A characterization of reciprocal processes via an integration by parts formula on the path space

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Abstract

We characterize in this paper the class of reciprocal processes associated to a Brownian diffusion (therefore not necessarily Gaussian) as the set of Probability measures under which a certain integration by parts formula holds on the path space $\mathcal{C}([0,1];\mathbb{R})$. This functional equation can be interpreted as a perturbed duality equation between Malliavin derivative operator and stochastic integration. An application to periodic Ornstein-Uhlenbeck process is presented. We also deduce from our integration by parts formula the existence of Nelson derivatives for general reciprocal processes.

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1 Introduction

The present paper deals with reciprocal processes which we characterize by a simple functional equation, an integration by parts formula, on the space of continuous paths. Reciprocal processes are Markovian fields with respect to the time parameter and therefore a generalization of Markov processes. The interest in these processes was motivated by a Conference of Schrödinger [24] about the most probable dynamics for a Brownian particle whose laws at two different times are given. Actually, Schrödinger was only concerned with Markovian reciprocal processes. His interpretation in terms of (large) deviations from an expected behavior was further developed by Föllmer, Cattiaux and Léonard, Gantert. Schrödinger processes were also analysed by Zambrini and Nagasawa for their possible connections to quantum mechanics. One year after Schrödinger, Bernstein noticed the importance of non-Markovian processes with given conditional dynamics, where the conditioning is made at two fixed times. This was the beginning of the study of general reciprocal processes.

Jamison [11] proved that the set of reciprocal processes is partitioned into classes; each subclass is characterized by a set of functions, called Reciprocal Characteristics ([4], [13]). The main result we obtain is that, for real-valued processes, each class of reciprocal processes with Reciprocal Characteristics $(1, F)$ coincides with the set of solutions of a functional equation in which the function F plays a similar role as the Hamilton function associated to a set of Gibbs measures ([21]). This functional equation is indeed an integration by parts formula on the path space $\mathcal{C}([0,1];\mathbb{R})$ and it exhibits a perturbed duality relation between the stochastic integration w.r.t. a reciprocal process and the Malliavin derivative operator along a class of test functions which is smaller than the usual one on the Wiener space.

Then, to illustrate our approach of reciprocal processes, we consider some Stochastic Differential Equations with time boundary conditions (initial and final times). Solutions of such stochastic equations form a wide class of non adapted (then anticipative) non Markovian processes and we hope that our way to identify their reciprocal properties will be a help in the analysis of such processes.

The search of a characterization of reciprocal processes as the set of solutions of some second order equation was proposed by Krener (cf [13]). It was achieved in the Gaussian case by Krener, Frezza and Levy in [15]. (For the Gaussian stationary case see also [2].) As far as we know, no such characterization was available in the non Gaussian case. Our result fills this gap in dimension 1.

Concerning the general non Gaussian case, one of the authors proved in [25] that reciprocal processes satisfy a stochastic Newton equation which involves Nelson derivatives, the reciprocal characteristics as well as a stochastic version of acceleration. At the end of section 4 of the present paper, we study the relationship between our result and the result of [25]. The integration by parts formula which we introduce provides sufficient conditions for a reciprocal process to be differentiable in Nelson's sense.

Reciprocal processes are time random fields defined on a compact time interval. When the time parameter belongs to an interval with infinite length, the problematic is closed to time Gibbs measure, or quasi-invariant measure on the space of continuous functions, as introduced in the seventies in the context of Euclidean Quantum Field theory by Courrège and Renouard [5] (see also [23]). Still a lot of problems in this direction remain open.

The paper is devided into the following sections.

1. Introduction.

2. Notations and framework.

3. Characterization of $\mathcal{R}(P)$, the reciprocal class associated to the Brownian motion.

4. Characterization of the reciprocal class associated to a Brownian diffusion.

5. Application to the periodic Ornstein-Uhlenbeck process.

2 Notations and framework

Let $\Omega = \mathcal{C}([0,1];\mathbb{R})$ be the canonical - polish - path space of continuous real-valued functions on $[0,1]$, endowed with $\mathcal{F}$, the canonical σ-field. Let $(X_t)_{t\in[0,1]}$ denote the family of canonical projections from Ω into $\mathbb{R}$.

$\mathcal{P}(\Omega)$ is the set of probability measures on Ω. We use equivalently the notation $Q(f)$ or $E_Q(f)$ for the integral of the function f under a probability measure Q.
Let $P \in \mathcal{P}(\Omega)$ denote the Wiener measure on Ω satisfying $P(X_0 = 0) = 1$.
More generally, for $x \in \mathbb{R}$, P^x is the shifted Wiener measure satisfying $P(X_0 = x) = 1$.

We define now the space of smooth cylindrical functionals on Ω :

$$\begin{aligned}\mathcal{S} \quad = \quad & \{\Phi, \Phi(\omega) = \varphi(\omega_{t_1}, \dots, \omega_{t_n}) \text{ where } \varphi \text{ is a bounded } \mathcal{C}^\infty\text{-function} \\ & \text{from } \mathbb{R}^n \text{ in } \mathbb{R} \text{ with bounded derivatives and } 0 \le t_1 \le \dots \le t_n \le 1\}.\end{aligned}$$

Clearly $\mathcal{S} \subset L^2(\Omega; P)$.

On $\mathcal{S}$ we define the *derivation operator* D in the direction $g \in L^2(0,1)$ by

$$\begin{aligned} D_g\Phi(\omega) &= \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\omega_{t_1}, \dots, \omega_{t_n}) \int_0^{t_i} g(t)dt \\ &= \int_0^1 g(t) D_t\Phi(\omega)dt \end{aligned}$$

where

$$D_t\Phi(\omega) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\omega_{t_1}, \dots, \omega_{t_n}) \mathbf{1}_{t\le t_i}.$$

It is clear that $D_g\Phi$ is also equal to the Gâteaux-derivative of Φ in the direction $\int_0^\cdot g(t)dt$, which is a typical element of the Cameron-Martin space.

We can now define the space $\mathbf{D}^{1,2}$ as the closure of $\mathcal{S}$ for the following norm :

$$\|\Phi\|_{1,2}^2 = E_P(\Phi^2) + E_P\Big(\int_0^1 D_t\Phi^2 dt\Big).$$

It is well known (see for example [1]) that the operator D (also called Malliavin derivation) is the dual operator on $\mathbf{D}^{1,2}$ of the stochastic integration operator δ defined on Ω by $\delta(g)(\omega) = \int_0^1 g(t)d\omega_t$:

$$\forall g \in L^2(0,1), \forall \Phi \in \mathbf{D}^{1,2}, \qquad E_P(D_g\Phi) = E_P\Big(\Phi\ \delta(g)\Big) \tag{1}$$

The main object we deal with in this paper are the so called reciprocal classes.
We consider a given Markov diffusion $\tilde{P} \in \mathcal{P}(\Omega)$ such that, for each $0 \le s < t \le 1$, the map $(x, y) \mapsto \tilde{P}(\,./X_s = x, X_t = y)$ is continuous on $\mathbb{R}^2$. The reciprocal class associated to $\tilde{P}$ is the subset $\mathcal{R}(\tilde{P})$ of $\mathcal{P}(\Omega)$ defined by :

$$\mathcal{R}(\tilde{P}) = \{Q \in \mathcal{P}(\Omega), \forall 0 \le s < t \le 1, Q(\,./\mathcal{F}_s \vee \hat{\mathcal{F}}_t) = \tilde{P}(\,./X_s, X_t)\} \tag{2}$$

where the forward (resp. backward) filtration $(\mathcal{F}_t)_{t\in[0,1]}$ (resp. $(\hat{\mathcal{F}}_t)_{t\in[0,1]}$) is given by

$$\mathcal{F}_t = \sigma(X_s, 0 \le s \le t), \quad (\text{resp. } \hat{\mathcal{F}}_t = \sigma(X_s, t \le s \le 1)).$$

Each element of $\mathcal{R}(\tilde{P})$ is called a reciprocal process associated to $\tilde{P}$.

From the definition (2) of a reciprocal class, it is clear that each reciprocal process Q is a *Markovian field* in the sense that, for $0 \le s < t \le 1$, $\mathcal{F}_s \vee \hat{\mathcal{F}}_t$ and $\sigma(X_r; s \le r \le t)$ are independent under Q conditionnally to $\sigma(X_s, X_t)$.

Nevertheless, a reciprocal process is not necessarily a Markov process. Jamison gave in [11] the following description of the subset $\mathcal{R}_M(\tilde{P})$ whose elements are the Markovian processes of $\mathcal{R}(\tilde{P})$:

$$\begin{aligned}\mathcal{R}_M(\tilde{P}) \;\; = \;\; & \{Q \in \mathcal{R}(\tilde{P}), \exists \nu_0, \nu_1 \quad \sigma\text{-finite measures on } \mathbb{R}, \\ & Q \circ (X_0, X_1)^{-1}(dx, dy) = \tilde{p}(0, x, 1, y)\nu_0(dx)\nu_1(dy)\}\end{aligned} \tag{3}$$

where $\tilde{p}(s, x, t, y)$ is the probability transition density of $\tilde{P}$ (which always exists and is regular in the cases treated in this paper). Due to historical reasons recalled in the introduction, the elements of $\mathcal{R}_M(\tilde{P})$ are called in the litterature "Schrödinger processes".

