STATIONARY STATES OF INTERACTING BROWNIAN MOTIONS

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In Memoriam Alfréd Rényi

Abstract. We are interested in a description of stationary states of gradient dynamics of interacting Brownian particles. In contrast to lattice models, this problem does not seem to be solvable at a formal level of the stationary Kolmogorov equation. We can only study stationary states of a well controlled Markov process. In space dimensions four or less, for smooth and superstable pair potentials of finite range the non-equilibrium dynamics of interacting Brownian particles can be constructed in an explicitely defined deterministic set of locally finite configurations, see [F3]. This set is of full measure with respect to any canonical Gibbs state for the interaction and every canonical state is stationary. Assuming translation invariance of a stationary measure, and also the finiteness of its specific entropy with respect to an equilibrium Gibbs state, it is shown that it is canonical Gibbs, too.

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1. Introduction

The main purpose of this paper is to identify a class of stationary states of the following system of interacting particles as the set of translation invariant canonical Gibbs states with interaction U. The evolution law is given by an infinite system of stochastic differential equations,

$$d\omega_k = -\frac{1}{2}\sum_{j\neq k} \mathrm{grad}\, U(\omega_k - \omega_j)\, dt + dw_k\,, \quad \omega_k(0) = \sigma_k\,, \quad k \in S \tag{1.1}$$

where S is a countable index set, $w = (w_k)_{k\in S}$ is a family of independent standard d-dimensional Wiener processes, and each $\omega_k = \omega_k(t)\,, t \geq 0$ is assumed to be a continuous trajectory in $\mathbb{R}^d$. The potential $U : \mathbb{R}^d \mapsto \mathbb{R}$ is symmetric and superstable with finite range, that is $U(x) = U(-x)$, there is an $R > 0$ such that $U(x) = 0$ if $|x| > R$, and we have constants $A \geq 0$, $B > 0$ such that for any finite sequence $q_1, q_2, ..., q_n$ of not necessarily distinct points from $\mathbb{R}^d$

$$nA + \sum_{k=1}^{n}\sum_{j\neq k} U(q_k - q_j) \geq BN \tag{1.2}$$

where N is the number of pairs $\{j,k\}$ such that $|q_k - q_j| \leq R$, see [Ru1]. Let Ω denote the set of configurations having no limit points. Although the right hand side of (1.1) is certainly well defined for such, locally finite configurations $\omega \in \Omega$, to develop a satisfactory existence theory we have to restrict the configuration space in a much more radical way. On the other hand, the set of allowed configurations should be large enough to support a possibly wide set probability measures including Gibbs states with various interactions.

The first mathematical results concerning this model go back to R. Lang, see [La1] and [La2], where the existence of equilibrium dynamics, and also the canonical Gibbs property of reversible measures is proven. These dynamics is defined almost surely with respect to a Gibbs state with interaction U, see also the more sophisticated argument of [Os]. For a study of stationary measures in general, we need a more direct construction involving explicit bounds on the rate of convergence of solutions to finite subsystems (partial dynamics) when the number of active particles tends to infinity, see Section 3 below. Indeed, the problem of stationary measures can not be solved at a formal level of the stationary Kolmogorov equation, see [FFL] and [FLO], we really need that that our measure is realized as a stationary state of a well controlled Markov process.

For a generic, locally finite configuration $\omega = (\omega_k)_{k\in S}$ let $H(\omega, m, r)$ denote total energy in the ball $B(m,r)$ of center $m \in \mathbb{R}^d$ and radius $r \geq 1$, and for $\alpha \geq 0$ define

$$\begin{aligned} \overline{H}_\alpha(\omega) :&= \sup_{m\in\mathbb{Z}^d}\ \sup_{r\in\mathbb{N}} \frac{H\big(\omega, m, r g_\alpha^{1/d}(m)\big)}{r^d g_\alpha(m)} \quad \text{where} \\ H(\omega, m, r) &:= \frac{1}{2}\sum_{\omega_k\in B(m,r)}\ \sum_{j\neq k, \omega_j\in B(m,r)} U(\omega_j - \omega_k) \\ g_\alpha(u) &:= 1 + |u|^\alpha \log(1+|u|) \quad \text{for } u \in \mathbb{R}, \mathbb{R}^d . \end{aligned} \tag{1.3}$$

The set of allowed configurations is now specified as $\overline{\Omega}_\alpha := \{\omega \in \Omega : \overline{H}_\alpha(\omega) < +\infty\}$; we shall see that for an effective a priori bound we need $\alpha \leq 2 - d/2$, thus $d \leq 4$. Let $C_0(\mathbb{R}^d)$ denote the space of continuous $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ of compact support. Spaces of k times continuously differentiable functions with compact supports are marked by a superscript k, while a subscript b in place of 0 refers to bounded functions without any support condition. For an open and bounded domain $\Lambda \subset \mathbb{R}^d$ the σ-field $\mathcal{F}_\Lambda$ is generated by the variables $\omega(\varphi) := \sum_{k\in S} \varphi(\omega_k)$ such that the support of $\varphi \in C_0(\mathbb{R}^d)$ is contained in Λ; the number of points in Λ will be denoted by $\omega(\Lambda)$. This means that configurations are interpreted as nonnegative, integer valued measures, and $\overline{\Omega}_\alpha$ is equipped with the associated weak topology and Borel structure. Observe that, due to superstability (1.2), the level sets $\overline{\Omega}_{\alpha,h} := [\overline{H}_\alpha(\omega) \leq h]$ are compact if h is large enough. The restriction of $\omega \in \overline{\Omega}_\alpha$ to Λ is ω_Λ, and Λ^c denotes the complement of Λ.

For any bounded domain $\Lambda \subset \mathbb{R}^d$ and $\sigma \in \overline{\Omega}_\alpha$, $n \in \mathbb{N}$ let $\Sigma_\Lambda(n|\sigma)$ denote the set of $\omega \in \overline{\Omega}_\alpha$ such that $\omega(\Lambda) = n$ and $\omega_{\Lambda^c} = \sigma_{\Lambda^c}$. A probability measure λ is a canonical Gibbs state (with unit temperature) for U if its conditional distribution $\lambda[d\omega_\Lambda|\omega_{\Lambda^c}, \omega(\Lambda) = n]$, given the configuration outside of Λ and the number of points in Λ, admits an nd-dimensional Lebesgue density $f_{\Lambda,n}$,

$$
\begin{aligned}
f_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c}) &:= \frac{\exp\big(-H_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c})\big)}{Z_{\Lambda,n}(\omega_{\Lambda^c})} \quad \text{if } \omega \in \Sigma_\Lambda(n|\omega_{\Lambda^c}), \\
H_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c}) &:= \frac{1}{2}\sum_{\omega_k\in\Lambda}\sum_{\omega_j\in\Lambda} U(\omega_k-\omega_j) + \sum_{\omega_k\in\Lambda}\sum_{\omega_j\in\Lambda^c} U(\omega_k-\omega_j)
\end{aligned}
\tag{1.4}
$$

where Σ_Λ denotes the summation for such pairs $\{j,k\}$ that at least one of ω_j and ω_k belongs to Λ, Z is the canonical partition function (normalization). Gibbs states are the extremal canonical measures, see e.g. [G]. In view of the superstability estimates of [R2], there exits at least one translation invariant Gibbs state λ such that $\lambda(\overline{\Omega}_0) = 1$, of course $\overline{\Omega}_\alpha \subset \overline{\Omega}_\beta$ if $\alpha > \beta$.

The unique strong solution $\boldsymbol{\omega} = \omega(t,\sigma)$ to the infinite system (1.1) with initial configuration $\sigma \in \overline{\Omega}_\alpha$ is constructed as the a.s limit of partial solutions $\boldsymbol{\omega}^\theta = \omega^\theta(t,\sigma)$ when a spatial cutoff θ is removed. To ensure the convergence of partial dynamics we have to assume that $\alpha \leq 2 - d/2$. Like in [F2], partial dynamics also preserve any canonical Gibbs measure because we keep particles within a bounded domain, while external particles are frozen. More precisely, for any $\theta \in C_0^1(\mathbb{R}^d)$ such that $0 \leq \theta \leq 1$ there is a differential operator $\mathcal{L}_\theta$,

$$
\mathcal{L}_\theta \phi := \frac{1}{2}\sum_{k\in S}\sum_{i=1}^{d} e^{H_k(\omega)}\partial_{k,i}\big(\theta(\omega_k)e^{-H_k(\omega)}\partial_{k,i}\phi(\omega)\big)
\tag{1.5}
$$

where $\partial_{k,i}$ denotes differentiation with respect to the i coordinate of ω_k and

$$
H_k(\omega) := \sum_{j\neq k} U(\omega_j - \omega_k).
\tag{1.6}
$$

We consider $\mathcal{L}_\theta$ as the (formal) generator of partial dynamics with cutoff θ, the infinite system (1.1) corresponds to $\theta \equiv 1$. All generators of this kind are certainly well defined on $C_0^2(\Omega)$, where $C_0^k(\Omega)$ is the space of test functions

$$\phi(\omega) = \psi\big(\omega(\varphi_1), \omega(\varphi_2), ..., \omega(\varphi_\ell)\big), \quad \psi \in C_b^k(\mathbb{R}^\ell),\ \varphi_j \in C_0^k(\mathbb{R}^d),\ \ell \in \mathbb{N} \tag{1.7}$$

The stochastic equations for cutoff θ read as

$$d\omega_k = \tfrac{1}{2}\, e^{H_k} \partial_k \big(\theta(\omega_k) e^{-H_k}\big)\, dt + \sqrt{\theta(\omega_k)}\, dw_k\,, \tag{1.8}$$

they have a unique strong solution $\boldsymbol{\omega}^\theta = \omega^\theta(t,\sigma)$ for each initial configuration σ. If $\Lambda \supset \operatorname{supp}\theta$ then the particle number in Λ is a constant of motion, that is $\omega^\theta(t,\sigma)(\Lambda) \equiv \sigma(\Lambda)$. Therefore (1.8) defines a fairly regular diffusion in each $\Sigma_\Lambda(n|\sigma)$, and it is easy to verify that the realizations of the canonical conditional distribution $\lambda[d\omega_\Lambda|\omega_{\Lambda^c}, \omega(\Lambda)=n]$ are all reversible measures of the associated (nd-dimensional) diffusion process for every $n \in \mathbb{N}$ and external configuration $\omega_{\Lambda^c} = \sigma_{\Lambda^c}$. The associated Markov semigroup will be denoted as $\mathcal{P}_\theta^t$, it is strongly continuous in $C(\overline{\Omega}_\alpha)$ and also in $L^2(\overline{\Omega}_\alpha, \lambda)$ whenever λ is a canonical Gibbs state.

In the paper [F2] it is shown that for every initial configuration $\sigma \in \overline{\Omega}_0$ a sequence of partial solutions $\omega^\theta(t,\sigma)$ converges almost surely to a strong solution $\boldsymbol{\omega} \equiv \omega^1(t,\sigma)$ of (1.1) as $\theta \to 1$. This limiting solution is distinguished by an a priori bound: $\overline{H}_0(\omega(t,\sigma)$ is bounded on finite intervals of time, and there is no other solution having this property. Following the lines of the proof we see that the result extends immediately to all $\alpha \le 2 - d/2$, see Proposition 1 in Section 3. The limiting semigroup, $\mathcal{P}^t$ is less regular than partial dynamics, the Hille-Yoshida theory is only available in a restricted form.

