An existence result for infinite-dimensional Brownian diffusions with non-regular and non-Markovian drift

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Abstract

We prove in this paper an existence result for infinite-dimensional stationary interactive Brownian diffusions. The interaction is supposed to be small in the norm $\|\cdot\|_\infty$ but otherwise is very general, being possibly non-regular and non-Markovian. Our method consists in using the characterization of such diffusions as space-time Gibbs fields so that we construct them by space-time cluster expansions in the small coupling parameter.

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1 Introduction

Let $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R})^{\mathbb{Z}^d}$ be the configuration space, and $\mathcal{F}$ be the canonical σ-field in it. For $\omega \in \Omega$ we write $\omega = (\omega_i(t))_{i \in \mathbb{Z}^d, t \in \mathbb{R}}$. Suppose we are given the following infinite-dimensional stochastic differential equation (s.d.e.)

$$dX_i(t) = \Big(-\frac{1}{2}\varphi'(X_i(t)) + \mathbf{b}(\theta_{i,t}X)\Big)dt + dB_i(t) \ , \quad i \in \mathbb{Z}^d, \ t \in \mathbb{R} \tag{1}$$

where

- φ is a suitable *self potential*, to be chosen in a class that will be defined in Section 2;
- $\mathbf{b} : \mathcal{C}((-\infty, 0], \mathbb{R})^{\mathbb{Z}^d} \to \mathbb{R}$ is a measurable bounded local function, say $\mathbf{b}(\omega) = \mathbf{b}(\omega_{V_0})$, where $\omega_{V_0} = (\omega_i(t))_{i \in \Lambda_0, t \in]-T_0, 0]}$ is the restriction of ω to a bounded subset $V_0 = \Lambda_0 \times]-T_0, 0] \subset \mathbb{Z}^d \times \mathbb{R}$ of space-time- coordinates, containing the origin;
- $\theta_{i,t}$ is the space-time translation on Ω given by $(\theta_{i,t}\omega)_j(s) = \omega_{i+j}(t+s)$;
- $(B_i)_{i \in \mathbb{Z}^d}$ is a sequence of independent real-valued Brownian motions.

Our aim in this paper is to prove the existence of a stationary weak solution of (1) with possibly non-Markovian and non-gradient drift in an infinite time-interval.

Such diffusions restricted to a finite time-interval (say $[0, 1]$), with $\mathbf{b}(\omega) = b(\omega(0))$ (*Markovian drift*), and when b is the gradient of a smooth Hamilton function, were described as lattice Gibbs states on $\mathcal{C}([0, 1], \mathbb{R})^{\mathbb{Z}^d}$ first by Deuschel in [5, 6] and later in [3]. We will use here the description of weak solutions of (1) as space-time Gibbs states on Ω. To be more precise, let $Q \in \mathcal{P}_s(\Omega)$ be a space-time translation invariant probability measure on $(\Omega, \mathcal{F})$, and $\mathbf{b}$ be a given function as above. We denote by P the reference measure in $\mathcal{P}_s(\Omega)$, law of the stationary solution of equation (1) with $\mathbf{b} \equiv 0$. We make assumptions on φ which guarantee existence and uniqueness of such "free" infinite-dimensional diffusion without interaction P. Under the integrability condition

$$\mathcal{H}(Q) < +\infty,$$

where $\mathcal{H}$ denotes the specific entropy of Q with respect to P (see formula (11)), the main result in [4] was the equivalence of the following assertions:

(i) Q is a stationary weak solution of the stochastic differential equation (1).

(ii) Q is a space-time invariant Gibbs state for a specification which is built on a Hamiltonian functional H defined on $\mathcal{C}(\mathbb{R}, \mathbb{R})^{\mathbb{Z}^d}$ and given in (10). This specification is defined as a perturbation of a *reference* specification, which in this model consists of stochastic bridges derived from P.

No existence result of solution of equation (1) was proved in [4].

When the drift $\mathbf{b}(\omega) = b(\omega(0))$ is a regular Markovian one, existence and uniqueness of strong solutions of (1) were proved in [7] and [26]. But it is not clear whether among the solutions there is one that is time stationary. Furthermore, not having assumed any smoothness on the drift $\mathbf{b}$ and no Markovianity, it is not known whether the s.d.e. (1) admits any weak solution. Indeed, we show here that a stationary solution of (1) with general drift $\mathbf{b}$ can be constructed by cluster expansion, provided $\|\mathbf{b}\|_\infty$ is sufficiently small.

Gibbs fields on the trajectory space $\mathcal{C}(\mathbb{R}, \mathbb{R})$ were introduced in the context of Euclidean quantum field theory as quasi-invariant measures (see Courrège and Renouard [2], Royer and

Yor [25] and references therein). One of the difficulties in dealing with Gibbs fields on path spaces comes from the fact that disjoint time regions are not independent under the reference measure. More recently Betz and Lörinczi [1] used D-L-R approach for constructing $P(\Phi)_1$-processes, and Osada and Spohn [21] used it for constructing a class of Gibbsian non Markovian real valued stochastic processes. One finds also in [15] and [16] the use of cluster expansion methods to solve existence problems of Brownian paths under different types of interaction.

In this paper, following the notions introduced by Minlos, Rœlly and Zessin in [19], we deal with Gibbs fields on $\mathcal{C}(\mathbb{R},\mathbb{R})^{\mathbb{Z}^d}$, that are parametrized by space and time $\mathbb{Z}^d \times \mathbb{R}$. Some of these fields are related with quantum Gibbs states through the Feynman-Kac-Nelson representation - cf. [20] for a very clear description of the relation with physical quantum models -. The originality of the model we present here comes from the generality of the Hamiltonian functional H, which is neither a quadratic or a polynomial one as in [19] (formula (35) or (74)), nor a bounded functional. It includes a stochastic integral term, and therefore is highly explosive.

The paper is divided into the following sections.

1. Introduction.
2. Infinite-dimensional diffusion as space-time Gibbs states.
3. Cluster representation and cluster estimates

2 Infinite-dimensional diffusion as space-time Gibbs state

First of all we introduce our one dimensional reference process, whose law on $\mathcal{C}(\mathbb{R},\mathbb{R})$ is denoted by W, as the time-stationary solution of the scalar differential equation

$$dx(t) = -\frac{1}{2}\varphi'(x(t))dt + dw(t) \tag{2}$$

where w is a real valued Brownian motion and the *self potential* φ is a $\mathcal{C}^2(\mathbb{R},\mathbb{R})$ function satisfying the following properties :

$$\lim_{|x|\to+\infty} \varphi(x) = +\infty, \quad \text{and } \exists C_0 \in \mathbb{R} \text{ such that } \tilde{\varphi} =: \varphi'' - \frac{1}{2}(\varphi')^2 \le C_0 \tag{3}$$

$$e^{-\varphi} \in L^1(\mathbb{R}). \tag{4}$$

$$0 < \liminf_{|x|\to+\infty} \varphi''(x) \quad \text{and } \exists M > 0, \int_{|x|>M} \frac{1}{\varphi'(x)} dx < +\infty. \tag{5}$$

Property (3) guarantees that, for any given initial condition, a unique non-exploding strong solution of (2) exists (see Theorem 2.2.19 in [24]). Property (4) is in fact a consequence of the first part of (5) since the assumption $\liminf_{|x|\to+\infty} \varphi''(x) > 0$ implies that the measure $e^{-\varphi(x)}dx$ has tails not bigger than Gaussian. It ensures that the measure $e^{-\varphi}dx$, which is invariant, is normalizable. Let $\mu(dx) = e^{-\varphi(x)}dx / \int e^{-\varphi(y)}dy$ denote this unique invariant Probability measure associated to (2). Property (5) ensures that the process $x(t)$ is sufficiently ergodic against μ, in the sense that its associated semi-group is ultracontractive.

2.1 The infinite-dimensional diffusion

Let $\Omega = \mathcal{C}(\mathbb{R},\mathbb{R})^{\mathbb{Z}^d}$ be the canonical configuration space, and $\mathcal{F}$ be the canonical σ-field. With $\mathcal{P}(\Omega)$ we denote the space of probability measures on Ω, and $\mathcal{P}_s(\Omega)$ is the subset of $\mathcal{P}(\Omega)$ consisting of those probabilities that are invariant for the space-time shift maps $(\theta_{i,t})_{i\in\mathbb{Z}^d,\, t\in\mathbb{R}}$.

