Noise sensitivity of Lévy driven SDEs: estimates and applications

T. Kosenkova[1]
joint work with J. Gairing[2], M. Högele[3], A. Kulik[4]

[1]Uni Potsdam [2]LMU Munich [3]Uniandes, Bogota [4]Institut of Mathematics, Kyiv

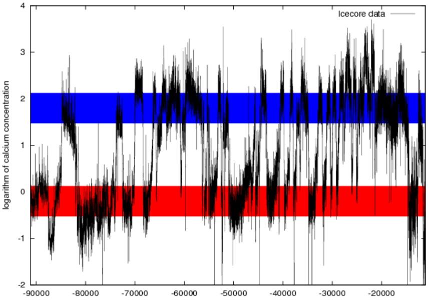
15/02/2018
Potsdam-Toulouse Workshop

Outline

Climate proxy data from Greenland ice core record

Climate proxy for the yearly average temperature of the oceanic flow during the last glacial period

log Calcium Signal



Fast transitions between cold and warm metastable states.

Model selection problem for climate dynamics

- It was proposed to use a Lévy process as a stochastic component of the above model

$$dX(t) = \left(-U'\right) dt + dZ(t),$$

where $U : \mathbb{R} \to \mathbb{R}$ is a double-well potential with two local minima, which corresponds to two climate meta-stable states, and $Z(t)$ is a Lévy process.

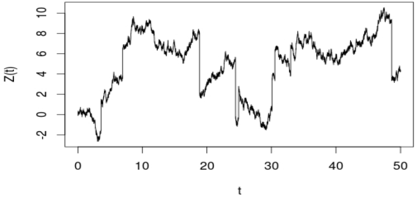
- Ditlevsen'99 conjectured that the jump component has a polynomial decay and, by means of the residual analysis, he had obtained the estimate for the decay parameter.
- In Hein, Imkeller, Pavlyukevich'2009 the problem of model selection in this setting was tackled by means of another technique based on the limiting behaviour of p-variations for the process X. The fitting test for the law of the noise component was established and also gave an approximation of the decay parameter.
- No quantification of the "distance between data and a model" !

- Ditlevsen'99 conjectured that the jump component has a polynomial decay and, by means of the residual analysis, he had obtained the estimate for the decay parameter.
- In Hein, Imkeller, Pavlyukevich'2009 the problem of model selection in this setting was tackled by means of another technique based on the limiting behaviour of p-variations for the process X. The fitting test for the law of the noise component was established and also gave an approximation of the decay parameter.
- No quantification of the "distance between data and a model" !

Lévy process: definition

Definition A standard Lévy process $(Z(t)), t > 0$ is a stochastic process with values in $\mathbb{R}$

- $Z(0) = 0$
- independent increments $Z(t_n) - Z(t_{n-1}); \ldots; Z(t_2) - Z(t_1); Z(t_1)$
- stationary increments $Z(t) - Z(s) \sim Z(t - s)$
- $t \to Z(t)$ is almost surely right continuous with left limits (cádlág)



A realisation of Lévy process[1]

[1] L. Schreiter

Lévy-Khinchin characterisation of the law

Theorem

Every Lévy process $(Z(t))_{t\geq 0}$ is uniquely determined by a characteristic triplet (a, b^2, Π)

- drift $a \in \mathbb{R}$
- covariance $b^2 > 0$ with $b \in \mathbb{R}^+$
- "Lévy measure" $\Pi : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $\int_{\mathbb{R}} \min(|u|^2, 1) < \infty$

by means of its *cumulant function* ψ

$$\mathbb{E}e^{iuZ(t)} = e^{t\psi(u)}, \; u \in \mathbb{R}$$

linked to the characteristic triplet via the Lévy-Khinchin formula

$$\psi(z) = iaz - \frac{1}{2}b^2z^2 + \int_{\mathbb{R}} \left[e^{izu} - 1 - izu\mathbf{1}_{|u|\leq 1}\right]\Pi(du) = iaz - \frac{1}{2}b^2z^2 + \int_{\mathbb{R}} \left[e^{izu} - 1 - izu\right]\Pi^H(du) + \int_{\mathbb{R}} \left[e^{izu} - 1\right]\Pi^T(du), \quad z \in \mathbb{R}. \tag{1}$$