As a general reference measure we choose a translation invariant Gibbs state λ with interaction U and unit temperature, it is also a reversible measure of each partial dynamics. Introduce $F_\lambda(\phi) := \log \lambda(e^\phi)$, then entropy of another probability measure μ relative to λ is just

$$I[\mu|\lambda] := \sup_\phi \{\mu(\phi) - F_\lambda(\phi)\ :\ \phi \in C_0(\Omega)\} = \int \log \frac{d\mu}{d\lambda}\, d\mu \tag{1.9}$$

if $\mu \ll \lambda$; $I[\mu|\lambda] = +\infty$ otherwise. It is easy to verify that $\mu(\phi) \le I[\mu|\lambda] + \log \lambda(e^\phi)$ whenever $\phi : \overline{\Omega}_\alpha \mapsto \mathbb{R}$ is measurable and $\mu(\phi) < +\infty$. The entropy of μ in $\Lambda \subset \mathbb{R}^d$ is

$$I_\Lambda[\mu|\lambda] := I[\mu_\Lambda|\lambda_\Lambda] = I[\mu_\Lambda\lambda|\lambda] = \sup_\phi \{\mu(\phi) - \log \lambda(e^\phi)\ :\ \phi \in \mathcal{F}_\Lambda \cap C_0(Om)\} \tag{1.10}$$

where μ_Λ is the restriction of μ to $\mathcal{F}_\Lambda$ and $\mu_\Lambda\lambda$ is the measure obtained by extending μ_Λ to the whole space by means of the conditional distribution of λ, that is $(\mu_\Lambda\lambda)(d\omega) := \lambda(d\omega_{\Lambda^c}|\omega_\Lambda)\mu_\Lambda(d\omega_\Lambda)$. If μ is translation invariant and Λ_n denotes the centered cubic box of side $2n$ then

$$\begin{aligned} \bar{I}[\mu|\lambda] &:= \lim_{n\to\infty} \frac{I_{\Lambda_n}[\mu|\lambda]}{|\Lambda_n|} = \sup_\phi \{\mu(\phi) - \overline{F}_\lambda(\phi) : \phi \in C_0(\Omega)\} \\ \overline{F}_\lambda(\phi) &:= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \int \exp\Big(\sum_{m \in \Lambda_n \cap \mathbb{Z}^d} \mathbf{s}^m \phi \Big)\, d\lambda\,, \end{aligned} \tag{1.11}$$

denotes the (relative) specific entropy of μ, see Section 5 in [OVY]. Here and alos later on, $\mathbf{s}^m$ is the shift by $m \in \mathbb{R}^d$, i.e. $\mathbf{s}^m\phi(\omega) \equiv \phi(\mathbf{s}^m)$. Observe that $\bar{I}[\mu|\lambda] < +\infty$ implies $\mu(\overline{\Omega}_1) = 1$, see e.g. [FLO]. Our main result is the following:

Theorem 1. *Suppose that μ^* is a translation invariant stationary distribution of the infinite system (1.1) such that $\bar{I}[\mu^*|\lambda] < +\infty$, then μ^* is a canonical Gibbs state of unit temperature with interaction U.*

The starting point of the argument is a nice, general entropy inequality for Markov processes in such situations when the initial distribution has finite entropy relative to a stationary reference measure. see [FLO]. This inequality and some of its first consequences are discussed in the next section. In Section 3 we develop some uniform estimates on the rate of convergence of partial dynamics to the full (infinite) one. In Section 4 these bounds are then used when the basic entropy inequality is extended to the infinite system, which completes the proof.

2. An Entropy Inequality and its Consequences

The idea that relative entropy with respect to a stationary measure is nice and effective tool of the study of ergodic properties of Markov processes goes back to A. Rényi [Re1,Re2], where ergodicity of irreducible Markov chains in a finite state space is shown by using entropy as a Liapunov function to show the convergence of the evolved measure. Let us first review this argument in a general context of discrete time Markov processes in a probability space $(X, \mathcal{X}, \lambda)$, see e.g. [Fo] for basic notions and results. Let $p = p(x, dy)$ denote a λ-a.s.defined transition function, it is a probability measure on $\mathcal{X}$ for almost each $x \in X$, and that the operator $\mathcal{P}$ of conditional expectation, $\mathcal{P}\varphi(x) := \int p(x, dy)\varphi(y)$ maps $L^\infty(\lambda)$ into itself. Given an initial distribution $\mu \ll \lambda$, the evolved measure at time $t \in \mathbb{N}$ is denoted as $\mu_t = \mu\mathcal{P}^t$, i.e. $\mu_0 = \mu$ and $\mu_1 = \mu\mathcal{P} = \mu\mathcal{P}^1$. We are assuming that $\lambda = \lambda\mathcal{P}$ is a stationary measure, then $I[\mu\mathcal{P}|\lambda] \le I[\mu|\lambda]$ by convexity. Moreover, as noticed by I. Csiszár [Cs], the difference is again a relative entropy:

$$I[\mu|\lambda] - I[\mu_t|\lambda] = I[\mu \circ \mathcal{P}^t | \mathcal{Q}^t \circ \mu_t] \tag{2.1}$$

where $\mu \circ \mathcal{P}$ and $\mathcal{Q} \circ \mu$ are probability measures on $X \times X$ characterized by

$$\begin{aligned} (\mu \circ \mathcal{P})(\phi) &= \int \mu(dx)\varphi(x)\mathcal{P}\psi(x) \quad \text{and} \\ (\mathcal{Q} \circ \mu)(\phi) &= \int \mu(dy)\psi(y)\mathcal{Q}\varphi(y) \end{aligned} \tag{2.2}$$

for $\phi(x, y) = \varphi(x)\psi(y)$ with measurable and bounded $\varphi, \phi : X \mapsto \mathbb{R}$. Here $q = q(y, dx)$ is the transition probability of the backward process reversed with respect to λ; the associated transition operator, $\mathcal{Q}$, $\mathcal{Q}\varphi(y) = \int q(y, dx)\varphi(x)$, is defined as the adjoint of $\mathcal{P}$ in $L^2(\lambda)$, i.e. $\lambda(\varphi\mathcal{P}\psi) = \lambda(\psi\mathcal{Q}\varphi)$ for $\varphi, \psi \in L^2(\lambda)$. Therefore $I[\mu|\lambda] = I[\mu\mathcal{P}|\lambda] < +\infty$ implies $\mu \circ \mathcal{P} = \mathcal{Q} \circ \mu\mathcal{P}$, thus μ is a stationary and reversible measure of the composed, reversible process $\mathcal{R} = \mathcal{P}\mathcal{Q}$, see [F1]. The following reformulation of results by Rényi and Csiszár demostrates an intrisic relationship of the notions of entropy and reversibility.

Theorem 2. *Every stationary measure $\bar{\mu} \ll \lambda$ is reversible with respect to $\mathcal{R}$. If $\mu \ll \lambda$ then so is $\mu\mathcal{P}^t$, and the sequence of densities, $f_t := d\mu\mathcal{P}^t/d\lambda$ is uniformly integrable with* *respect to* λ. *Moreover, if* $\mu\mathcal{P}^{t_n}(\varphi) \to \bar{\mu}(\varphi)$ *for all* $\varphi \in L^\infty(\lambda)$ *as* $t_n \to +\infty$ *then* $\bar{\mu}$ *is a reversible measure of* $\mathcal{R}$, *that is we have a weak convergence to the set of* $\mathcal{R}$*-reversible measures.*

Proof: Suppose first that $I[\mu|\lambda] < +\infty$, then $I[\mu\mathcal{P}^t|\lambda] \leq I[\mu|\lambda]$ implies the uniform integrability of f_t, $t \in \mathbb{N}$, thus the Dunford–Pettis Theorem applies. We have to show that every weak limit point $\bar{\mu}$ satisfies $I[\bar{\mu}|\lambda] = I[\bar{\mu}\mathcal{P}|\lambda]$.

If $\bar{\mu}(\varphi) = \lim_n \mu\mathcal{P}^{t_n}(\varphi)$ for all $\varphi \in L^\infty(\lambda)$ and $\phi : X \times X \mapsto \mathbb{R}$ is measurable and bounded, then

$$\begin{aligned}(\bar{\mu} \circ \mathcal{P})(\phi) - \log(\mathcal{Q} \circ \bar{\mu})(e^\phi) &= \lim_{n\to\infty} (\mu_{t_n} \circ \mathcal{P})(\phi) - \log(\mathcal{Q} \circ \mu_{t_n+1})(e^\phi) \\ &\leq \lim_{n\to\infty} \big(I[\mu_{t_n}|\lambda] - I[\mu_{t_n+1}|\lambda]\big) = 0 \qquad (2.3)\end{aligned}$$

Taking the supremum on the left hand side we get $I[\bar{\mu} \circ \mathcal{P}|\mathcal{Q} \circ \bar{\mu}\mathcal{P}] = 0$, whence $\bar{\mu} \circ \mathcal{P} = \mathcal{Q} \circ \bar{\mu}\mathcal{P}$, i.e. $\bar{\mu} = \bar{\mu}\mathcal{P}\mathcal{Q}$. Replacing $\mathcal{P}$ by $\mathcal{R}$ in the argument above, we get $\bar{\mu} \circ \mathcal{R} = \mathcal{R} \circ \bar{\mu}$, the condition of reversibility of $\bar{\mu}$ with respect to $\mathcal{R} = \mathcal{P}\mathcal{Q}$.

The general case of $\mu \ll \lambda$ follows by a direct approximation procedure. For each $\varepsilon > 0$ we have some μ^ε such that $I[\mu^\varepsilon|\lambda] < +\infty$ and $|\mu - \mu^\varepsilon|_1 < \varepsilon$, where $|\cdot|_1$ denotes the variational distance. Set $f_t^\varepsilon := d\mu^\varepsilon\mathcal{P}^t/d\lambda$ and $|x|_+ := \max\{0, x\}$; since $\mathcal{P}$ is a contraction of $L^1(\lambda)$,

$$\begin{aligned}\int |f_t - a|_+ \, d\lambda &\leq \int |f_t^\varepsilon - a|_+ \, d\lambda + \int |f_t - f_t^\varepsilon| \, d\lambda \\ &\leq \int |f_0^\varepsilon - a|_+ \, d\lambda + \int |f_0 - f_0^\varepsilon| \, d\lambda \leq 2\varepsilon\end{aligned}$$

if a is large enough, thus f_t is still a uniformly integrable sequence. Consider now a weak limit point $\bar{\mu}$ of $\mu\mathcal{P}^t$, $t_n \to +\infty$ is the subsequence along which $\mu\mathcal{P}^t$ converges to $\bar{\mu}$, and let $\bar{\mu}^\varepsilon$ denote a limit point of $\mu^\varepsilon\mathcal{P}^{t_n}$. Therefore we have a subsequence $\{t'_n\} \subset \{t_n\}$ such that for any $\varphi \in L^\infty(\lambda)$

$$|\bar{\mu}(\varphi) - \bar{\mu}^\varepsilon(\varphi)| = \lim_{n\to\infty} |\mu\mathcal{P}^{t'_n}(\varphi) - \mu^\varepsilon\mathcal{P}^{t'_n}(\varphi)| \leq \varepsilon|\varphi|_\infty$$

so that $\bar{\mu} - \bar{\mu}^\varepsilon|_1 \leq \varepsilon$ implying $\bar{\mu}(\varphi\mathcal{R}\psi) = \bar{\mu}(\psi\mathcal{R}\varphi)$ for $\varphi, \psi \in L^\infty(\lambda)$. □

This result is useful because usually it is easier to identify the reversible measures than the stationary ones. Of course, the set of reversible measures of $\mathcal{R} = \mathcal{P}\mathcal{Q}$ can be much larger than the set of stationary measures of $\mathcal{P}$, then a next, more specific step is needed.