In what follows we let P be the law of the reference non-interacting infinite system, i.e.

$$P = \otimes_{\mathbb{Z}^d} W \ \in \mathcal{P}_s(\Omega).$$

The main object of this paper is an infinite-dimensional diffusion that is obtained by perturbing through an interaction a system of independent particles each evolving with dynamics given by (2): we fix a bounded subset $V_0 = \Lambda_0 \times]-T_0, 0] \subset \mathbb{Z}^d \times \mathbb{R}^-$, and assume we are given a measurable bounded V_0-local function $\mathbf{b}(\omega) = \mathbf{b}(\omega_{V_0})$ on $\mathcal{C}([-T_0,0],\mathbb{R})^{\Lambda_0}$, where this path space is provided with the topology of uniform convergence, and the corresponding Borel σ-field.

We consider the associated stochastic differential system

$$dX_i(t) = \Big(-\frac{1}{2}\varphi'(X_i(t)) + \mathbf{b}(\theta_{i,t}X)\Big)dt + dB_i(t)\ , \quad i \in \mathbb{Z}^d,\ t \in \mathbb{R}^+. \tag{6}$$

Remark that, in the time-stationary situation, we can also consider the above system for any time $t \in \mathbb{R}$.

We recall that a *weak solution* of the s.d.e. (6) is a probability measure Q on Ω such that the scalar processes

$$\Big(X_i(\cdot) - \int_0^{\cdot} \Big(-\frac{1}{2}\varphi'(X_i(s)) + \mathbf{b}(\theta_{i,s}X)\Big) ds \Big)_{i\in\mathbb{Z}^d}$$

are Q-independent Brownian motions, where X is the canonical process on Ω: $X_i(t,\omega) = \omega_i(t)$ for $\omega \in \Omega$, $i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$.

2.2 Its characteristics as Gibbs field

For $Q \in \mathcal{P}(\Omega)$, and $\mathcal{G}$ sub-σ-field of $\mathcal{F}$, we denote by $Q(\cdot/\mathcal{G})$ a regular version of Q conditioned to $\mathcal{G}$, while $Q_{|\mathcal{G}}$ denotes the restriction of Q to $\mathcal{G}$. To define space-time Gibbs fields, we need to introduce different filtrations on the space-time structure. Let $\mathcal{V}$ be the set of space-time volumes V having the form $V = \Lambda \times I$ where $\Lambda \subset \mathbb{Z}^d$ is finite, and $I =]a_1, a_2[$ is a bounded open interval. For a space-volume $\Lambda \subset \mathbb{Z}^d$ we define its enlargement Λ^+ and its boundary $\partial\Lambda$ by

$$\Lambda^+ = \{i \in \mathbb{Z}^d : (\Lambda_0 + i) \cap \Lambda \neq \emptyset\}, \text{ and } \partial\Lambda = (\Lambda^+)^+ \setminus \Lambda.$$

For a time-volume $I =]a_1, a_2[\subset \mathbb{R}$ we define its enlargement I^+ by $I^+ = [a_1 - T_0, a_2]$.
For $V = \Lambda \times I \in \mathcal{V}$ the *forward* σ-field $\mathcal{F}_V$ and the *backward* σ-field $\hat{\mathcal{F}}_V$ are defined by

$$\mathcal{F}_V = \sigma\{\omega_i(t) : i \in \Lambda^{++}, t \in I^+\}, \text{ and } \hat{\mathcal{F}}_V = \sigma\{\omega_i(t) : (i,t) \notin V\}.$$

The *boundary* σ-field $\partial\mathcal{F}_V$ is given by

$$\partial\mathcal{F}_V = \mathcal{F}_V \cap \hat{\mathcal{F}}_V.$$

For future use, we also let

$$\mathcal{B}_V = \sigma\{\omega_i(t) : i \in \Lambda, t \in I\}.$$

Now, the *reference specification* Π^0 we consider is the following kernel based on P :

$$\forall V \in \mathcal{V}, A \in \mathcal{F}_V,\ \Pi^0_V(A) = P(A/\hat{\mathcal{F}}_V) \quad P-\text{a.s.} \tag{7}$$

It is easy to see that, for $V = \Lambda \times]a_1, a_2[$, Π^0_V is given by

$$\Pi^0_V(\omega, d\omega') = \otimes_{(i,t)\notin V}\delta_{\omega_i(t))}(d\omega'_i(t)) \otimes \otimes_{i\in\Lambda} W^{\omega_i(a_1),\omega_i(a_2)}_{[a_1,a_2]}(d\omega'_i) \tag{8}$$

where $W^{x,y}_{[a_1,a_2]}$ is the law of the stochastic bridge on $[a_1, a_2]$ obtained by conditioning W to be x at time a_1 and y at time a_2. That Π^0 is a space-time specification in the Gibbsian sense has been proved in [19], Example 2, Section 1.4.2.

On the path level, the *interaction* functional is given through a *potential* $\Phi = (\Phi_V)_{V\in\mathcal{V}}$ which is defined on a subset $\Omega' \subset \Omega$ as follows :

$$\left\{\begin{array}{rcll} \Phi_{\Lambda\times I} & \equiv & 0 & \text{if } \nexists i \in \mathbb{Z}^d : \Lambda = i + \Lambda_0 \\ \Phi_{(i+\Lambda_0)\times I}(\omega) & = & -\int_I \mathbf{b}(\theta_{i,t}\omega)d\omega_i(t) + \frac{1}{2}\int_I \Big[\mathbf{b}(\theta_{i,t}\omega)(\mathbf{b}(\theta_{i,t}\omega) - \varphi'(\omega_i(t)))\Big]dt & \text{otherwise} \\ & = & -\int_I \mathbf{b}(\theta_{i,t}\omega)d\tilde{B}_i(t) + \frac{1}{2}\int_I \mathbf{b}^2(\theta_{i,t}\omega)dt & \end{array}\right. \tag{9}$$

where the process $\tilde{B}$ is defined by

$$\tilde{B}_i(t) = \omega_i(t) + \frac{1}{2}\int_{a_1}^t \varphi'(\omega_i(s))ds,\ t \in]a_1, a_2[,$$

and satisfies that $\tilde{B}_i(a_1+.) - \tilde{B}_i(a_1)$'s are independent Brownian motions under P. Note that the potential Φ is not defined a priori on the whole Ω, but only for $\omega \in \Omega'$ for which the stochastic integral $\int_I \mathbf{b}(\theta_{i,t}\omega)d\omega_i(t)$ makes sense (in particular, $P(\Omega') = 1$). Anyway $\Phi_{(i+\Lambda_0)\times I} \in L^2(P)$, and therefore is finite P-almost surely. We make the convention that it is always chosen in such a way that it does not assume the value $-\infty$.

The associated *Hamiltonian* is defined on Ω' for $V = \Lambda \times I$ by

$$H_V(\omega) = \sum_{\Lambda'\cap\Lambda\neq\emptyset} \Phi_{\Lambda'\times I}(\omega) = -\sum_{i\in\Lambda^+}\left[\int_I \mathbf{b}(\theta_{i,t}\omega)d\tilde{B}_i(t) - \frac{1}{2}\int_I \mathbf{b}^2(\theta_{i,t}\omega)dt\right], \quad \omega \in \Omega'. \tag{10}$$

We observe that Φ and H are space-time translation invariant, and that H_V is $\mathcal{F}_V$-measurable.

We can now define for $V \in \mathcal{V}$, $\omega \in \Omega$ the specification $\Pi^H_V(\omega, .)$ as the following Probability measure on Ω with support included into Ω'

$$\Pi^H_V(\omega, d\omega') = \left\{\begin{array}{ll} \frac{1}{Z^H_V(\omega)}\mathbf{1}_{\Omega'}(\omega')\exp(-H_V(\omega'))\Pi^0_V(\omega, d\omega') & \text{if } 0 < Z^H_V(\omega) < +\infty \\ 0 & \text{otherwise,} \end{array}\right.$$

where

$$Z^H_V(\omega) = \int_{\Omega'} \exp(-H_V(\omega'))\Pi^0_V(\omega, d\omega')$$

is the ($\partial\mathcal{F}_V$-measurable) normalization factor.

