Penrose-stable Interactions in Classical Statistical Mechanics

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in memoriam Jean Ginibre

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Abstract

For a pair potential Φ in a general underlying space X satisfying some natural and sufficiently general conditions in the sense of Penrose [29] and Poghosyan and Ueltschi [30] together with a locally finite measure ρ on X we define by means of the socalled Ursell kernel a function r which is shown to be the correlation function of a unique process G, the limiting Gibbs process for (Φ, ρ) with empty boundary conditions. This process is exhibited as a Gibbs process in the sense of Dobrushin, Lanford and Ruelle for a class of pair potentials, which contains classical stable and hard-core potentials that are called Penrose potentials here. Particularly, a class of positive potentials is included. Finally, for some class of Penrose potentials we show that G is the unique Gibbs process for Φ. We use the classical method of Kirkwood-Salsburg equations. A decisive role is played by a generalization of Ruelle's estimate for correlation functions.

Introduction

The main impetus for this paper is the development of Ruelle's approach to the uniqueness of Gibbs processes in [34], which was done under assumptions of superstability and lower regularity, in the framework of Penrose-stable pair potentials.[1] Moreover, Ginibre's uniqueness theorem (Proposition 3.5 in [6]), already presented under assumptions of Penrose-stability, strengthened our motivation.

This is done in a general framework of point process theory. Both works had been published around 1970. Another motivation is the development of these topics following Minlos' approach to Gibbsian theory [22] from 1967. Minlos, like Ruelle, used

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[1] This explains the choice of our title.

already tools of cluster expansions and Kirkwood-Salsburg equations. The two quantum mechanical models presented below are inspired by Ginibre's work in [4, 6].

Let X denote an underlying Polish space together with some locally finite measure ρ on X, $\mathscr{M}^{\cdot\cdot}(X)$ the collection of all locally finite particle configurations and $\mathfrak{X}$ the subset of finite configurations. Φ is a stable, regular pair potential on X in the sense as specified later in Section 3. E_Φ denotes the associated energy, $B = \mathrm{e}^{-E_\Phi}$ the Boltzmann factor and $D_\xi B(\nu) = B(\nu + \xi)$ its algebraic derivative. B is an element of a function algebra $(\mathscr{A}, \star)$ having an inverse with respect to the $\star$-multiplication, which is denoted by $B_\star^{-1}$.

We define by means of Φ the *Ursell kernel*

$$G(\xi,\eta) = \left(B_\star^{-1} \star D_\xi B\right)(\eta), \qquad \xi,\eta \in \mathfrak{X},$$

and consider the function (cf. [33], Section 4.4.5)

$$r(\xi) = \int_{\mathfrak{X}} G(\xi,\eta)\,\Lambda_\rho(\mathrm{d}\eta), \qquad \xi \in \mathfrak{X}.$$

Here Λ_ρ is the measure ρ lifted to $\mathfrak{X}$ and defined in Part I, Section 1. If $\mathfrak{X}$ is replaced by the set $\mathfrak{X}(A)$ of configurations in some bounded Borel set A, then the right hand side is the well defined correlation function of the Gibbs process in A specified by (Φ,ρ).

Our first effort in Chapter 2 leads to Theorem 1 which shows that r is the well defined limiting correlation function. It satisfies a *Ruelle bound*, which will play a fundamental role in the sequel. This chapter is based on the work of Minlos and Poghosyan [24] and Poghosyan and Ueltschi [30] and contains tools from the theory of cluster expansions. Here we use the notion of weak $\mathscr{P}$-stable pair potentials which is used systematically in this paper.[2]

The next aim is Theorem 2, which contains the construction of a (point) process G in X for which r is the correlation function. This process G is the limiting Gibbs process in the sense of Minlos [22] with empty boundary conditions. Theorem 1 and Theorem 2 are proved under so called *standard conditions* (denoted below by $(\mathfrak{W}_\rho)$) on weak $\mathscr{P}$-stable potentials Φ. For its construction we do not use the work of Lenard [15] because it is not clear how to verify his positivity condition. We instead use a theorem of Zessin (cf. [39], Theorem 2.2.) and have to restrict the class of underlying spaces X to locally, compact, second countable Hausdorff topological spaces. Under the conditions made on (Φ,ρ), the process G is uniquely determined by r on account of Ruelle's bound. Recently Sabine Jansen gave in [8] a construction of Gibbs processes in the case of positive pair potentials.

For a subclass $(\mathfrak{W}_\rho^\sharp)$ of the standard conditions we then show in Theorem 3 that G is a Gibbs process in the DLR[3]-sense. Minlos' work on limiting Gibbs processes foreshadowed the probabilistic nature of Gibbs processes discovered shortly after in the works of Dobrushin [1] and Lanford and Ruelle [13] in 1968 and 1969. For the proof of Theorem 3 we do not use the DLR-approach but the one of Nguyen Xuan Xanh and Zessin [28] from 1976. The difficult proof of this result uses basic ideas of Benjamin Nehring [26] as main tools.