For example, if X is a countable set then $\mathcal{P}$ is given by a stochastic matrix $p = p(x,y)$, and $q(y,x) := \lambda(x)p(x,y)/\lambda(y)$ is the associated backward transition probability; $\lambda(x) > 0$ for all $x \in X$ may be assumed. From (2.3) with $\mathcal{P}^t$ in place of $\mathcal{P}$ we get $\bar{\mu} \circ \mathcal{P}^t = \mathcal{Q}^t \circ \bar{\mu}\mathcal{P}^t$ for any limit distribution $\bar{\mu}$, which reads as

$$\frac{\mu(x)}{\lambda(x)} p^t(x,y) = p^t(x,y) \frac{\bar{\mu}_t(y)}{\lambda(y)}$$

in the present context. Therefore if the chain is aperiodic in the sense that for each $x \in X$ there exists an integer $t(x) > 0$ such that $p^t(x,x) > 0$ whenever $t \geq t(x)$, then $\bar{\mu}(x) = \bar{\mu}_t(x)$ for $t \geq t(x)$. Similarly, $\bar{\mu}_1(x) = \bar{\mu}_{t+1}(x)$ if $t \geq t(x)$, consequently $\bar{\mu}(x) = \bar{\mu}_1(x)$ for all $x \in X$, i.e. $\bar{\mu} = \bar{\mu}\mathcal{P}$. Uniqueness of the stationary measure follows immediately from a condition of irreducibiliy: if for each pair $x, y \in X$ we have some $t = t(x,y)$ such that $p^t(x,y) > 0$ then $\bar{\mu}(x)/\lambda(x) = \bar{\mu}(y)/\lambda(y)$, whence $\bar{\mu}(x) = \lambda(x)$ for all $x \in X$, consequently we have $\mu_t(x) \to \lambda(x)$ for all $x \in X$ as $t \to \infty$.

In the case of continuous time it is natural to assume that X is a complete and separable metric space, and both $\mathcal{P}^t$ and its adjoint in $L^2(\lambda)$, $\mathcal{Q}^t$ form strongly continuous semigroups in $C_b(X)$; basic notations are the same as above. To obtain a lower bound for $I[\mu|\lambda] - I[\mu\mathcal{P}^t|\lambda]$ consider an auxiliary distribution $\nu \ll \lambda$ such that $\psi := d\nu/d\lambda > 0$; then $\mu \ll \nu$ and $I[\mu|\nu] = I[\mu|\lambda] - \nu(\log\psi)$, while $I[\mu\mathcal{P}^t|\nu\mathcal{P}^t] = I[\mu\mathcal{P}^t|\lambda] - \mu(\log\mathcal{Q}^\psi)$ as $d\mu\mathcal{P}^t/d\lambda = \mathcal{Q}^t\psi$. Since $I[\mu\mathcal{P}^t|\nu\mathcal{P}^t] \leq I[\mu|\nu]$ by convexity,

$$\begin{aligned} I[\mu|\lambda] - I[\mu\mathcal{P}^t|\lambda] &\geq \mu(\log\psi) - \mu\mathcal{P}^t(\log\mathcal{Q}^t\psi) \\ &\geq \mu(\log\psi) - \mu(\log\mathcal{R}^t\psi) \geq \int \frac{\psi - \mathcal{R}^t\psi}{\psi}\, d\mu \end{aligned} \tag{2.4}$$

as $\log x - \log y \geq (x-y)/x$. Observe that $\lambda \circ \mathcal{R}^t$ is a symmetric measure, thus with $f = d\mu/d\lambda$ we get

$$\begin{aligned} \int \frac{\mathcal{R}^t\psi}{\psi}\, d\mu &= \frac{1}{2}\iint (\lambda\circ\mathcal{R}^t)(dx,dy)\Big(\frac{f(x)\psi(y)}{\psi(x)} + \frac{f(y)\psi(x)}{\psi(y)}\Big) \\ &\geq \iint (\lambda\circ\mathcal{R}^t)(dx,dy)\sqrt{f(x)}\sqrt{f(y)} \end{aligned} \tag{2.6}$$

This means that the right hand side is maximal if $\psi = \sqrt{f}$.

Consider now the Donsker–Varadhan rate D,

$$D[\mu|\mathcal{G}] := \sup_\psi \Big\{ -\int \frac{\mathcal{G}\psi}{\psi}\, d\mu \;:\; \psi \in \mathrm{Dom}\ \mathcal{G}\,,\ \inf\psi > 0 \Big\} \tag{2.7}$$

where $\mathcal{G}$ is any semigroup generator, and notice that Dom $\mathcal{G}$ in the definition of D can be replaced by any core of $\mathcal{G}$ in $C_b(X)$. Moreover, if $\mathcal{G}$ is self-adjoint in $L^2(\lambda)$ and $f = d\mu/d\lambda$, then $D[\mu|\mathcal{G}] < +\infty$ implies $\sqrt{f} \in \mathrm{Dom}\ (-\mathcal{G})^{1/2}$ and

$$D[\mu|\mathcal{G}] = \int \big(\sqrt{-\mathcal{G}}\sqrt{f}\big)^2\, d\lambda \tag{2.8}$$

see (2.6) and Theorem 5 in [DV]. Observe now that $\mathcal{R}^t$ is self-adjoint in $L^2(\lambda)$, thus so is its generator, $\mathcal{G}$, too. By a formal calculation we get $\mathcal{G} = \mathcal{L} + \mathcal{L}^*$, where $\mathcal{L}$ is the generator of the semigroup $\mathcal{P}^t$, while its adjoint $\mathcal{L}^*$ denotes that of $\mathcal{Q}^t$. For small t the right hand side of (2.3) becomes approximately $-t\mu(\mathcal{G}\psi/\psi) + o(t)$, thus we have

Proposition 1. *Suppose that we have a dense $C^* \subset C_b(X)$ such that it is a common core of $\mathcal{L}$ and $\mathcal{L}^*$ with respect to either $C_b(X)$ and $L^2(\lambda)$, then*

$$I[\mu\mathcal{P}^t|\lambda] + 2tD[\bar{\mu}_t|\mathcal{G}] \leq I[\mu|\lambda]\,; \quad \bar{\mu}_t := \frac{1}{t}\int_0^t \mu\mathcal{P}^s\,ds\,.$$

For a more detailed proof see [FLO]. Therefore if $I[\mu|\lambda] < +\infty$, then $D[\mu|\mathcal{G}] = 0$ in a stationary regime with $I[\mu|\lambda] < +\infty$ implying the reversibility of μ with respect to $\mathcal{R}^t$, that is $\mu(\varphi\mathcal{G}\psi) = \mu(\psi\mathcal{G}\varphi)$ for all $\varphi, \psi \in \mathrm{Dom}\,\mathcal{G}$. In the case of reversible diffusion processes the verification of the conditions of Proposition 1 amounts to establishing smooth dependence of solutions on initial values. Assuming the smoothness of the coefficients of the underlying stochastic equations, a standard argument shows that twice continuously differentiable functions with compact supports form a core of the generator. If the diffusion matrix is positive then $D[\mu|\mathcal{G}] = 0$ yields $\mu = \lambda$, thus $\mu_0\mathcal{P}^t \to \lambda$ as $t \to \infty$ for all $\mu_0 \ll \lambda$.

Our next task is to extend these results to infinity volumes, this is done by means of a familiar argument of Holley [Ho]; in translation invariant situations we can pass to the thermodynamic limit. This procedure can not be carried out in a general framework, see e.g. [FFL] and [FLO]; technical requirements are summarized in the next section.

3. On Locality of Dynamics

Results of [F2] are not directly applicable in the present situation, that is why we review some parts of the argument. A convenient collection Θ of cutoff functions is defined for $m \in \mathbb{R}^d$ and $\ell \geq 1$ by $\theta_m^\ell = \theta_m^\ell(x) := \theta_0(||x-m| - \ell|_+)$, where $\theta_0 \in C_0^3(\mathbb{R}_+)$ satisfies $0 \leq \theta_0(u) \leq 1\ \forall u \geq 0$ while $\theta_0(u) = 1$ if $u \leq 1$ and $\theta_0(u) = 0$ if $u \geq 2$; thus Θ is the set of such functions including also $\theta \equiv 1$, that is the case of full (infinite) dynamics. The limiting solution will be denoted as $\boldsymbol{\omega} = \omega^1(t,\sigma)$. The basic a priori bound of [F2] can be reformulated as follows, see Proposition 2 and (3.18) there. Let $N_k(\omega)$ denote the number of points of ω in $B(\omega_k, 1)$, and

$$\overline{N}_\theta(t,\sigma) := 1 + \sup_{k\in S}\max_{s\leq t}\frac{N_k(\omega^\theta(s))}{\sqrt{g_\alpha(\omega_k^\theta(s))}}\,. \tag{3.1}$$

Exploiting superstability of the interaction, by means of the argument of [F2] we get

Proposition 2. *If $\alpha \leq 2 - d/2$ then for each $t > 0$ and $h > 0$*

$$\lim_{\rho\to\infty}\sup_{\theta\in\Theta}\sup_{\sigma\in\overline{\Omega}_{\alpha,h}} P[\overline{N}_\theta(t,\sigma) > \rho] = 0\,.$$

First we derive a uniform bound on the localization of particles. From the stochastic equations

$$\begin{aligned}|\omega_k^\theta(t,\sigma) - \sigma_k| \leq K_1 &\int_0^t \overline{N}_\theta(t,\sigma)\sqrt{g_\alpha(\omega_k^\theta(s,\sigma))}\,ds \\ &+ \left|\int_0^t \sqrt{\theta(\omega_k^\theta(s,\sigma))}\,dw_k\right|\,.\end{aligned} \tag{3.2}$$

Let $g_*(u) := (1+|u|)^{4/5}$, by a direct calculation

$$\begin{aligned}\delta_{\theta,k}(t,\sigma) &:= \max_{s\le t} |\omega_k^\theta(s,\sigma) - \sigma_k| \le \xi_\theta(t,\sigma)\big(g_*(\sigma_k) + g_*(\delta_{\theta,k}(t,\sigma)\,,\\ \xi_\theta(t,\sigma) &:= K_2 \int_0^t \overline{N}_\theta(s,\sigma)\,ds + \sup_{k\in S}\max_{s\le t} \frac{K_2}{g_*(\sigma_k)}\Big|\int_0^t \sqrt{\theta(\omega_k^\theta(s,\sigma))}\,dw_k\Big|\end{aligned} \tag{3.3}$$

whence by assuming $\delta_{\theta,k} \ge g_*(\sigma_k)$ we get

$$\delta_{\theta,k}(t,\sigma) \le \eta_\theta(t,\sigma) g_*(\sigma_k)\,, \tag{3.4}$$

where the explicit form $\eta = K_3\xi^5$ is not relevant, we only need

$$\lim_{y\to\infty} \sup_{\theta\in\Theta} \sup_{\sigma\in\overline{\Omega}_{\alpha,h}} P[\eta_\theta(t,\sigma) > y] = 0 \tag{3.5}$$

for all $t, h > 0$, which is a direct consequence of the definition of ξ .