Let us mention the following equivalent definition of $\mathcal{R}(\tilde{P})$ as the class of processes having the same bridges as $\tilde{P}$ (see [11]) :

$$\mathcal{R}(\tilde{P}) = \{Q \in \mathcal{P}(\Omega), \exists\, m \in \mathcal{P}(\mathbb{R}^2), Q = \int_{\mathbb{R}^2} \tilde{P}(\,/X_0 = x, X_1 = y) m(dx, dy)\}. \tag{4}$$

Remark that from the above definition (4) any reciprocal process Q in $\mathcal{R}(\tilde{P})$ is a mixture of bridges of $\tilde{P}$.

3 Characterization of $\mathcal{R}(P)$, the reciprocal class associated to the Brownian motion

3.1 Duality under the Brownian bridge

We recalled in the above equality (1) the duality between Malliavin derivative D and stochastic integration δ under the Wiener measure P. In fact, (1) remains valid if P is replaced by any other Wiener measure P^x, $x \in \mathbb{R}$, and therefore, by linearity of this equation with respect to the integrator, equality (1) is also true under P^μ, a μ-mixture of (P^x, $x \in \mathbb{R}$):

$$P^\mu = \int_{\mathbb{R}} P^x \quad \mu(dx), \qquad \mu \in \mathcal{P}(\mathbb{R}). \tag{5}$$

What is more surprising is the fact that the duality between D and δ holds also under any desintegration of the Wiener measure in Brownian bridges, if we restrict the class of test functions g in (1) to a smaller space than $L^2(0, 1)$. So let us introduce the function space

$$L^2_0(0, 1) = \big\{g \in L^2(0, 1), \quad \int_0^1 g(r)\, dr = 0\big\}.$$

It is the orthogonal subspace in $L^2(0,1)$ to the constant functions.

Let us stress the following remark: for the characterization based on integration by parts formula developed in the rest of the paper, it is enough to consider the class of step functions $g \in L^2_0(0,1)$. For these functions, $\delta(g)$ is intrinsically and trivially defined; in particular the stochastic integral does not depend on the reference probability measure on Ω.

We have

Proposition 3.1 *Let $(x,y) \in \mathbb{R}^2$ and $P^{x,y} \in \mathcal{P}(\Omega)$ be the law of the Brownian bridge on $[0,1]$ from x to y. Then*

$$\forall g \text{ step function in } L^2_0(0,1), \forall \Phi \in \mathcal{S}, \quad P^{x,y}(D_g\Phi) = P^{x,y}\Big(\Phi\ \delta(g)\Big). \tag{6}$$

Proof : The duality formula (6) has been proved by Driver in [8] even for the Brownian bridge on a Riemannian manifold. His proof relies on the absolute continuity of $P^{x,y}$ with respect to P^x on $\mathcal{F}_\tau$, with $0<\tau<1$. However for the sake of completeness, let us sketch an alternative proof of this duality. As noticed at the beginning of the section 3.1, the duality

$$P^\mu\Big(\Phi\ \delta(g)\Big) = P^\mu(D_g\Phi) \tag{7}$$

holds for any g step function, $\Phi \in \mathcal{S}$ and $\mu \in \mathcal{P}(\mathbb{R})$.
Taking $\Phi(\omega) = \phi_0(\omega_0)\phi_1(\omega_1)\tilde{\Phi}(\omega)$ for ϕ_0, $\phi_1 \in \mathcal{C}^\infty(\mathbb{R})$, and $\tilde{\Phi} \in \mathcal{S}$, one obtains from (7)

$$\begin{aligned}&P^\mu\Big(\phi_0(X_0)\phi_1(X_1)P^\mu(\tilde{\Phi}\delta(g)/X_0,X_1)\Big) = \\ &P^\mu\Big(\phi_0(X_0)\phi_1(X_1)P^\mu(D_g\tilde{\Phi}/X_0,X_1)\Big) + P^\mu\Big(\phi_0(X_0)\phi_1'(X_1)\tilde{\Phi}\Big)\int_0^1 g(r)\,dr.\end{aligned}$$

So, for g step function in $L^2_0(0,1)$, the last term vanishes and this implies

$$P^{X_0,X_1}\Big(\tilde{\Phi}\delta(g)\Big) = P^\mu\Big(\tilde{\Phi}\delta(g)/X_0,X_1\Big) = P^{X_0,X_1}\Big(D_g\tilde{\Phi}\Big) \quad \text{for } a.s.(X_0,X_1) \text{ under } P^\mu.$$

By continuity of the map $(x,y) \mapsto P^{x,y}$ the duality formula (6) holds for all $(x,y) \in \mathbb{R}^2$. ■

Remark 3.2 : To prove the duality equation (6) under $P^{0,0}$ we could also use the correspondence between the Gauss space of the Brownian bridge $P^{0,0}$ and the Wiener space (Ω, P), based on the isomorphism α between $L^2_0(0,1)$ and $L^2(0,1)$ defined by :

$$\forall g \in L^2_0(0,1), \quad \alpha(g)(r) = g(r) + \frac{1}{1-r}\int_0^r g(s)\,ds.$$

In fact, following Gosselin and Wurzbacher ([10], Proposition 2.2), if X is a Brownian motion under P, the image process of X under the transformation

$$\Theta : \omega \rightarrow \Big(t \rightarrow (\Theta\omega)_t = (1-t)\int_0^t \frac{d\omega_s}{1-s}\Big)_{0\le t<1}$$

is a Brownian bridge with law $P^{0,0}$; the stochastic integral $\delta(g)(\Theta X) = \int_0^1 g(r)d(\Theta X)_r$ is well defined for $g \in L^2_0(0,1)$ and moreover :

$$\delta(g)(\Theta X) = \delta(\alpha(g))(X) \quad P-a.s..$$

So, to deduce (6) from (1) it is enough to remark that, for $g \in L^2_0(0,1)$ and $\Phi \in \mathcal{S}$,

$$D_g\Phi \circ \Theta = D_{\alpha(g)}(\Phi \circ \Theta).$$

3.2 Characterization of the conditional probabilities

The natural question is now to analyse if the duality under a measure Q between D and δ tested on all $(g, \Phi) \in L_0^2(0,1) \times \mathcal{S}$ characterizes the bridges of Q. The positive answer is the object of the following :

Proposition 3.3 *Let $Q \in \mathcal{P}(\Omega)$ such that $Q(\sup_{t\in[0,1]} |X_t|) < +\infty$. If*

$$\begin{array}{cc} \forall g \text{ step function in } L_0^2(0,1), \forall \Phi \in \mathcal{S}, & Q(D_g\Phi) = Q\Big(\Phi\ \delta(g)\Big) \\ \text{then} & Q(\,./X_0, X_1) = P^{X_0,X_1} \quad Q-a.s.. \end{array} \tag{8}$$

Proof :

First, following the same argument as in Proposition 3.1, it is clear that (8) also holds under $Q(\,./X_0, X_1)\, Q-a.s.$. For simplicity, let us denote by $Q^{x,y} \in \mathcal{P}(\Omega)$ the law of the bridge of Q on $[0,1]$ between x and y, $(x,y) \in \mathbb{R}^2$. Let $\tilde{g}$ a fixed step function on $[0,1]$, and for $\lambda \in \mathbb{R}$, define

$$\psi(\lambda) = Q^{x,y}\Big(\exp(i\lambda\delta(\tilde{g}))\Big). \tag{9}$$

By recentering $\tilde{g}$, we also introduce the step function

$$g = \tilde{g} - \int_0^1 \tilde{g}(r)\,dr\ \in L_0^2(0,1). \tag{10}$$

Now, remarking that ψ is differentiable on $\mathbb{R}$, we obtain

$$\begin{aligned} \psi'(\lambda) &= iQ^{x,y}\Big(\delta(\tilde{g})\exp(i\lambda\delta(\tilde{g}))\Big) \\ &= iQ^{x,y}\Big(\Big(\delta(g) + (y-x)\int_0^1 \tilde{g}(r)\,dr\Big)\exp(i\lambda\delta(\tilde{g}))\Big) \\ &= ie^{i\lambda(y-x)\int_0^1 \tilde{g}(r)\,dr} Q^{x,y}\Big(\delta(g)\exp(i\lambda\delta(g))\Big) + i(y-x)\int_0^1 \tilde{g}(r)\,dr\,\psi(\lambda). \end{aligned}$$