Lévy-Itô representation of the path for Lévy process

Theorem For any Lévy process $(Z(t))_{t \geqslant 0}$ with characteristic triplet (a, b^2, Π)

$$Z(t) = at + bB(t) + Z^H(t) + Z^T(t) \quad \text{a.s. for all } t > 0,$$

where

- B is a standard Brownian motion in $\mathbb{R}$
- Z^T is a compound Poisson Process in $\mathbb{R}$ with the Lévy measure $\Pi^T(\cdot) := \Pi(\{u \in \mathbb{R} : \ |u| > 1\} \cap \cdot)$
- Z^H is a pure jump process with the Lévy measure $\Pi^H(\cdot) := \Pi(\{u \in \mathbb{R} : \ |u| \leq 1\} \cap \cdot)$ of possibly infinite intensity with jumps bounded by 1.

The model selection problem for Lévy driven SDEs with additive noise

Consider the SDE with Lévy noise, i.e. the process of the following type

$$X(t) = x + \int_0^t V\left(X(s)\right) ds + Z(t) \quad t \geq 0,\ x \in \mathbb{R}, \tag{2}$$

with a "good" function $V : \mathbb{R} \to \mathbb{R}$ and a Lévy process $(Z(t))_{t\geq 0}$, which is uniquely determined by a characteristic triplet (a, b^2, Π).

How the law of the process X depends on the characteristic triplet?

The model selection problem for Lévy driven SDEs with additive noise

Consider the SDE with Lévy noise, i.e. the process of the following type

$$X(t) = x + \int_0^t V\left(X(s)\right) ds + Z(t) \quad t \geq 0,\ x \in \mathbb{R}, \tag{2}$$

with a "good" function $V : \mathbb{R} \to \mathbb{R}$ and a Lévy process $(Z(t))_{t\geq 0}$, which is uniquely determined by a characteristic triplet (a, b^2, Π).

How the law of the process X depends on the characteristic triplet?

Heads and tails for Lévy measure

- For a given Lévy measure Π and given $r > 0$ we make a decomposition of Π into two σ-finite measures $\Pi^{H,r}$ and $\Pi^{T,r}$ of the form $\Pi = \Pi^{H,r} + \Pi^{T,r}$, such that
- the total mass of $\Pi^{T,r}$ is r,
- there exists $\varepsilon^r \geq 0$ for which $\mathrm{supp}(\Pi^{H,r}) = \{u : |u| \leq \varepsilon^r\}$ and $\mathrm{supp}(\Pi^{T,r}) = \{u : |u| \geq \varepsilon^r\}$.
- For a Lévy measure Π and $r > 0$ define the probability measure

$$\pi^r = 1/r\Pi^{T,r}.$$

Heads and tails for Lévy measure

- For a given Lévy measure Π and given $r > 0$ we make a decomposition of Π into two σ-finite measures $\Pi^{H,r}$ and $\Pi^{T,r}$ of the form $\Pi = \Pi^{H,r} + \Pi^{T,r}$, such that
- the total mass of $\Pi^{T,r}$ is r,
- there exists $\varepsilon^r \geq 0$ for which $\mathrm{supp}(\Pi^{H,r}) = \{u : |u| \leq \varepsilon^r\}$ and $\mathrm{supp}(\Pi^{T,r}) = \{u : |u| \geq \varepsilon^r\}$.
- For a Lévy measure Π and $r > 0$ define the probability measure

$$\pi^r = 1/r\Pi^{T,r}.$$

Heads and tails for Lévy measure

- For a given Lévy measure Π and given $r > 0$ we make a decomposition of Π into two σ-finite measures $\Pi^{H,r}$ and $\Pi^{T,r}$ of the form $\Pi = \Pi^{H,r} + \Pi^{T,r}$, such that
- the total mass of $\Pi^{T,r}$ is r,
- there exists $\varepsilon^r \geq 0$ for which $\operatorname{supp}(\Pi^{H,r}) = \{u : |u| \leq \varepsilon^r\}$ and $\operatorname{supp}(\Pi^{T,r}) = \{u : |u| \geq \varepsilon^r\}$.
- For a Lévy measure Π and $r > 0$ define the probability measure

$$\pi^r = 1/r\Pi^{T,r}.$$

Coupling distance

- Recall that on a metric space (S, d) the *Wasserstein-Kantorovich-Rubinstein metric* of order 2, between two probability measures μ, ν on (S, d) is defined by

$$W_{2,d}(\mu, \nu) := \inf_{(\xi,\eta)\in\mathcal{C}(\mu,\nu)} \left(\mathbf{E}\, d^2(\xi,\ \eta)\right)^{1/2},$$

  where $\mathcal{C}(\mu, \nu)$ denotes the set of all (μ, ν)-couplings.
- Let the metric ρ on $\mathbb{R}$ be defined by

$$\rho(x, y) = |x - y| \wedge 1.$$

- Define

$$T_r(\Pi_1, \Pi_2) := r^{1/2} W_{2,\rho}(\pi_1^r, \pi_2^r), r > 0,$$
$$T(\Pi_1, \Pi_2) := \sup_{r>0} T_r(\Pi_1, \Pi_2).$$