Now we are in a position to estimate the rate of convergence of partial dynamics $\boldsymbol{\omega}^\theta$ to its limit $\boldsymbol{\omega}$ as $\theta \to 1$. For any initial configuration $\sigma \in \overline{\Omega}_\alpha$ let $S(m,r,\sigma)$ denote the set of $k \in S$ such that $|\sigma_k - m| \le r$, and consider

$$\Delta_{m,\ell}(t,r,\sigma) := \max_{k\in S(m,r,\sigma)} \max_{s\le t} |\omega_k^\theta(s,\sigma) - \omega_k(s,\sigma)| \quad \text{with } \theta = \theta_m^\ell\,. \tag{3.6}$$

For any fixed $T > 0$ and $r_0, \ell \ge 1$ define r_κ , $\kappa = 0,1,...,\chi,...$ by

$$\begin{aligned} r_{\kappa+1} &= r_\kappa + 2g_*(|m|+\ell)\,\tilde{\eta}_{m,\ell}(T,\sigma) + R + 1\,, \quad \text{where}\\ \tilde{\eta}_{m,\ell}(T,\sigma) &:= \max\{\eta_\theta(T,\sigma), \eta_1(T,\sigma)\}\,, \quad \theta = \theta_m^\ell\,. \end{aligned} \tag{3.7}$$

In view of (3.4) this means that before time T the particles starting from $B(m, r_\kappa)$ can not interact with those starting from outside of $B(m, r_{\kappa+1})$, therefore

$$\begin{aligned} \Delta_{m,\ell}(t,r_\kappa,\sigma) &\le Lg(|m|+\ell)\,\widetilde{N}_{m,\ell}(t,\sigma)\int_0^t \Delta_{m,\ell}(s,r_{\kappa+1},\sigma)\,ds\,, \quad \text{where}\\ \widetilde{N}_{m,\ell}(t,\sigma) &:= \max\{\overline{N}_{\theta_m^\ell}(t,\sigma), \overline{N}_1(t,\sigma)\}\,, \end{aligned} \tag{3.8}$$

provided that $r_{\kappa+1} + R \le \ell$.

Suppose that (3.8) can χ times be iterated, then for $t < T$

$$\Delta_{m,\ell}(t,r_0,\sigma) \le 2(\ell+1)\frac{(Lt)^\chi}{\chi!}\big(g_*(|m|+\ell)\widetilde{N}_{m,\ell}(t,\sigma)\big)^\chi \tag{3.9}$$

where $\chi = O(\ell(|m|+\ell)^{-4/5})$ is a random number. Of course, this inequality implies the a.s. convergence of partial solutions; this was shown in [F2] when $m = 0$ and $\ell \to +\infty$. Here we need a more delicate result: $\boldsymbol{\omega}^{m,\ell}$ converges even if $|m|$ increases together with ℓ . More precisely, for any $r_0, t, h, \varepsilon > 0$ we have

$$\lim_{n\to\infty} \sup_{\sigma,m}\big\{P[\widetilde{N}_{m,\ell}(t,\sigma)\Delta_{m,\ell}(t,r_0,\sigma) > \varepsilon] : \sigma \in \overline{\Omega}_{\alpha,h}\,,\ m,\ell \in M_n\big\} = 0\,, \tag{3.10}$$

where $M_n := \{m,\ell : |m| + \ell + R < n\,,\ \ell > n^{5/6}\}$. Indeed, in this situation χ of (3.9) goes a.s. to $+\infty$ as $n \to \infty$.

In the next section the following consequence of (3.10) will be needed. Suppose that we are given a translation invariant probability measure μ such that $\mu(\overline{\Omega}_\alpha) = 1$, and set $\hat{\mu}_n := \mu_{\Lambda_n}\lambda$. The above calculations are summarized in

Lemma 1. *For any $\phi \in C_0^1(\Omega)$ and $t > 0$ we have*

$$\lim_{n\to\infty} \sup_{s,m}\big\{|\hat{\mu}_n\mathcal{P}^s\mathbf{s}^m\phi - \mu\mathcal{P}^s\phi| : s \leq t\,,\ |m| < n - n^{5/6}\big\} = 0\,.$$

Proof: Since $\hat{\mu}_n(s^m\varphi) = \mu(\varphi)$ if $\varphi, \mathbf{s}^m\varphi \in \mathcal{F}_{\Lambda_n}$, it is natural to set $\varphi_\ell = \mathcal{P}^s_{\theta^\ell_0}\phi$, then $\mathbf{s}^m\varphi_\ell = \mathcal{P}^s_{\theta^\ell_m}\phi$. Since ϕ is Lipschitz continuous by assumption, we can compare $\mathbf{s}^m\varphi_\ell$ and $\mathcal{P}^s\mathbf{s}^m\phi$ via (3.10), at least if $|m| < n - n^{5/6}$. The missing part of the bound,

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}}\,, \hat{\mu}_n(\overline{\Omega}_{\alpha,h}) = 0 \tag{3.11}$$

follows from the basic superstability estimate of Ruelle [Ru2]. Indeed, for any box Λ of given shape and size we have $\lambda[\omega(\Lambda) > \nu|\mathcal{F}_{\Lambda^c}] \leq Ce^{-c\nu^2}$, where c and C do not depend on ω_{Λ^c}. In view of (1.2) this yields $\lambda(\overline{\Omega}_\alpha) = 1$ by the Borel-Cantelli lemma. Since $\mu(\overline{\Omega}_\alpha) = 1$ by assumption, estimating the contribution of particles from Λ_n^c to $\overline{H}_\alpha$ via superstability, we get (3.11) by a similar computation. □

Remark: Since the level sets of $\overline{H}$ are compact, the Stone-Weierstrass theorem allows us to extend Lemma 1 to continuous and bounded local functions.

4. Passage to the Thermodynamic Limit

Now we are in a position to prove Theorem 1 by extending Proposition 1 to infinite volumes. Using the notation $\hat{\mu}_n^* = \mu^*_{\Lambda_n}\lambda$ of Lemma 1, we have

$$I[\hat{\mu}_n^*\mathcal{P}^t_\theta|\lambda] + 2tD[\bar{\mu}^*_{n,\theta,t}|\mathcal{L}_\theta] \leq I[\hat{\mu}^*_n|\lambda] = I_{\Lambda_n}[\mu^*|\lambda] \tag{4.1}$$

for any smooth cutoff θ, where $\bar{\mu}^*_{n,\theta,t}$ is the time average of the evolved measures $\hat{\mu}^*_n\mathcal{P}^s_\theta$ from $s = 0$ through $s = t$. In view of (2.6) D is subadditive in the following sense. Suppose that $J^\ell_\theta(n) \subset \mathbb{Z}^d$ satisfies $\theta \geq \theta^\ell_m$ and $\theta^\ell_m\theta^\ell_k = 0$ for $m, k \in J^\ell_\theta(n)$, $k \neq m$ then

$$D[\bar{\mu}^*_{n,\theta,t}|\mathcal{L}_\theta] \geq \sum_{m\in J^\ell_\theta(n)} D[\bar{\mu}^*_{n,\theta,t}|\mathcal{L}_{\theta^\ell_m}] \geq \sum_{m\in J^\ell_\theta(n)} \int \frac{-\mathcal{L}_{\theta^\ell_m}\mathbf{s}^m\psi}{\mathbf{s}^m\psi}\, d\bar{\mu}^*_{n,\theta,t} \tag{4.2}$$

for smooth $\psi > 0$. Similarly, for all $\varphi \in C_0(\Omega)$

$$I[\hat{\mu}^*_n\mathcal{P}^t_\theta|\lambda] \geq \int S_n\big(\mathcal{P}^t_\theta(\varphi)\big)\, d\hat{\mu}^*_n - F_\lambda\big(S_n(\varphi)\big)\,, \quad \text{with}$$
$$S_n(\varphi) := \sum_{m\in\Lambda_n\cap\mathbb{Z}^d} \mathbf{s}^m\varphi \tag{4.3}$$

Now we can remove the cutoff of dynamics, keeping $J^\ell_\theta(n) = J^\ell(n) \subset \Lambda_{n-n^{5/6}}$ fixed during this procedure we get

$$\sum_{m\in J^\ell(n)} \int_0^t ds \int \frac{-\mathcal{L}_{\theta^\ell_m}\mathbf{s}^m\psi}{\mathbf{s}^m\psi}\, d\hat{\mu}^*_n\mathcal{P}^s \leq I[\hat{\mu}^*_n|\lambda] + F_\lambda\big(S_n(\varphi)\big) - \int S_n(\mathcal{P}^t\varphi)\, d\hat{\mu}^*_n \tag{4.4}$$

As far as ℓ is fixed, we may assume that Card $J^\ell(n) \geq c_\ell|\Lambda_n|$ with some $c_\ell > 0$; thus dividing both sides by $|\Lambda_n|$ we can pass to a thermodynamic limit. Indeed, in view of Lemma 1 all terms of $\hat{\mu}_n^* \mathcal{P}^t\big(S_n(\varphi)\big)$ become asymptotically identical when $n \to \infty$. Since $\mathcal{L}_{\theta_m^\ell} = \mathbf{s}^m \mathcal{L}_{\theta_0^\ell}$, the same holds true on the left hand side, thus for all $\theta \in \Theta$ with compact support we have some $c_\theta > 0$ such that

$$c_\theta t \int \frac{-\mathcal{L}_\theta \psi}{\psi}\, d\mu^* \leq \bar{I}[\mu^*|\lambda] + \bar{F}_\lambda(\varphi) - \mu^*(\varphi), \tag{4.6}$$

where $\varphi \in C_0(\Omega)$ is arbitrary, thus from (1.11) $\mu^*(\mathcal{L}_\theta\psi/\psi) \geq 0$ for all smooth $\psi > 0$. Since $\mathcal{L}_\theta \log\psi \geq \mathcal{L}\psi/\psi$ by convexity, and $\phi = \log\psi$ is still quite general, the resulting stationary Kolmogorov equation $\mu^*(\mathcal{L}_\theta\phi) = 0$ implies that μ^* is a stationary and reversible measure of every partial evolution $\mathcal{P}_\theta$, which completes the proof of Theorem 1.

