Limit behaviour of random walks with a sticky point

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Abstract

Let $\tilde{S}(n)$ be a random walk which behaves as a symmetric random walk everywhere except for the point 0. Upon hitting 0 the random walk is arrested there for a random amount of time $\eta_i \geq 0$ (i.i.d.); and then continues its way as usual. We study the limit behaviour of this process scaled as in the Donsker theorem. In case of $\mathbb{E}\eta_i < \infty$, it is proved convergence towards a Wiener process. We also consider a sequence of processes whose arrest times are geometrically distributed and grow with n. We prove that possible limits for the last model are a Wiener process, a Wiener process stopped at 0 and a Wiener process with a sticky point.

1 Introduction

Let $\{S(n), n \in \mathbb{Z}\}$ be a random walk on $\mathbb{Z}$ and $S(0) = 0$ with centred and square integrable jumps with variance equals to σ^2. We linearly interpolate the sequence S for all $t \geq 0$. Set

$$X_n(t) = \frac{S(nt)}{\sigma\sqrt{n}},\ n \in \mathbb{N}.$$

A well-known Donsker theorem (e.g. [1]) states weak convergence of stochastic processes in $\mathrm{C}([0,T])$

$$X_n(t) \xrightarrow{w} W(t),\ n \to \infty,$$

where W is a Wiener process.

Upon changing transition probabilities at one point or a set of points (e.g. [2, 3, 4]) one could obtain limit processes connected to Brownian motion, for example, skew Brownian motion, Brownian motion with a sticky point, Brownian motion with bouncing.

Semi-Markov random walks with continuous-time and non-exponential arrests give rise to equations with fractional derivatives [5, 6, 7]. For example, a process with jumps in $\mathbb{R}$ and lagged at each point for a random amount of time with a "heavy tail" distribution constitutes a sub-diffusion model. As remarked in [8] the processes with a sticky point could be used for modelling behaviour on a financial market with governmental control. Sticky Brownian motion also arises while discussing storage processes that have different intensities in and out of zero, [9].

We consider a modified discrete random walk which is arrested for a random amount of time at each visit of zero. We show that if an expectation of the arrest time is finite then naturally the limiting process is a Brownian motion. We also consider a triangular array of random walks with geometrically distributed times of arrest whose expectations depend on n. This construction let Brownian motion with a sticky point to appear. For further discussion of this process check [8, 9, 10, 11, 12].

2 Problem statement and results

Let $\{S(n)\}$ be a random walk generated by independent identically distributed random variables $\{\xi_n\}_{n=1}^{\infty}$

$$S(n) = \sum_{i=1}^{n} \xi_i,\ n \in \mathbb{N} \text{ and } S(0) = 0.$$

Moreover $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}\xi_1^2 = \sigma^2 < \infty$.

Extend S for all positive $t > 0$ by linearity:

$$S(t) = S(n) + (t - n)(S(n+1) - S(n)),\ t \in [n, n+1].$$

Let also $\{\eta_n\}_{n=1}^{\infty}$ be a sequence of non-negative integer-valued i.i.d. that is independent of $\{\xi_i\}$.

We construct a modified random walk $\{\tilde{S}(n)\}$ as follows. Let the excursions of $\tilde{S}(\cdot)$ be equal to those of $S(\cdot)$. Insert η_i amount of time between i-th and $i+1$-st excursion of $\tilde{S}(\cdot)$. Check pictures 1, 2.