From (8), using the fact that $\Phi = \exp(i\lambda\delta(g)) \in \mathcal{S}$, we deduce that for $Q\circ(X_0,X_1)^{-1}a.a.(x,y)$,

$$Q^{x,y}\Big(\delta(g)\exp(i\lambda\delta(g))\Big) = Q^{x,y}\Big(D_g(\exp(i\lambda\delta(g))\Big)$$

which is equivalent to

$$Q^{x,y}\Big(\delta(g)\exp(i\lambda\delta(g))\Big) = i\lambda\int_0^1 g^2(r)\,dr \quad Q^{x,y}\Big(\exp(i\lambda\delta(g))\Big)$$

So,

$$\psi'(\lambda) = \Big(i(y-x)\int_0^1 \tilde{g}(r)\,dr - \lambda\Big(\int_0^1 \tilde{g}^2(r)\,dr - (\int_0^1 \tilde{g}(r)\,dr)^2\Big)\Big)\psi(\lambda).$$

The unique solution of this differential equation with initial condition $\psi(0) = 1$ is

$$\psi(\lambda) = \exp\Big(-\frac{\lambda^2}{2}\Big(\int_0^1 \tilde{g}^2(r)\,dr - (\int_0^1 \tilde{g}(r)\,dr)^2\Big) + i\lambda(y-x)\int_0^1 \tilde{g}(r)\,dr\Big). \tag{11}$$

Thus, for $Q \circ (X_0, X_1)^{-1}$ almost all (x, y), equality (11) holds true for all $\tilde{g}$ in the following countable set of step functions : $\{\sum_{i=0}^{p} \alpha_i \mathbf{1}_{[s_i, s_{i+1}[}, 0 = s_0 < \ldots \le s_p < s_{p+1} = 1,\ p \in \mathbb{N},\ s_i, \alpha_i \in \mathbb{Q}\}$. This set is dense in $L^2(0,1)$, so equality (11) holds also true for each $g \in L^2(0,1)$, since its both sides are $L^2(0,1)$-continuous functionals of $\tilde{g}$ under the assumption that $Q(\sup_{t\in[0,1]} |X_t|) < +\infty$.

Next step is to identify the process with the above characteristic functional. Let us indicate two possibilities :

Either one verifies that the following process

$$Y_t = x(1-t) + B_t + t(y - B_1)$$

where B is a Brownian motion, is indeed a Brownian bridge with law $P^{x,y}$ and admits ψ as characteristic functional (cf. for example Theorem IV.40.3 in [22]).

Or one remarks that ψ is associated to a Gaussian process : by taking $\lambda = 1$ and

$$\tilde{g} = \sum_{i=0}^{p} \alpha_i \mathbf{1}_{[t_{i-1}, t_i[},\ 0 = t_0 < t_1 < \ldots < t_{p-1} < t_p = 1,$$

it is clear that $Q^{x,y}\Big(\exp(i\delta(\tilde{g}))\Big)$ is the exponential of a bilinear form in (α_i). Moreover, taking now $\tilde{g} = \mathbf{1}_{[s,t]}$, we obtain the first two moments of this Gaussian process :

$$Q^{x,y}\Big(\exp(i\lambda\delta(\mathbf{1}_{[s,t]}))\Big) = e^{-\frac{\lambda^2}{2}(t-s-(t-s)^2) + i\lambda(y-x)(t-s)}$$

implies

$$Q^{x,y}(X_t) = ty + (1-t)x \text{ and } Cov(X_s, X_t) = s(1-t),\ s \le t.$$

These moments also characterize the law of the Brownian bridge. ■

3.3 The class $\mathcal{R}(P)$ as the set of solutions of a duality equation

It is known that the duality (1) characterizes the set of Wiener measures $\{P^\mu, \mu \in \mathcal{P}(\mathbb{R})\} \subset \mathcal{P}(\Omega)$ (see [21], Theorem 1.2). By restricting the class of test functions g to those with vanishing integral on $[0;1]$, it is clear that the set of Probability measures under which the duality holds is larger. The following theorem does explicit this subset of $\mathcal{P}(\Omega)$.

Theorem 3.4 *Let $Q \in \mathcal{P}(\Omega)$ such that $Q(\sup_{t\in[0,1]} |X_t|) < +\infty$.*
The following two assertions are equivalent :

i) $\forall g$ *step function in* $L_0^2(0,1), \forall \Phi \in \mathcal{S}, \quad Q(D_g\Phi) = Q\Big(\Phi\ \delta(g)\Big)$

ii) $Q \in \mathcal{R}(P)$, *i.e.* Q *is a reciprocal process in the same class as the Brownian motion.*

Proof :

By Proposition 3.3, *i)* implies the a.s. equality between the bridges of Q and those of P. But

$$Q = \int_{\mathbb{R}^2} Q(\,/X_0 = x, X_1 = y)\, m(dx, dy)$$

where $m = Q \circ (X_0, X_1)^{-1}$.Then using the definition of $\mathcal{R}(P)$ given in (4) we obtain directly assertion *ii)*.

Reciprocally, if $Q \in \mathcal{R}(P)$, the desintegration (4) holds. So Q is a mixture in (x, y) of bridges $P^{x,y}$. But, by Proposition 3.1, under each bridge the duality between D and δ holds. This property remains valid by mixing the underlying measure. So *i)* holds. ■

4 Characterization of the reciprocal class associated to a Brownian diffusion.

In this section we want to extend the results obtained in the previous section for other classes of reciprocal processes than $\mathcal{R}(P)$. So we take as reference process no more a Brownian motion but a Markovian Brownian semi-martingale, also called Brownian diffusion, and defined as solution of the stochastic differential equation :

$$\left\{\begin{array}{l} dX_t = dB_t + b(t, X_t)\, dt \\ X_0 = x \end{array}\right. \tag{12}$$

where B is a Brownian motion and the drift b satisfies the following regularity assumptions :

$$b \in \mathcal{C}^{1,2}([0,1] \times \mathbb{R}\,;\mathbb{R}) \tag{13}$$

$$\exists K > 0,\, \forall (t,x) \in [0,1] \times \mathbb{R}, \quad x\, b(t,x) \leq K(1+x^2). \tag{14}$$

Since condition (13) implies that b is locally lipschitz continuous uniformly on time, both conditions (13) and (14) ensure existence and uniqueness of a strong solution to equation (12) (see for example [3] p.234). We denote by $\tilde{P} \in \mathcal{P}(\Omega)$ the law of this solution.

We introduce the following supplementary regularity assumption on the probability transition density associated to $\tilde{P}$ - it will be useful when we compute the reciprocal characteristics of bridges of $\tilde{P}$ - :

$$\begin{array}{l}\tilde{p}(s,x,t,y) = \tilde{P}(X_t \in dy / X_s = x)/dy \text{ is strictly positive for any } s,t \in [0,1], x,y \in \mathbb{R} \text{ and} \\ \text{belongs, as function of } (s,x)(\text{resp.}(t,y)), \text{ to } \mathcal{C}^{1,3}(]0,1] \times \mathbb{R}\,;\mathbb{R})(\text{resp.}\mathcal{C}^{1,3}([0,1[\times \mathbb{R}\,;\mathbb{R})).\end{array} \tag{15}$$

Let us now introduce a space-time function F defined on $[0,1] \times \mathbb{R}$ and derived from b by :

$$F(t,x) = \frac{\partial}{\partial t} b(t,x) + b(t,x) \frac{\partial}{\partial x} b(t,x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} b(t,x). \tag{16}$$

This function together with the diffusion coefficient 1 (due to the fact that the martingale part of X is a Brownian motion) are the so-called local reciprocal characteristics associated to $\tilde{P}$ (cf [4] and [13]). The function F, as functional of the drift, is invariant on the set $\mathcal{R}_M(\tilde{P})$ and moreover the pair $(1, F)$ characterizes completely the reciprocal class $\mathcal{R}(\tilde{P})$ (see Theorem 1 in [4] when b is bounded and Theorem 4.7 in [26] under less restrictive assumptions).

4.1 An integration by parts formula

Let us now investigate how the duality equation $i)$ in Theorem 3.4 satisfied by every reciprocal process in the Brownian class $\mathcal{R}(P)$ is perturbated when the reference process admits a drift $b \neq 0$.

