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Some properties of the multitype measure branching process

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Qualitative properties of the multitype measure branching process and its occupation time process are investigated, including martingale properties, Hausdorff dimension of supports, existence of densities and stochastic equations.

multitype measure branching process * random measure * Hausdorff dimension * martingale measure

1. Introduction

The *multitype measure branching process* (MMBP), or *multitype Dawson-Watanabe process*, is a vector measure-valued process which arises as a small particle limit of a system of particles of several types undergoing random migration, branching and mutation. It is a natural generalization of the monotype case, the main new feature being the interaction of types produced by the mutations. The existence and characterization of this process was established by Gorostiza and López-Mimbela [10] following the martingale approach used by Roelly-Coppoletta [24] in the monotype case. Continuous state multitype measure branching processes (without motion) have been considered by Rhyzhov and Skorokhod [23] and Watanabe [26].

The purpose of the present paper is to investigate some properties of the MMBP and its corresponding occupation time process, generalizing the known results for the monotype case. The scaling involved in the approximating branching particle system is such that in the limit each particle produces offspring of its own type only. However, the approximation allows the presence of an interaction of types in the limit; this effect is represented by the *interaction of types matrix* $\boldsymbol{D}$, which appears in the linear part of the non linear equation (2.5) for the cumulant semigroup of

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the MMBP. Hence it is reasonable to expect that at least some properties of the components of the MMBP are the same as in the monotype case. We will show that this is true for support properties (Theorems 3.1 and 3.5). The main ideas of the proofs are similar to the monotype case [3, 8, 24, 27], but new technical problems arise due to the interaction effect. In contrast, the interaction has non trivial consequences on other aspects of the MMBP, such as persistence properties [12].

We remark that even in the monotype case our model is somewhat more general than those previously studied, and hence some of our results in this case may be considered new; e.g. the conditions for the existence of a density (Theorem 3.5) do not require spatial homogeneity of the processes.

In Section 2 we recall the approximating particle system, the scaling which yields the MMBP and the characterization of the process, and we define and characterize the occupation time process. Section 3 begins with a brief summary of the properties of the monotype process which we wish to extend, and then the properties of the MMBP are given. Since some of the theorems are natural extensions of the monotype case, their proofs will be omitted or only outlined. Detailed proofs and additional information are available in the technical report [11].

In the remainder of this section we give some necessary technical background and notation.

For $p>0$, let

$$K_p(\mathbb{R}^d)=C_c^\infty(\mathbb{R}^d)\cup\{\varphi_p\},$$

where $C_c^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable real-valued functions on $\mathbb{R}^d$ with compact support, and

$$\varphi_p(x)=(1+|x|^2)^{-p},\quad x\in\mathbb{R}^d,$$

with $|\cdot|$ the usual norm on $\mathbb{R}^d$. Let $C_p(\mathbb{R}^d)$ designate the Banach space of real continuous functions φ on $\mathbb{R}^d$ with norm

$$\|\varphi\|_p=\sup_{x\in\mathbb{R}^d}|\varphi(x)/\varphi_p(x)|<\infty.$$

Note that $K_p(\mathbb{R}^d)\subset C_p(\mathbb{R}^d)$. By $C_p(\mathbb{R}^d)_+$ we denote the set of non-negative elements of $C_p(\mathbb{R}^d)$.

Let $\mathcal{M}_p(\mathbb{R}^d)$ denote the space of non-negative Radon measures μ on $\mathbb{R}^d$ such that $\int\varphi_p\,\mathrm{d}\mu<\infty$. The Lebesgue measure belongs to $\mathcal{M}_p(\mathbb{R}^d)$ for $p>\frac{1}{2}d$. The spaces $C_p(\mathbb{R}^d)$ and $\mathcal{M}_p(\mathbb{R}^d)$ are in duality, and we write

$$\langle\mu,\varphi\rangle=\int\varphi\,\mathrm{d}\mu,\quad \mu\in\mathcal{M}_p(\mathbb{R}^d),\ \ \varphi\in C_p(\mathbb{R}^d).$$

Any $\mu\in\mathcal{M}_p(\mathbb{R}^d)$ is uniquely determined by $\{\langle\mu,\varphi\rangle\colon\varphi\in K_p(\mathbb{R}^d)\}$.

The spherically symmetric stable process on $\mathbb{R}^d$ with exponent α, $0<\alpha\leqslant 2$, is a homogeneous Markov process with infinitesimal generator given by $\Delta_\alpha\equiv-(-\Delta)^{\alpha/2}$. We denote by $\{S_t^\alpha,\ t\geqslant 0\}$ the semigroup generated by Δ_α. If $p>\frac{1}{2}d$, and in addition $p<\frac{1}{2}(d+\alpha)$ in case $\alpha<2$, then $K_p(\mathbb{R}^d)\subset\mathcal{D}(\Delta_\alpha)$, where $\mathcal{D}(\Delta_\alpha)$ is the domain of Δ_α

in $C_p(\mathbb{R}^d)$, the operators Δ_α and S_t^α for each t map $K_p(\mathbb{R}^d)$ continuously into $C_p(\mathbb{R}^d)$, and $t \to S_t^\alpha \varphi$ is a continuous curve in $C_p(\mathbb{R}^d)$ for each $\varphi \in C_p(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} \varphi(x)/\varphi_p(x)$ exists. (See [2] for details.)

Due to the above facts, all the forthcoming expressions (in particular time integrals) are well-defined.

We define, for $k = 1, 2, \ldots,$

$$\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle = \sum_{i=1}^{k} \langle \mu_i, \varphi_i \rangle,$$

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k) \in (\mathcal{M}_p(\mathbb{R}^d))^k, \qquad \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k) \in (C_p(\mathbb{R}^d))^k.$$

For a set $\mathscr{C} \subset \mathbb{R}^d \times \mathbb{R}_+$ and $t \in \mathbb{R}_+$ we denote $\mathscr{C}_t = \{x \in \mathbb{R}^d, (x, t) \in \mathscr{C}\}$, and for $\boldsymbol{\mathscr{C}} = (\mathscr{C}_1, \ldots, \mathscr{C}_k) \in (\mathbb{R}^d \times \mathbb{R}_+)^k$, we write $\boldsymbol{\mathscr{C}}_t = (\mathscr{C}_{1,t}, \ldots, \mathscr{C}_{k,t})$.

Λ_C stands for the Lebesgue measure restricted to a Borel set $C \subset \mathbb{R}^d$, and we write

$$\langle \boldsymbol{\beta} \boldsymbol{\Lambda}_{\boldsymbol{C}}, \boldsymbol{\varphi} \rangle = \sum_{i=1}^{k} \beta_i \langle \Lambda_{C_i}, \varphi_i \rangle,$$

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in \mathbb{R}_+^k, \qquad \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_k),$$

$$\boldsymbol{C} = (C_1, \ldots, C_k), \quad C_i \text{ Borel sets of } \mathbb{R}^d.$$

$\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$ is the space of functions from $\mathbb{R}_+$ into $(\mathcal{M}_p(\mathbb{R}^d))^k$ which are right-continuous and possess left limits, with a Skorokhod topology.