Definition 1 *A probability measure Q on Ω is said to be a* space-time Gibbs state *with specification Π^H if there exists a subset $\Omega' \subset \Omega$ such that $Q(\Omega') = 1$, H is well defined on Ω' and, for all $V \in \mathcal{V}$ and $A \in \mathcal{F}_V$*

$$Q(A/\hat{\mathcal{F}}_V) = \Pi^H_V(A) \quad Q-a.s.$$

The set of space-time Gibbs states for Π^H will be denoted by either $\mathcal{G}(\Pi^H)$ or $\mathcal{G}(H, \Pi^0)$. Moreover we let $\mathcal{G}_s(\Pi^H)$ denote the set of space-time invariant Gibbs states, i.e.

$$\mathcal{G}_s(\Pi^H) = \mathcal{G}(\Pi^H) \cap \mathcal{P}_s(\Omega).$$

For $Q \in \mathcal{P}_s(\Omega)$ we also define its specific entropy with respect to P by

$$\mathcal{H}(Q) = \int h(Q(\cdot/\mathcal{B}^-)_{|\mathcal{B}_1}; P(\cdot/\mathcal{B}^-)_{|\mathcal{B}_1})\, dQ \tag{11}$$

where $h(\cdot;\cdot)$ denotes the relative entropy between two measures,

$$\mathcal{B}_1 = \mathcal{B}_{\{0\}\times]0,1[}$$

and

$$\mathcal{B}^- = \sigma\{\omega_i(t) : (t \leq 0, i \in \mathbb{Z}^d) \text{ or } (0 < t \leq 1 \text{ and } i < 0)\}.$$

Here we use as order in $\mathbb{Z}^d$ (denoted also by "$<$") the lexicographic order.

Let us now enounce the key point on which our existence theorem is based (it is a condensed version of Proposition 1, Theorems 1 and 2 in [4]).

Proposition 2 *Let $Q \in \mathcal{P}_s(\Omega)$ be a weak solution of the s.d.e. (6). Then $Q \in \mathcal{G}_s(\Pi^H)$ where the Hamiltonian H is given by (10). Reciprocally, if $Q \in \mathcal{G}_s(\Pi^H)$ is such that $\mathcal{H}(Q) < +\infty$, then Q is a weak solution of the s.d.e. (6).*

2.3 The Existence theorem

We are now able to state our main result.

Theorem 3 *If the drift $\mathbf{b}$ has a norm $\|\mathbf{b}\|_\infty$ sufficiently small, then there exists a stationary weak solution Q of the s.d.e. (6) . This Probability measure Q admits a cluster expansion and is invariant with respect to space-time translations. Moreover it satisfies the property of short range correlations, i.e. for every $\Lambda \subset \mathbb{Z}^d$ finite, I bounded interval of $\mathbb{R}$ and $F, G : \Omega \to \mathbb{R}$ $\mathcal{F}_{\Lambda\times I}$-measurable, we have*

$$\lim_{|i|+|t|\to+\infty} Q[F(G \circ \theta_{i,t})] = Q(F)Q(G).$$

The proof of this theorem is based on the following convergence result.

Without loss of generality, we may assume that $\Lambda_0 = \{i \in \mathbb{Z}^d : |i| \leq r_0\}$. Define, recursively, $\Lambda_{n+1} = \Lambda_n^+$. Moreover, let $I(n)$ be an increasing sequence of bounded intervals whose union is $\mathbb{R}$. Note that $V_n = \Lambda_n \times I(n) \subset \mathbb{Z}^d \times \mathbb{R}$ is an increasing sequence of bounded sets which tends to $\mathbb{Z}^d \times \mathbb{R}$ as n tends to infinity. Finally, for $\eta \in \Omega$, we define $\eta^{(n)}$ by

$$\eta_i^{(n)}(t) = \begin{cases} \eta_i(t) & \text{for } t \geq \inf I(n) \\ \eta_i(\inf I(n)) & \text{otherwise.} \end{cases}$$

Lemma 4 *Let Q_n be the Probability measure on Ω defined by :*

$$Q_n(d\eta) = \exp\left[-H_{V_n}(\eta^{(n)})\right] P(d\eta).$$

If, for each local bounded measurable function F,

$$\lim_n Q_n(F) =: Q(F),$$

then the weak limit Probability measure Q belongs to $\mathcal{G}(\Pi^H)$.

Proof : For $\eta \in \Omega$, we write η_V the restriction of η to $(i,t) \in V$. Observe that

$$Q_n(d\eta) = \Pi^H_{V_n}(\eta^{(n)}_{V_n^c}, d\eta) Z^H_{V_n}(\eta^{(n)}_{V_n^c}) P_{|\mathcal{B}_{V_n^c}}(d\eta_{V_n^c}),$$

since $\Pi^0_{V_n}(\eta^{(n)}_{V_n^c}, d\eta) = \Pi^0_{V_n}(\eta_{V_n^c}, d\eta)$. Moreover, by Girsanov Theorem, it is easy to check that

$$\int Z^H_{V_n}(\eta^{(n)}_{V_n^c}) P_{|\mathcal{B}_{V_n^c}}(d\eta_{V_n^c}) = 1.$$

Thus, Q_n is a *mixture* of the local specifications $\Pi^H_{V_n}(\eta_{V_n^c}, d\eta)$. The conclusion then follows as in Proposition 1 in [19]. □

The next section is devoted to the proof of the convergence of Q_n which is based on the method of cluster expansion.

3 Cluster representation and cluster estimates

3.1 The cluster representation of statistical sums

The different steps of the proof to obtain a cluster representation of the measures Q_n are the same as in [19] Section 4. The new difficulty comes in the next subsection for the computation of the cluster estimates since the Hamiltonian functional is highly non regular.

The main idea is to discretize in time the volume V_n, to have a better understanding of the local structure of Q_n. After having performed the discretization, the typical strategy in cluster expansion is to expand the partition function

$$Z_n = \int \exp\left[-H_{V_n}(\eta^{(n)})\right] \; P(d\eta),$$

that in our case equals 1, due to the P-martingale property of $\exp -H_{V_n}(\eta)$. The cluster estimates for the integral above are, however, exactly what is needed for our purposes. Let $a > 0$ be a real number which we will choose later and $\mathbb{Z}_a \subset \mathbb{R}$ the one-dimensional time lattice with step length a. We let, for $j \in \mathbb{Z}$, $I_j = [ja, (j+1)a]$. Moreover, we let

$$\mathbb{Z}^{d+1}_a = \mathbb{Z}^d \times \mathbb{Z}_a$$

be the space-time lattice with scale a for the time. We call <u>temporal edge</u> in

$\mathbb{Z}^{d+1}_a$ a pair of the form (i, I_j), $i \in \mathbb{Z}^d, j \in \mathbb{Z}$. The points $(i, ja), (i, (j+1)a)$ are called the <u>vertices</u> of the edge.

We call <u>contour</u> on the interval I_j a sequence of <u>Λ_0-connected</u> temporal edges γ^j of the following type :

$$\gamma^j = \{(i_1, I_j), \dots, (i_m, I_j)\},$$

where Λ_0-connected means that, for $k = 1, \dots, m-1$, $(i_{k+1} + \Lambda_0) \cap (i_k + \Lambda_0) \neq \emptyset$. If the meaning is clear, then we write sometimes $k \in \gamma^j$ instead of $(k, I_j) \in \gamma^j$.

For every set B of temporal edges we denote by $[B] \subset \mathbb{Z}^{d+1}_a$ the set of vertices of the elements of B. For example, $[\{(i, I_j)\}] = \{(i,j), (i, j+1)\} \subset \mathbb{Z}^{d+1}_a$.

We now assume that the time interval $I(n)$ which appears in Q_n is of the form

$$I(n) = [-Na, Na] = \bigcup_{j=-N}^{N-1} I_j,$$

where $N = pn$ for some $p \in \mathbb{Z}$, $p > 0$.