Proposition 4.1 *Let $Q \in \mathcal{P}(\Omega)$ a reciprocal process in the class $\mathcal{R}(\tilde{P})$. Suppose moreover that*

$$Q(\sup_{t\in[0,1]} |X_t|^2) < +\infty \text{ and } Q\Big(\int_0^1 |F(t,X_t)|^2 dt\Big) < +\infty. \tag{17}$$

Then the following integration by parts formula is satisfied under Q :

$$\forall g \text{ step function in } L_0^2(0,1), \forall \Phi \in \mathcal{S}, Q(D_g\Phi) = Q\Big(\Phi\ \delta(g)\Big) + Q\Big(\Phi \int_0^1 g(r) \int_r^1 F(t,X_t)\, dt dr\Big). \tag{18}$$

As anounced below, the perturbation term - the second term of the r.h.s. - is given by F. In the course of the proof we will need the following

Lemma 4.2 *Let $P_\beta \in \mathcal{P}(\mathcal{C}([0,\tau];\mathbb{R}))$ be the law of a Brownian diffusion with initial value x and drift β, for some $0 < \tau \le 1$. We assume the following :*

$$\beta \in \mathcal{C}^{1,2}([0,\tau]\times\mathbb{R}\,;\mathbb{R}) \text{ and } \beta(\tau, X_\tau) \in L^2(P_\beta)$$
$$F_\beta(t, X_t) \in L^2(dt \otimes dP_\beta) \text{ where } F_\beta = \frac{\partial}{\partial t}\beta + \beta\frac{\partial}{\partial x}\beta + \frac{1}{2}\frac{\partial^2}{\partial x^2}\beta.$$

Then, for $g \in L^2(0,\tau)$ and Φ any $\mathcal{F}_\tau$-measurable function in $\mathcal{S}$,

$$P_\beta(D_g\Phi) = P_\beta\Big(\Phi\ \delta(g)\Big) + P_\beta\Big(\Phi\int_0^\tau g(r)\int_r^\tau F_\beta(t,X_t)\,dtdr\Big) - \int_0^\tau g(r)dr\,P_\beta\Big(\Phi\,\beta(\tau,X_\tau)\Big). \quad (19)$$

Proof of Lemma 4.2: Let us denote by M^β the density of P_β with respect to P^x,

$$M^\beta = \exp\Big(\int_0^\tau \beta(t,X_t)dX_t - \frac{1}{2}\int_0^\tau \beta^2(t,X_t)dt\Big).$$

We denote by $M^{n,\beta}$ the r.v. defined by $M^{n,\beta} = \exp\Big(\chi_n(\log M^\beta)\Big)$ where χ_n is a smooth bounded function with bounded derivative on $\mathbb{R}$ satisfying

$$\begin{cases} \chi_n \mathbf{1}_{[-n-1,n+1]^c} = -(n+1)\mathbf{1}_{]-\infty,-n-1[} + (n+1)\mathbf{1}_{]n+1,+\infty[} \\ \chi_n\mathbf{1}_{[-n,n]} = Id.\mathbf{1}_{[-n,n]}. \end{cases}$$

Such a cut-off for M^β appears in [7]. Remark that $0 \le M^{n,\beta} \le M^\beta + 1$.

Let $P^n_\beta \in \mathcal{P}(\mathcal{C}([0,\tau];\mathbb{R}))$ be the positive measure with Radon-Nikodym derivative $M^{n,\beta}$ with respect to the Wiener measure P^x. By definition of P^n_β,

$$P^n_\beta(D_g\Phi) = P^x\Big(M^{n,\beta}D_g\Phi\Big) = P^x\Big(D_g(\Phi\,M^{n,\beta})\Big) - P^x\Big(\Phi D_g M^{n,\beta}\Big)$$

which implies, by integration by parts formula under the Wiener measure, that

$$P^x\Big(M^{n,\beta}D_g\Phi\Big) = P^x\Big(\Phi\,M^{n,\beta}\delta(g)\Big) - P^x\Big(\Phi D_g M^{n,\beta}\Big).$$

By dominated convergence, the l.h.s. of the above identity converges to $P^x\Big(M^\beta D_g\Phi\Big) = P_\beta\Big(D_g\Phi\Big)$. The same argument applies to $P^x\Big(\Phi\,M^{n,\beta}\delta(g)\Big)$ which therefore converges to $P^x\Big(\Phi\,M^\beta\delta(g)\Big) = P_\beta\Big(\Phi\delta(g)\Big)$. By definition,

$$D_g M^{n,\beta} = M^{n,\beta}\chi_n'(\log M^\beta)D_g(\log M^\beta).$$

Morever,

$$\begin{aligned} D_g(\log M^\beta) \ &= \int_0^\tau g(r)\Big(\beta(r,X_r) + \int_r^\tau \frac{\partial}{\partial x}\beta(t,X_t)dX_t \\ &\quad - \int_r^\tau \beta(t,X_t)\frac{\partial}{\partial x}\beta(t,X_t)dt\Big)dr \end{aligned}$$

which, by Ito's formula, is equal to

$$\beta(\tau, X_\tau) \int_0^\tau g(r)dr - \int_0^1 g(r) \int_r^\tau F_\beta(p, X_p)\, dpdr.$$

The last term for which we have to study the convergence is therefore

$$P^x\Big(\Phi\, M^{n,\beta}\chi_n'(\log M^\beta)\Big(\beta(\tau, X_\tau) \int_0^\tau g(r)dr - \int_0^1 g(r) \int_r^\tau F_\beta(p, X_p)\, dpdr\Big)\Big).$$

We conclude since

$$\begin{aligned} |M^{n,\beta}\chi_n'(\log M^\beta)| &\leq M^{n,\beta}\mathbf{1}_{[-(n+1),n+1]}(\log M^\beta) \\ &\leq M^\beta \mathbf{1}_{[-(n+1),n+1]}(\log M^\beta) \end{aligned}$$

and, by assumption, the r.v.

$$M^\beta\Big(\beta(\tau, X_\tau) \int_0^\tau g(r)dr - \int_0^1 g(r) \int_r^\tau F_\beta(p, X_p)\, dpdr\Big).$$

is in $L^1(P^x)$ since the r.v. into parenthesis is in $L^2(P_\beta)$. ■

Proof of Proposition 4.1:

Let us denote by μ the law of (X_0, X_1) under Q. It is sufficient to prove identity (18) under $Q^{x,y}$ for μ-a.a. (x, y), since it will remain true by reintegration under μ.

Obviously, assumption (17) remains true under $Q^{x,y}$ for μ-a.a. (x, y). In the sequel of the present proof, we fix such an (x, y). Moreover, since $Q \in \mathcal{R}(\tilde{P})$, $Q^{x,y}$ coincides with $\tilde{P}^{x,y}$ and is therefore the law of a Brownian diffusion on each $[0, \tau], \tau < 1$ whose drift β satisfies

$$\beta(t, z) = b(t, z) + \frac{\partial}{\partial z} \log \tilde{p}(t, z, 1, y)$$

where $\tilde{p}$ is the transition probability density of $\tilde{P}$. Let us first notice that, when $\int_0^\tau g(r)dr = 0$, it is easy to verify that in the proof of Lemma 4.2 the assumption $\beta(\tau, X_\tau) \in L^1(P_\beta)$ is no more required. The remaining assumptions of Lemma 4.2 on β and $F_\beta \equiv F$ are direct consequences of assumptions (15) and (17). Therefore, for all $\Phi \in \mathcal{S}, \mathcal{F}_\tau$-measurable and all step functions $g \in L_0^2(0, \tau)$, one has

$$Q^{x,y}(D_g\Phi) = Q^{x,y}\Big(\Phi\ \delta(g)\Big) + Q^{x,y}\Big(\Phi \int_0^\tau g(r) \int_r^\tau F(t, X_t)\, dtdr\Big). \tag{20}$$

Let us now fix $\Phi \in \mathcal{S}, \mathcal{F}_1$-measurable, and g a step function in $L_0^2(0, 1)$. These are the testing objects which we need in order to prove (18). Since $\Phi \in \mathcal{S}$, there exists a function φ and a real number $\tau < 1$ such that $\Phi(X) = \varphi(x, X_{t_1}, \cdots, X_\tau, y)$, $Q^{x,y}$-a.s.. We also fix n large enough so that $\tau < 1 - \frac{1}{n}$ and g is constant on $[1 - \frac{2}{n}; 1[$. Let us set

$$g_n = g\mathbf{1}_{[0,1-\frac{2}{n}[} + n(\int_{1-\frac{2}{n}}^1 g(r)dr)\mathbf{1}_{[1-\frac{2}{n},1-\frac{1}{n}]}.$$

By construction $g_n \in L_0^2(0, 1 - \frac{1}{n})$ since $g \in L_0^2(0, 1)$. From Lemma 4.2, we deduce the identity

$$Q^{x,y}(D_{g_n}\Phi) = Q^{x,y}\Big(\Phi\ \delta(g_n)\Big) + Q^{x,y}\Big(\Phi \int_0^{1-\frac{1}{n}} g_n(r) \int_r^{1-\frac{1}{n}} F(t, X_t)\, dtdr\Big).$$

It remains to verify that each term converges when n tends to infinity towards the corresponding term in (18) written under $Q^{x,y}$. We have the followinginequalities :

- $|Q^{x,y}(D_{g_n}\Phi - D_g\Phi)| \le \|D\Phi\|_\infty \|g_n - g\|_1 = 2\frac{C}{n}\|D\Phi\|_\infty$
  where C is the constant value of g on $[1-\frac{2}{n}, 1[$.