Occasionally we will need to refer to the probability space where our processes are defined, and we denote it by $(\Omega, \mathscr{F}, P)$.

Functions of time are written $f(t)$ or f_t according to notational convenience.

κ stands for a positive constant which may vary from place to place.

2. The multitype measure branching process and its occupation time process

The ingredients of the MMBP are understood from the approximating particle system and the scaling, which we describe presently.

The system consists of particles of $k \geq 1$ types in $\mathbb{R}^d$, $d \geq 1$, which evolve in the following manner. At time $t = 0$ particles of type i are distributed according to a random $\mathcal{M}_p(\mathbb{R}^d)$-valued point measure μ_i, $i = 1, \ldots, k$, independently of the other types. In addition particles of type i immigrate according to a Poisson random field on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\beta_i \Lambda_{\mathscr{C}_i}$, where $\beta_i \geq 0$ and $\mathscr{C}_i$ is a Borel set of $\mathbb{R}^d \times \mathbb{R}_+$, $i = 1, \ldots, k$, these random fields being independent of each other and of the initial random measures. (We assume the $\mathscr{C}_i$ are sufficiently smooth so that the integrals over the t-sections of these sets are well behaved).

Each particle of type i independently migrates as a symmetric stable process in $\mathbb{R}^d$ with exponent α_i, $0 < \alpha_i \leq 2$, and at the end of an exponentially distributed lifetime with parameter V_i it produces offspring of each type according to a branching law

$$\{p_i(j_1, \ldots, j_k), j_1, \ldots, j_k = 0, 1, \ldots\}, \quad i = 1, \ldots, k,$$

(i.e. $p_i(j_1, \ldots, j_k)$ is the probability that j_h particles of type h are produced, $h = 1, \ldots, k$). The mean and the second and third factorial moments of the branching law p_i, which we assume to be finite, are given by

$$m_i^{(1)}(h) = \sum_{j_1,\ldots,j_k \geq 0} p_i(j_1, \ldots, j_k) j_h,$$

$$m_i^{(2)}(h, l) = \sum_{j_1,\ldots,j_k \geq 0} p_i(j_1, \ldots, j_k) j_h (j_l - \delta_{hl}),$$

$$m_i^{(3)}(h, l, n) = \sum_{j_1,\ldots,j_k \geq 0} p_i(j_1, \ldots, j_k) j_h (j_l - \delta_{hl})(j_n - \delta_{nh} - \delta_{nl}),$$

$i, h, l, n = 1, \ldots, k$, where δ_{ij} is the Kronecker delta. We define the mean matrix

$$\boldsymbol{M}^{(1)} = [m_i^{(1)}(h)]_{i,h=1,\ldots,k},$$

and the bilinear functions

$$M_i^{(2)}(x, y) = \sum_{h,l=1}^{k} m_i^{(2)}(h, l) x_h y_l,$$

$$\boldsymbol{x} = (x_1, \ldots, x_k),\ \boldsymbol{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k,\quad i = 1, \ldots, k.$$

Let $N_i(t, A)$ denote the number of particles of type i present in the Borel set $A \subset \mathbb{R}^d$ at time t, $i = 1, \ldots, k$, and consider the process

$$\boldsymbol{N} \equiv \{\boldsymbol{N}(t), t \geq 0\} \equiv \{(N_1(t), \ldots, N_k(t)), t \geq 0\}.$$

Under the conditions above $\boldsymbol{N}$ has a version in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$ for $p > \frac{1}{2}d$ [19].

We now introduce the scaling, indexed by $K \geq 1$, which yields the MMBP.

The initial random measures are assumed to converge: $\mu_i^K \Rightarrow \mu_i$ as $K \to \infty$, where μ_i is a deterministic element of $\mathcal{M}_p(\mathbb{R}^d)$, $i = 1, \ldots, k$. (Random limits μ_i can also be considered.) The Poisson intensities of the immigrant particles are $K\beta_i$, $i = 1, \ldots, k$. The lifetime distribution parameter of a particle of type i is KV_i, $i = 1, \ldots, k$. The moments of the branching law $\{p_i^K\}_{i=1,\ldots,k}$ satisfy the conditions

$$\begin{aligned} & m_i^{K,(1)}(h) = \delta_{ih} + d_{ih}^K / K, \quad \lim_{K\to\infty} d_{ih}^K = d_{ih}, \\ & \lim_{K\to\infty} m_i^{K,(2)}(h, l) = m_i^{(2)}(h, l), \\ & \sup_{K\geq 1} m_i^{K,(3)}(h, l, n) < +\infty, \quad i, h, l, n = 1, \ldots, k. \end{aligned} \tag{2.1}$$

(The condition on the third moments may be replaced by a weaker one of Lindeberg type, as in [16, Theorem 4.2.2].)

It is important to observe the meaning of the conditions (2.1). It is easy to show (see [19]) that these conditions imply that the branching law is asymptotically critical with mean matrix $\boldsymbol{I} = [\delta_{ij}]$, and

$$m_i^{(2)}(h, l) = 0 \quad \text{if } h \neq i \text{ or } l \neq i. \tag{2.2}$$

Hence in the limit the particles of each type produce only offspring of their own type, and the average number of offspring of each particle is one. However, the matrix $\boldsymbol{D}^K \equiv [d_{ij}^K]$, which multiplied by K^{-1} measures the discrepancy of the mean matrix $\boldsymbol{M}^{K,(1)} = [m_i^{K,(1)}(j)]$ from the critical mean matrix $\boldsymbol{I}$, converges to a matrix $\boldsymbol{D} \equiv [d_{ij}]$ as $K \to +\infty$. We call $\boldsymbol{D}$ the *interaction of types matrix* because it represents the asymptotic effect of mutations of types in the system (if it is not diagonal).

