\right).$$

log-Gauss Cox process (Coles, Jones 91; Møller, Syversveen, Waagepetersen 98)

$P \sim \mathrm{lGC}(\mu, c)$ if P is a Cox process driven by e^Y, where Y is Gaussian with mean μ and covariance c.

Reduced Palm measures of log-Gauss Cox processes (Cœurjolly, Møller, Waagepetersen 15)

If $P \sim \mathrm{lGC}(\mu, c)$, then its reduced Palm measure $P^!_\nu$ for a simple and finite point measure ν is log-Gauss Cox with parameters

$$\mu + \int c_{x,\cdot}\, \nu(\mathrm{d}x), \quad c.$$

Thinning

If $P \sim \mathrm{lGC}(\mu, c)$, then its q-thinning is log-Gauss Cox $P \sim \mathrm{lGC}(\mu + \ln q, c)$.

Splitting

If $P \sim \mathrm{lGC}(\mu, c)$ a finite process, then its q-splitting kernel is

$$\Upsilon(\nu, \cdot\,) = \frac{(1-q)^N}{Z_\nu} P^!_\nu,$$

i.e. is log-Gauss Cox process with parameters

$$\mu + \int c_{x,\cdot}\,\nu(\mathrm{d}x) + \ln(1-q), \quad c.$$

Gauss-Poisson process (Newman 70; Milne, Westcott 72; Macchi 72)

$P \sim \mathrm{GP}(\lambda, H)$ if P has Laplace fransform

$$\mathcal{L}(f) = \exp\left(- \int 1 - e^{-f(x)}\,\lambda(\mathrm{d}x) \right.$$
$$\left. + \frac{1}{2} \iint \left[1 - e^{-f(x)}\right]\left[1 - e^{-f(y)}\right] H(\mathrm{d}x, \mathrm{d}y) \right).$$

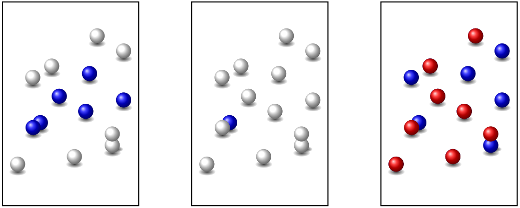
Thinning (Milne, Westcott 72)

If $P \sim \mathrm{GP}(\lambda, H)$, then its q-thinning is Gauss-Poisson $P^q \sim \mathrm{GP}(q\lambda, q^2 H)$.

Extensions

- replace independent thinning by dependent thinning
  - pairs of thinning and condensing kernels
  - integration by parts
- relation between birth-and-death process and thinned birth-and-death process

- described point processes in three different ways: DLR equations, integration by parts, splittings/dependent convolutions



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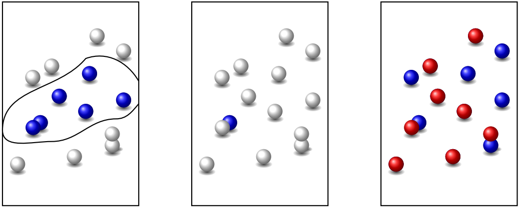

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