  We shall call T_r and T *coupling (semi)distances* on the set of Lévy measures.
- **Proposition** The function (T_r) T is a (semi)metric on the set of Lévy measures. (Gairing, Högele, K., Kulik'15)

One illustrative example: the α-stable measure

Let us consider the one sided α-*stable* Lévy measure which is defined as follows

$$\Pi(du) = \alpha c u^{-\alpha-1}\, du, \quad u > 0, \qquad \text{for } \alpha \in (0,2), c > 0.$$

Proposition

1. Let $\Pi_j, j = 1,2$ be two one-sided α-stable measures with the same shape parameter α and different scale parameters $c_1 \neq c_2$. In this case

$$T^2(\Pi_1, \Pi_2) \leqslant \left(\frac{2}{2-\alpha}\right) \left|c_1^{1/\alpha} - c_2^{1/\alpha}\right|^{\alpha}. \tag{3}$$

2. Let $\Pi_j, j = 1,2$ be two one-sided α-stable measures with the same scale parameter c, but different shape parameters $0 < \alpha_1 < \alpha_2 < 2$, say. In this case

$$T^2(\Pi_1, \Pi_2) \leqslant c h(\alpha_2 - \alpha_1), \tag{4}$$

where $h(\alpha_2 - \alpha_1) \to 0, \alpha_2 - \alpha_1 \to 0.$

Sensitivity bounds for the solutions of the Lévy driven SDEs with additive noise

Introduce the following metric ζ on $\mathbb{D}(0,1)$ by

$$\zeta(x,y) = \sup_{t\in[0,1]} \rho(x(t), y(t)).$$

Theorem (Gairing, Högele, K., Kulik'2013)
Let (a_j, b_j^2, Π_j) be two Lévy characteristics and $x_j \in \mathbb{R}$ given initial values, $j = 1, 2$. And the function $V \in \mathcal{C}^2$ and satisfies for some constant $L > 0$ the condition $(V(x) - V(y))(x-y) \leqslant L(x-y)^2, \quad x, y \in \mathbb{R}$. Then for any two solutions X_j of equation (2) and any $r > 0$ there exist constants $Q_r^1, Q_r^2 > 0$ such that the following estimate holds true

Sensitivity bounds for the solutions of the Lévy driven SDEs with additive noise

Introduce the following metric ζ on $\mathbb{D}(0,1)$ by

$$\zeta(x,y) = \sup_{t\in[0,1]} \rho(x(t), y(t)).$$

Theorem (Gairing, Högele, K., Kulik'2013)
Let (a_j, b_j^2, Π_j) be two Lévy characteristics and $x_j \in \mathbb{R}$ given initial values, $j = 1, 2$. And the function $V \in \mathcal{C}^2$ and satisfies for some constant $L > 0$ the condition $(V(x) - V(y))(x-y) \leqslant L(x-y)^2, \quad x, y \in \mathbb{R}$. Then for any two solutions X_j of equation (2) and any $r > 0$ there exist constants $Q_r^1, Q_r^2 > 0$ such that the following estimate holds true

Bound

$$W_{2,\zeta}^2\Big(\mathrm{Law}(X_1), \mathrm{Law}(X_2)\Big) \leqslant Q_r^1 e^{L/\arctan(1/2)} + Q_r^2. \tag{5}$$

$$Q_r^1 = 2\rho^2(x_1, x_2) + \frac{4}{\pi}\Big(\frac{3^{3/4}}{2}|a_1 - a_2| + (b_1 - b_2)^2 + U_r(\Pi_1) + U_r(\Pi_2) +$$
$$+ (\pi + 3^{3/4}) T_r^2(\Pi_1, \Pi_2) + 3^{3/4}\min(\Pi_1(|u| > 1) + \Pi_2(|u| > 1), r)^{1/2} T_r(\Pi_1, \Pi_2)\Big),$$

$$Q_r^2 = \frac{4}{\pi}\sqrt{3^{3/2}(b_1 - b_2)^2 + (2\pi)^2(U_r(\Pi_1) + U_r(\Pi_2) + T_r^2(\Pi_1, \Pi_2))},$$

and

$$U_r(\Pi_j) = \int_{|u|\leqslant \varepsilon_j^r} u^2 \Pi_j(du), \qquad j = 1, 2.$$