In addition to $\boldsymbol{D} = [d_{ij}]$ we will use the following notation:

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k), \quad \boldsymbol{\Delta}_\alpha = \mathrm{diag}(\Delta_{\alpha_1}, \ldots, \Delta_{\alpha_k}), \quad V = \mathrm{diag}(V_1, \ldots, V_k),$$

$$\boldsymbol{M}^{(2)} = (M_1^{(2)}, \ldots, M_k^{(2)}), \quad M_i^{(2)}(x, x) = m_i^{(2)} x_i^2,$$

$$m_i^{(2)} = m_i^{(2)}(i, i), \quad i = 1, \ldots, k.$$

Let $\boldsymbol{N}^K$ denote the process $\boldsymbol{N}$ defined above subject to the previous scaling. In addition we assume that each particle of every type has a mass equal to $1/K$, and we consider the mass process $\boldsymbol{X}^K \equiv K^{-1}\boldsymbol{N}^K$. The main result proved in [10] (assuming the μ_i^K are Poisson random fields) is the following:

Proposition 2.1. *$\boldsymbol{X}^K \Rightarrow \boldsymbol{X}$ in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$ as $K \to \infty$, where $\boldsymbol{X}$ is an $(\mathcal{M}_p(\mathbb{R}^d))^k$-valued Markov process which is the unique continuous solution of the following martingale problem: For each $\boldsymbol{\varphi} \in (K_p(\mathbb{R}^d))^k$ the process*

$$\langle \boldsymbol{X}(t), \boldsymbol{\varphi}\rangle - \langle \boldsymbol{\mu}, \boldsymbol{\varphi}\rangle - \int_0^t (\langle \boldsymbol{X}(s), (\boldsymbol{\Delta}_\alpha + \boldsymbol{VD})\boldsymbol{\varphi}\rangle + \langle \boldsymbol{\beta}\Lambda_{\mathscr{C}_s}, \boldsymbol{\varphi}\rangle)\,\mathrm{d}s, \quad t \geq 0, \tag{2.3}$$

is a martingale with increasing process

$$\int_0^t \langle \boldsymbol{X}(s), \boldsymbol{VM}^{(2)}(\boldsymbol{\varphi}, \boldsymbol{\varphi})\rangle\,\mathrm{d}s, \quad t \geq 0. \qquad \square$$

Remark. $\boldsymbol{X}^K \Rightarrow \boldsymbol{X}$ actually takes place in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\dot{\mathbb{R}}^d))^k)$, where $\dot{\mathbb{R}}^d$ is a one-point compactification of $\mathbb{R}^d$ (note that $\mathcal{M}_p(\dot{\mathbb{R}}^d)$ is locally compact [12]; on this point see also [7]). p is restricted to $p > \frac{1}{2}d$, and in addition $p < \frac{1}{2}(d + \min\{\alpha_1, \ldots, \alpha_k\})$ if $\min\{\alpha_1, \ldots, \alpha_k\} < 2$. Henceforth these conditions on p are assumed. It is shown in [19] that $\boldsymbol{X}$ has a version in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$.

The MMBP is the solution $\boldsymbol{X}$ of the above martingale problem. Another characterization of $\boldsymbol{X}$, also given in [10], is a continuous $(\mathcal{M}_p(\mathbb{R}^d))^k$-valued Markov process whose transition Laplace functional is given by

$$\begin{aligned} L_t(\boldsymbol{\varphi}) &\equiv E[\exp\{-\langle \boldsymbol{X}(t), \boldsymbol{\varphi}\rangle\} \mid \boldsymbol{X}(0) = \boldsymbol{\mu}] \\ &= \exp\left\{-\langle \boldsymbol{\mu}, \boldsymbol{H}(t)\rangle - \int_0^t \langle \boldsymbol{\beta}\Lambda_{\mathscr{C}_s}, \boldsymbol{H}(t-s)\rangle\,\mathrm{d}s\right\}, \\ &\boldsymbol{\varphi} \in (C_p(\mathbb{R}^d)_+)^k, \quad \boldsymbol{\mu} \in (\mathcal{M}_p(\mathbb{R}^d))^k, \quad t \geq 0, \end{aligned} \tag{2.4}$$

where $\boldsymbol{H}(t, x)(\equiv \boldsymbol{H}_t(x) \equiv \boldsymbol{H}(\boldsymbol{\varphi}, t, x) \equiv \boldsymbol{H}_t(\boldsymbol{\varphi}, x))$ is a non-linear semigroup, the so-called *cumulant semigroup*, which is the unique global (classical) solution of the initial value problem

$$\begin{aligned}&\frac{\partial}{\partial t} \boldsymbol{H}(t) = (\boldsymbol{\Delta}_\alpha + \boldsymbol{V}\boldsymbol{D})\boldsymbol{H}(t) - \tfrac{1}{2}\boldsymbol{V}\boldsymbol{M}^{(2)}(\boldsymbol{H}(t), \boldsymbol{H}(t)), \quad t \geqslant 0,\\ &\boldsymbol{H}(0) = \boldsymbol{\varphi},\end{aligned} \tag{2.5}$$

when $\boldsymbol{\varphi} \in \mathscr{D}(\boldsymbol{\Delta}_\alpha) = \mathscr{D}(\Delta_{\alpha_1}) \times \cdots \times \mathscr{D}(\Delta_{\alpha_k})$, or the unique global (mild) solution of the equation

$$\boldsymbol{H}(t) = T_t\boldsymbol{\varphi} - \tfrac{1}{2}\int_0^t \boldsymbol{V}T_{t-s}\boldsymbol{M}^{(2)}(\boldsymbol{H}(s), \boldsymbol{H}(s))\,\mathrm{d}s, \quad t \geqslant 0, \tag{2.6}$$

when $\boldsymbol{\varphi} \in (C_p(\mathbb{R}^d))^k$, where $\{T_t, t \geqslant 0\}$ is the semigroup generated by $\boldsymbol{\Delta}_\alpha + \boldsymbol{V}\boldsymbol{D}$.

(Equation (2.5) has a unique solution on a maximal interval $[0, T_{\max}]$ [22]. For a diagonal matrix $\boldsymbol{D}$, $T_{\max} = +\infty$ [5]. For a general matrix $\boldsymbol{D}$, also $T_{\max} = +\infty$ because this corresponds to a bounded linear perturbation of the equation with diagonal $\boldsymbol{D}$.)

We observe that since $\boldsymbol{\mu}$ can be chosen arbitrarily, (2.4) implies that if $\boldsymbol{H}(0)$ has non-negative components then $\boldsymbol{H}(t)$ also has non-negative components for all $t > 0$. We also note that if the initial measures $\mu_1, \ldots, \mu_k$ are finite, and the spatial parts of the immigration sets $\mathscr{C}_1, \ldots, \mathscr{C}_k$ are bounded, then the components of the MMBP are finite measure-valued for all $t > 0$ (see [24]).

The *occupation time process* of the MMBP $\boldsymbol{X}$ is the $(\mathscr{M}_p(\mathbb{R}^d))^k$-valued process $\boldsymbol{Y}$ defined by

$$\langle \boldsymbol{Y}(t), \boldsymbol{\varphi}\rangle = \int_0^t \langle \boldsymbol{X}(s), \boldsymbol{\varphi}\rangle\,\mathrm{d}s, \quad t \geqslant 0, \ \boldsymbol{\varphi} \in (C_p(\mathbb{R}^d))^k.$$

In the (critical) monotype case this process was introduced and studied by Iscoe [13], and its properties (in particular limit behaviour) have been investigated by Cox and Griffeath [1], Dynkin [4], Fleischmann [8], Fleischmann and Gärtner [9] and Iscoe [14, 15].