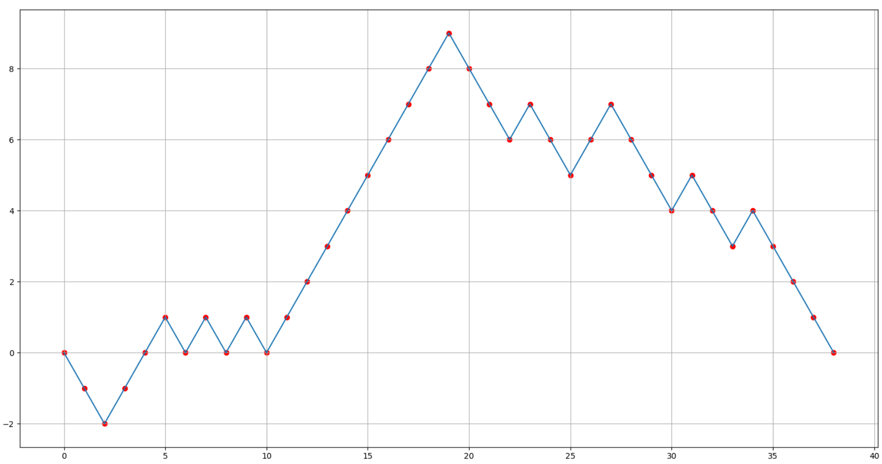

Figure 1: $S(t)$

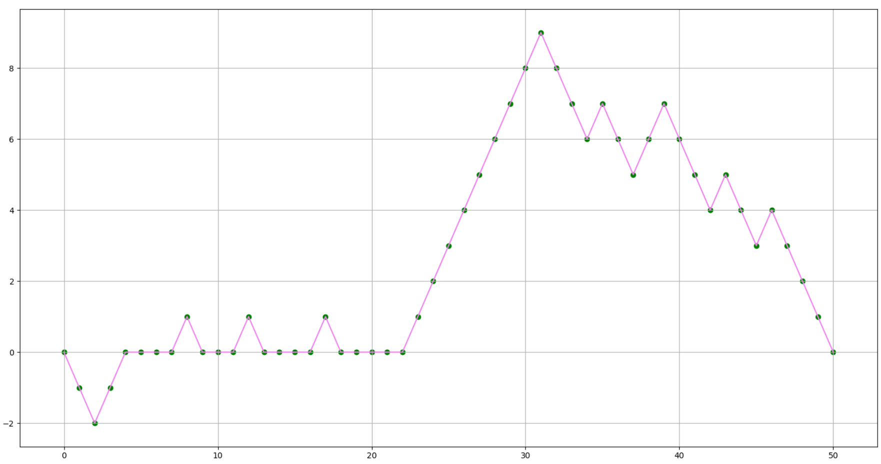

Figure 2: $\tilde{S}(t)$

The modification $\{\tilde{S}(n)\}$ could be defined more formally. Define firstly $\alpha(t)$:

$$\alpha(t) = t + \sum_{i=1}^{\tau_0(t)} \eta_i, \; t \geq 0.$$

where $\tau_0(t) = \#\{k : S(k) = 0, \; 1 \leq k \leq t\}$ is a number of visits to zero of the random walk $S(\cdot)$ before the time t.

Set a generalised inverse

$$\alpha^{(-1)}(t) = Inv[\alpha(\cdot)](t) = \inf\{x : \alpha(x) \geq t\}, \; t \geq 0.$$

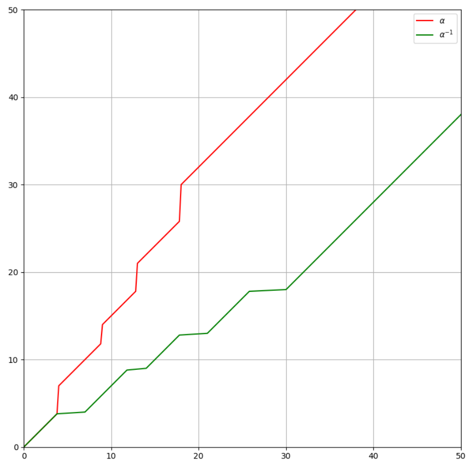

Figure 3: Plots of $\alpha(t)$ and $\alpha^{(-1)}(t)$

The process $\tilde{S}(t)$ is defined by

$$\tilde{S}(t) = S(\alpha^{(-1)}(t)).$$

Our goal is to study the limit behaviour of a sequence $\left\{\frac{\tilde{S}(nt)}{\sqrt{n}}\right\}$ as $n \to \infty$. Denote by $\mathrm{C}[0, \infty)$ a space of continuous functions endowed with a topology of uniform convergence on finite intervals.

Theorem 1. *Let $\{\tilde{S}(n)\}$ be a modified random walk, where $\mathbb{E}\eta_1 < \infty$. For a sequence of processes $\{\tilde{X}_n(\cdot) = \frac{\tilde{S}(n\cdot)}{\sigma\sqrt{n}},\ n \geq 1\}$ weak convergence in $\mathrm{C}[0, \infty)$ holds:*

$$\tilde{X}_n(\cdot) \xrightarrow{w} W(\cdot),\ n \to \infty,$$

where W is a Wiener process.