We define the transition density $q_t(x; y)$ of the one-dimensional reference process $x(t)$ solution of (2) with respect to its invariant probability measure μ by the following equation :

$$W(x(t) = dx/x_0 = y) = q_t(x; y)\mu(dx).$$

Since the process $x(t)$ is Markovian, we have :

$$W(\cdot/x(ja) = y_j, j = -N, \dots, N) = \otimes_{j=-N}^{N-1} W_{I_j}^{y_j, y_{j+1}}(\cdot).$$

Let also denote by y_n the $(2N+1)|\Lambda_{n+2}|$-dimensional vector $(y_{i,j})_{(i,j)\in\Lambda_{n+2}\times\{-N,\dots,N\}}$, where $|\Lambda_{n+2}|$ is the number of elements in Λ_{n+2}; let also $\partial I(n) = \{\inf(I(n)), \sup(I(n))\}$. With these notations, and after having noticed that $H_{V_n}(\eta^{(n)})$ depends only on $\eta_i(t)$, $i \in \Lambda_{n+2}$, $t \in I(n)$, one can write the partition functions Z_n as follows

$$\begin{aligned} Z_n &=: \int_\Omega \int_\Omega \exp\left[-H_{V_n}(\eta^{(n)})\right] \ \Pi^0_{\Lambda_{n+2}\times I(n)}(\omega, d\eta) P_{|\mathcal{B}_{\Lambda_{n+2}\times\partial I(n)}}(d\omega) \\ &= \int_{\mathbb{R}^{|\Lambda_{n+2}|(2N+1)}} Z_n(y_n) \prod_{\substack{i\in\Lambda_{n+2} \\ j=-N,..,N-1}} q_a(y_{i,j+1}; y_{i,j}) \otimes_{\substack{i\in\Lambda_{n+2} \\ j=-N,..,N}} \mu(dy_{i,j}) \end{aligned} \tag{12}$$

where

$$Z_n(y_n) = \int_\Omega \exp\left[-H_{V_n}(\eta^{(n)})\right] \otimes_{\substack{i\in\Lambda_{n+2} \\ j=-N,..,N}} W_{I_j}^{y_{i,j}, y_{i,j+1}}(d\eta_i). \tag{13}$$

Since the Hamilton functional $H_{\Lambda\times I}$ is additive with respect to the time interval I,

$$Z_n(y_n) = \int_\Omega \prod_{j=-N}^{j=N-1} \left[-H_{\Lambda_n\times I_j}(\eta^{(n)})\right] \otimes_{\substack{i\in\Lambda_{n+2} \\ j=-N,..,N}} W_{I_j}^{y_{i,j}, y_{i,j+1}}(d\eta_i).$$

For a step length of the time-lattice a sufficiently large, $I_j^+ = [ja - T_0, (j+1)a] \subset \overline{I_{j-1} \cup I_j}$ and the coefficient $Z_n(y_n)$ decomposes into the following product of integrals :

$$\begin{aligned} Z_n(y_n) &= \prod_{j=-N}^{j=N-1} \int_\Omega \exp\left[-H_{\Lambda_n\times I_j}(\eta^{(n)})\right] \otimes_{i\in\Lambda_{n+2}} W_{I_{j-1}}^{y_{i,j-1}, y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j}, y_{i,j+1}}(d\eta_i) \\ &= \prod_{j=-N}^{j=N-1} \int_\Omega \prod_{k\in\Lambda_{n+1}} \exp\left[-\Phi_{(k+\Lambda_0)\times I_j}(\eta^{(n)})\right] \otimes_{i\in\Lambda_{n+2}} W_{I_{j-1}}^{y_{i,j-1}, y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j}, y_{i,j+1}}(d\eta_i) \end{aligned} \tag{14}$$

For simplification, let us denote by

$$\Phi_{k,j}(\eta) =: \Phi_{(k+\Lambda_0)\times I_j}(\eta^{(n)}).$$

We first analyze the product on the space-lattice in the last expression, in order to exchange it later with the integration on Ω.

$$\begin{aligned} \prod_{k\in\Lambda_{n+1}} \exp(-\Phi_{k,j}(\eta)) &= \prod_{k\in\Lambda_{n+1}} \Big(1 + \exp(-\Phi_{k,j}(\eta)) - 1\Big) \\ &= 1 + \sum_L \prod_{k\in L} \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big) \\ &= 1 + \sum_{s\geq 1} \sum_{\gamma_1^j,\dots,\gamma_s^j} \prod_{m=1}^s \prod_{k\in\gamma_m^j} \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big) \end{aligned} \tag{15}$$

where the summation $\sum_L$ takes into account all non empty subsets of Λ_{n+1}, and the summation $\sum_{\gamma_1^j,\dots,\gamma_s^j}$ takes into account all maximal "Λ_0-connected" components of (L, I_j), id est $(L, I_j) \equiv \{(k, I_j), k \in L\} = \gamma_1^j \cup \cdots \cup \gamma_s^j$ and this decomposition is the finest one such that $\gamma_1^j, \dots, \gamma_s^j$ are disjoint sets satisfying for $m \neq m'$,$(\gamma_m^j + \Lambda_0) \cap (\gamma_{m'}^j + \Lambda_0) = \emptyset$ - in an obvious way, $\gamma^j + \Lambda_0 \equiv \{(k, I_j), k = k' + i$ where $(k', I_j) \in \gamma^j$ and $i \in \Lambda_0\}$ -. So

$$Z_n(y_n) = \prod_{j=-N}^{j=N-1} \int_\Omega \Big(1 + \sum_{s\geq 1} \sum_{\gamma_1^j,\dots,\gamma_s^j} \prod_{m=1}^{s} \prod_{k\in\gamma_m^j} \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big)\Big) \otimes_{i\in\Lambda_{n+2}} W_{I_{j-1}}^{y_{i,j-1},y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j},y_{i,j+1}}(d\eta_i). \tag{16}$$

To avoid an excessively heavy notation, we do not write the condition that the contours γ_k appearing in the sum above are formed by temporal edges of the type (i, I_j), with $i \in \Lambda_{n+1}$. Before inserting the expression (16) into (12), we analyse the time-product :

$$\begin{aligned} \prod_{j=-N,\dots,N-1} q_a(y_{i,j+1}; y_{i,j}) &= \prod_{j=-N,\dots,N-1} \Big(1 + q_a(y_{i,j+1}; y_{i,j}) - 1\Big) \\ &= 1 + \sum_\tau \prod_{I_j\in\tau} \Big(q_a(y_{i,j+1}; y_{i,j}) - 1\Big) \\ &= 1 + \sum_{p\geq 1} \sum_{\tau_1^i\dots,\tau_p^i} \prod_{u=1}^{p} \prod_{I_j\in\tau_u^i} \Big(q_a(y_{i,j+1}; y_{i,j}) - 1\Big) \end{aligned} \tag{17}$$

where the summation $\sum_\tau$ is over all non ordered collections of intervals of the type I_j included in $I(n)$ and the summation $\sum_{\tau_1^i\dots,\tau_u^i}$ is over all pairwise non intersecting collections of consecutive (connected) time intervals $\tau_u^i = (I_j, I_{j+1}, \dots, I_{j+r})$. The τ_u^i's, called temporal series, are then the connected components of τ and can also be represented by the following collection of temporal edges:

$$\tau_u^i = \{(i, I_j), \dots, (i, I_{j+r})\}.$$

Then, inserting expressions (16) and (17) into (12), we obtain

$$\begin{aligned} Z_n = &\int_{\mathbb{R}^{|\Lambda_{n+2}|(2N+1)}} \\ &\prod_{j=-N}^{j=N-1} \int_\Omega \Big(1 + \sum_{s\geq 1} \sum_{\gamma_1^j,\dots,\gamma_s^j} \prod_{m=1}^{s} \prod_{k\in\gamma_m^j} \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big)\Big) \otimes_{i\in\Lambda_{n+2}} W_{I_{j-1}}^{y_{i,j-1},y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j},y_{i,j+1}}(d\eta_i) \\ &\prod_{i\in\Lambda_{n+2}} \Big(1 + \sum_{p\geq 1} \sum_{\tau_1^i\dots,\tau_p^i} \prod_{u=1}^{p} \prod_{I_j\in\tau_u^i} \Big(q_a(y_{i,j+1}; y_{i,j}) - 1\Big)\Big) \otimes_{\substack{i\in\Lambda_{n+2} \\ j=-N,..,N}} \mu(dy_{i,j}). \end{aligned}$$

So,

$$Z_n = 1 + \sum_{v\geq 1} \sum_{\substack{\Gamma_1,\dots,\Gamma_v \\ \Gamma_u\in B_n}} \prod_{l=1}^{v} K_{\Gamma_l}, \tag{18}$$

where the last summation is taken on all non ordered collections of pairwise non intersecting aggregates Γ_l, an aggregate Γ being a non empty collection

$$\Gamma = \{\gamma_1^{j_1}, \dots, \gamma_s^{j_s}; \tau_1^{i_1}, \dots, \tau_p^{i_p}\}$$

of Λ_0-connected contours and temporal series satisfying : for $m \neq m'$, $(\gamma^j_m + \Lambda_0) \cap (\gamma^j_{m'} + \Lambda_0) = \varnothing$ and for $u \neq u'$, $\tau^i_u \cap \tau^i_{u'} = \varnothing$. Moreover, B_n is the set of aggregates corresponding to the volume V_n, i.e. $\Gamma = \{\gamma^{j_1}_1, \ldots, \gamma^{j_s}_s; \tau^{i_1}_1, \ldots, \tau^{i_p}_p\} \in B_n$ if the temporal edges in the $\gamma^{j_h}_h$ are of the form (i, I_j) with $i \in \Lambda_{n+1}$, $I_j \subset I(n)$, and the ones in the $\tau^{i_k}_k$ are of the form (i, I_j) with $i \in \Lambda_{n+2}$, $I_j \subset I(n)$.