We will describe the process $\boldsymbol{Y}$ by generalizing the method of El Karoui and Roelly [7] in the monotype case, i.e. using the exponential martingale characterization of $\boldsymbol{X}$ and making a change of probability on the distribution of $\boldsymbol{X}$.

Proposition 2.2. *The Laplace functional of $\boldsymbol{Y}(t)$ is given by*

$$\begin{aligned}&E\exp\{-\langle \boldsymbol{Y}(t), \boldsymbol{\varphi}\rangle\} = \exp\left\{-\langle \boldsymbol{\mu}, \tilde{\boldsymbol{H}}_t(\boldsymbol{\varphi}, \boldsymbol{0})\rangle - \int_0^t \langle \boldsymbol{\beta}\boldsymbol{\Lambda}_{\mathscr{C}_s}, \tilde{\boldsymbol{H}}_{t-s}(\boldsymbol{\varphi}, \boldsymbol{0})\rangle\,\mathrm{d}s\right\},\\ &\boldsymbol{\varphi} \in (C_p(\mathbb{R}^d)_+)^k,\end{aligned} \tag{2.7}$$

where $\boldsymbol{X}(0) = \boldsymbol{\mu} \in (\mathscr{M}_p(\mathbb{R}^d))^k$, and $\tilde{\boldsymbol{H}}(t) \equiv \tilde{\boldsymbol{H}}_t(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is the non-linear cumulant semigroup which solves

$$\begin{aligned}&\frac{\partial}{\partial t} \tilde{\boldsymbol{H}}(t) = \boldsymbol{\varphi} + (\boldsymbol{\Delta}_\alpha + \boldsymbol{V}\boldsymbol{D})\tilde{\boldsymbol{H}}(t) - \tfrac{1}{2}\boldsymbol{V}\boldsymbol{M}^{(2)}(\tilde{\boldsymbol{H}}(t), \tilde{\boldsymbol{H}}(t)), \quad t \geqslant 0,\\ &\tilde{\boldsymbol{H}}(0) = \boldsymbol{\psi} \in (K_p(\mathbb{R}^d))^k.\end{aligned} \tag{2.8}$$

Proof. By the proof of Proposition 5.5 of [10] we know that X is characterized by the following martingale property: if P denotes the distribution of X on $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$, then for each $\psi \in (K_p(\mathbb{R}^d))^k$ the process

$$W_t(\psi) \equiv \exp\left\{ -\left(\langle X(t), \psi\rangle - \int_0^t (\langle X(s), (\Delta_\alpha + VD)\psi\rangle + \langle \beta \Lambda_{\mathscr{C}_s}, \psi\rangle - \langle X(s), \tfrac{1}{2} VM^{(2)}(\psi, \psi)\rangle)\, \mathrm{d}s \right)\right\}, \quad t \geq 0, \tag{2.9}$$

is a P-local martingale.

Let Q be the probability on $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_p(\mathbb{R}^d))^k)$ defined on $\mathscr{F}_t$, the σ-algebra generated by $\{X(s), s \leq t\}$, by

$$Q = \exp\left\{ - \int_0^t \langle X(s), \varphi\rangle \, \mathrm{d}s \right\} P.$$

Then Q is the distribution of an MMBP associated to the non-linear cumulant semigroup $\tilde{H}_t(\varphi, \cdot)$. In particular, if ζ is the lifetime of X, which is equal to $+\infty$ P-a.s., then from (2.9) applied to $\psi = 0$ we have

$$\begin{aligned} &E_P \exp\left\{ - \int_0^t \langle X(s), \varphi\rangle \, \mathrm{d}s \right\} \\ &= E_Q 1_{\{t<\zeta\}} \\ &= \exp\left\{ -\langle \mu, \tilde{H}_t(\varphi, 0)\rangle - \int_0^t \langle \beta \Lambda_{\mathscr{C}_s}, \tilde{H}_{t-s}(\varphi, 0)\rangle \, \mathrm{d}s \right\}. \qquad \square \end{aligned}$$

The occupation time process can be extended to a linear functional of X which acts on time-dependent functions (see [5, Theorem 6; 11, Proposition 2.3]).

3. Properties of the multitype measure branching process

The following results are known in the monotype case without immigration: For fixed $t > 0$, $X(t)$ has a random support B such that $\dim(B) = \min\{\alpha, d\}$ a.s. [3, 27]. Assuming spatial homogeneity, i.e. the initial measure μ is proportional to Lebesgue measure, then for fixed $t > 0$, if $d = 1$, $X(t)$ has a density if and only if $1 < \alpha \leq 2$ [24], and if $d \leq 3$, $Y(t)$ has a density if and only if $\frac{1}{2}d < \alpha \leq 2$ [8]. The process X satisfies a stochastic evolution equation where the driving term is a martingale measure [21], and in the case of existence of a density process, this process satisfies a stochastic differential equation [18, 21]. In this section we will extend these results for the multitype case.

Clearly, if the interaction matrix D is diagonal, the components of the MMBP are independent monotype measure branching processes, and in this case their properties are just those of the monotype case. The new problems arise from the non-zero off-diagonal elements of $\boldsymbol{D}$.

We recall that the MMBP and the occupation time process are denoted by $\boldsymbol{X}=(X_1,\ldots,X_k)$ and $\boldsymbol{Y}=(Y_1,\ldots,Y_k)$, respectively.

3.1. *Hausdorff dimension of a random support*

We will assume here that there is no immigration ($\boldsymbol{\beta}=0$).

We will prove that even in the presence of interaction of types (i.e. the matrix $\boldsymbol{D}$ is not diagonal), the results obtained by Zähle [27] in the monotype case ($k=1$, $D\in\mathbb{R}$) can be extended to the multitype case.

Theorem 3.1. *For each $t>0$, $\boldsymbol{X}(t)$ has a random support*

$$\boldsymbol{B}(\omega)=B_1(\omega)\times\cdots\times B_k(\omega)\subset(\mathbb{R}^d)^k,\quad \omega\in\Omega,$$

such that for $i=1,\ldots,k$,

$$\dim(B_i)=\min\{\alpha_i,d\}\quad a.s.$$

and

$$X_i(t,B_i\cap K)=X_i(t,K)\quad a.s.$$

for every compact subset K of $\mathbb{R}^d$.