Remark 1. Consider a Markov chain

$$p_{ij} = \mathbb{P}(\xi = j - i) \text{ and } p_{00} = p, p_{0j} = (1 - p)\mathbb{P}(\xi = j),$$

where $\mathbb{E}\xi = 0, \mathbb{E}\xi^2 < \infty$. Theorem 1 may be applied to this case for $\{\eta_i\}$ being independent geometrically distributed random variables with $\mathbb{E}\eta_i = \frac{1}{p}$.

Let us consider more closely the random walk from the remark above. Denote it as $S^{(p)}(\cdot)$. The sequence of processes

$$X_n^{(p_n)}(t) = \frac{S^{(p_n)}(nt)}{\sigma\sqrt{n}}$$

with

$$p_n = \frac{\rho}{n^\gamma}$$

has different limits with respect to γ. Theorem 2 describes all possible modes.

Denote by $W_{\beta\text{-sticky}}(t)$ a Brownian motion with a sticky point defined by

$$W_{\beta\text{-sticky}}(t) = W(A_\beta^{(-1)}(t)),$$

where

$$A_\beta(t) = t + \beta L(t),\ A_\beta^{(-1)}(t) \text{ is a generalised inverse}$$

and

$$L(t) = \mathbb{P}\text{-}\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t 1_{\{W(s)\in[-\varepsilon,\varepsilon]\}}ds$$

a local time of a Brownian motion at zero. As opposed to a usual Brownian motion, this one stays at zero for a positive amount of time, yet there is no interval of positive length that it is there.

Theorem 2. *Convergence in distribution in* $\mathrm{C}[0,\infty)$ *holds:*

$$\begin{aligned}
&\textit{if } 0 \le \gamma < 0.5,\ \textit{then } X_n^{(p_n)}(t) \xrightarrow{w} W(t),\ n\to\infty,\\
&\textit{if } \gamma > 0.5,\ \textit{then } X_n^{(p_n)}(t) \xrightarrow{w} 0,\ n\to\infty,\\
&\textit{if } \gamma = 0.5,\ \textit{then } X_n^{(p_n)}(t) \xrightarrow{w} W_{\rho^{-1}\text{-}sticky}(t),\ n\to\infty.
\end{aligned}$$

3 Proofs

The following two lemmas may be found in [13] (proposition 3.2).

Lemma 1. *Let* $\{\xi_n(t)\}_{n\ge1}$*,* $t\in[0,T]$ *be a sequence of random processes such that*

(a) for each n *the process* $\xi_n(t)$ *is monotonous a.s.;*

(b) for every t

$$\xi_n(t) \xrightarrow{\mathbb{P}} \xi(t),\ n\to\infty;$$

(c) the limiting process $\xi(t)$ *is continuous a.s.*

Then uniform convergence in probability holds

$$\sup_{t\in[0,T]} |\xi_n(t)-\xi(t)| \xrightarrow{\mathbb{P}} 0,\ n\to\infty.$$

Lemma 2. *Let* $\{\xi_n(t)\}_{n\ge1}$*,* $t\in[0,T]$ *be a sequence of random processes such that (a), (b), (c) are satisfied and*

(d) for each n

$$\xi_n(0)=0,\ \xi_n(\infty)=\infty.$$

Then for any $T>0$ *uniform convergence in probability holds*

$$\sup_{t\in[0,T]} |\xi_n^{(-1)}(t)-\xi^{(-1)}(t)| \xrightarrow{\mathbb{P}} 0,\ n\to\infty.$$

3.1 Proof of Theorem 1

Set

$$h_n(t) = \frac{\alpha^{(-1)}(nt)}{n}.$$

From the definition of $\tilde{S}(n)$ one has

$$\tilde{X}_n(t) = \frac{\tilde{S}(nt)}{\sqrt{n}} = \frac{S(\alpha^{(-1)}(nt))}{\sigma\sqrt{n}} = \frac{S(n\frac{\alpha^{(-1)}(nt)}{n})}{\sigma\sqrt{n}} = X_n(h_n(t)).$$