Main tools of the proof

$$F(Y(t)) = F(Y(0)) + \int_0^t g(X_1(s), X_2(s))\, ds + M_t, \tag{6}$$

where

$$g(z_1, z_2) =$$

$$(V(z_1) - V(z_2))F'(z_1 - z_2) + (a_1 - a_2)F'(z_1 - z_2) + \frac{1}{2}(\sigma_1 - \sigma_2)^2 F''(z_1 - z_2)$$
$$+ \int_{\mathbb{R}^2} \Big[F(z_1 - z_2) + (u_1 - u_2)) - F(z_1 - z_2) - F'(z_1 - z_2)(u_1 - u_2)\Big]\hat{\Pi}(du)$$
$$+ \int_{\mathbb{R}^2} \Big[F((z_1 - z_2) + (u_1 - u_2)) - F(z_1 - z_2) - F'(z_1 - z_2)\Big(\tau(u_1) - \tau(u_2)\Big)\Big]\Pi^{T,r}(du) \tag{7}$$

Wasserstein statistic

- Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with common law μ (here we suppose that μ has a finite second moment)
- Then the *empirical distribution* based on the sample size n, μ_n, is given by

$$\mu_n(E) := \frac{1}{n}\sum_{i=1}^{n}\delta_{Y_i}(E) = \frac{\#\{Y_i \in E\}}{n}.$$

- The corresponding *empirical distribution function* is defined as $F_n(y) := \mu_n((-\infty, y]),\ y \in \mathbb{R}$.
- Then the Wasserstein statistic is defined as follows $\mathrm{w}_n(\mu_n, \mu) := W^2_{2,|\cdot|}(\mu_n, \mu)$
- By the optimal coupling property $\mathrm{w}_n(\mu_n, \mu) = \int_0^1 |F_n^{-1}(u) - F^{-1}(u)|^2 du.$

How to compute quantile functions for empirical distributions?

- Consider $Y_{i:n}$ the i-th order statistic of a sample size n, i.e. the ordered sample

$$Y_{1:n} \leq Y_{2:n} \leq \ldots \leq Y_{n:n}.$$

- For $0 < u \leq 1$, by the definition

$$F^{-1}(u) = \inf\{y \in \mathbb{R} : F_n \geq u\} = \inf\{y \in \mathbb{R} : \#\{Y_{i:n} \leq y\} \geq nu\}$$
$$= \min\{y \in \mathbb{R} : \#\{Y_{i:n} \leq y\} \geq nu\} =: Y_{\lceil nu \rceil :n}.$$

Back to the Wasserstein statistics

- Having the latter representation we get

$$\mathrm{w}_n(\mu_n, \mu) = \int_0^1 |F_n^{-1}(u) - F^{-1}(u)|^2 du =$$

$$\sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |Y_{i:n} - F^{-1}(u)|^2 du = \sum_{i=1}^{n} \left(\beta_i^1 Y_{i:n}^2 + \beta_i^2 Y_{i:n}\right) + \beta^3.$$

- $\beta_i^1 = \frac{1}{n}$, $\beta_i^2 = -2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} F^{-1}(u) du$, $\beta^3 = \int_0^1 \left(F^{-1}(u)\right)^2 du.$

Empirical coupling distance

Assumption for Data

- Given a model $X_t = x + \int_0^t V(X_s)ds + Z_s$
- Z (pure jump) Lévy process with LM Π, V one-sided Lipschitz
- For a given Data $(x_i)_{i=1,\ldots,n}$ assume that
  $x_i = X_{t_i}(\omega)$ for times $t_1 < t_2 < \ldots < t_n$ and $\omega \in \Omega$.

  Model hypotheses

  For a given cutoff $\varepsilon > 0$
- The observation frequency is high:
  In each time interval $[t_{i-1}, t_i)$ occurs at most one jump $> \varepsilon$.
- Small jumps or continuous increments are benign: they cannot accumulate over $[t_{i-1}, t_i)$ and appear as a large jump.