The decomposition (18) is called a cluster representation of the statistical sum Z_n and the coefficient K_Γ is given by the following expression :

$$K_\Gamma = \int \prod_{m=1}^{s} \int_\Omega \prod_{k \in \gamma^{j_m}_m} \Big(\exp(-\Phi_{k,j_m}(\eta)) - 1 \Big) \otimes_{i \in \gamma^{j_m}_m + \Lambda_o} W^{y_{i,j_m-1}, y_{i,j_m}}_{I_{j_m-1}}(d\eta_i) W^{y_{i,j_m}, y_{i,j_m+1}}_{I_{j_m}}(d\eta_i)$$
$$\prod_{u=1}^{p} \prod_{I_j \in \tau^{i_u}_u} \Big(q_a(y_{i_u,j+1}; y_{i_u,j}) - 1 \Big) \otimes_{(i,j) \in [\bar\Gamma]} \mu(dy_{i,j}) \tag{19}$$

where $\bar\Gamma$ is the set of all temporal edges which compose Γ.

3.2 The cluster estimates

The following proposition is the key point of the convergence proof of the measures Q_n .

Proposition 5 *Under a suitable choice of the time scale a there exists some constant $\lambda(\varepsilon)$ which tends to 0 as ε goes to 0 such that, if the norm $\|\mathbf{b}\|_\infty$ of the interaction is smaller than ε, the weight K_Γ of the aggregate $\Gamma = \{\gamma^{j_1}_1, \ldots, \gamma^{j_s}_s; \tau^{i_1}_1, \ldots, \tau^{i_p}_p\}$ satisfies the estimate :*

$$\big| K_\Gamma \big| < \lambda(\varepsilon)^{|\Gamma|} \tag{20}$$

where $|\Gamma| = |\bar\Gamma|$ is the number of temporal edges which compose Γ.

Proof of Proposition 5 : To estimate the coefficient K_Γ defined by (19), we need to commute several times integration and products. To this aim, the following abstract integration lemma, which generalizes Hölder inequalities, will be very useful. It is proved in [20] Lemma 5.2 :

Lemma 6 *Let $(\mu_x)_{x \in \mathcal{X}}$ be a family of Probability measures, each one defined on a space $\mathbf{E}_x$, where the elements x belong to some finite set $\mathcal{X}$. Let us also define a finite family $(f_i)_i$ of functions on $\mathbf{E}_\mathcal{X} = \times_{x \in \mathcal{X}} \mathbf{E}_x$ such that each f_i is $\mathcal{X}_i$-local for a certain $\mathcal{X}_i \subset \mathcal{X}$, in the sense that*

$$f_i(e) = f_i(e_{|\mathcal{X}_i}), \ \textit{for } e = (e_x)_{x \in \mathcal{X}} \in \mathbf{E}_\mathcal{X}.$$

Let $\rho_i > 1$ be numbers satisfying the following conditions :

$$\forall x \in \mathcal{X}, \sum_{\mathcal{X}_i \ni x} \frac{1}{\rho_i} \leq 1. \tag{21}$$

Then

$$\Big| \int_{\mathbf{E}_\mathcal{X}} \prod_i f_i \otimes_{x \in \mathcal{X}} d\mu_x \Big| \leq \prod_i \Big(\int_{\mathbf{E}_{\mathcal{X}_i}} |f_i|^{\rho_i} \otimes_{x \in \mathcal{X}_i} d\mu_x \Big)^{1/\rho_i} \tag{22}$$

Lemma 6 allows us to exchange the second product with the integral over Ω in the expression (19). Let us define

$$K(\gamma_m^j) = \int_\Omega \prod_{k\in\gamma_m^j} \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big) \otimes_{i\in\gamma_m^j+\Lambda_o} W_{I_{j-1}}^{y_{i,j-1},y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j},y_{i,j+1}}(d\eta_i).$$

Then, by Lemma 6 applied to $\mathcal{X} = \gamma_m^j+\Lambda_o$, $\mathcal{X}_i = i+\Lambda_o$, $\mathbf{E}_x = \mathcal{C}(\mathbb{R},\mathbb{R})$, $\mu_i = W_{I_{j-1}}^{y_{i,j-1}}\otimes W_{I_j}^{y_{i,j},y_{i,j+1}}$, $f_i = \exp(-\Phi_{i,j}) - 1$, we get

$$\big|K(\gamma_m^j)\big| \le \prod_{k\in\gamma_m^j} \Bbbk_{k,j}(y_n) \tag{23}$$

where

$$\Bbbk_{k,j}(y_n) \equiv \left(\int_\Omega \Big(\exp(-\Phi_{k,j}(\eta)) - 1\Big)^{\rho_1} \otimes_{i\in k+\Lambda_o} W_{I_{j-1}}^{y_{i,j-1},y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j},y_{i,j+1}}(d\eta_i)\right)^{1/\rho_1}$$

and ρ_1 is an even natural number greater than $|\Lambda_0|$, in such a way that the condition (21) holds: for every fixed i

$$|\{k, k+\Lambda_0 \ni i\}|\frac{1}{\rho_1} = \frac{|\Lambda_0|}{\rho_1} \le 1.$$

So, returning to (19), we obtain

$$\begin{aligned} |K_\Gamma| &= \Big|\int \prod_{m=1}^{s} K(\gamma_m^{j_m}) \prod_{u=1}^{p} \prod_{I_j\in\tau_u^{iu}} (q_a(y_{i_u,j+1};y_{i_u,j}) - 1) \otimes_{(i,j)\in[\bar\Gamma]} \mu(dy_{i,j})\Big| \\ &\le \int \prod_{m=1}^{s} \prod_{k\in\gamma_m^{j_m}} \Bbbk_{k,j_m}(y_n) \prod_{u=1}^{p} \prod_{I_j\in\tau_u^{iu}} \big|q_a(y_{i_u,j+1};y_{i_u,j}) - 1\big| \otimes_{(i,j)\in[\bar\Gamma]} \mu(dy_{i,j}) \end{aligned} \tag{24}$$

Applying once more Lemma 6, we obtain

$$\begin{aligned} |K_\Gamma| \le& \prod_{m=1}^{s} \prod_{k\in\gamma_m^{j_m}} \left(\int \Bbbk_{k,j_m}(y_n)^{\rho_1} \otimes_{(i,j)\in k+\Lambda_0\times\{j_m-1,j_m,j_m+1\}} \mu(dy_{i,j})\right)^{1/\rho_1} \\ & \prod_{u=1}^{p} \prod_{I_j\in\tau_u^{iu}} \left(\int \big|q_a(y_{i_u,j+1};y_{i_u,j}) - 1\big|^{\rho_2} \mu(dy_{i_u,j})\mu(dy_{i_u,j+1})\right)^{1/\rho_2} \end{aligned} \tag{25}$$

where ρ_1 and ρ_2 have to satisfy the adapted condition (21) : for every fixed (k, ja)

$$|\{(k',j'),(k'+\Lambda_0, I_{j'}) \ni (k,ja)\}|\frac{1}{\rho_1} + \frac{2}{\rho_2} \le 1,$$

which is equivalent to

$$\frac{2|\Lambda_0|}{\rho_1} + \frac{2}{\rho_2} \le 1.$$

The choice $(\rho_1,\rho_2) = (4|\Lambda_0|, 4)$ is a possible one and we will take it. Then

$$|K_\Gamma| \le M_1^{\sum_{m=1}^{s}|\gamma_m^{j_m}|} M_2^{\sum_{u=1}^{p}|\tau_u^{iu}|} \tag{26}$$

where $M_1^{4|\Lambda_0|}$ is an upper bound independent of k and j of

$$\int\int_\Omega \Big(\exp(-\Phi_{k,j}(\eta))-1\Big)^{4|\Lambda_0|} \otimes_{i\in k+\Lambda_0} W_{I_{j-1}}^{y_{i,j-1},y_{i,j}}(d\eta_i) W_{I_j}^{y_{i,j},y_{i,j+1}}(d\eta_i)\mu(dy_{i,j-1})\mu(dy_{i,j})\mu(dy_{i,j+1})$$
$$= \int_\Omega \Big(\exp(-\Phi_{k,j}(\eta))-1\Big)^{4|\Lambda_0|}\Big(\prod_{i\in k+\Lambda_0} q_{2a}(\eta_i((j+1)a);\eta_i((j-1)a))\Big)^{-1} \otimes_{i\in k+\Lambda_0} W_{\overline{I_{j-1}\cup I_j}}(d\eta_i)$$
$$= \int_\Omega \Big(\exp(-\Phi_{k,j}(\eta))-1\Big)^{4|\Lambda_0|}\Big(\prod_{i\in k+\Lambda_0} q_{2a}(\eta_i((j+1)a);\eta_i((j-1)a))\Big)^{-1} P(d\eta)$$

and M_2 is an upper –bound of

$$\Big(\int (q_a(y;x)-1)^4\mu(dy)\mu(dx)\Big)^{1/4}. \tag{27}$$

Lemma 7 *If $\|\mathbf{b}\|_\infty \le \varepsilon$ and for a time unit a sufficiently large, we have*

$$\Big(\int_\Omega \Big(\exp(-\Phi_{k,j}(\eta))-1\Big)^{4|\Lambda_0|}\Big(\prod_{i\in k+\Lambda_0} q_{2a}(\eta_i((j+1)a);\eta_i((j-1)a))\Big)^{-1} P(d\eta)\Big)^{1/(4|\Lambda_0|)} \le C\varepsilon^{1/2}$$

where C is a positive constant independent of k, j and n.