Hence we prove that

$$X_n(h_n(\cdot)) \xrightarrow{w} W(\cdot),\ n \to \infty. \tag{1}$$

We are interested in the behaviour of $h_n(t)$ as $n \to \infty$. Note that the function $\frac{\alpha(nt)}{n}$ is a generalised inverse for $h_n(t)$. That is because for any $a \neq 0$ one has

$$\begin{aligned} Inv[ah(\cdot)](t) &= Inv[h(\cdot)](t/a), \\ Inv[h(a\cdot)](t) &= \frac{1}{a}Inv[h(\cdot)](t). \end{aligned} \tag{2}$$

Let us show that for any $t \geq 0:$

$$\frac{\alpha(nt)}{n} \xrightarrow{\mathbb{P}} t,\ n \to \infty. \tag{3}$$

This is obvious if $t = 0$. For $t > 0$

$$\frac{\alpha(nt)}{n} = t + \frac{1}{n}\sum_{i=1}^{\tau_0(nt)} \eta_i = t + \frac{\tau_0(nt)}{n}\frac{1}{\tau_0(nt)}\sum_{i=1}^{\tau_0(nt)} \eta_i. \tag{4}$$

For a fixed $t > 0$ one has $\mathbb{P}\{\tau_0(nt) \underset{n\to\infty}{\to} \infty\} = 1$, thus due to the law of large numbers

$$\frac{1}{\tau_0(nt)}\sum_{i=1}^{\tau_0(nt)} \eta_i \to \mathbb{E}\eta_1 < \infty,\ n \to \infty,\ \text{a.s.}$$

It is well known that $\frac{\tau_0(nt)}{\sqrt{n}}$ converges weakly towards an absolute value of a Gaussian random variable as $n \to \infty$. So

$$\frac{\tau_0(nt)}{n} \xrightarrow{\mathbb{P}} 0,\ n \to \infty.$$

And thus

$$\frac{\alpha(nt)}{n} \xrightarrow{\mathbb{P}} t,\ n \to \infty. \tag{5}$$

Since $\{\frac{\alpha(n\cdot)}{n}\}_{n\geq 1}$ are monotonous and converge towards the continuous limit, we invoke Lemmas 1 and 2 to see that

$$\sup_{t\in[0,T]} |h_n(t) - t| = \sup_{t\in[0,T]} \left| \frac{\alpha^{(-1)}(nt)}{n} - t \right| \xrightarrow{\mathbb{P}} 0, \ n \to \infty. \tag{6}$$

The following is well-known, e.g. theorem 4.4 in [1].

Lemma 3. *Let E be a Polish space, $\{X_n, n \geq 1\}, X, \{h_n, n \geq 1\}$ be random elements with values in E, and $h \in E$ be non-random. Assume that $X_n \xrightarrow{w} X$ and $h_n \xrightarrow{w} h$. Then the pairs of random variables converge weakly*

$$(X_n, h_n) \xrightarrow{w} (X, h), \ n \to \infty.$$

As $X_n(\cdot) \xrightarrow{w} W(\cdot)$ and $h_n(\cdot) \xrightarrow{\mathbb{P}} h(\cdot)$ for any finite interval and, furthermore, the function h is non-random, Lemma 3 yields $(X_n, h_n) \xrightarrow{w} (W, h)$. Due to the Skorokhod representation theorem [1] there exist a probability space and random elements $\bar{X}_n, \bar{h}_n$ there such that in $\mathrm{C}[0,\infty)$:

$$(\bar{X}_n, \bar{h}_n) \stackrel{w}{=} (X_n, h_n),$$

and for any $T > 0$ uniform convergence on $[0, T]$ holds

$$\bar{X}_n(t) \rightrightarrows \bar{W}(t) \ \text{ and } \ \bar{h}_n(t) \rightrightarrows t \ \text{ as } n \to \infty, \ \text{a.s.}$$