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STATIONARY STATES OF INTERACTING BROWNIAN MOTIONS

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In Memoriam Alfréd Rényi

Abstract. We are interested in a description of stationary states of gradient dynamics of interacting Brownian particles. In contrast to lattice models, this problem does not seem to be solvable at a formal level of the stationary Kolmogorov equation. We can only study stationary states of a well controlled Markov process. In space dimensions four or less, for smooth and superstable pair potentials of finite range the non-equilibrium dynamics of interacting Brownian particles can be constructed in an explicitely defined deterministic set of locally finite configurations, see [F3]. This set is of full measure with respect to any canonical Gibbs state for the interaction and every canonical state is stationary. Assuming translation invariance of a stationary measure, and also the finiteness of its specific entropy with respect to an equilibrium Gibbs state, it is shown that it is canonical Gibbs, too.

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1. Introduction

The main purpose of this paper is to identify a class of stationary states of the following system of interacting particles as the set of translation invariant canonical Gibbs states with interaction U. The evolution law is given by an infinite system of stochastic differential equations,

$$d\omega_k = -\frac{1}{2}\sum_{j\neq k} \text{grad}\, U(\omega_k - \omega_j)\, dt + dw_k\,, \quad \omega_k(0) = \sigma_k\,, \quad k \in S \tag{1.1}$$

where S is a countable index set, $w = (w_k)_{k\in S}$ is a family of independent standard d-dimensional Wiener processes, and each $\omega_k = \omega_k(t)\,, t \geq 0$ is assumed to be a continuous trajectory in $\mathbb{R}^d$. The potential $U : \mathbb{R}^d \mapsto \mathbb{R}$ is symmetric and superstable with finite range, that is $U(x) = U(-x)$, there is an $R > 0$ such that $U(x) = 0$ if $|x| > R$, and we have constants $A \geq 0$, $B > 0$ such that for any finite sequence $q_1, q_2, ..., q_n$ of not necessarily distinct points from $\mathbb{R}^d$

$$nA + \sum_{k=1}^{n}\sum_{j\neq k} U(q_k - q_j) \geq BN \tag{1.2}$$

where N is the number of pairs $\{j,k\}$ such that $|q_k - q_j| \leq R$, see [Ru1]. Let Ω denote the set of configurations having no limit points. Although the right hand side of (1.1) is certainly well defined for such, locally finite configurations $\omega \in \Omega$, to develop a satisfactory existence theory we have to restrict the configuration space in a much more radical way. On the other hand, the set of allowed configurations should be large enough to support a possibly wide set probability measures including Gibbs states with various interactions.

The first mathematical results concerning this model go back to R. Lang, see [La1] and [La2], where the existence of equilibrium dynamics, and also the canonical Gibbs property of reversible measures is proven. These dynamics is defined almost surely with respect to a Gibbs state with interaction U, see also the more sophisticated argument of [Os]. For a study of stationary measures in general, we need a more direct construction involving explicit bounds on the rate of convergence of solutions to finite subsystems (partial dynamics) when the number of active particles tends to infinity, see Section 3 below. Indeed, the problem of stationary measures can not be solved at a formal level of the stationary Kolmogorov equation, see [FFL] and [FLO], we really need that that our measure is realized as a stationary state of a well controlled Markov process.

For a generic, locally finite configuration $\omega = (\omega_k)_{k\in S}$ let $H(\omega, m, r)$ denote total energy in the ball $B(m, r)$ of center $m \in \mathbb{R}^d$ and radius $r \geq 1$, and for $\alpha \geq 0$ define

$$\begin{aligned} \overline{H}_\alpha(\omega) :&= \sup_{m\in\mathbb{Z}^d}\ \sup_{r\in\mathbb{N}} \frac{H\big(\omega, m, r g_\alpha^{1/d}(m)\big)}{r^d g_\alpha(m)} \quad \text{where} \\ H(\omega, m, r) &:= \frac{1}{2}\sum_{\omega_k\in B(m,r)}\ \sum_{j\neq k, \omega_j\in B(m,r)} U(\omega_j - \omega_k) \\ g_\alpha(u) &:= 1 + |u|^\alpha \log(1 + |u|) \quad \text{for } u \in \mathbb{R}, \mathbb{R}^d. \end{aligned} \tag{1.3}$$

The set of allowed configurations is now specified as $\overline{\Omega}_\alpha := \{\omega \in \Omega : \overline{H}_\alpha(\omega) < +\infty\}$; we shall see that for an effective a priori bound we need $\alpha \le 2 - d/2$, thus $d \le 4$. Let $C_0(\mathbb{R}^d)$ denote the space of continuous $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ of compact support. Spaces of k times continuously differentiable functions with compact supports are marked by a superscript k, while a subscript b in place of 0 refers to bounded functions without any support condition. For an open and bounded domain $\Lambda \subset \mathbb{R}^d$ the σ-field $\mathcal{F}_\Lambda$ is generated by the variables $\omega(\varphi) := \sum_{k\in S} \varphi(\omega_k)$ such that the support of $\varphi \in C_0(\mathbb{R}^d)$ is contained in Λ; the number of points in Λ will be denoted by $\omega(\Lambda)$. This means that configurations are interpreted as nonnegative, integer valued measures, and $\overline{\Omega}_\alpha$ is equipped with the associated weak topology and Borel structure. Observe that, due to superstability (1.2), the level sets $\overline{\Omega}_{\alpha,h} := [\overline{H}_\alpha(\omega) \le h]$ are compact if h is large enough. The restriction of $\omega \in \overline{\Omega}_\alpha$ to Λ is ω_Λ, and Λ^c denotes the complement of Λ.

For any bounded domain $\Lambda \subset \mathbb{R}^d$ and $\sigma \in \overline{\Omega}_\alpha$, $n \in \mathbb{N}$ let $\Sigma_\Lambda(n|\sigma)$ denote the set of $\omega \in \overline{\Omega}_\alpha$ such that $\omega(\Lambda) = n$ and $\omega_{\Lambda^c} = \sigma_{\Lambda^c}$. A probability measure λ is a canonical Gibbs state (with unit temperature) for U if its conditional distribution $\lambda[d\omega_\Lambda|\omega_{\Lambda^c}, \omega(\Lambda) = n]$, given the configuration outside of Λ and the number of points in Λ, admits an nd-dimensional Lebesgue density $f_{\Lambda,n}$,

$$
\begin{aligned}
f_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c}) &:= \frac{\exp\big(-H_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c})\big)}{Z_{\Lambda,n}(\omega_{\Lambda^c})} \quad \text{if } \omega \in \Sigma_\Lambda(n|\omega_{\Lambda^c}), \\
H_{\Lambda,n}(\omega_\Lambda|\omega_{\Lambda^c}) &:= \frac{1}{2} \sum_{\omega_k\in\Lambda} \sum_{\omega_j\in\Lambda} U(\omega_k - \omega_j) + \sum_{\omega_k\in\Lambda} \sum_{\omega_j\in\Lambda^c} U(\omega_k - \omega_j)
\end{aligned}
\tag{1.4}
$$

where Σ_Λ denotes the summation for such pairs $\{j,k\}$ that at least one of ω_j and ω_k belongs to Λ, Z is the canonical partition function (normalization). Gibbs states are the extremal canonical measures, see e.g. [G]. In view of the superstability estimates of [R2], there exits at least one translation invariant Gibbs state λ such that $\lambda(\overline{\Omega}_0) = 1$, of course $\overline{\Omega}_\alpha \subset \overline{\Omega}_\beta$ if $\alpha > \beta$.

The unique strong solution $\boldsymbol{\omega} = \omega(t,\sigma)$ to the infinite system (1.1) with initial configuration $\sigma \in \overline{\Omega}_\alpha$ is constructed as the a.s limit of partial solutions $\boldsymbol{\omega}^\theta = \omega^\theta(t,\sigma)$ when a spatial cutoff θ is removed. To ensure the convergence of partial dynamics we have to assume that $\alpha \le 2 - d/2$. Like in [F2], partial dynamics also preserve any canonical Gibbs measure because we keep particles within a bounded domain, while external particles are frozen. More precisely, for any $\theta \in C_0^1(\mathbb{R}^d)$ such that $0 \le \theta \le 1$ there is a differential operator $\mathcal{L}_\theta$,

$$
\mathcal{L}_\theta \phi := \frac{1}{2} \sum_{k\in S} \sum_{i=1}^{d} e^{H_k(\omega)} \partial_{k,i} \big(\theta(\omega_k) e^{-H_k(\omega)} \partial_{k,i} \phi(\omega)\big) \tag{1.5}
$$

where $\partial_{k,i}$ denotes differentiation with respect to the i coordinate of ω_k and

$$
H_k(\omega) := \sum_{j\neq k} U(\omega_j - \omega_k). \tag{1.6}
$$

We consider $\mathcal{L}_\theta$ as the (formal) generator of partial dynamics with cutoff θ, the infinite system (1.1) corresponds to $\theta \equiv 1$. All generators of this kind are certainly well defined on $C_0^2(\Omega)$, where $C_0^k(\Omega)$ is the space of test functions

$$\phi(\omega) = \psi\big(\omega(\varphi_1), \omega(\varphi_2), ..., \omega(\varphi_\ell)\big)\,, \quad \psi \in C_b^k(\mathbb{R}^\ell)\,,\ \varphi_j \in C_0^k(\mathbb{R}^d)\,,\ \ell \in \mathbb{N} \tag{1.7}$$

The stochastic equations for cutoff θ read as

$$d\omega_k = \tfrac{1}{2}\, e^{H_k}\partial_k\big(\theta(\omega_k)e^{-H_k}\big)\, dt + \sqrt{\theta(\omega_k)}\, dw_k\,, \tag{1.8}$$

they have a unique strong solution $\boldsymbol{\omega}^\theta = \omega^\theta(t,\sigma)$ for each initial configuration σ. If $\Lambda \supset \operatorname{supp}\theta$ then the particle number in Λ is a constant of motion, that is $\omega^\theta(t,\sigma)(\Lambda) \equiv \sigma(\Lambda)$. Therefore (1.8) defines a fairly regular diffusion in each $\Sigma_\Lambda(n|\sigma)$, and it is easy to verify that the realizations of the canonical conditional distribution $\lambda[d\omega_\Lambda|\omega_{\Lambda^c}, \omega(\Lambda){=}n]$ are all reversible measures of the associated (nd-dimensional) diffusion process for every $n \in \mathbb{N}$ and external configuration $\omega_{\Lambda^c} = \sigma_{\Lambda^c}$. The associated Markov semigroup will be denoted as $\mathcal{P}_\theta^t$, it is strongly continuous in $C(\overline{\Omega}_\alpha)$ and also in $L^2(\overline{\Omega}_\alpha, \lambda)$ whenever λ is a canonical Gibbs state.