Proof of Lemma 7 : First note that under assumption (5) given in section 2 on the interaction of the one dimensional reference process W, its semi-group is ultracontractive and, in particular, $q_t(x;y)$ converges, when t tends to infinity, towards 1 uniformly in x and y -the precise rate will be computed later to estimate M_2, see (29) -. So, for t large enough, $q_t(x;y)$ is bounded from below uniformly in x and y by some strictly positive constant, that is :

$$\exists A>0, \exists a_0\in\mathbb{R}^+, \forall a>a_0, \forall x,y,\ q_a(x;y)^{-1}\le A.$$

We now have to estimate for a large enough the $4|\Lambda_0|$-moment of $(e^{-\Phi_{k,j}}-1)$ under the Probability P. To simplify, let us use the notation $\rho =: 4|\Lambda_0|$.

$$\int_\Omega \Big(e^{-\Phi_{k,j}(\eta)}-1\Big)^\rho\Big(\prod_{i\in k+\Lambda_0} q_{2a}(\eta_i((j+1)a);\eta_i((j-1)a))\Big)^{-1} P(d\eta)$$
$$\le A^{|\Lambda_0|}\int\Big(\int_0^1 \Phi_{k,j}(\eta)e^{-\tau\Phi_{k,j}(\eta)}d\tau\Big)^\rho P(d\eta)$$

and

$$\int\Big(\int_0^1 \Phi_{k,j}(\eta)e^{-\tau\Phi_{k,j}(\eta)}d\tau\Big)^\rho P(d\eta)$$
$$= \int_\Omega \Phi_{k,j}(\eta)^\rho \int_{[0,1]^\rho} e^{-(\tau_1+\cdots+\tau_\rho)\Phi_{k,j}(\eta)}d\tau_1\dots d\tau_\rho P(d\eta)$$
$$= \int_{[0,1]^\rho}\int_\Omega \Phi_{k,j}(\eta)^\rho e^{-(\tau_1+\cdots+\tau_\rho)\Phi_{k,j}(\eta)}P(d\eta)d\tau_1\dots d\tau_\rho$$
$$= \int_{[0,1]^\rho} \frac{d^\rho}{dz^\rho}\mathcal{S}(z)_{|z=\tau_1+\cdots+\tau_\rho}d\tau_1\dots d\tau_\rho \tag{28}$$

where

$$\mathcal{S}(z) = \int_\Omega e^{-z\Phi_{k,j}(\eta)} P(d\eta)$$

is the Laplace transform of the r.v. $\Phi_{k,j}$. Extending $\mathcal{S}$ to complex numbers, we have, for any r such that $\mathcal{S}$ is well defined on $B(z,r)$,

$$\left|\frac{d^\rho}{dz^\rho}\mathcal{S}(z)\right| \le \frac{\rho!}{r^\rho} \sup_{\{\zeta\in\mathbb{C}, |\zeta - z| = r\}} \left|\mathcal{S}(\zeta)\right|.$$

But, for $\zeta = x + iy,\ x, y \in \mathbb{R}$,

$$\left|\mathcal{S}(\zeta)\right| \le \int_\Omega \left|e^{-\zeta\Phi_{k,j}(\eta)}\right| P(d\eta) = \int_\Omega e^{-x\Phi_{k,j}(\eta)} P(d\eta).$$

To bound this exponential moment of $\Phi_{k,j}$ under P, we will use the fundamental property that, for each $\Lambda \subset \mathbb{Z}^d$ finite and $a_1 \in \mathbb{R}$ the process $\Big(\exp(-H_{\Lambda\times]a_1,a_2[})\Big)_{a_2>a_1}$ is a P-martingale for the filtration $(\mathcal{F}_{\Lambda\times]a_1,a_2[})_{a_2>a_1}$. In particular, $\exp(-\Phi_{k,j})$ is the value at time $(j+1)a$ of a P-martingale which equals 1 at time ja. So we have

$$e^{-x\Phi_{k,j}(\eta)} = \exp\Big(x\int_{I_j} \mathbf{b}(\theta_{k,t}\eta^{(n)})d\tilde{B}_k(t) - \frac{x^2}{2}\int_{I_j} \mathbf{b}^2(\theta_{k,t}\eta^{(n)})dt\Big) \exp\Big(\frac{x^2 - x}{2}\int_{I_j} \mathbf{b}^2(\theta_{k,t}\eta^{(n)})dt\Big)$$

where the first term in the product of the R.H.S. is the value at time $(j+1)a$ of a P-martingale which equals 1 at time ja.
To bound (independently of η) the second term in the product of the above R.H.S. note that, since x is the real part of a complex number ζ satisfying $|\zeta - (\tau_1 + \cdots + \tau_\rho)| = r$, x is bounded above by $\rho + r$. So

$$\exp\Big(\frac{x^2 - x}{2}\int_{I_j} \mathbf{b}^2(\theta_{k,t}\eta^{(n)})dt\Big) \le \exp\Big(\frac{(\rho+r)^2}{2} a\|\mathbf{b}\|_\infty^2\Big).$$

This implies that,

$$\begin{aligned}\sup_{\{\zeta\in\mathbb{C}, |\zeta - z| = r\}} \left|\mathcal{S}(\zeta)\right| &\le \exp\Big(\frac{(\rho+r)^2}{2} a\|\mathbf{b}\|_\infty^2\Big)\\ &\quad \int_\Omega \exp\Big(x\int_{I_j} \mathbf{b}(\theta_{k,t}\eta^{(n)})d\tilde{B}_k(t) - \frac{x^2}{2}\int_{I_j} \mathbf{b}^2(\theta_{k,t}\eta^{(n)})dt\Big) P(d\eta)\\ &\le \exp\Big(\frac{(\rho+r)^2}{2} a\|\mathbf{b}\|_\infty^2\Big).\end{aligned}$$

Returning to (28), we obtain for $r \ge 0$,

$$\begin{aligned}\int_\Omega \Big(e^{-\Phi_{k,j}(\eta)} - 1\Big)^\rho P(d\eta) &\le \frac{\rho!}{r^\rho} exp\Big(\frac{(\rho+r)^2}{2} a\|\mathbf{b}\|_\infty^2\Big)\\ &\le \frac{\rho!}{r^\rho} \exp\Big(2a\|\mathbf{b}\|_\infty^2\, r^2\Big).\end{aligned}$$

Since this last upper bound holds for any $r \ge \rho$, we choose the r which minimizes the R.H.S., and obtain

$$\begin{aligned}\int_\Omega \Big(e^{-\Phi_{k,j}(\eta)} - 1\Big)^\rho P(d\eta) &\le \rho! \exp(\rho/2) \Bigg(\frac{4a\|\mathbf{b}\|_\infty^2}{\rho}\Bigg)^{\rho/2}\\ &\le C^\rho a^{\rho/2} \|\mathbf{b}\|_\infty^\rho.\end{aligned}$$

Taking now the time scale $a(\varepsilon) = 1/\varepsilon$ and being $\rho \geq 1$, we obtain the desired bound of Lemma 7. □