Thus $\bar{X}_n(\bar{h}_n(\cdot)) \to \bar{W}(\cdot), \ n \to \infty,$ a.s. So

$$X_n(h_n(\cdot)) \xrightarrow{w} \bar{W}(\cdot).$$

3.2 Proof of Theorem 2

As previously we introduce $\alpha_n(t), \ \alpha_n^{(-1)}(t)$,

$$h_n(t) = \frac{\alpha_n^{(-1)}(nt)}{n},$$

and

$$X_n^{(p_n)}(t) = \frac{S^{(p_n)}(nt)}{\sqrt{n}} = \frac{S(\alpha_n^{(-1)}(nt))}{\sigma\sqrt{n}} = \frac{S(n\frac{\alpha_n^{(-1)}(nt)}{n})}{\sigma\sqrt{n}} = X_n(h_n(t)).$$

Let us start with discussing the behaviour of

$$\frac{\alpha_n(nt)}{n} = t + \frac{1}{n} \sum_{i=1}^{\tau_0^{(n)}(nt)} \eta_i^{(n)}, \tag{7}$$

where $\eta_i^{(n)}$ are geometrically distributed with parameter p_n and $\tau_0^{(n)}(t)$ is the number of visits to zero of $S^{(p_n)}$ before the time t.

The last expression may be rewritten

$$t + \frac{n^\gamma}{\sqrt{n}} \frac{\tau_0^{(n)}(nt)}{\sqrt{n}} \frac{1}{\tau_0^{(n)}(nt)} \sum_{i=1}^{\tau_0^{(n)}(nt)} \frac{\eta_i^{(n)}}{n^\gamma}. \tag{8}$$

Theorem 3 ([14]). *Let $W(t)$ be a Brownian motion in $\mathbb{R}$, $L(t)$ be its local time. Then in* $\mathrm{C}[0,\infty)$

$$\Big(\frac{\tau_0(nt)}{\sqrt{n}},\ \frac{S(nt)}{\sigma\sqrt{n}}\Big) \xrightarrow{w} (L(t),\ W(t)),\ n \to \infty.$$

With this and the Skorokhod theorem we construct a probability space and random variables there such that in $\mathrm{C}[0,\infty)$:

$$\Big(\frac{\bar\tau_0^{(n)}(nt)}{\sqrt{n}},\ \frac{\bar S^{(n)}(nt)}{\sqrt{n}}\Big)_{t\ge 0} \stackrel{w}{=} \Big(\frac{\tau_0^{(n)}(nt)}{\sqrt{n}},\ \frac{S^{(n)}(nt)}{\sqrt{n}}\Big)_{t\ge 0}, \tag{9}$$

and for any $T > 0$ uniform convergence on $[0, T]$ holds

$$\frac{\bar\tau_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar L(t) \text{ and } \frac{\bar S^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar W(t) \text{ as } n \to \infty, \text{ a.s.} \tag{10}$$

To ease notation we omit the upper index. We define $\{\eta_i^{(n)}\}_i$ independently of $\bar\tau_0(\cdot)$ and $\bar L(\cdot)$ on the same probability space.

Theorem 4. *For every $T > 0$*

$$\sup_{t\in[0,T]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}\bar L(t)} \frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar L(t)}{\rho} \right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty, \tag{11}$$

where $\sum_{i=1}^{x}$ means $\sum_{i=1}^{[x]}$.

Proposition 1. *For any fixed $t \ge 0$ we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} \xrightarrow{\mathbb{P}} \frac{t}{\rho},\ n \to \infty.$$

Proof. Since

$$\mathbb{E}\frac{1}{\sqrt{n}} \sum_{i=1}^{\sqrt{n}t} \frac{\eta_i^{(n)}}{n^\gamma} = \frac{t}{\rho},$$

it suffices to check that the variance of the sequence converges to 0. The summands are independent, thus

$$\mathbb{V}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}t}\frac{\eta_i^{(n)}}{n^\gamma}\right)=\frac{1}{n}\sum_{i=1}^{\sqrt{n}t}\frac{\mathbb{V}\eta_i^{(n)}}{n^{2\gamma}}.$$

Recall that $\{\eta_i^{(n)}\}$ are geometrically distributed random variables. So

$$\frac{1}{n}\sum_{i=1}^{\sqrt{n}t}\frac{1-\frac{\rho}{n^\gamma}}{\frac{\rho^2}{n^{2\gamma}}n^{2\gamma}}=\frac{t}{\sqrt{n}}\frac{1-\frac{\rho}{n^\gamma}}{\rho^2}.$$

This proves that for $\gamma>0$ one has convergence towards 0 of the mentioned. ■