In the paper [F2] it is shown that for every initial configuration $\sigma \in \overline{\Omega}_0$ a sequence of partial solutions $\omega^\theta(t,\sigma)$ converges almost surely to a strong solution $\boldsymbol{\omega} \equiv \omega^1(t,\sigma)$ of (1.1) as $\theta \to 1$. This limiting solution is distinguished by an a priori bound: $\overline{H}_0(\omega(t,\sigma)$ is bounded on finite intervals of time, and there is no other solution having this property. Following the lines of the proof we see that the result extends immediately to all $\alpha \leq 2 - d/2$, see Proposition 1 in Section 3. The limiting semigroup, $\mathcal{P}^t$ is less regular than partial dynamics, the Hille-Yoshida theory is only available in a restricted form.

As a general reference measure we choose a translation invariant Gibbs state λ with interaction U and unit temperature, it is also a reversible measure of each partial dynamics. Introduce $F_\lambda(\phi) := \log\lambda(e^\phi)$, then entropy of another probability measure μ relative to λ is just

$$I[\mu|\lambda] := \sup_\phi\{\mu(\phi) - F_\lambda(\phi)\ :\ \phi \in C_0(\Omega)\} = \int \log\frac{d\mu}{d\lambda}\, d\mu \tag{1.9}$$

if $\mu \ll \lambda$; $I[\mu|\lambda] = +\infty$ otherwise. It is easy to verify that $\mu(\phi) \leq I[\mu|\lambda] + \log\lambda(e^\phi)$ whenever $\phi : \overline{\Omega}_\alpha \mapsto \mathbb{R}$ is measurable and $\mu(\phi) < +\infty$. The entropy of μ in $\Lambda \subset \mathbb{R}^d$ is

$$I_\Lambda[\mu|\lambda] := I[\mu_\Lambda|\lambda_\Lambda] = I[\mu_\Lambda\lambda|\lambda] = \sup_\phi\{\mu(\phi) - \log\lambda(e^\phi)\ :\ \phi \in \mathcal{F}_\Lambda \cap C_0(Om)\} \tag{1.10}$$

where μ_Λ is the restriction of μ to $\mathcal{F}_\Lambda$ and $\mu_\Lambda\lambda$ is the measure obtained by extending μ_Λ to the whole space by means of the conditional distribution of λ, that is $(\mu_\Lambda\lambda)(d\omega) := \lambda(d\omega_{\Lambda^c}|\omega_\Lambda)\mu_\Lambda(d\omega_\Lambda)$. If μ is translation invariant and Λ_n denotes the centered cubic box of side $2n$ then

$$\begin{aligned} \bar{I}[\mu|\lambda] &:= \lim_{n\to\infty}\frac{I_{\Lambda_n}[\mu|\lambda]}{|\Lambda_n|} = \sup_\phi\{\mu(\phi) - \overline{F}_\lambda(\phi) : \phi \in C_0(\Omega)\} \\ \overline{F}_\lambda(\phi) &:= \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\log\int\exp\Big(\sum_{m\in\Lambda_n\cap\mathbb{Z}^d}\mathbf{s}^m\phi\Big)\, d\lambda\,, \end{aligned} \tag{1.11}$$

denotes the (relative) specific entropy of μ, see Section 5 in [OVY]. Here and alos later on, $\mathbf{s}^m$ is the shift by $m \in \mathbb{R}^d$, i.e. $\mathbf{s}^m\phi(\omega) \equiv \phi(\mathbf{s}^m)$. Observe that $\bar{I}[\mu|\lambda] < +\infty$ implies $\mu(\overline{\Omega}_1) = 1$, see e.g. [FLO]. Our main result is the following:

Theorem 1. *Suppose that μ^* is a translation invariant stationary distribution of the infinite system (1.1) such that $\bar{I}[\mu^*|\lambda] < +\infty$, then μ^* is a canonical Gibbs state of unit temperature with interaction U.*

The starting point of the argument is a nice, general entropy inequality for Markov processes in such situations when the initial distribution has finite entropy relative to a stationary reference measure. see [FLO]. This inequality and some of its first consequences are discussed in the next section. In Section 3 we develop some uniform estimates on the rate of convergence of partial dynamics to the full (infinite) one. In Section 4 these bounds are then used when the basic entropy inequality is extended to the infinite system, which completes the proof.

2. An Entropy Inequality and its Consequences

The idea that relative entropy with respect to a stationary measure is nice and effective tool of the study of ergodic properties of Markov processes goes back to A. Rényi [Re1,Re2], where ergodicity of irreducible Markov chains in a finite state space is shown by using entropy as a Liapunov function to show the convergence of the evolved measure. Let us first review this argument in a general context of discrete time Markov processes in a probability space $(X, \mathcal{X}, \lambda)$, see e.g. [Fo] for basic notions and results. Let $p = p(x, dy)$ denote a λ-a.s.defined transition function, it is a probability measure on $\mathcal{X}$ for almost each $x \in X$, and that the operator $\mathcal{P}$ of conditional expectation, $\mathcal{P}\varphi(x) := \int p(x, dy)\varphi(y)$ maps $L^\infty(\lambda)$ into itself. Given an initial distribution $\mu \ll \lambda$, the evolved measure at time $t \in \mathbb{N}$ is denoted as $\mu_t = \mu\mathcal{P}^t$, i.e. $\mu_0 = \mu$ and $\mu_1 = \mu\mathcal{P} = \mu\mathcal{P}^1$. We are assuming that $\lambda = \lambda\mathcal{P}$ is a stationary measure, then $I[\mu\mathcal{P}|\lambda] \leq I[\mu|\lambda]$ by convexity. Moreover, as noticed by I. Csiszár [Cs], the difference is again a relative entropy:

$$I[\mu|\lambda] - I[\mu_t|\lambda] = I[\mu \circ \mathcal{P}^t | \mathcal{Q}^t \circ \mu_t] \tag{2.1}$$

where $\mu \circ \mathcal{P}$ and $\mathcal{Q} \circ \mu$ are probability measures on $X \times X$ characterized by

$$\begin{aligned} (\mu \circ \mathcal{P})(\phi) &= \int \mu(dx)\varphi(x)\mathcal{P}\psi(x) \quad \text{and} \\ (\mathcal{Q} \circ \mu)(\phi) &= \int \mu(dy)\psi(y)\mathcal{Q}\varphi(y) \end{aligned} \tag{2.2}$$

for $\phi(x, y) = \varphi(x)\psi(y)$ with measurable and bounded $\varphi, \phi : X \mapsto \mathbb{R}$. Here $q = q(y, dx)$ is the transition probability of the backward process reversed with respect to λ; the associated transition operator, $\mathcal{Q}$, $\mathcal{Q}\varphi(y) = \int q(y, dx)\varphi(x)$, is defined as the adjoint of $\mathcal{P}$ in $L^2(\lambda)$, i.e. $\lambda(\varphi\mathcal{P}\psi) = \lambda(\psi\mathcal{Q}\varphi)$ for $\varphi, \psi \in L^2(\lambda)$. Therefore $I[\mu|\lambda] = I[\mu\mathcal{P}|\lambda] < +\infty$ implies $\mu \circ \mathcal{P} = \mathcal{Q} \circ \mu\mathcal{P}$, thus μ is a stationary and reversible measure of the composed, reversible process $\mathcal{R} = \mathcal{P}\mathcal{Q}$, see [F1]. The following reformulation of results by Rényi and Csiszár demostrates an intrisic relationship of the notions of entropy and reversibility.

Theorem 2. *Every stationary measure $\bar{\mu} \ll \lambda$ is reversible with respect to $\mathcal{R}$. If $\mu \ll \lambda$ then so is $\mu\mathcal{P}^t$, and the sequence of densities, $f_t := d\mu\mathcal{P}^t/d\lambda$ is uniformly integrable with* *respect to* λ. *Moreover, if* $\mu\mathcal{P}^{t_n}(\varphi) \to \bar{\mu}(\varphi)$ *for all* $\varphi \in L^\infty(\lambda)$ *as* $t_n \to +\infty$ *then* $\bar{\mu}$ *is a reversible measure of* $\mathcal{R}$, *that is we have a weak convergence to the set of* $\mathcal{R}$*-reversible measures.*

Proof: Suppose first that $I[\mu|\lambda] < +\infty$, then $I[\mu\mathcal{P}^t|\lambda] \le I[\mu|\lambda]$ implies the uniform integrability of f_t, $t \in \mathbb{N}$, thus the Dunford–Pettis Theorem applies. We have to show that every weak limit point $\bar{\mu}$ satisfies $I[\bar{\mu}|\lambda] = I[\bar{\mu}\mathcal{P}|\lambda]$.

If $\bar{\mu}(\varphi) = \lim_n \mu\mathcal{P}^{t_n}(\varphi)$ for all $\varphi \in L^\infty(\lambda)$ and $\phi : X \times X \mapsto \mathbb{R}$ is measurable and bounded, then

$$\begin{aligned}(\bar{\mu}\circ\mathcal{P})(\phi) - \log(\mathcal{Q}\circ\bar{\mu})(e^\phi) &= \lim_{n\to\infty}(\mu_{t_n}\circ\mathcal{P})(\phi) - \log(\mathcal{Q}\circ\mu_{t_n+1})(e^\phi)\\ &\le \lim_{n\to\infty}\big(I[\mu_{t_n}|\lambda] - I[\mu_{t_n+1}|\lambda]\big) = 0 \end{aligned}\tag{2.3}$$

Taking the supremum on the left hand side we get $I[\bar{\mu}\circ\mathcal{P}|\mathcal{Q}\circ\bar{\mu}\mathcal{P}] = 0$, whence $\bar{\mu}\circ\mathcal{P} = \mathcal{Q}\circ\bar{\mu}\mathcal{P}$, i.e. $\bar{\mu} = \bar{\mu}\mathcal{P}\mathcal{Q}$. Replacing $\mathcal{P}$ by $\mathcal{R}$ in the argument above, we get $\bar{\mu}\circ\mathcal{R} = \mathcal{R}\circ\bar{\mu}$, the condition of reversibility of $\bar{\mu}$ with respect to $\mathcal{R} = \mathcal{P}\mathcal{Q}$.

The general case of $\mu \ll \lambda$ follows by a direct approximation procedure. For each $\varepsilon > 0$ we have some μ^ε such that $I[\mu^\varepsilon|\lambda] < +\infty$ and $|\mu - \mu^\varepsilon|_1 < \varepsilon$, where $|\cdot|_1$ denotes the variational distance. Set $f_t^\varepsilon := d\mu^\varepsilon\mathcal{P}^t/d\lambda$ and $|x|_+ := \max\{0, x\}$; since $\mathcal{P}$ is a contraction of $L^1(\lambda)$,

$$\begin{aligned}\int |f_t - a|_+\, d\lambda &\le \int |f_t^\varepsilon - a|_+\, d\lambda + \int |f_t - f_t^\varepsilon|\, d\lambda\\ &\le \int |f_0^\varepsilon - a|_+\, d\lambda + \int |f_0 - f_0^\varepsilon|\, d\lambda \le 2\varepsilon\end{aligned}$$

if a is large enough, thus f_t is still a uniformly integrable sequence. Consider now a weak limit point $\bar{\mu}$ of $\mu\mathcal{P}^t$, $t_n \to +\infty$ is the subsequence along which $\mu\mathcal{P}^t$ converges to $\bar{\mu}$, and let $\bar{\mu}^\varepsilon$ denote a limit point of $\mu^\varepsilon\mathcal{P}^{t_n}$. Therefore we have a subsequence $\{t'_n\} \subset \{t_n\}$ such that for any $\varphi \in L^\infty(\lambda)$

$$|\bar{\mu}(\varphi) - \bar{\mu}^\varepsilon(\varphi)| = \lim_{n\to\infty} |\mu\mathcal{P}^{t'_n}(\varphi) - \mu^\varepsilon\mathcal{P}^{t'_n}(\varphi)| \le \varepsilon|\varphi|_\infty$$

so that $\bar{\mu} - \bar{\mu}^\varepsilon|_1 \le \varepsilon$ implying $\bar{\mu}(\varphi\mathcal{R}\psi) = \bar{\mu}(\psi\mathcal{R}\varphi)$ for $\varphi, \psi \in L^\infty(\lambda)$. □

This result is useful because usually it is easier to identify the reversible measures than the stationary ones. Of course, the set of reversible measures of $\mathcal{R} = \mathcal{P}\mathcal{Q}$ can be much larger than the set of stationary measures of $\mathcal{P}$, then a next, more specific step is needed.