Zähle [27] proved this result in the monotype case using the basic ideas of Dawson and Hochberg [3] for the upper bound and a new criterion for the lower bound. The upper bound for the Hausdorff dimension of the support of a (general) random measure is estimated by a probabilistic generalization of the dimension of the Cantor set, that is, by means of a local analysis of the number of subcubes of a fine subdivision charged by the random measure. Moreover, by using the fact that the measure branching process X_t is infinitely divisible, it is decomposed as a random sum of random measures whose spatial diffusion is better controlled. The lower bound results from an estimate of the Campbell measure.

However, we cannot apply directly Zähle's criteria, since he studies random measures, and in the multitype case we have random vector measures whose components in addition interact. We will give here a proof following the same steps as in [27], but which is not a straightforward generalization of the monotype case.

$\boldsymbol{X}(t)$ is infinitely divisible, and, by a direct generalization of Kallenberg [17], its Lévy–Khintchine decomposition is

$$\begin{aligned}
&E\exp\{-\langle \boldsymbol{X}(t),\boldsymbol{\varphi}\rangle\}\\
&\quad=\exp\{-\langle \boldsymbol{X}(0),\boldsymbol{H}(\boldsymbol{\varphi},t)\rangle\},\\
&\quad=\exp\left\{-\sum_{j=1}^{k}\int_{\mathbb{R}^d}\int_{(\mathcal{M}_p-\{0\})^k}(1-\exp\{-\langle \boldsymbol{m},\boldsymbol{\varphi}\rangle\})Q_j(t,x,\mathrm{d}\boldsymbol{m})X_j(0)(\mathrm{d}x)\right\}.
\end{aligned}$$

We will firstly derive some results we shall need. Letting $\boldsymbol{X}(0)=(0,\dots,\delta_x,\dots,0)$, δ_x being the jth component, we have

$$H_j(\boldsymbol{\varphi},t)(x)=\int_{(\mathcal{M}_p-\{0\})^k}(1-\exp\{-\langle\boldsymbol{m},\boldsymbol{\varphi}\rangle\})Q_j(t,x,\mathrm{d}\boldsymbol{m}),$$

and $H_j^i(\varphi,t)(x)\equiv H_j(\boldsymbol{\varphi},t)(x)$ with $\boldsymbol{\varphi}=(0,\dots,\varphi,\dots,0)$, φ being the ith component. For the constant function $\varphi=\lambda$,

$$H_j^i(\lambda,t)(x)=\int_{(\mathcal{M}_p-\{0\})^k}(1-\exp\{-\langle m,\lambda\rangle\})Q_j(t,x,\mathrm{d}(0,\dots,m,\dots,0)), \quad (3.1)$$

where $\{H_j^i(\lambda,t)\}_j$ satisfies the system of equations (see (2.5))

$$\frac{\partial}{\partial t}H_j^i(t)=V_j\sum_{l=1}^k d_{jl}H_l^i(t)-\tfrac{1}{2}V_jM_jH_j^i(t)^2,\quad M_j\equiv m_j^{(2)},$$

$$H_j^i(0)=\lambda\,\delta_{ij}.$$

Unlike the monotype case, we cannot solve this system explicitly. However, all we will need is the approximate behaviour of the solution for small t. Since $H_j^i(t)\to 0$ as $t\to 0$ for $j\neq i$, then $H_j^j(t)$ satisfies approximately the same equation as in the monotype case for small t, and therefore

$$\begin{aligned}H_j^j(\lambda,t)&\cong\lambda V_jd_{jj}\,\mathrm{e}^{V_jd_{jj}t}(\lambda\tfrac{1}{2}V_jM_j(\mathrm{e}^{V_jd_{jj}t}-1)+V_jd_{jj})^{-1}\\&\cong\lambda/(\lambda\tfrac{1}{2}V_jM_jt+1)\end{aligned}$$

for small t. On the other hand, for $j\neq i$,

$$H_j^i(\lambda,t)\cong V_jd_{ji}tH_i^i(\lambda,t)\ll H_i^i(\lambda,t)$$

for small t. From (3.1) we have

$$\begin{aligned}Q_j^i(t,x,\mathcal{M}_p-\{0\})&\equiv Q_j(t,x,(0,\dots,\mathcal{M}_p-\{0\},\dots,0))\\&\equiv\lim_{\lambda\to\infty}H_j^i(\lambda,t)(x),\end{aligned}\quad (3.2)$$

and therefore

$$Q_j^j(t,x,\mathcal{M}_p-\{0\})\cong 2/V_jM_jt \quad (3.3)$$

and

$$Q_j^i(t,x,\mathcal{M}_p-\{0\})\ll 2/V_iM_it,\quad i\neq j, \quad (3.4)$$

for all x and small t. (The exchange of limits in λ and t is possible because $H_j^i(\lambda,t)$, as a cumulant, is an increasing function of λ).

We will consider the support of $X_i(t)$ for fixed i, and may take without loss of generality $i=1$ (for notational convenience). Since the matrix $\boldsymbol{D}$ is not diagonal in general, $X_1(t)$ depends on $X_j(0)$, $j=1,\dots,k$.

For $\varphi\in C_p(\mathbb{R}^d)$ we denote $\boldsymbol{\varphi}=(\varphi,\dots,0)$, and then

$$\langle X_1(t),\varphi\rangle=\langle\boldsymbol{X}(t),\boldsymbol{\varphi}\rangle.$$

The following lemma, which generalizes Step 2 of the proof of Theorem 7.1 [27] gives a Poissonian decomposition of $X_1(t)$.

Lemma 3.2. *$X_1(t)$ has the same distribution as*

$$\sum_{j=1}^{k}\sum_{i=1}^{W_t^j} C_t^{Z_i^j,j,i},$$

where

(i) *W_t^j, $j=1,\dots,k$, are independent Poisson random variables with respective parameters*

$$w_t^j=\int_{\mathbb{R}^d} Q_j^1(t,x,\mathcal{M}_p-\{0\})X_j(0)(\mathrm{d}x), \quad j=1,\dots,k, \tag{3.5}$$

where $Q_j^1(t)$ is defined in (3.2).

(ii) *$\{C_t^{x,j,i}\}_{i,j}$ are independent random measures. For each i, $C_t^{x,j,i}$ has distribution $Q_j^1(t,x,\cdot)/Q_j^1(t,x,\mathcal{M}_p-\{0\})$ and Laplace functional $1-H_j^1(\cdot,t)(x)/Q_j^1(t,x,\mathcal{M}_p-\{0\})$.*

(iii) *$\{Z_i^j\}_{i,j}$ are independent $\mathbb{R}^d$-valued r.v.'s, and for each i, Z_i^j has distribution $X_j(0)/\langle X_j(0),1\rangle$.* □

We will show, similarly as in Step 4 of the proof of Theorem 7.1 [27], that $X_1(t)$ charges essentially small balls centered at the points Z_i^j.