- $\Big|Q^{x,y}\Big(\Phi\left(\delta(g_n - g)\right)\Big)\Big| \le \|\Phi\|_\infty Q^{x,y}\Big(|X_1 - X_{1-\frac{2}{n}}|\Big)$
  which converges to 0 by a.s. continuity of paths and dominated convergence theorem thanks to assumption (17).

- $\Big|Q^{x,y}\Big(\Phi(\int_0^{1-\frac{1}{n}} g_n(r)\int_r^{1-\frac{1}{n}} F(t,X_t)dtdr - \int_0^1 g(r)\int_r^1 F(t,X_t)dtdr)\Big)\Big|$
  which vanishes thanks to assumption (17).

■

4.2 Characterization of the reciprocal class $\mathcal{R}(\tilde{P})$.

We are now interested by the converse statement of Proposition 4.1. More precisely, our main result is to show that the integration by parts formula (18) characterizes the regular elements of $\mathcal{R}(\tilde{P})$. More precisely, recall that in the previous section, we introduced the regularity assumptions (13) and (15) in order to define the reciprocal characteristic F. In the same way, in order to prove a converse statement to Proposition 4.1, we have to consider probabilities on Ω which a priori satisfy the following set of regularity conditions which will be denoted by (A) in the sequel:

- (A1) $\forall t < u, y, z$ there exists a density function $q(t,z,u,.,1,y)$ such that

$$Q(X_u \in dw/\ X_t = z, X_1 = y) = q(t,z,u,w,1,y)dw$$

- (A2) $\forall x,y,u,w, \qquad q(0,x,u,w,1,y)$ is strictly positive

- (A3) $\forall u,w,y, \quad (t,z) \mapsto q(t,z,u,w,1,y)$ belongs to $\mathcal{C}^{1,2}([0,1[\times\mathbb{R};\mathbb{R})$ and for all (t,z) there exists a neighborhood $\mathcal{V}$ of (t,z) and a function $\phi_{\mathcal{V}}(u,w,1,y)$ such that

$$\sup_{(s,\xi)\in\mathcal{V}} |\partial_t q(s,\xi,u,w,1,y)| + |\partial_z q(s,\xi,u,w,1,y)| + |\partial_{zz} q(s,\xi,u,w,1,y)| \le \phi_{\mathcal{V}}(u,w,1,y)$$

and $\qquad \int_0^1\int_{\mathbb{R}} \phi_{\mathcal{V}}(u,w,1,y)|F(u,w)|dudw < +\infty.$

Theorem 4.3 *Let $Q \in \mathcal{P}(\Omega)$ satisfying (A) and such that*

$$Q(\sup_{t\in[0,1]} |X_t|^2) < +\infty \text{ and } Q\Big(\int_0^1 |F(t,X_t)|^2 dt\Big) < +\infty. \tag{21}$$

If the following integration by parts formula is satisfied under Q :

$$\begin{aligned}&\forall g \text{ step function in } L_0^2(0,1), \forall \Phi \in \mathcal{S},\\ &Q(D_g\Phi) = Q\Big(\Phi\ \delta(g)\Big) + Q\Big(\Phi \int_0^1 g(r)\int_r^1 F(t,X_t)\,dtdr\Big)\end{aligned} \tag{22}$$

then Q is a reciprocal process in the class $\mathcal{R}(\tilde{P})$.

Proof :

The proof of this theorem divides in three steps.

Step 1 : We first prove that $(X_t, t\in[0,1])$ is a Q-quasi-martingale on $[0,1]$.

This amounts to verify that

$$\sup Q\Big(\sum_{i=0}^{n-1}\big|Q(X_{t_{i+1}}-X_{t_i}/\mathcal{F}_{t_i})\big|\Big) < +\infty$$

where the supremum is taken over all the finite partitions $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$. Let us fix such a partition, and take

$$g_i = \mathbf{1}_{[t_i,t_{i+1}]} + \frac{t_{i+1}-t_i}{1-t_i}\mathbf{1}_{[t_i,1]}.$$

The integration by parts formula (22), applied to g_i and any Φ $\mathcal{F}_{t_i}$-measurable, implies that, for $0 \le i \le n-1$,

$$\begin{aligned} Q\Big(X_{t_{i+1}}-X_{t_i}/\mathcal{F}_{t_i}\Big) &= (t_{i+1}-t_i)Q\Big(\frac{X_1-X_{t_i}}{1-t_i}/\mathcal{F}_{t_i}\Big) - Q\Big(\int_{t_i}^{t_{i+1}}\int_r^1 F(t,X_t)\,dtdr/\mathcal{F}_{t_i}\Big) \\ &+ \frac{t_{i+1}-t_i}{1-t_i}Q\Big(\int_{t_i}^1\int_r^1 F(t,X_t)\,dtdr/\mathcal{F}_{t_i}\Big). \end{aligned}$$

We thus have the following inequality

$$Q\Big(\sum_{i=0}^{n-1}\big|Q(X_{t_{i+1}}-X_{t_i}/\mathcal{F}_{t_i})\big|\Big) \le \sum_{i=0}^{n-1}(t_{i+1}-t_i)\frac{Q(|X_1-X_{t_i}|)}{1-t_i} + 2Q\Big(\int_0^1 |F(t,X_t)|\,dt\Big).$$

To prove the boundedness of the r.h.s. on all partitions it is sufficient to control it for partitions which mesh goes to zero. But then, we identify the sum in the r.h.s. as a Riemann sum associated to the integral $\int_0^1 \frac{Q(|X_1-X_s|)}{1-s}ds$. The convergence of this integral is a direct consequence of the following

Lemma 4.4 *Let $Q \in \mathcal{P}(\Omega)$ satisfying the assumptions*

$$\sup_{t\in[0,1]} Q(|X_t|^2) < +\infty \textit{ and } Q\Big((\int_0^1 |F(t,X_t)|dt)^2\Big) < +\infty. \tag{23}$$

If the integration by parts formula (22) is satisfied under Q for all $\Phi \in \mathcal{S}$, then it holds also for the unbounded functional defined by $\Phi(X) = X_t - X_s, 0 \le s < t \le 1$.
Moreover, there exists a positive constant C such that

$$\forall s \in [0,1],\ Q\Big((X_1-X_s)^2\Big) \le C(1-s).$$

Proof of Lemma 4.4 : Let χ_n be the cut-off function defined in the proof of Lemma 4.2. The integration by parts formula (22) holds true for any step function $g \in L_0^2(0,1)$ and $\Phi_n(X) = \chi_n(X_t - X_s)$. Due to the assumptions (23), the dominated convergence theorem applies to each term and then, (22) holds also for $\Phi(X) = X_t - X_s$.

For proving the second assertion, let us set $g = \frac{1}{1-s}\mathbf{1}_{[s,1]} - 1$ and $\Phi(X) = X_1 - X_s$ for $s \in [0,1]$. Taking $t = 1$ in the first assertion, one deduces the identity :

$$\begin{aligned} s &= \frac{Q\Big((X_1-X_s)^2\Big)}{1-s} - Q\Big((X_1-X_s)(X_1-X_0)\Big) \\ &+ Q\Big((X_1-X_s)(\frac{1}{1-s}\int_s^1\int_r^1 F(t,X_t)dtdr - \int_0^1\int_r^1 F(t,X_t)dtdr)\Big). \end{aligned}$$

We thus conclude that

$$\frac{Q\Big((X_1-X_s)^2\Big)}{1-s} \leq 1+4\sup_{t\in[0,1]} Q(|X_t|^2)+4\Big(\sup_{t\in[0,1]} Q(|X_t|^2)\Big)^{\frac{1}{2}}\Big(Q\Big(\int_0^1 |F(t,X_t)|dt\Big)^2\Big)^{\frac{1}{2}}$$

which is finite by assumption (23). ■

Remarking that assumptions (23) are weaker than assumptions (21), this completes the proof of step 1. By Rao's theorem (cf. [6] Chapitre VII), since $(X_t, t\in[0,1])$ is a continuous Q-quasi-martingale, it is then a continuous Q-semi-martingale.

Step 2 : We now identify the local characteristics of the continuous Q-semi-martingale $(X_t, t\in[0,1])$.

- Let us denote by A the bounded variation part of X.