For example, if X is a countable set then $\mathcal{P}$ is given by a stochastic matrix $p = p(x, y)$, and $q(y, x) := \lambda(x)p(x, y)/\lambda(y)$ is the associated backward transition probability; $\lambda(x) > 0$ for all $x \in X$ may be assumed. From (2.3) with $\mathcal{P}^t$ in place of $\mathcal{P}$ we get $\bar{\mu}\circ\mathcal{P}^t = \mathcal{Q}^t\circ\bar{\mu}\mathcal{P}^t$ for any limit distribution $\bar{\mu}$, which reads as

$$\frac{\mu(x)}{\lambda(x)}p^t(x, y) = p^t(x, y)\frac{\bar{\mu}_t(y)}{\lambda(y)}$$

in the present context. Therefore if the chain is aperiodic in the sense that for each $x \in X$ there exists an integer $t(x) > 0$ such that $p^t(x,x) > 0$ whenever $t \geq t(x)$, then $\bar\mu(x) = \bar\mu_t(x)$ for $t \geq t(x)$. Similarly, $\bar\mu_1(x) = \bar\mu_{t+1}(x)$ if $t \geq t(x)$, consequently $\bar\mu(x) = \bar\mu_1(x)$ for all $x \in X$, i.e. $\bar\mu = \bar\mu\mathcal{P}$. Uniqueness of the stationary measure follows immediately from a condition of irreducibiliy: if for each pair $x, y \in X$ we have some $t = t(x,y)$ such that $p^t(x,y) > 0$ then $\bar\mu(x)/\lambda(x) = \bar\mu(y)/\lambda(y)$, whence $\bar\mu(x) = \lambda(x)$ for all $x \in X$, consequently we have $\mu_t(x) \to \lambda(x)$ for all $x \in X$ as $t \to \infty$.

In the case of continuous time it is natural to assume that X is a complete and separable metric space, and both $\mathcal{P}^t$ and its adjoint in $L^2(\lambda)$, $\mathcal{Q}^t$ form strongly continuous semigroups in $C_b(X)$; basic notations are the same as above. To obtain a lower bound for $I[\mu|\lambda] - I[\mu\mathcal{P}^t|\lambda]$ consider an auxiliary distribution $\nu \ll \lambda$ such that $\psi := d\nu/d\lambda > 0$; then $\mu \ll \nu$ and $I[\mu|\nu] = I[\mu|\lambda] - \nu(\log\psi)$, while $I[\mu\mathcal{P}^t|\nu\mathcal{P}^t] = I[\mu\mathcal{P}^t|\lambda] - \mu(\log\mathcal{Q}^\psi)$ as $d\mu\mathcal{P}^t/d\lambda = \mathcal{Q}^t\psi$. Since $I[\mu\mathcal{P}^t|\nu\mathcal{P}^t] \leq I[\mu|\nu]$ by convexity,

$$\begin{aligned} I[\mu|\lambda] - I[\mu\mathcal{P}^t|\lambda] &\geq \mu(\log\psi) - \mu\mathcal{P}^t(\log\mathcal{Q}^t\psi) \\ &\geq \mu(\log\psi) - \mu(\log\mathcal{R}^t\psi) \geq \int \frac{\psi - \mathcal{R}^t\psi}{\psi}\, d\mu \end{aligned} \tag{2.4}$$

as $\log x - \log y \geq (x-y)/x$. Observe that $\lambda \circ \mathcal{R}^t$ is a symmetric measure, thus with $f = d\mu/d\lambda$ we get

$$\begin{aligned} \int \frac{\mathcal{R}^t\psi}{\psi}\, d\mu &= \frac{1}{2}\iint (\lambda\circ\mathcal{R}^t)(dx,dy)\Big(\frac{f(x)\psi(y)}{\psi(x)} + \frac{f(y)\psi(x)}{\psi(y)}\Big) \\ &\geq \iint (\lambda\circ\mathcal{R}^t)(dx,dy)\sqrt{f(x)}\sqrt{f(y)} \end{aligned} \tag{2.6}$$

This means that the right hand side is maximal if $\psi = \sqrt{f}$.

Consider now the Donsker–Varadhan rate D,

$$D[\mu|\mathcal{G}] := \sup_\psi \Big\{ -\int \frac{\mathcal{G}\psi}{\psi}\, d\mu \;:\; \psi \in \mathrm{Dom}\;\mathcal{G},\; \inf\psi > 0 \Big\} \tag{2.7}$$

where $\mathcal{G}$ is any semigroup generator, and notice that Dom $\mathcal{G}$ in the definition of D can be replaced by any core of $\mathcal{G}$ in $C_b(X)$. Moreover, if $\mathcal{G}$ is self-adjoint in $L^2(\lambda)$ and $f = d\mu/d\lambda$, then $D[\mu|\mathcal{G}] < +\infty$ implies $\sqrt{f} \in \mathrm{Dom}\;(-\mathcal{G})^{1/2}$ and

$$D[\mu|\mathcal{G}] = \int \big(\sqrt{-\mathcal{G}}\sqrt{f}\big)^2\, d\lambda \tag{2.8}$$

see (2.6) and Theorem 5 in [DV]. Observe now that $\mathcal{R}^t$ is self-adjoint in $L^2(\lambda)$, thus so is its generator, $\mathcal{G}$, too. By a formal calculation we get $\mathcal{G} = \mathcal{L} + \mathcal{L}^*$, where $\mathcal{L}$ is the generator of the semigroup $\mathcal{P}^t$, while its adjoint $\mathcal{L}^*$ denotes that of $\mathcal{Q}^t$. For small t the right hand side of (2.3) becomes approximately $-t\mu(\mathcal{G}\psi/\psi) + o(t)$, thus we have

Proposition 1. *Suppose that we have a dense $C^* \subset C_b(X)$ such that it is a common core of $\mathcal{L}$ and $\mathcal{L}^*$ with respect to either $C_b(X)$ and $L^2(\lambda)$, then*

$$I[\mu\mathcal{P}^t|\lambda] + 2tD[\bar{\mu}_t|\mathcal{G}] \leq I[\mu|\lambda]\,; \quad \bar{\mu}_t := \frac{1}{t}\int_0^t \mu\mathcal{P}^s\,ds\,.$$

For a more detailed proof see [FLO]. Therefore if $I[\mu|\lambda] < +\infty$, then $D[\mu|\mathcal{G}] = 0$ in a stationary regime with $I[\mu|\lambda] < +\infty$ implying the reversibility of μ with respect to $\mathcal{R}^t$, that is $\mu(\varphi\mathcal{G}\psi) = \mu(\psi\mathcal{G}\varphi)$ for all $\varphi, \psi \in \mathrm{Dom}\,\mathcal{G}$. In the case of reversible diffusion processes the verification of the conditions of Proposition 1 amounts to establishing smooth dependence of solutions on initial values. Assuming the smoothness of the coefficients of the underlying stochastic equations, a standard argument shows that twice continuously differentiable functions with compact supports form a core of the generator. If the diffusion matrix is positive then $D[\mu|\mathcal{G}] = 0$ yields $\mu = \lambda$, thus $\mu_0\mathcal{P}^t \to \lambda$ as $t \to \infty$ for all $\mu_0 \ll \lambda$.

Our next task is to extend these results to infinity volumes, this is done by means of a familiar argument of Holley [Ho]; in translation invariant situations we can pass to the thermodynamic limit. This procedure can not be carried out in a general framework, see e.g. [FFL] and [FLO]; technical requirements are summarized in the next section.

3. On Locality of Dynamics

Results of [F2] are not directly applicable in the present situation, that is why we review some parts of the argument. A convenient collection Θ of cutoff functions is defined for $m \in \mathbb{R}^d$ and $\ell \geq 1$ by $\theta_m^\ell = \theta_m^\ell(x) := \theta_0(||x-m| - \ell|_+)$, where $\theta_0 \in C_0^3(\mathbb{R}_+)$ satisfies $0 \leq \theta_0(u) \leq 1\ \forall u \geq 0$ while $\theta_0(u) = 1$ if $u \leq 1$ and $\theta_0(u) = 0$ if $u \geq 2$; thus Θ is the set of such functions including also $\theta \equiv 1$, that is the case of full (infinite) dynamics. The limiting solution will be denoted as $\boldsymbol{\omega} = \omega^1(t,\sigma)$. The basic a priori bound of [F2] can be reformulated as follows, see Proposition 2 and (3.18) there. Let $N_k(\omega)$ denote the number of points of ω in $B(\omega_k, 1)$, and

$$\overline{N}_\theta(t,\sigma) := 1 + \sup_{k\in S}\max_{s\leq t} \frac{N_k(\omega^\theta(s))}{\sqrt{g_\alpha(\omega_k^\theta(s))}}\,. \tag{3.1}$$

Exploiting superstability of the interaction, by means of the argument of [F2] we get

Proposition 2. *If $\alpha \leq 2 - d/2$ then for each $t > 0$ and $h > 0$*

$$\lim_{\rho\to\infty}\sup_{\theta\in\Theta}\sup_{\sigma\in\overline{\Omega}_{\alpha,h}} P[\overline{N}_\theta(t,\sigma) > \rho] = 0\,.$$

First we derive a uniform bound on the localization of particles. From the stochastic equations

$$|\omega_k^\theta(t,\sigma) - \sigma_k| \leq K_1 \int_0^t \overline{N}_\theta(t,\sigma)\sqrt{g_\alpha(\omega_k^\theta(s,\sigma))}\,ds + \left|\int_0^t \sqrt{\theta(\omega_k^\theta(s,\sigma))}\,dw_k\right|. \tag{3.2}$$

Let $g_*(u) := (1+|u|)^{4/5}$, by a direct calculation

$$\begin{aligned} \delta_{\theta,k}(t,\sigma) &:= \max_{s\le t} |\omega_k^\theta(s,\sigma)-\sigma_k| \le \xi_\theta(t,\sigma)\big(g_*(\sigma_k)+g_*(\delta_{\theta,k}(t,\sigma)\big), \\ \xi_\theta(t,\sigma) &:= K_2\int_0^t \overline{N}_\theta(s,\sigma)\,ds + \sup_{k\in S}\max_{s\le t}\frac{K_2}{g_*(\sigma_k)}\Big|\int_0^t \sqrt{\theta(\omega_k^\theta(s,\sigma))}\,dw_k\Big| \end{aligned} \tag{3.3}$$

whence by assuming $\delta_{\theta,k} \ge g_*(\sigma_k)$ we get

$$\delta_{\theta,k}(t,\sigma) \le \eta_\theta(t,\sigma)g_*(\sigma_k)\,, \tag{3.4}$$

where the explicit form $\eta = K_3\xi^5$ is not relevant, we only need

$$\lim_{y\to\infty}\sup_{\theta\in\Theta}\sup_{\sigma\in\overline{\Omega}_{\alpha,h}} P[\eta_\theta(t,\sigma)>y]=0 \tag{3.5}$$

for all $t,h>0$, which is a direct consequence of the definition of ξ .