Lemma 3.3. *Let $\{B_{ji}^n\}_{j,i}$ be balls of radius $n/2^n$ centered at the points Z_i^j, $i=1,\dots,W_{t_n}^j$, $j=1,\dots,k$, with $t_n=2^{-\alpha_1 n}$. Then, for large n,*

$$E\left(X_1\left(t_n,\left(\bigcup_{j=1}^{k}\bigcup_{i=1}^{W_{t_n}^j} B_{ji}^n\right)^{\mathrm{c}}\right)\right)^2 \leqslant \kappa\sum_{j=1}^{k}(\langle X_j(0),1\rangle+\langle X_j(0),1\rangle^2)(2^{-\alpha_1 n}+n^{-2\alpha_1}).$$

Proof. Using the decomposition given in Lemma 3.2, we have

$$E\left(X_1\left(t_n,\left(\bigcup_{j=1}^{k}\bigcup_{i=1}^{W_{t_n}^j} B_{ji}^n\right)^{\mathrm{c}}\right)\right)^2 \leqslant \kappa\sum_{j=1}^{k}\left(EW_{t_n}^j E(C_{t_n}^{Z_1^j,j,1}((B_{j1}^n)^{\mathrm{c}}))^2+E(W_{t_n}^j)^2(EC_{t_n}^{Z_1^j,j,1}((B_{j1}^n)^{\mathrm{c}}))^2\right)$$

with $EW_{t_n}^j=w_{t_n}^j$.

The moments of $C_t^{x,j,1}$ are derived from its Laplace functional (see the proof of Theorem 3.5 below for a detailed computation of this sort):

$$E\langle C_t^{x,j,1},\varphi\rangle=\langle X_j(0),1\rangle(w_t^j)^{-1}f_{j1}(t)S_t^{\alpha_1}\varphi(x)$$

and

$$E\langle C_t^{x,j,1},\varphi\rangle^2=\langle X_j(0),1\rangle(w_t^j)^{-1}V_j\int_0^t\sum_{l=1}^{k}f_{jl}(t-s)S_{t-s}^{\alpha_l}M_l(f_{l1}(s)S_s^{\alpha_1}\varphi)^2(x)\,\mathrm{d}s,$$

where $f_{ji}(t)=(\exp\{\boldsymbol{VD}t\})_{ji}$.

The same estimates used in Step 5 of Theorem 7.1 in [27] give the desired result. □

In the next lemma we show that the hypotheses of Lemma 3.1 of [3] are fulfilled.

Lemma 3.4. *If $\beta < \alpha_1$, there exists sequences $\eta_n, \delta_n \to 0+$ such that for large n,*

$$P\left(\frac{\log N_n^{1/n^\beta}(X_1(t_n))}{\log 2^n} > \alpha_1 + \eta_n\right) \leqslant \kappa \sum_{j=1}^{k} (\langle X_j(0), 1\rangle + \langle X_j(0), 1\rangle^2)\delta_n. \tag{3.6}$$

Proof. By Lemma 3.3,

$$\begin{aligned}
&P(N_n^{1/n^\beta}(X_1(t_n)) > (2n)^d n^2 2^{\alpha_1 n}) \\
&\leqslant P\left(\sum_{j=1}^{k} W_{t_n}^j > n^2 2^{\alpha_1 n}\right) \\
&\quad + P\left(\sum_{j=1}^{k} W_{t_n}^j \leqslant n^2 2^{\alpha_1 n}, X_1\left(t_n, \left(\bigcup_{j=1}^{k} \bigcup_{i=1}^{W_{t_n}^j} B_{ji}^n\right)^c\right) > 1/n^\beta\right) \\
&\leqslant \sum_{j=1}^{k} w_{t_n}^j / n^2 2^{\alpha_1 n} + \kappa \sum_{j=1}^{k} (\langle X_j(0), 1\rangle + \langle X_j(0), 1\rangle^2) n^{2\beta}(2^{-\alpha_1 n} + n^{-2\alpha_1})
\end{aligned}$$

for large n, and by (3.2), (3.3) and (3.5),

$$w_{t_n}^1 \leqslant \kappa \langle X_1(0), 1\rangle 2^{\alpha_1 n}$$

and

$$w_{t_n}^j \ll \kappa \langle X_j(0), 1\rangle 2^{\alpha_1 n}, \quad j = 2, \ldots, k. \qquad \square$$

Proof of Theorem 3.1. The upper bound results from Lemmas 3.2, 3.3 and 3.4. The lower bound is a simple generalization of Theorem 7.2 in [27]. $\square$

3.2. Existence of densities

In this section we shall assume that the initial measures μ_i satisfy the condition

$$\sup_{0 \leqslant t \leqslant T} \sup_{y \in R^d} \int_{\mathbb{R}^d} p_t^{\alpha_j}(x, y)\mu_i(\mathrm{d}x) < \infty \tag{3.7}$$

for fixed $T > 0$, where $p_t^{\alpha_j}(x, y)$ denotes the transition density of the symmetric stable process with exponent α_j, for $i, j = 1, \ldots, k$. This condition holds in particular for measures that are dominated by a constant times the Lebesgue measure and for finite sums of Dirac measures.

We will prove that the conditions for existence of densities of the components of $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ are the same with and without interaction of types, and hence the same as in the monotype case (see [8, 24] for the spatially homogeneous monotype case without immigration).

Theorem 3.5. (i) *If $d = 1$, then for each $i = 1, \ldots, k$ and fixed $t > 0$, $X_i(t)$ has an $L^2(\Omega)$-density if and only if $1 < \alpha_i \leqslant 2$, and this density is $L^2(\Omega)$-continuous on $\mathbb{R}^d$.*

(ii) *If $d \leqslant 3$, then for each $i = 1, \ldots, k$ and fixed $t > 0$, $Y_i(t)$ has an $L^2(\Omega)$-density if and only if $\frac{1}{2}d < \alpha_i \leqslant 2$, and this density is $L^2(\Omega)$-continuous on $\mathbb{R}^d$.*