Proposition 2. *For every interval* $[0,T]$ *and for any* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}t}\frac{\eta_i^{(n)}}{n^\gamma}-\frac{t}{\rho}\right|>\varepsilon\right)=0.$$

Proof. The sum is monotonous in t and due to proposition 1 it has a continuous limit. Thus this proposition follows from lemma 1. ■

Proof of the theorem 4. Let $\delta>0$ be a fixed number. Find T' such that the set $\Omega_\delta=\{\bar{L}(T)<T'\}$ satisfies $\mathbb{P}(\Omega_\delta)>1-\delta$. Note that for any $t\in[0,T]$ it holds that $\bar{L}(t)\le\bar{L}(T)$. Hence on the set Ω_δ

$$\sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}\bar{L}(t)}\frac{\eta_i^{(n)}}{n^\gamma}-\frac{\bar{L}(t)}{\rho}\right|\le\sup_{y\in[0,T']}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}y}\frac{\eta_i^{(n)}}{n^\gamma}-\frac{y}{\rho}\right|.$$

Denote by

$$A_{n,\varepsilon}=\left\{\sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}\bar{L}(t)}\frac{\eta_i^{(n)}}{n^\gamma}-\frac{\bar{L}(t)}{\rho}\right|>\varepsilon\right\}$$

And write

$$\mathbb{P}(A_{n,\varepsilon})=\mathbb{P}(A_{n,\varepsilon}\cap\Omega_\delta)+\mathbb{P}(A_{n,\varepsilon}\cap\bar{\Omega}_\delta).$$

From proposition 2

$$\overline{\lim_{n\to\infty}}\,\mathbb{P}(A_{n,\varepsilon})\le 0+\delta.$$

As δ and ε were arbitrary, the last inequality proves the theorem. ■

Now suppose that Ω is a set where (10) holds with probability 1. Let ε be fixed, then for N large enough find the set $\Omega_\delta \subset \Omega$ such that the event

$$\sup_{t\in[0,T]}\left|\bar{L}(t) - \frac{\bar{\tau}_0(nt)}{\sqrt{n}}\right| < \varepsilon$$

holds for each $n > N$ and $\mathbb{P}(\Omega_\delta) > 1-\delta$.

Consider the difference

$$\sup_{t\in[0,T]}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{\sqrt{n}\bar{L}(t)}\frac{\eta_i^{(n)}}{n^\gamma} - \sum_{i=1}^{\bar{\tau}_0(nt)}\frac{\eta_i^{(n)}}{n^\gamma}\right|. \tag{12}$$

We show that (12) converges to 0 and so the limits of the summands should coincide. Since $\{\eta_i^{(n)}\}$ are independent of $(\bar{L}, \bar{\tau}_0)$, the last expression is equal in distribution to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}\sup_{t\in[0,T]}|\bar{L}(t)-\frac{\bar{\tau}_0(nt)}{\sqrt{n}}|}\frac{\eta_i^{(n)}}{n^\gamma}.$$

Now on the set Ω_δ for $n > N$ this is less or equal to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}\varepsilon}\frac{\eta_i^{(n)}}{n^\gamma}.$$

Proposition 2 entails its convergence to $\frac{\varepsilon}{\rho}$. Since the probability of the complement of Ω_δ is small and ε was arbitrary, one sees that (12) converges in probability to 0. Now due to Theorem 4

$$\sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\bar{\tau}_0(nt)}\frac{\eta_i^{(n)}}{n^\gamma} - \frac{\bar{L}(t)}{\rho}\right| \xrightarrow{\mathbb{P}} 0,\ n\to\infty. \tag{13}$$

3.2.1 Proof of the theorem in case $\gamma < 0.5$

Recall (8):

$$\frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_0(nt)}\frac{\eta_i^{(n)}}{n^\gamma}. \tag{14}$$

In case $\gamma < 0.5$ the right hand side of (14) converges to t in probability. Now lemmas 1 and 2 assure that for every $T > 0$:

$$\sup_{t\in[0,T]}|h_n(t) - t| = \sup_{t\in[0,T]}\left|\frac{\alpha_n^{(-1)}(nt)}{n} - t\right| \xrightarrow{\mathbb{P}} 0,\ n\to\infty. \tag{15}$$