We first prove that for any $t\in[0,1]$, the (random) measure $Q(dA/\mathcal{F}_t)$ on $[t,1]$ is absolutely continuous with respect to Lebesgue measure, with density $\beta_t(.)$ satisfying

$$\beta_t(r)=Q\Big(\frac{A_r-A_t}{r-t}/\mathcal{F}_t\Big)+\frac{1}{r-t}\int_t^r\int_s^r Q\Big(F(p,X_p)/\mathcal{F}_t\Big)dpds.$$

To this aim, let us take $u>t$ and, as test function, a step function g with support in $[t,u]$. We first show that

$$Q\Big(\int_t^u g(r)dA_r/\mathcal{F}_t\Big)=\int_t^u g(r)\beta_t(r)dr.$$

Equation (22) applied to $\Phi=\Phi_t$, $\mathcal{F}_t$-measurable and to $\tilde{g}=g-\frac{1}{u-t}(\int_t^u g(r)dr)\mathbf{1}_{[t,u]}$ yields

$$\begin{aligned} Q\Big(\int_t^u g(r)dA_r/\mathcal{F}_t\Big) &= \frac{1}{u-t}\Big(\int_t^u g(r)dr\Big)Q\Big(A_u-A_t/\mathcal{F}_t\Big)-\int_t^u g(r)\int_r^u Q\Big(F(p,X_p)/\mathcal{F}_t\Big)dpdr \\ &\quad +\frac{1}{u-t}\Big(\int_t^u g(r)dr\Big)\int_t^u\int_s^u Q\Big(F(p,X_p)/\mathcal{F}_t\Big)dpds. \end{aligned} \tag{24}$$

Assumption (21) implies that $Q(\int_0^1 |dA_s|)<+\infty$); so we can apply Fubini's theorem to the l.h.s. of the above equality. Taking $u=1$ in (24), we obtain that $Q(dA/\mathcal{F}_t)$ is absolutely continuous with respect to Lebesgue measure on $[t,1]$, and its density is given by

$$\beta_t(r)=Q(\frac{A_1-A_t}{1-t}/\mathcal{F}_t)-\int_r^1 Q(F(p,X_p)/\mathcal{F}_t)dp+\frac{1}{1-t}\int_t^1\int_s^1 Q(F(p,X_p)/\mathcal{F}_t)dpds. \tag{25}$$

From this expression we obtain the continuity and even the a.s. derivability of the function β_t from [t,1[to $L^1(Q)$. Moreover, for all $u>r$, using the expression given in (24), we also have

$$\beta_t(r)=Q(\frac{A_u-A_t}{u-t}/\mathcal{F}_t)-\int_r^u Q(F(p,X_p)/\mathcal{F}_t)dp+\frac{1}{u-t}\int_t^u\int_s^u Q(F(p,X_p)/\mathcal{F}_t)dpds \tag{26}$$

For r fixed, letting u tend to r, one obtains from (26) the desired form for β_t :

$$\beta_t(r)=Q(\frac{A_r-A_t}{r-t}/\mathcal{F}_t)+\frac{1}{r-t}\int_t^r\int_s^r Q(F(p,X_p)/\mathcal{F}_t)dpds,\ t<r<1.$$

From the expression of $Q(A/\mathcal{F}_t)$, we now want to deduce the value of A. First we prove the following equality as processes in $L^1(dr\times Q)$:

$$\beta_t(.)=Q(\beta_.(.)/\mathcal{F}_t). \tag{27}$$

Since $s \mapsto \beta_r(s)$ is continuous from [r,1[to $L^1(Q)$, then $\beta_r(r) = \lim_{s\searrow r} Q(\frac{A_s - A_r}{s-r}/\mathcal{F}_r)$, and we have

$$\begin{aligned} Q(\beta_r(r)/\mathcal{F}_t) &= Q\Big(\lim_{s\searrow r} Q(\frac{A_s - A_r}{s-r}/\mathcal{F}_r)/\mathcal{F}_t\Big) \\ &= \lim_{s\searrow r} Q\Big(\frac{A_s - A_r}{s-r}/\mathcal{F}_t\Big) \end{aligned}$$

But

$$\begin{aligned} Q\Big(\frac{A_s - A_r}{s-r}/\mathcal{F}_t\Big) &= Q\Big(\frac{A_s - A_t}{s-r} - \frac{A_r - A_t}{s-r}/\mathcal{F}_t\Big) \\ &= \frac{s-t}{s-r}\Big(\beta_t(s) - \frac{1}{s-t}\int_t^s \int_u^s Q(F(p,X_p)/\mathcal{F}_t)dpdu\Big) \\ &\quad -\frac{r-t}{s-r}\Big(\beta_t(r) - \frac{1}{r-t}\int_t^r \int_u^r Q(F(p,X_p)/\mathcal{F}_t)dpdu\Big) \\ &= \beta_t(s) + (r-t)\frac{\beta_t(s) - \beta_t(r)}{s-r} \\ &\quad -\frac{1}{s-r}\int_r^s \int_u^s Q(F(p,X_p)/\mathcal{F}_t)dpdu - \frac{1}{s-r}\int_t^r \int_r^s Q(F(p,X_p)/\mathcal{F}_t)dpdu. \end{aligned}$$

When s tends to r the first term of the r.h.s tends to $\beta_t(r)$; the third term of the r.h.s. tends to 0 ; the limits of the second term and the forth are opposite since, from (25), for almost all r, β_t is differentiable and $\beta_t'(r) = Q(F(r,X_r)/\mathcal{F}_t)$. This completes the proof of (27).

Now we conclude observing that the process

$$\Big(A_u - A_t - \int_t^u \beta_r(r)dr\Big)_{u\in[t,1]}$$

is both a bounded variation process and a continuous Q-martingale due to (27). It is then equal to the constant 0, which means that dA_r is indeed absolutely continuous with respect to Lebesgue measure dr and its density is equal to $\beta_r(r)$.

So the semi-martingale decomposition of $(X_t, t\in[0,1])$ under Q is the following :

$$dX_t = dM_t + \beta(t,X)dt$$

where M is a Q-martingale and $\beta(r,X) =: \beta_r(r)(X)$ is given for $r<1$ by

$$\beta(r,X) = Q(\frac{X_1 - X_r}{1-r}/\mathcal{F}_r) - \int_r^1 Q(F(p,X_p)/\mathcal{F}_r)dp + \frac{1}{1-r}\int_r^1 \int_s^1 Q(F(p,X_p)/\mathcal{F}_r)dpds. \quad (28)$$

– Let us show that the martingale M is in fact a Brownian motion. The assumption (21) and formula (28) imply that $\sup_{t\in[0,\tau]} |M_t| \in L^2(\Omega)$, $\forall \tau \in [0;1[$. So, following Meyer's terminology, M belongs to the class (D) on $[0;\tau]$ and, in order to verify that M is a Brownian motion, it is enough to show that

$$\lim_{h\searrow 0} \int_0^\tau Q\Big(\frac{(X_{t+h} - X_t)^2}{h}/\mathcal{F}_t\Big)dt = \tau$$

in $L^1(Q)$ (cf. [16], Theorems T 28 and T 29 p.156).

With the same arguments as in the proof of Lemma 4.4 we can verify that (22) holds also for $\Phi(X) = \Phi_t(X)(X_{t+h} - X_t)$, where $t \in [0,1[, h > 0$, and Φ_t is $\mathcal{F}_t$-measurable, and for $g = \frac{\mathbf{1}_{[t,t+h]}}{h} - \frac{\mathbf{1}_{[t,1]}}{1-t}$; we obtain

$$\begin{aligned}
Q\Big(\frac{(X_{t+h}-X_t)^2}{h}/\mathcal{F}_t\Big) &= 1 - \frac{h}{1-t} + Q\Big((X_{t+h}-X_t)\frac{X_1 - X_t}{1-t}/\mathcal{F}_t\Big) \\
&\quad - Q\Big((X_{t+h}-X_t)\frac{1}{h}\int_t^{t+h}\int_r^1 F(s,X_s)dsdr/\mathcal{F}_t\Big) \\
&\quad + Q\Big((X_{t+h}-X_t)\frac{1}{1-t}\int_t^1\int_r^1 F(s,X_s)dsdr/\mathcal{F}_t\Big).
\end{aligned}$$

The r.h.s. converges in $L^1(Q)$ to 1 when h tends to 0 uniformly in $t \in [0,\tau]$ thanks to assumptions (21) and Lemma 4.4, so Q is a Brownian semi-martingale.

Step 3: In the last step, we show that the coordinate process under Q is reciprocal, and we identify its reciprocal class.

Since Q is the mixture of its bridges under $Q \circ (X_0, X_1)^{-1}$, it is sufficient to prove that for $Q \circ (X_0, X_1)^{-1}$-almost all (x,y) the bridge $Q^{x,y}$ belongs to the reciprocal class $\mathcal{R}(\tilde{P})$.