Proof. (i) The second moment of $\langle \boldsymbol{X}(t), \boldsymbol{\varphi}\rangle$ is derived from the first and second derivatives with respect to λ of the Laplace functional $L_t(\lambda\boldsymbol{\varphi})$ given by (2.4), setting $\lambda = 0$. We find, since $\boldsymbol{H}(\boldsymbol{0}, t) = 0$,

$$E\langle \boldsymbol{X}(t), \boldsymbol{\varphi}\rangle^2 = \left(\langle \boldsymbol{\mu}, \boldsymbol{H}'(t)\rangle + \int_0^t \langle \boldsymbol{\beta}\boldsymbol{\Lambda}_{\mathscr{C}_s}, \boldsymbol{H}'(t-s)\rangle\, \mathrm{d}s\right)^2$$
$$-\langle \boldsymbol{\mu}, \boldsymbol{H}''(t)\rangle - \int_0^t \langle \boldsymbol{\beta}\boldsymbol{\Lambda}_{\mathscr{C}_s}, H''(t-s)\rangle\, \mathrm{d}s, \tag{3.8}$$

where

$$\boldsymbol{H}'(t) \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} \boldsymbol{H}(\lambda\boldsymbol{\varphi}, t)|_{\lambda=0} \quad \text{and} \quad \boldsymbol{H}''(t) \equiv \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \boldsymbol{H}(\lambda\boldsymbol{\varphi}, t)|_{\lambda=0}$$

satisfy

$$\frac{\partial}{\partial t} \boldsymbol{H}'(t) = (\boldsymbol{\Delta}_\alpha + \boldsymbol{VD})\boldsymbol{H}'(t),$$
$$\boldsymbol{H}'(0) = \boldsymbol{\varphi},$$

and

$$\frac{\partial}{\partial t} \boldsymbol{H}''(t) = (\boldsymbol{\Delta}_\alpha + \boldsymbol{VD})\boldsymbol{H}''(t) - \boldsymbol{VM}^{(2)}(\boldsymbol{H}'(t), \boldsymbol{H}'(t)),$$
$$\boldsymbol{H}''(0) = \boldsymbol{0},$$

respectively. Hence

$$\boldsymbol{H}'(t) = T_t\boldsymbol{\varphi} \quad \text{and} \quad \boldsymbol{H}''(t) - \int_0^t T_{t-s}\boldsymbol{VM}^{(2)}(T_s\boldsymbol{\varphi}, T_s\boldsymbol{\varphi})\, \mathrm{d}s.$$

Writing $f(t) \equiv \exp\{\boldsymbol{VD}t\}$, we have

$$(T_t\boldsymbol{\varphi})_j = \sum_{l=1}^k f_{jl}(t) S_t^{\alpha_l}\varphi_l, \quad j = 1, \ldots, k;$$

in particular, for $\boldsymbol{\varphi} = (0, \ldots, \varphi, \ldots, 0)$, φ being the ith component,

$$(T_t\boldsymbol{\varphi})_j = f_{ji}(t) S_t^{\alpha_i}\varphi, \quad j = 1, \ldots, k.$$

Substituting into (3.8) we obtain, for each $i = 1, \ldots, k$,

$$E\langle X_i(t), \varphi\rangle^2 = \left(\sum_{j=1}^k f_{ji}(t) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t^{\alpha_i}(x, y)\varphi(y)\, \mathrm{d}y\, \mu_j(\mathrm{d}x)\right.$$
$$\left. + \sum_{j=1}^k \beta_j \int_0^t f_{ji}(t-r) \int_{\mathscr{C}_{j,r}} \int_{\mathbb{R}^d} p_{t-r}^{\alpha_i}(x, y)\varphi(y)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}r\right)^2$$
$$+ \sum_{j=1}^k \sum_{l=1}^k V_l m_l^{(2)} \int_0^t f_{jl}(t-s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}^{\alpha_l}(x, y)$$
$$\cdot \left(f_{li}(s) \int_{\mathbb{R}^d} p_s^{\alpha_i}(y, z)\varphi(z)\, \mathrm{d}z\right)^2 \mathrm{d}y\, \mu_j(\mathrm{d}x)\, \mathrm{d}s$$

$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j V_l m_l^{(2)}\int_0^t\int_0^{t-r} f_{jl}(t-r-s)\int_{\mathscr{C}_{j,r}}\int_{\mathbb{R}^d} p_{t-r-s}^{\alpha_l}(x,y)$$

$$\cdot\left(f_{li}(s)\int_{\mathbb{R}^d} p_s^{\alpha_i}(y,z)\varphi(z)\,\mathrm{d}z\right)^2 \mathrm{d}y\,\mu_j(\mathrm{d}x)\,\mathrm{d}s\,\mathrm{d}r, \quad (3.9)$$

which can be written as

$$E\langle X_i(t),\varphi\rangle^2=\iint_{\mathbb{R}^d\times\mathbb{R}^d} k_t^i(y,z)\varphi(y)\varphi(z)\,\mathrm{d}y\,\mathrm{d}z, \quad (3.10)$$

where $k_t^i(y,z)$ is a measurable kernel on $\mathbb{R}^d\times\mathbb{R}^d$.

We will apply the following criterion: a random field whose second moment is given by an expression like (3.10) possesses an $L^2(\Omega)$-density if and only if the function $y\to(k_t^i(y,y))^{1/2}$ is locally integrable (see [20, Theorem 3, in the case $d=1$]). We observe from (3.9) that the local integrability of $(k_t^i(y,y))^{1/2}$ is determined by the behaviour of $p_s^{\alpha_i}(y,y)$ near $s=0$. Indeed, using condition (3.7) and the Chapman–Kolmogorov relation, from (3.9) we have

$$|k_t^i(y,y)|\leqslant\kappa\left(\sum_{j=1}^{k}|f_{ji}(t)|^2+\sum_{j=1}^{k}\left(\int_0^t|f_{ji}(r)|\,\mathrm{d}r\right)^2\right.$$

$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^t|f_{jl}(t-s)|f_{li}^2(s)p_{2s}^{\alpha_i}(y,y)\,\mathrm{d}s$$

$$\left.+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^t\int_0^{t-r}|f_{jl}(t-r-s)|f_{li}^2(s)p_{2s}^{\alpha_i}(y,y)\,\mathrm{d}s\,\mathrm{d}r\right). \quad (3.11)$$

But the scaling property of the symmetric stable density implies $p_u^{\alpha_i}(y,y)=u^{-d/\alpha_i}p_1^{\alpha_i}(0,0)$. Hence $|k_t^i(y,y)|^{1/2}$ is locally integrable if and only if the right hand side of (3.11), which is independent of y, is finite, i.e. if and only if $\alpha_i>1$, since $d=1$.

Thus $X_i(t)$ has an $L^2(\Omega)$-density if and only if $\alpha_i>1$, and this density is $L^2(\Omega)$-continuous by Theorem 5 in [20].