The last limit is non random, thus we use Lemma 3 and the Skorokhod theorem to construct a probability space and random variables there such that in C$[0,\infty)$:

$$\Big(\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}}, \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}}, \frac{\bar{h}_n(nt)}{n}\Big)_{t\geq 0} \overset{w}{=} \Big(\frac{\tau_0^{(n)}(nt)}{\sqrt{n}}, \frac{S^{(n)}(nt)}{\sqrt{n}}, \frac{h_n(nt)}{n}\Big)_{t\geq 0},$$

and for every $T>0$ uniform convergence on $[0,T]$ holds

$$\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{L}(t), \ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{W}(t) \ \text{ and } \ \frac{\bar{h}_n(nt)}{n} \rightrightarrows t \ \text{ as } \ n\to\infty, \ \text{a.s.}$$

Recall that in Theorem 1 we had the similar situation. So analogously one obtains that the limit is a Brownian motion

$$X_n^{(p_n)}(\cdot) \xrightarrow{w} W(\cdot), \ n\to\infty.$$

3.2.2 Proof of the theorem in case $\gamma > 0.5$

In case $\gamma > 0.5$ the expression (14) converges to ∞ in probability for every $t>0$. Since for any $n\geq 1$ functions $\frac{\alpha_n(n\cdot)}{n}$ are monotonous, we have

$$\forall\delta>0 \ \forall M \ \exists N \ \forall t\in[\delta,\infty) \ \forall n>N \quad \mathbb{P}\Big(\frac{\alpha_n(nt)}{n} > M\Big) > 1-\delta.$$

This ensures that uniform convergence on $[0,\infty)$ in probability holds

$$h_n(t) = \frac{\alpha_n^{(-1)}(nt)}{n} \overset{\mathbb{P}}{\rightrightarrows} 0, \ n\to\infty.$$

Once again this limit is non random. By Lemma 3 and the Skorokhod theorem we construct a probability space and random variables there such that in C$[0,\infty)$:

$$\Big(\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}}, \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}}, \bar{h}_n(t)\Big)_{t\geq 0} \overset{w}{=} \Big(\frac{\tau_0^{(n)}(nt)}{\sqrt{n}}, \frac{S^{(n)}(nt)}{\sqrt{n}}, h_n(t)\Big)_{t\geq 0},$$

and uniform convergence on $[0,\infty)$ holds

$$\frac{\bar{\tau}_0^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{L}(t), \ \frac{\bar{S}^{(n)}(nt)}{\sqrt{n}} \rightrightarrows \bar{W}(t) \ \text{ and } \ \frac{\bar{h}_n(nt)}{n} \rightrightarrows 0 \ \text{ as } \ n\to\infty, \ \text{a.s.}$$

Thus

$$X_n(h_n(t)) \xrightarrow{w} 0, \ n\to\infty.$$

3.2.3 Proof of the theorem in case $\gamma = 0.5$

In this case $\frac{n^\gamma}{\sqrt{n}} = 1$ and so from (13) one sees that (14) has a non-trivial limit

$$h_n(t) = \frac{\alpha_n(nt)}{n} \xrightarrow{w} t + L(t)/\rho,\ n \to \infty.$$

Furthermore, we may consider the copies of random variables that we constructed after stating Theorem 3 and for which we proved (13). For them convergence towards the limit is uniform for any $T > 0$

$$\sup_{t\in[0,T]} \left| \frac{\bar{\alpha}_n(nt)}{n} - t - \frac{\bar{L}(t)}{\rho} \right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty. \tag{16}$$

For each n the functions $\frac{\bar{\alpha}_n(n\cdot)}{n}$ are monotone and their limit is continuous (because the local time is continuous). Thus from Lemma 2 we have

$$\sup_{t\in[0,T]} \left| \frac{\bar{\alpha}_n^{(-1)}(nx)}{n} - Inv[t + \bar{L}(t)/\rho](x) \right| \xrightarrow{\mathbb{P}} 0,\ n \to \infty. \tag{17}$$

And hence convergence in $\mathrm{C}[0,\infty)$ is proved

$$\bar{X}_n(\bar{h}_n(\cdot)) \xrightarrow{w} \bar{W}(Inv[t + L(t)/\rho](\cdot)),\ n \to \infty.$$

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