Now we are in a position to estimate the rate of convergence of partial dynamics $\boldsymbol{\omega}^\theta$ to its limit $\boldsymbol{\omega}$ as $\theta\to 1$. For any initial configuration $\sigma\in\overline{\Omega}_\alpha$ let $S(m,r,\sigma)$ denote the set of $k\in S$ such that $|\sigma_k - m|\le r$, and consider

$$\Delta_{m,\ell}(t,r,\sigma) := \max_{k\in S(m,r,\sigma)}\max_{s\le t}|\omega_k^\theta(s,\sigma)-\omega_k(s,\sigma)| \quad \text{with } \theta=\theta_m^\ell\,. \tag{3.6}$$

For any fixed $T>0$ and $r_0,\ell\ge 1$ define r_κ , $\kappa=0,1,...,\chi,...$ by

$$\begin{aligned} r_{\kappa+1} &= r_\kappa + 2g_*(|m|+\ell)\,\tilde{\eta}_{m,\ell}(T,\sigma)+R+1\,, \quad \text{where} \\ \tilde{\eta}_{m,\ell}(T,\sigma) &:= \max\{\eta_\theta(T,\sigma),\eta_1(T,\sigma)\}\,, \quad \theta=\theta_m^\ell\,. \end{aligned} \tag{3.7}$$

In view of (3.4) this means that before time T the particles starting from $B(m,r_\kappa)$ can not interact with those starting from outside of $B(m,r_{\kappa+1})$, therefore

$$\begin{aligned} \Delta_{m,\ell}(t,r_\kappa,\sigma) &\le Lg(|m|+\ell)\,\widetilde{N}_{m,\ell}(t,\sigma)\int_0^t \Delta_{m,\ell}(s,r_{\kappa+1},\sigma)\,ds\,, \quad \text{where} \\ \widetilde{N}_{m,\ell}(t,\sigma) &:= \max\{\overline{N}_{\theta_m^\ell}(t,\sigma),\overline{N}_1(t,\sigma)\}\,, \end{aligned} \tag{3.8}$$

provided that $r_{\kappa+1}+R\le\ell$.

Suppose that (3.8) can χ times be iterated, then for $t<T$

$$\Delta_{m,\ell}(t,r_0,\sigma)\le 2(\ell+1)\frac{(Lt)^\chi}{\chi!}\big(g_*(|m|+\ell)\widetilde{N}_{m,\ell}(t,\sigma)\big)^\chi \tag{3.9}$$

where $\chi = O(\ell(|m|+\ell)^{-4/5})$ is a random number. Of course, this inequality implies the a.s. convergence of partial solutions; this was shown in [F2] when $m=0$ and $\ell\to+\infty$. Here we need a more delicate result: $\boldsymbol{\omega}^{m,\ell}$ converges even if $|m|$ increases together with ℓ . More precisely, for any $r_0,t,h,\varepsilon>0$ we have

$$\lim_{n\to\infty}\sup_{\sigma,m}\big\{P[\widetilde{N}_{m,\ell}(t,\sigma)\Delta_{m,\ell}(t,r_0,\sigma)>\varepsilon]:\sigma\in\overline{\Omega}_{\alpha,h}\,,\ m,\ell\in M_n\big\}=0\,, \tag{3.10}$$

where $M_n := \{m,\ell : |m|+\ell+R<n\,,\ \ell>n^{5/6}\}$. Indeed, in this situation χ of (3.9) goes a.s. to $+\infty$ as $n\to\infty$.

In the next section the following consequence of (3.10) will be needed. Suppose that we are given a translation invariant probability measure μ such that $\mu(\overline{\Omega}_\alpha)=1$, and set $\hat{\mu}_n := \mu_{\Lambda_n}\lambda$. The above calculations are summarized in

Lemma 1. *For any* $\phi \in C_0^1(\Omega)$ *and* $t > 0$ *we have*

$$\lim_{n\to\infty} \sup_{s,m} \big\{ |\hat{\mu}_n \mathcal{P}^s \mathbf{s}^m \phi - \mu \mathcal{P}^s \phi| : s \le t\,,\ |m| < n - n^{5/6} \big\} = 0\,.$$

Proof: Since $\hat{\mu}_n(s^m \varphi) = \mu(\varphi)$ if $\varphi, \mathbf{s}^m\varphi \in \mathcal{F}_{\Lambda_n}$, it is natural to set $\varphi_\ell = \mathcal{P}^s_{\theta_0^\ell}\phi$, then $\mathbf{s}^m \varphi_\ell = \mathcal{P}^s_{\theta_m^\ell}\phi$. Since ϕ is Lipschitz continuous by assumption, we can compare $\mathbf{s}^m\varphi_\ell$ and $\mathcal{P}^s\mathbf{s}^m\phi$ via (3.10), at least if $|m| < n - n^{5/6}$. The missing part of the bound,

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}}\,,\, \hat{\mu}_n(\overline{\Omega}_{\alpha,h}) = 0 \tag{3.11}$$

follows from the basic superstability estimate of Ruelle [Ru2]. Indeed, for any box Λ of given shape and size we have $\lambda[\omega(\Lambda) > \nu|\mathcal{F}_{\Lambda^c}] \le Ce^{-c\nu^2}$, where c and C do not depend on ω_{Λ^c}. In view of (1.2) this yields $\lambda(\overline{\Omega}_\alpha) = 1$ by the Borel-Cantelli lemma. Since $\mu(\overline{\Omega}_\alpha) = 1$ by assumption, estimating the contribution of particles from Λ_n^c to $\overline{H}_\alpha$ via superstability, we get (3.11) by a similar computation. □

Remark: Since the level sets of $\overline{H}$ are compact, the Stone-Weierstrass theorem allows us to extend Lemma 1 to continuous and bounded local functions.

4. Passage to the Thermodynamic Limit

Now we are in a position to prove Theorem 1 by extending Proposition 1 to infinite volumes. Using the notation $\hat{\mu}_n^* = \mu_{\Lambda_n}^*\lambda$ of Lemma 1, we have

$$I[\hat{\mu}_n^*\mathcal{P}_\theta^t|\lambda] + 2tD[\bar{\mu}_{n,\theta,t}^*|\mathcal{L}_\theta] \le I[\hat{\mu}_n^*|\lambda] = I_{\Lambda_n}[\mu^*|\lambda] \tag{4.1}$$

for any smooth cutoff θ, where $\bar{\mu}_{n,\theta,t}^*$ is the time average of the evolved measures $\hat{\mu}_n^*\mathcal{P}_\theta^s$ from $s = 0$ through $s = t$. In view of (2.6) D is subadditive in the following sense. Suppose that $J_\theta^\ell(n) \subset \mathbb{Z}^d$ satisfies $\theta \ge \theta_m^\ell$ and $\theta_m^\ell\theta_k^\ell = 0$ for $m, k \in J_\theta^\ell(n)$, $k \ne m$ then

$$D[\bar{\mu}_{n,\theta,t}^*|\mathcal{L}_\theta] \ge \sum_{m\in J_\theta^\ell(n)} D[\bar{\mu}_{n,\theta,t}^*|\mathcal{L}_{\theta_m^\ell}] \ge \sum_{m\in J_\theta^\ell(n)} \int \frac{-\mathcal{L}_{\theta_m^\ell}\mathbf{s}^m\psi}{\mathbf{s}^m\psi}\, d\bar{\mu}_{n,\theta,t}^* \tag{4.2}$$

for smooth $\psi > 0$. Similarly, for all $\varphi \in C_0(\Omega)$

$$\begin{aligned} I[\hat{\mu}_n^*\mathcal{P}_\theta^t|\lambda] &\ge \int S_n\big(\mathcal{P}_\theta^t(\varphi)\big)\, d\hat{\mu}_n^* - F_\lambda\big(S_n(\varphi)\big)\,, \quad \text{with} \\ S_n(\varphi) &:= \sum_{m\in\Lambda_n\cap\mathbb{Z}^d} \mathbf{s}^m\varphi \end{aligned} \tag{4.3}$$

Now we can remove the cutoff of dynamics, keeping $J_\theta^\ell(n) = J^\ell(n) \subset \Lambda_{n-n^{5/6}}$ fixed during this procedure we get

$$\sum_{m\in J^\ell(n)} \int_0^t ds \int \frac{-\mathcal{L}_{\theta_m^\ell}\mathbf{s}^m\psi}{\mathbf{s}^m\psi}\, d\hat{\mu}_n^*\mathcal{P}^s \le I[\hat{\mu}_n^*|\lambda] + F_\lambda\big(S_n(\varphi)\big) - \int S_n(\mathcal{P}^t\varphi)\, d\hat{\mu}_n^* \tag{4.4}$$

As far as ℓ is fixed, we may assume that Card $J^\ell(n) \geq c_\ell|\Lambda_n|$ with some $c_\ell > 0$; thus dividing both sides by $|\Lambda_n|$ we can pass to a thermodynamic limit. Indeed, in view of Lemma 1 all terms of $\hat{\mu}_n^*\mathcal{P}^t\big(S_n(\varphi)\big)$ become asymptotically identical when $n \to \infty$. Since $\mathcal{L}_{\theta_m^\ell} = \mathbf{s}^m\mathcal{L}_{\theta_0^\ell}$, the same holds true on the left hand side, thus for all $\theta \in \Theta$ with compact support we have some $c_\theta > 0$ such that

$$c_\theta t \int \frac{-\mathcal{L}_\theta\psi}{\psi}\, d\mu^* \leq \bar{I}[\mu^*|\lambda] + \bar{F}_\lambda(\varphi) - \mu^*(\varphi), \tag{4.6}$$

where $\varphi \in C_0(\Omega)$ is arbitrary, thus from (1.11) $\mu^*(\mathcal{L}_\theta\psi/\psi) \geq 0$ for all smooth $\psi > 0$. Since $\mathcal{L}_\theta \log\psi \geq \mathcal{L}\psi/\psi$ by convexity, and $\phi = \log\psi$ is still quite general, the resulting stationary Kolmogorov equation $\mu^*(\mathcal{L}_\theta\phi) = 0$ implies that μ^* is a stationary and reversible measure of every partial evolution $\mathcal{P}_\theta$, which completes the proof of Theorem 1.

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