(ii) The proof is similar to the previous one, so we will only give a sketch. From (2.7) and (2.8) we obtain

$$E\langle \boldsymbol{Y}(t),\boldsymbol{\varphi}\rangle^2=\left(\langle\boldsymbol{\mu},\tilde{\boldsymbol{H}}'(t)\rangle+\int_0^t\langle\boldsymbol{\beta}\boldsymbol{\Lambda}_{\mathscr{C}_s},\tilde{\boldsymbol{H}}'(t-s)\rangle\,\mathrm{d}s\right)^2$$

$$-\langle\boldsymbol{\mu},\tilde{\boldsymbol{H}}''(t)\rangle-\int_0^t\langle\boldsymbol{\beta}\boldsymbol{\Lambda}_{\mathscr{C}_s},\tilde{\boldsymbol{H}}''(t-s)\rangle\,\mathrm{d}s,$$

where $\tilde{\boldsymbol{H}}'(t)$ and $\tilde{\boldsymbol{H}}''(t)$ satisfy

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{H}}'(t)=\boldsymbol{\varphi}+(\boldsymbol{\Delta}_\alpha+\boldsymbol{VD})\tilde{\boldsymbol{H}}'(t),$$

$$\tilde{\boldsymbol{H}}'(0)=\boldsymbol{0},$$

and

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{H}}''(t)=(\boldsymbol{\Delta}_\alpha+\boldsymbol{VD})\tilde{\boldsymbol{H}}''(t)-\boldsymbol{VM}^{(2)}(\tilde{\boldsymbol{H}}'(t),\tilde{\boldsymbol{H}}'(t)),$$

$$\tilde{\boldsymbol{H}}''(0)=\boldsymbol{0},$$

respectively. Hence

$$\tilde{\boldsymbol{H}}'(t)=\int_0^t T_s\boldsymbol{\varphi}\quad \mathrm{d}s$$

and

$$\tilde{\boldsymbol{H}}''(t)=-\int_0^t T_{t-s}\boldsymbol{VM}^{(2)}\left(\int_0^s T_r\boldsymbol{\varphi}\,\mathrm{d}r,\int_0^s T_r\boldsymbol{\varphi}\,\mathrm{d}r\right)\mathrm{d}s.$$

Therefore the computations are the same as in (i) with $T_t\boldsymbol{\varphi}$ replaced by $\int_0^t T_s\boldsymbol{\varphi}\,\mathrm{d}s$. Then the kernel will have a bound analogous to (3.11), which will be finite if and only if

$$\int_0^t\int_0^s\int_0^s p^{\alpha_i}_{u+v}(y,y)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s=p_1^{\alpha_i}(0,0)\int_0^t\int_0^s\int_0^s(u+v)^{-d/\alpha_i}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s<\infty,$$

i.e. if and only if $\alpha_i>\frac{1}{2}d$. $\square$

Remarks. (a) It can also be shown that the $L^2(\Omega)$-densities of $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ are $L^2(\Omega)$-continuous functions of t.

(b) We observe that the immigration has no effect on the conditions for existence of a density, and time integration of the MMBP yields a gain (with respect to dimension) in absolute continuity. For the overlapping dimensions $\alpha_i\leqslant d<2\alpha_i$ there is a sharp contrast between the supports of the MMBP and its occupation time process.

(c) In the monotype case with spatial homogeneity and no immigration, Fleischmann [8] has obtained similar results when the system has a certain branching law which belongs to the domain of normal attraction of a stable law with exponent <2; in this case the process does not have finite second moments.

3.3. Stochastic equations

Méléard and Roelly-Coppoletta [21] proved that the monotype measure branching process satisfies a stochastic evolution equation where the driving term is a martingale measure. We refer to [6, 25] for martingale measures. The following results are direct extensions of those for the monotype case.

Theorem 3.6. (i) *The MMBP* $\boldsymbol{X}$ *satisfies the stochastic evolution equation*

$$\begin{aligned}&\mathrm{d}\boldsymbol{X}(t)=((\boldsymbol{\Delta}_\alpha+\boldsymbol{VD})^*\boldsymbol{X}(t)+\boldsymbol{\beta}\Lambda_{\mathscr{C}_t})\,\mathrm{d}t+\mathrm{d}\boldsymbol{M}(t),\quad t\geqslant 0,\\&\boldsymbol{X}(0)=\boldsymbol{\mu},\end{aligned}\tag{3.12}$$

where $\boldsymbol{M}$ is an $(L^2(\Omega, \mathcal{F}, P))^k$-valued continuous orthogonal martingale measure with $(\mathcal{M}_p(\mathbb{R}^d))^k$-valued intensity measure $\boldsymbol{V}\boldsymbol{M}^{(2)}\boldsymbol{X}(t, \mathrm{d}x)\,\mathrm{d}t$. Moreover, $\boldsymbol{M} = (M^1, \ldots, M^k)$, where for each $i = 1, \ldots, k$, M^i is an $L^2(\Omega, \mathcal{F}, P)$-valued continuous orthogonal martingale measure with $\mathcal{M}_p(\mathbb{R}^d)$-valued intensity measure $V_i m_i^{(2)} X_i(t, \mathrm{d}x)\,\mathrm{d}t$, and $M^1, \ldots, M^k$ are orthogonal to each other.

(ii) *If $X_i(t)$ admits a density $x_t^i(x)$ for each $i = 1, \ldots, k$, and $t \in \mathbb{R}_+$, then the martingale measure M_t^i can be represented as $M_t^i = (V_i m_i^{(2)} x_t^i)^{1/2} W_t^i$, $i = 1, \ldots, k$, where $W^1, \ldots, W^k$ are independent white noises on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $\mathrm{d}x\,\mathrm{d}t$. In this case the density process $\boldsymbol{x}_t \equiv (x_t^1, \ldots, x_t^k)$ satisfies the stochastic evolution equation*

$$\mathrm{d}\boldsymbol{x}_t = ((\boldsymbol{\Delta}_\alpha + \boldsymbol{V}\boldsymbol{D})^* \boldsymbol{x}_t + \boldsymbol{\beta} \Lambda_{\mathscr{C}_t})\,\mathrm{d}t + (\boldsymbol{V}\boldsymbol{m}^{(2)} \boldsymbol{x}_t)^{1/2}\,\mathrm{d}\boldsymbol{W}_t, \tag{3.13}$$

where

$$(\boldsymbol{V}\boldsymbol{m}^{(2)}\boldsymbol{x}_t)^{1/2}\,\mathrm{d}\boldsymbol{W}_t \equiv ((V_1 m_1^{(2)} x_t^1)^{1/2}\,\mathrm{d}W_t^1, \ldots, (V_k m_k^{(2)} x_t^k)^{1/2}\,\mathrm{d}W_t^k). \qquad \square$$

See [21] concerning the interpretation of the formal equation (3.12) as a variation of constants equation.

Equation (3.13) is compatible with the result of Konno and Shiga [18] in the critical monotype case without immigration, who give an equation for x_t in the space of Schwartz distributions.

Final remark. Theorem 3.1 shows that the mutations do not produce an interaction of the dimensions of the supports of the components of the MMBP. An interesting question is if the supports themselves interact.

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