Following the same argument as in the proof of Proposition 3.3, for $Q \circ (X_0, X_1)^{-1}$-almost all (x,y), the integration by parts formula (22) holds true under $Q^{x,y}$. Let us fix such an $(x,y) \in \mathbb{R}^2$ and $s \in]0,1]$. We now show that $Q^{x,y}$ is a Markovian semi-martingale. More precisely, we prove that the law of $(X_r, r \in [s,1])$ is the same under $Q^{x,y}(\,./\mathcal{F}_s)$ and $Q^{x,y}(\,./X_s)$. Let us denote for simplicity $Q^{x,y}(\,./\mathcal{F}_s)$ by $Q^{x,y}_{\mathcal{F}_s}$ and $Q^{x,y}(\,./X_s)$ by $Q^{x,y}_{X_s}$. These two probabilities satisfy also equation (22) for test functions g with support in $[s,1]$. By the same arguments as in Steps 1 and 2, we deduce that $(X_r, r \in [s,1])$ is a Brownian semi-martingale under both probabilities whose drifts at time $r < 1$, computed as in (28), are respectively given by $Q^{x,y}_{\mathcal{F}_s}(U(r,X)/\mathcal{F}_r)$ and $Q^{x,y}_{X_s}(U(r,X)/\mathcal{F}_r)$, where

$$U(r,X) = \frac{y - X_r}{1-r} - \int_r^1 F(u,X_u)du + \frac{1}{1-r}\int_r^1\int_s^1 F(u,X_u)duds. \tag{29}$$

But, for $r \geq s$,

$$Q^{x,y}_{\mathcal{F}_s}(\,./\mathcal{F}_r) = Q^{x,y}_{X_s}(\,./\mathcal{F}_r) = Q^{x,y}(./\mathcal{F}_r).$$

Then both drifts coincide a.s. which implies that $Q^{x,y}$ is Markovian. In particular its drift process is the following function $\beta^{x,y}$ on time and space :

$$\beta^{x,y}(r,z) = \frac{y-z}{1-r} - \int_r^1 Q^{x,y}(F(u,X_u)/X_r = z)\,du + \frac{1}{1-r}\int_r^1\int_s^1 Q^{x,y}(F(u,X_u)/X_r = z)\,duds. \tag{30}$$

By the same arguments as above, $Q^y =: Q(./X_1 = y)$ is a Markovian semi-martingale. Therefore,

$$\begin{aligned}
Q^{x,y}(F(u,X_u)/X_r = z) &= Q^y(F(u,X_u)/X_0 = x, X_r = z) \\
&= Q^y(F(u,X_u)/X_r = z) \\
&= \int_{\mathbb{R}} F(u,w)q(r,z,u,w,1,y)dw.
\end{aligned}$$

Thanks to hypotheses (A),$(r,z) \mapsto \beta^{x,y}(r,z)$ belongs to $\mathcal{C}^{1,2}([0,1[\times\mathbb{R}\,;\mathbb{R})$ and the reciprocal characteristics associated to $Q^{x,y}$ are $(1, F^{x,y})$, where $F^{x,y}$ is derived from $\beta^{x,y}$ as was F from b

in (16). Let us now prove that $F^{x,y} = F$ for all $x, y \in \mathbb{R}$. From (30) and assumptions (21), the process $\beta^{x,y}(r, X_r)$ admits a forward derivative defined by

$$\lim_{h\to 0} Q^{x,y}\Big(\frac{\beta^{x,y}(r+h, X_{r+h}) - \beta^{x,y}(r, X_r)}{h} / \mathcal{F}_r\Big).$$

Moreover this derivative is equal to $F(r, X_r)$. Indeed,

$$\begin{aligned}
& \lim_{h\to 0} Q^{x,y}\Big(\frac{\beta^{x,y}(r+h, X_{r+h}) - \beta^{x,y}(r, X_r)}{h} / \mathcal{F}_r\Big) \\
= & \lim_{h\to 0} Q^{x,y}\Big(\frac{Q^{x,y}(U(r+h, X)/\mathcal{F}_{r+h}) - Q^{x,y}(U(r, X)/\mathcal{F}_r)}{h} / \mathcal{F}_r\Big) \\
= & \lim_{h\to 0} Q^{x,y}\Big(\frac{U(r+h, X) - U(r, X)}{h} / \mathcal{F}_r\Big) \\
= & \frac{y}{(1-r)^2} - \frac{X_r}{(1-r)^2} - \lim_{h\to 0} Q^{x,y}\Big(\frac{X_{r+h} - X_r}{h(1-r)} + \frac{1}{h}\int_{r+h}^{r} F(p, X_p)dp / \mathcal{F}_r\Big) \\
& + Q^{x,y}\Big(- \frac{1}{1-r}\int_r^1 F(p, X_p)dp + \frac{1}{(1-r)^2}\int_r^1\int_s^1 F(p, X_p)dpds / \mathcal{F}_r\Big) \\
= & F(r, X_r)
\end{aligned}$$

since all the terms of the r.h.s. vanish except $-\lim_{h\to 0} Q^{x,y}(\frac{1}{h}\int_{r+h}^{r} F(p, X_p)dp/\mathcal{F}_r)$ which tends to the desired expression. Since $Q(|\beta^{x,y}(r, X_r)|) < +\infty$ and $Q(\int_0^1 |F^{x,y}(r, X_r)|dr) < +\infty$, the martingale part of the semi-martingale $\beta^{x,y}(r, X_r)$ is a true martingale. This property enables us to identify the forward derivative of $\beta^{x,y}(r, X_r)$ with the finite variation part of $\beta^{x,y}(r, X_r)$ computed by using Ito's formula, that is

$$F(r, X_r) = F^{x,y}(r, X_r).$$

The strict positivity of $q(0, x, r, z, 1, y)$ assumed in (A) implies $F = F^{x,y}$. This completes the proof of Theorem 4.3. ∎

Remark 4.5 : Let us make some comments about the results of section 4.

- If $Q \in \mathcal{P}(\Omega)$ belongs to the class $\mathcal{R}(\tilde{P})$ and satisfies the assumptions of Proposition 4.1, we can see that, as in step 2 of the proof of Theorem 4.3,

$$Q(\frac{X_{t+h} - X_t}{h} / \mathcal{F}_t) = \frac{1}{h}\int_t^{t+h} \beta_t(r)dr$$

  where β_t is given by (25). Thanks to a result of Föllmer (cf [9], Proposition 2.5), we conclude that for almost every $t \in [0, 1[$, the forward Nelson derivative defined as $d_+X_t := L^1(\Omega) - \lim_{h\to 0}\frac{1}{h}E(X_{t+h} - X_t/\mathcal{F}_t)$ exists and is equal to β_t. By symmetry we also obtain the existence of $d_-X_t := L^1(\Omega) - \lim_{h\to 0}\frac{1}{h}E(X_t - X_{t-h}/\hat{\mathcal{F}}_t)$.

- Our integration by parts formula enables us to recover a particular case of Theorem 8.1 in [25]: if $Q \in \mathcal{P}(\Omega)$ is a reciprocal process in the class $\mathcal{R}(\tilde{P})$, satisfies the assumptions of Proposition 4.1 and is also such that for all $t \in]0, 1[$, the first and second order derivatives $d_+X_t, d_-X_t, d_+d_+X_t, d_-d_-X_t$ exist then, for allmost all $t \in]0, 1[$,

$$Q(d_+d_+X_t/X_t) = Q(d_-d_-X_t/X_t) = F(t, X_t). \tag{31}$$

This implies that $Q\Big(\frac{1}{2}(d_+d_+X_t+d_-d_-X_t)/X_t\Big) = F(t,X_t)$. The term $\frac{1}{2}(d_+d_+X_t+d_-d_-X_t)$ can be interpreted as an acceleration in Stochastic mechanics. This is why such an equation may be called Newton equation (cf. [27]).

- Krener in [14] has also proved two results of second order nature concerning reciprocal processes. In the first he establishes what he calls "second order Feller postulates", which provide a moment estimate of infinitesimal second order increments of the form $Q((X_{t+h} + X_{t-h} - 2X_t)^k / X_{t-h}, X_{t+h})$. The estimates only depend on the reciprocal characteristics. In his second result he gives a meaning to a second order s.d.e. whose coefficients are the reciprocal characteristics. For details and rigourous statements, we refer the reader to [14].

- As corollary of Steps 1 and 2 of the above proof, we obtain the fact that any reciprocal process with reciprocal characteristics $(1, F)$ satisfying assumptions (21) is a semi-martingale.

5 Application to the periodic Ornstein-Uhlenbeck process.

Let us denote by $\overline{P}$ the law of the real-valued stationary Ornstein-Uhlenbeck process, which, for $\lambda > 0$ fixed, is the solution of the stochastic differential equation :

$$\begin{cases} dX_t = dB_t - \lambda X_t\, dt \\ X_0 \sim \mathcal{N}(0; \frac{1}{\lambda}). \end{cases} \tag{32}$$

This is a particular case of the Brownian diffusion $\tilde{P}$ defined in the last section, taking b independent of time and linear with respect to space. This process is Markovian, Gaussian, and admits as reciprocal characteristics the function

$$F(t,x) = \lambda^2\, x.$$

In the present section we are interested in the solution of the following s.d.e. with periodic boundary conditions :

$$\begin{cases} dX_t = dB_t - \lambda X_t\, dt \\ X_0 = X_1. \end{cases} \tag{33}$$

This process is called periodic Ornstein-Uhlenbeck process, and we denote its law by $\overline{P}^{per}$.