Estimating procedure

For given cutoff $\varepsilon > 0$

- Estimate jumps by increments: $X_{t_i} - X_{t_{i-1}} \approx Z_s^{T,\varepsilon} - Z_{s-}^{T,\varepsilon}$
- For the data we put: $y_i := (x_i - x_{i-1})\mathbf{1}\{|x_i - x_{i-1}| > \varepsilon\}$.
- Then modeling hypotheses yield that $(y_i)_{i=1,\dots N}$ is a sample of i.i.d. random variables $(Y_i)_{i=1,\dots n}$.
- Let μ_n be the empirical measure for $(Y_i)_{i=1,\dots n}$ and μ be the true common distribution.
- Note that:
    - μ is the distribution of the jumps for the true Lévy measure $\Pi_*^{T,\varepsilon}$
    - μ_n is the distribution of the jumps for the empirical Lévy measure $\tilde{\Pi}_n^{T,\varepsilon}$
- The distance between the Lévy measure, suggested by the model $(\Pi^{T,\varepsilon})$ and the data $(\Pi_*^{T,\varepsilon})$ is

$$T_r(\Pi^{T,\varepsilon}, \Pi_*^{T,\varepsilon}) \leq T_r(\Pi^{T,\varepsilon}, \tilde{\Pi}_n^{T,\varepsilon}) + T_r(\tilde{\Pi}_n^{T,\varepsilon}, \Pi_*^{T,\varepsilon}) =$$

$$(r_\varepsilon \mathrm{w}_{n,\rho}(\pi^{T,\varepsilon}, \mu_n))^{1/2} + T_r(\tilde{\Pi}_n^{T,\varepsilon}, \Pi_*^{T,\varepsilon}).$$

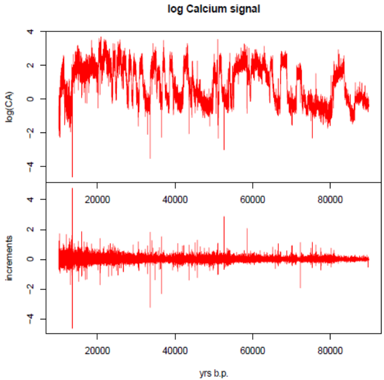
Empirical coupling distance for a paleoclimate time series

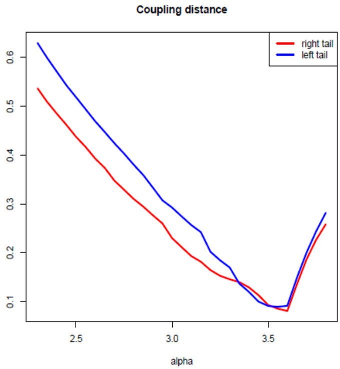
- One-sided Lévy measures

$$\Pi^1(dy) = \frac{r_1 dy}{|y|^{\alpha_1+1}} \mathbf{1}\{y < -\varepsilon_1\}, \quad \Pi^2(dy) = \frac{r_2 dy}{y^{\alpha_2+1}} \mathbf{1}\{y > \varepsilon_2\}$$

-

i	tail	ε_i	n_i	r_{ε_i}	α_i	T^*
1	left	0.34	530	11	3.55	0.089
2	right	0.36	894	8	3.6	0.081





References

P.D. Ditlevsen.
Observation of a stable noise induced millennial climate changes from an ice-core record.
Geophysical Research Letters, 26 (10):1441–1444, 1999.

J. Gairing, M. Högele, T. Kosenkova, A. Kulik: Coupling distances between Lévy measures and applications to noise sensitivity of SDE. Stochastics and Dynamics, 15 (2), 2015.

J. Gairing, M. Högele, T. Kosenkova, A. Kulik: On the calibration of Lévy driven time series with coupling distances with an application in paleoclimate. Springer INdAM volume "Mathematical Paradigms of Climate Sciences", ISSN: 2281-518X

J. Gairing, M. Högele, T. Kosenkova,, "Transportation distances and noise sensitivity of multiplicative Lévy SDE with applications", In Press, SPA 2017.

C. Hein, P. Imkeller, and I. Pavlyukevich.
Limit theorems for p-variations of solutions of SDEs driven by additive stable Lévy noise and model selection for paleo-climatic data.
Interdisciplinary Math. Sciences, 8:137-150, 2009.

Members NGRIP. NGRRIP data. Nature, 431:147-151, 2004.