Let us now give a bound called $M_2(\varepsilon)$ for the expression (27), when the time scale a tends to infinity with the rate $a(\varepsilon) = 1/\varepsilon$. We work under the assumptions (5) given in section 2 on the interaction of the one dimensional reference process and will now use the ultracontractivity of $x(t)$. So, to compute the rate of convergence to 0 of

$$\int (q_{1/\varepsilon}(y;x) - 1)^4 \mu(dy)\mu(dx)$$

as a function of ε, we will directly bound $|q_{1/\varepsilon}(y;x) - 1|$ uniformly in x and y, using similar arguments as in the appendix of [4] :

$$\begin{aligned}\sup_{y,x\in\mathbb{R}} |q_t(y;x) - 1| &= \sup_{y,x\in\mathbb{R}} |q_1 * (q_{t-2} - 1) * q_1(y;x)| \\ &\leq \sup_{x\in\mathbb{R}} \int \sup_{y\in\mathbb{R}} q_1 * |q_{t-2} - 1|(y;w) q_1(w;x)\mu(dw).\end{aligned}$$

where $q_t * q_s$ is defined by

$$q_t * q_s(y;x) = \int q_t(y;w) q_s(w;x)\mu(dw).$$

By Theorem 1.4 in [13] under assumptions (5), the semigroup associated to $x(t)$ is *ultracontractive*, i.e. it maps $L^2(\mu)$ into $L^\infty(\mu)$; so there exists $C_1 > 0$ such that

$$\sup_{y\in\mathbb{R}} q_1 * |q_{t-2} - 1|(y;w) \leq C_1 \|(q_{t-2} - 1)(\cdot;w)\|_{L^2(\mu)}.$$

So

$$\begin{aligned}\sup_{y,x\in\mathbb{R}} |q_t(y;x) - 1| &\leq C_1 \sup_{x\in\mathbb{R}} \int q_1(w;x) \|(q_{t-2} - 1)(\cdot;w)\|_{L^2(\mu)} \mu(dw) \\ &\leq C_1^2 \|w \to \|(q_{t-2} - 1)(\cdot,w)\|_{L^2(\mu)}\|_{L^2(\mu)}.\end{aligned}$$

Now, is it known that ultracontractivity implies L^2-contractivity. Thus, denoting by α the spectral gap,

$$\|(q_{t-2} - 1)(\cdot;w)\|_{L^2(\mu)} = \left\| q_{t-2}(\cdot;w) - \int q_{t-2}(z;w)\mu(dw) \right\|_{L^2(\mu)} \leq e^{-(t-3)\alpha} \|q_1(\cdot;w) - 1\|_{L^2(\mu)}$$

which implies

$$\sup_{x,y\in\mathbb{R}} |q_t(y;x) - 1| \leq C_1^2 e^{-(t-3)\alpha} \left(\int\int (q_1(x,y) - 1)^2 \mu(dx)\mu(dy) \right)^{1/2} \tag{29}$$

which converges exponentially to zero as t tends to infinity. Then, there exists a constant C_2 such that, for ε small enough,

$$\left(\int (q_{1/\varepsilon}(y;x) - 1)^4 \mu(dy)\mu(dx) \right)^{1/4} \leq C_2\, e^{-\alpha/\varepsilon} \leq C_2\, \varepsilon^{1/2}.$$

Introducing this last estimate together with Lemma 7 into inequation (26) we obtain the following cluster estimate :

$$|K_\Gamma| \leq C_3\, \varepsilon^{1/2|\Gamma|}. \tag{30}$$

This concludes the proof of Proposition 5 with $\lambda(\varepsilon) = \varepsilon^{1/2}$.

□

3.3 The cluster expansion of the measures Q_n

As usually when techniques of cluster expansions are used (cf. [18]) the representation (18) of Z_n and the estimates (20) allow to obtain in a canonical way an expansion for the measures Q_n. In our particular situation, the expansion of Q_n is very similar of that computed in Section 4 of [19] for polynomial interaction. Nevertheless, by care of completeness, we sketch here the important steps of this method.

We shall now get a representation for the integral $\int F_B dQ_n$, where F_B is a local bounded measurable function on Ω localized on B, and for n large enough, B is included in the set B_n of all temporal edges of $V_n = \Lambda_n \times I(n)$.

First we formulate some important consequence of the cluster representation (18).

Let τ be a finite set of temporal edges and let us introduce the partition function

$$Z_\tau = 1 + \sum_{\Gamma_1,\ldots,\Gamma_v} \prod_{l=1}^{v} K_{\Gamma_l},$$

where the summation is taken over all non ordered non empty collections $\{\Gamma_1, \ldots, \Gamma_v\}$ of pairwise non intersecting aggregates Γ_l such that $\bar{\Gamma}_l \subset \tau$, and K_{Γ_l} is defined by (19).

For any set of temporal edges $\tau' \supset \tau$ we define

$$f_\tau^{\tau'} = \frac{Z_{\tau' \setminus \bar{\tau}}}{Z_{\tau'}} \tag{31}$$

where $\bar{\tau}$ is the set of edges which have common points with edges from τ.

The following lemma, which can be found in [18], Chapter 3, holds :

Lemma 8 *For ε small enough,*

i) there exists a constant $C_4 > 0$ independent on τ and τ' such that

$$\left| f_\tau^{\tau'} \right| < C_4 \, 2^{|\tau|} \tag{32}$$

ii) the following expansion holds :

$$f_\tau^{\tau'} = 1 + \sum_{\Xi = \{\Gamma_1,\ldots,\Gamma_v\}, \bar{\Gamma}_i \subset \tau'}^{\tau} D_\tau(\Xi) \prod_{\Gamma_i \in \Xi} K_{\Gamma_i} \tag{33}$$

where the summation is over collections Ξ of aggregates Γ_i such that Ξ is connected, $\tau \cap \cup_i \bar{\Gamma}_i \neq \emptyset$ and $\cup_i \bar{\Gamma}_i \subset \tau'$. The coefficients $D_\tau(\Xi)$ do not depend on τ' and the serie is absolutely convergent.

iii) there exists a limit for the expansion (33) when τ' tends to the set of all temporal edges in $\mathbb{Z}_a^{d+1}$:

$$f_\tau = \lim_{\tau' \uparrow \bar{\mathbb{Z}}_a^{d+1}} f_\tau^{\tau'} = 1 + \sum_{\Xi = \{\Gamma_1,\ldots,\Gamma_v\}}^{\tau} D_\tau(\Xi) \prod_{\Gamma_i \in \Xi} K_{\Gamma_i} \tag{34}$$

iv) there exists a constant $C_5 > 0$ such that the following estimate holds :

$$\left| f_\tau^{\tau'} - f_\tau \right| < C_5 \, \frac{2^{|\bar{\tau}|}}{2^{d(\tau, \tau'^c)}} \tag{35}$$

where $d\,(\tau, \tau'^c)$ is the length of the smallest path which goes from τ to the complement of τ' in $\bar{\mathbb{Z}}_a^{d+1}$.

v) there exists a constant $C_6 > 0$ such that the following estimate holds : for τ_1, $\tau_2 \subset \tau'$

$$\begin{aligned} &\left| f^{\tau'}_{\tau_1 \cup \tau_2} - f^{\tau'}_{\tau_1} \cdot f^{\tau'}_{\tau_2} \right| < C_6 \; 3^{|\tau_1|+|\tau_2|} \lambda(\varepsilon)^{d(\tau_1,\tau_2)} \\ \textit{and} \quad & \\ &\left| f_{\tau_1 \cup \tau_2} - f_{\tau_1} \cdot f_{\tau_2} \right| < C_6 \; 3^{|\tau_1|+|\tau_2|} \lambda(\varepsilon)^{d(\tau_1,\tau_2)}. \end{aligned} \tag{36}$$

We now return to the expansion of the integral of the functional F_B. We have

$$\begin{aligned} \int F_B dQ_n &= \int_\Omega F_B(\eta) \; \exp\left[-H_{V_n}(\eta^{(n)}) \right] \; P(d\eta) \\ &=: \; Z_n(F_B) \end{aligned}$$

which has the following representation :

$$Z_n(F_B) = \sum_{\substack{\Theta = \{\Gamma_l\} \\ \bar{\Gamma}_l \subset B_n}} K_\Theta(F_B) \left(1 + \sum_{\substack{\Xi = \{\Gamma_i\} \\ \bar{\Gamma}_i \subset B_n \setminus (B \cup \bar{\Theta})}} \prod_{\Gamma_i \in \Xi} K_{\Gamma_i} \right) \tag{37}$$

and, modifying in the right way equation (19),

$$\begin{aligned} &K_\Theta(F_B) = \\ &\int \int_\Omega F_B(\eta) \prod_{\Gamma_l} \left(\prod_{m=1}^{s} \prod_{k \in \gamma_m^{j_m}} \left(\exp(-\Phi_{k,j_m}(\eta)) - 1 \right) \prod_{u=1}^{p} \prod_{I_j \in \tau_u^{i_u}} \left(q_a(y_{i_u,j+1}; y_{i_u,j}) - 1 \right) \right) \\ &\otimes_{i \in \gamma_m^{j_m} + \Lambda_o} W_{I_{j_m-1}}^{y_{i,j_m-1}, y_{i,j_m}}(d\eta_i) W_{I_{j_m}}^{y_{i,j_m}, y_{i,j_m+1}}(d\eta_i) \otimes_{(i,j) \in [\bar{\Gamma}_l]} \mu(dy_{i,j}). \end{aligned}$$