This type of processes has been already studied by several authors with various motivations. First, Kwakernaak [12] studied the moments of such Gaussian processes and related filtering problems. Then, the fact that the solution of (33) is a reciprocal process has been proved from the analysis of the covariance kernel in [2]. Nevertheless, we propose here an alternative proof of the reciprocal property of the periodic Ornstein-Uhlenbeck process based on the integration by parts formula (22). Our method enables us to prove that the periodic Ornstein-Uhlenbeck process is reciprocal, and simultaneously, to identify its reciprocal class. In this sense, it makes complete, in this very particular case, the result of Ocone and Pardoux [19], who study the Markov field property of solutions of general linear s.d.e. with boundary conditions, but without any identification of their reciprocal classes. We conjecture that our method, which essentially relies on Girsanov theorem, will extend to more general s.d.e. with boundary conditions than (33) (see [18] for a description of such a general class).

The method of variation of constants yields the following form for the unique solution of (33):

$$\begin{aligned} X_t &= e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)}dB_s \\ &= \int_0^t \frac{e^{-\lambda(t-s)}}{1-e^{-\lambda}}dB_s + \int_t^1 \frac{e^{-\lambda(1+t-s)}}{1-e^{-\lambda}}dB_s \\ &= \Psi(B)_t \end{aligned} \tag{34}$$

where Ψ is the map on Ω defined by :

$$\Psi(\omega)_t = \int_0^t \frac{e^{-\lambda(t-s)}}{1-e^{-\lambda}}d\omega_s + \int_t^1 \frac{e^{-\lambda(1+t-s)}}{1-e^{-\lambda}}d\omega_s.$$

It is then straighforward to verify that X is also the well known *hyperbolic cosine process*, i.e. a zero mean Gaussian process with covariance function given by

$$Cov(X_s, X_t) = \frac{\cosh\Big(\lambda(|t-s|-\frac{1}{2})\Big)}{2\lambda\sinh(\frac{\lambda}{2})} =: R(t,s)$$

which implies, in particular, that X is stationary.

From the explicit expression of R it is easy to verify that it solves in a weak sense the second order partial differential equation $-\frac{\partial^2 R}{\partial t^2}(t,s) + \lambda^2 R(t,s) = \delta(t-s)$. Carmichael, Masse and Theodorescu characterize in [2] the covariance of stationary gaussian reciprocal processes as solutions of such partial differential equations and in [15], a generalisation to the non stationary case is proved.

Theorem 5.1 *The law $\overline{P}^{per}$ of the solution of (33) is a reciprocal process associated to the stationary Ornstein-Uhlenbeck process, that is in the reciprocal class $\mathcal{R}(\overline{P})$.*

Proof : To prove the theorem we now show that $\overline{P}^{per}$ satisfies the integration by parts formula (22) with $F(t,x) = \lambda^2 x$. Let $g \in L_0^2(0,1)$ and $\Phi \in \mathcal{S}$. By definition,

$$\begin{aligned} \overline{P}^{per}(D_g\Phi) &= \overline{P}^{per}\Big(\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\Phi(X+\epsilon\int_0^{\cdot} g(s)ds) - \Phi(X))\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\overline{P}^{per}\Big(\Phi(X+\epsilon\int_0^{\cdot} g(s)ds) - \Phi(X)\Big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\overline{P}^{per}_{\epsilon}(\Phi) - \overline{P}^{per}(\Phi)\Big) \end{aligned}$$

where $\overline{P}^{per}_{\epsilon}$ is the image of $\overline{P}^{per}$ under the shift on Ω by the deterministic path $\epsilon\int_0^{\cdot} g(s)ds$. It is also the law of the solution of the periodic s.d.e.

$$\begin{cases} dX_t = dB_t^{\epsilon} - \lambda X_t\,dt \\ X_0 = X_1 \end{cases} \tag{35}$$

where $B_t^{\epsilon} = B_t + \epsilon\int_0^t \tilde{g}(s)ds$ and $\tilde{g}(s) = g(s) + \lambda\int_0^s g(r)dr$. By the method of variation of constants we deduce that the solution of (35) is equal to $\Psi(B^{\epsilon})$ in the same way as the solution of (33) was equal to $\Psi(B)$. We thus have

$$\overline{P}^{per}(D_g\Phi) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}P\Big((\mathcal{E}(\epsilon\tilde{g}) - 1)\Phi\circ\Psi\Big)$$

where P is the Wiener measure and $\mathcal{E}(\epsilon\tilde{g})$ denotes the Girsanov density :

$$\mathcal{E}(\epsilon\tilde{g}) = \exp\Big(\int_0^1 \epsilon\tilde{g}(s)dB_s - \frac{\epsilon^2}{2}\int_0^1 \tilde{g}^2(s)ds\Big).$$

Therefore

$$\overline{P}^{per}(D_g\Phi) = P\Big(\int_0^1 \tilde{g}(s)dB_s \quad \Phi\circ\Psi\Big).$$

We can now go back to an expectation under $\overline{P}^{per}$ for the right-hand side using again the fact that $\Psi(B) = X$ solves $\overline{P}^{per}$-a.s. equation (33). This yields

$$\overline{P}^{per}(D_g\Phi) = \overline{P}^{per}\Big(\Phi(X)\Big(\int_0^1 \tilde{g}(s)dX_s + \int_0^1 \tilde{g}(s)\lambda X_s ds\Big)\Big).$$

It remains to substitute for $\tilde{g}(s)$ into its expression $g(s) + \lambda\int_0^s g(r)dr$ and to show that $\int_0^1\int_0^s g(r)dr\, dX_s + \int_0^1 g(s)X_s\, ds$ vanishes. Fubini's theorem applies to the double integral since $\overline{P}^{per}\int_0^1 |X_s|ds < \infty$. We thus obtain that

$$\int_0^1\int_0^s g(r)dr\, dX_s + \int_0^1 g(s)X_s\, ds = X_1\int_0^1 g(r)dr = 0.$$

This completes the proof. ■

The law $\overline{P}^{per}$ of the periodic Ornstein-Uhlenbeck process being in $\mathcal{R}(\overline{P})$ it admits the following decomposition $\overline{P}^{per} = \int \overline{P}^{x,y}\,\mu(dx,dy)$ where μ is the law of (X_0, X_1) under $\overline{P}^{per}$. Here μ is supported by the diagonal. Thus

$$\overline{P}^{per} = \int \overline{P}^{x,x}\, m(dx)$$

where m is the law of X_0 under $\overline{P}^{per}$, equal to $\mathcal{N}(0; \frac{1}{2\lambda}\coth(\frac{\lambda}{2}))$. In this simple case, it is possible to explicit the semi-martingale decomposition of the bridge $\overline{P}^{x,x}$, since it solves the following s.d.e.

$$\begin{cases} dX_t = dB_t - \lambda X_t\, dt + \frac{\lambda}{\sinh(\lambda(1-t))}(x - e^{-\lambda(1-t)}X_t)dt \\ X_0 = x. \end{cases} \tag{36}$$

Indeed the additional term in the drift of $\overline{P}^{x,x}$ with respect to the drift of $\overline{P}$ is equal to $\frac{\partial}{\partial z}\log\overline{p}(t, X_t, 1, x)$ where $\overline{p}(t, z, 1, .)$ is the density of the Gaussian law $\overline{P}(X_1 \in ./X_t = z)$. To compute this density it is sufficient to compute $E(X_1/X_t = z)$ and $E(X_1^2/X_t = z)$, which come directly from the equality :

$$X_1 = e^{-\lambda(1-t)}X_t + \int_t^1 e^{-\lambda(1-s)}dB_s.$$

This completes the description of the desintegration of $\overline{P}^{per}$ into bridges.

Let us also mention the work of Recoules who proved in [20] that $\overline{P}^{per}$ is the law of the process solution of

$$\begin{cases} dX_t = dB_t - \lambda\Big(\frac{X_0}{\sinh(\lambda(1-t))} - \frac{X_t}{\tanh(\lambda(1-t))}\Big)dt \\ X_0 \sim \mathcal{N}(0; \frac{1}{2\lambda}\coth(\frac{\lambda}{2})). \end{cases} \tag{37}$$

Let us notice that equation (37) is a randomized version, for X_0 no longer deterministic, of equation (36), which exactly reflects at the level of the semi-martingale property the above desintegration

$$\overline{P}^{per} = \int \overline{P}^{x,x} \mathcal{N}(0; \frac{1}{2\lambda} \coth(\frac{\lambda}{2}))(dx).$$

Under $\overline{P}^{per}$, $\mathcal{F}_0$ is not degenerated and the drift of X at time t in (37) is a function of (X_0, X_t). So $\overline{P}^{per}$ is not Markovian while clearly $\overline{P}^{x,x}$ is Markovian.

From the point of view of entropy, Recoules remarked also that $\overline{P}^{per}$ is, among Gaussian stationary periodic processes, the unique one which minimizes the Kullback information with respect to the Brownian bridge with initial law $\mathcal{N}(0; \frac{1}{\lambda^2})$.

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