From (37) and (33) we find

$$\begin{aligned} \int F_B \; dQ_n &= \sum_\Theta K_\Theta(F_B) f^{B_n}_{B \cup \bar{\Theta}} \\ &= \sum_{\Theta, \Xi} K_\Theta(F_B) D_{B \cup \bar{\Theta}}(\Xi) \prod_{\Gamma \in \Xi} K_\Gamma. \end{aligned}$$

Using estimates (32) and (34), we can conclude that for ε small enough (which implies $\lambda(\varepsilon)$ small enough) the above serie converges absolutely and uniformly in n, so that

$$\begin{aligned} \lim_{V_n \uparrow \mathbb{Z}^d \times \mathbb{R}} \int F_B \; dQ_n &= \sum_\Theta K_\Theta(F_B) \; f_{B \cup \bar{\Theta}} \\ &=: \int F_B \; dQ. \end{aligned}$$

The functional $F_B \longmapsto \int F_B dQ$ is linear bounded and positive on the algebra of bounded local functions. Then there exists a unique probability measure Q such that

$$Q = \lim_{V_n \uparrow \mathbb{Z}^d \times \mathbb{R}} Q_n \, .$$

By computing the cluster expansion for $\int (F_B - F_B \circ \theta_{i,t}) \; dQ_n$ we can also conclude that Q is space-time shift invariant.

Furthermore, the fact that Q satisfies the property of short range correlations is a consequence of a cluster representation for

$$\int F_{B_1} F_{B_2} dQ_n - \int F_{B_1} dQ_n \int F_{B_2} dQ_n$$

and (36).

By Lemma 4 and Proposition 2, in order to complete the proof of Theorem 3, we only have to show that the measure Q we have just constructed has a finite specific entropy with respect to P. We first note that the map $Q \to \mathcal{H}(Q)$ is well defined by (11) for all $Q \in \mathcal{P}(\Omega)$, and it is lower semicontinuous. In particular,

$$\mathcal{H}(Q) \leq \liminf_n \mathcal{H}(Q_n) \leq \sup_n \mathcal{H}(Q_n). \tag{38}$$

Under Q_n and for $t \in I(n)$, the canonical process is a weak solution of the s.d.e.

$$\begin{array}{rcll} dX_i(t) &=& \left[-\frac{1}{2}\varphi'(X_i(t)) + \mathbf{b}(\theta_{i,t} X^{(n)})\right] dt + dB_i(t) & \text{for } i \in \Lambda_{n+1} \\ dX_i(t) &=& -\frac{1}{2}\varphi'(X_i(t)) dt + dB_i(t) & \text{for } i \notin \Lambda_{n+1}. \end{array}$$

Consider the σ-field

$$\hat{\mathcal{B}}^- = \sigma\{\omega_i(t) : (t \leq 0,\ i = 0) \text{ or } (t \leq 1,\ i \neq 0)\}.$$

We may assume that n is large enough so that $[0,1] \subset I(n)$ and $(\{0\}^+)^+ \subset \Lambda_n$. By the same argument of Lemma 2 in [4],

$$\left.\frac{dQ_n(\cdot/\hat{\mathcal{B}}^-)}{dP(\cdot/\hat{\mathcal{B}}^-)}\right|_{\mathcal{B}_1} = \frac{1}{Z(n)} \exp\left[\sum_{i \in \{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) d\tilde{B}_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)}) dt\right]\right],$$

where $Z(n)$ is a normalization factor. - Let us remark that $P(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1}(\omega) = W_{[0,1]}^{\omega_0(0)}(\cdot)$, where $W_{[0,1]}^x$ is the law on $[0,1]$ of the one dimensional reference process W defined in (2) conditioned to be x at time 0.- Define

$$\hat{\mathcal{H}}(Q_n) = \int h(Q_n(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1}; P(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1})\, dQ_n.$$

It is the local entropy defined by Föllmer et al. in [10] definition 2.1. Since $\mathcal{B}^- \subset \hat{\mathcal{B}}^-$, by Jensen's inequality, $\mathcal{H}(Q_n) \leq \hat{\mathcal{H}}(Q_n)$. Thus, we are left to show that $\hat{\mathcal{H}}(Q_n)$ is bounded in n. But

$$\hat{\mathcal{H}}(Q_n) = \int \left[\sum_{i \in \{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) d\tilde{B}_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)}) dt\right]\right] dQ_n - \int \log Z(n) dQ_n. \tag{39}$$

We analyze separately the two summands in (39).
Using the fact that

$$B_i(t) = \tilde{B}_i(t) - \int_0^t \mathbf{b}(\theta_{i,s}\omega^{(n)}) ds, \quad i \in \{0\}^+,$$

are independent Brownian motions under Q_n, we get

$$\int \left[\sum_{i\in\{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) d\tilde{B}_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] \right] dQ_n$$
$$= \int \left[\sum_{i\in\{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) dB_i(t) + \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] \right] dQ_n$$
$$= \int \left[\sum_{i\in\{0\}^+} \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] dQ_n$$
$$\leq |\Lambda_0| \|\mathbf{b}\|_\infty^2.$$

Moreover,

$$\begin{aligned}
-\log Z(n) &= -\log \int \exp \left[\sum_{i\in\{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) d\tilde{B}_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] \right] dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} \\
&\leq -\int \left[\sum_{i\in\{0\}^+} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) d\tilde{B}_i(t) - \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] \right] dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} \\
&= -\int \left[\sum_{i\in\{0\}^+\setminus\{0\}} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) dB_i(t) + \frac{1}{2}\int_0^1 \mathbf{b}^2(\theta_{i,t}\omega^{(n)})dt \right] \right] dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} \\
&\quad +\frac{1}{2}\int\int_0^1 \mathbf{b}^2(\theta_{0,t}\omega^{(n)})dt\, dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} \\
&\leq -\int \left[\sum_{i\in\{0\}^+\setminus\{0\}} \int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) dB_i(t) \right] dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} \\
&\quad +\frac{1}{2}\int\int_0^1 \mathbf{b}^2(\theta_{0,t}\omega^{(n)})dt\, dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1}
\end{aligned}$$

where we have used the fact that $\tilde{B}_0(\cdot)$ is a Brownian motion under $dP(\cdot/\hat{\mathcal{B}}^-)$. Thus, proceeding as above,

$$\begin{aligned}
-\int \log Z(n) dQ_n &\leq -\int \sum_{i\in\{0\}^+\setminus\{0\}} \left[\int_0^1 \mathbf{b}(\theta_{i,t}\omega^{(n)}) dB_i(t) \right] dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1} dQ_n + \frac{1}{2}\|\mathbf{b}\|_\infty^2 \quad (40) \\
&= \frac{1}{2}\|\mathbf{b}\|_\infty^2.
\end{aligned}$$

The last equality is due to the fact that the first term in the R.H.S. of (40) vanishes; indeed :

$$\begin{aligned}
P(d\omega/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1}\, Q_n(d\omega) &= W_{[0,1]}^{\omega_0(0)}(d\omega_0)\, Q_n(d\omega) \\
&= W_{[0,1]}^{\omega_0(0)}(d\omega_0) \exp\left[-H_{V_n}(\omega^{(n)}) \right]\, P(d\omega) \\
&= W_{[0,1]}^{x}(d\omega_0) \exp\left[-H_{V_n}(\omega^{(n)}) \right] W^x(d\omega_0)\mu(dx) \otimes_{i\neq 0} W(d\omega_i) \\
&= \exp\left[-H_{V_n}(\omega^{(n)}) \right] W^x(d\omega_0)\mu(dx) \otimes_{i\neq 0} W(d\omega_i) \\
&= Q_n(d\omega)\ ,
\end{aligned}$$

which implies that the stochastic integrals under B_i's have a $dP(\cdot/\hat{\mathcal{B}}^-)_{|\mathcal{B}_1}\, dQ_n$- mean equal to 0.

This completes the proof. □

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