We then investigate the problem under which assumptions on (Φ,ρ) the process G is the unique Gibbs process for (Φ,ρ). This is done for a slightly smaller class of $(\mathfrak{W}_\rho^\sharp)$ of Penrose-stable interactions and within a collection $\mathscr{G}_t(\Phi,\rho)$ of *tempered*

[2]This notion is weaker than Penrose-stability and stronger than the classical notion of stability.

[3]i.e. Dobrushin, Lanford and Ruelle

Gibbs processes. Since the work of Ruelle [34] one knows that a description of an *infinite volume equilibrium state* P, i.e. a process described by correlation functions satisfying the Kirkwood-Salsburg equations and interacting via some potential Φ which is not completely repulsive but has also an attractive part, requires a temperedness condition. This means that P is supported by some collection of tempered locally finite configurations. Our approach to the temperedness problem is different from the one of Ruelle.

To obtain uniqueness we need to work under Penrose-stability. This means that there exists a measurable function $\mathrm{c} : X \to [0, +\infty)$ such that

$$W_\Phi(x, \xi) \geq -\mathrm{c}(x), \qquad x \in X, \xi \in \mathfrak{V}_\Phi.$$

Here $W_\Phi(x, \xi)$ denotes the conditional energy of a particle in x given the surrounding particles ξ and $\mathfrak{V}_\Phi$ the set of all finite configurations having finite energy. Temperedness of P has the following meaning here:

(t1) for every x the conditional energy $W_\Phi(x, \mu)$ is well defined P-almost surely with respect to μ;

(t2) P is *visible*, i.e. P is supported by configurations of locally finite energy.

In view of $\mathscr{P}$-stability for c this implies that P-almost surely

$$W_\Phi(x, \mu) \geq -\mathrm{c}(x).$$

This infinite extendability of Penrose-stability to some full set for the process P is of central importance for our approach to uniqueness of Gibbs processes. In this connection we present the sufficient condition (15) for property (t1) and a general exclusion principle which implies property (t2).

In this context we show in Theorem 5 that the collection of tempered Gibbs processes is a singleton given by the limiting Gibbs process G. Historically the first who considered conditions of $\mathscr{P}$-stability (with constant stability function) under which one-dimensional classical systems show no phase transition were Van Hove [38] (cf. also Ruelle [33], Theorem 5.6.7) and Gallavotti, Miracle-Sole and Ruelle [2, 3]. A more-dimensional version has been presented by Ruelle in [34] in the setting of superstable and lower regular pair potentials. In the case of quantum systems in $\mathbb{R}^d$ Ginibre [5, 6] gives a proof that under $\mathscr{P}$-stability there is no phase transition. We also mention the important recent paper of Sabine Jansen [8] which contains a uniqueness result for positive pair potentials.

The results are applied to examples of classical and quantum statistical mechanics. Classical systems are given by interacting particles in Euclidean space and quantum systems by interacting finite clusters of particles in Euclidean space.

Part I
Preliminaries: Methods, assumptions and examples

1 Notations.

We use the notations and tools of point process theory from the books of Krickeberg [12] and J. Mecke [21]. (Cf. also [10, 18]) The underlying space $(X, \mathscr{B}(X), \mathscr{B}_0(X))$ is assumed to be Polish, where $\mathscr{B}(X)$ denotes its Borel σ-field and $\mathscr{B}_0(X)$ its bounded Borel sets. $\mathscr{M}(X)$ is the space of positive Radon measures on X, i.e. positive measures being finite on $\mathscr{B}_0(X)$. Here we use this term in a more general setting. [4] $\mathscr{M}(X)$ is Polish with respect to vague convergence, i.e. the topology generated by all mappings $\zeta_f : \mu \to \mu(f)$, where f is non-negative, continuous with bounded support. $\mathscr{M}^{\cdot\cdot}(X)$ is the closed subspace of point measures and the space $\mathscr{M}^{\cdot}(X)$ of simple point measures on X is a G_δ-set in $\mathscr{M}^{\cdot\cdot}(X)$. Point measures are Radon measures μ with $\mu(B) \in \mathbb{N}$ for all $B \in \mathscr{B}_0(X)$. Simple point measures may be considered as locally finite subsets of X. If $R > 0$ the subset $\mathscr{M}^{\cdot}_R(X)$ of μ having the property

$$(x, y \in \mu, x \neq y \Rightarrow d(x, y) > R)$$

is a G_δ-set in $\mathscr{M}^{\cdot}(X)$. Here d denotes some metric in X which generates the topology. We assume that $o \in \mathscr{M}^{\cdot}_R(X)$ and all singletons ε_x. (o is the zero-measure and ε_x the Dirac measure.) $\mathfrak{X}$ or $\mathfrak{X}(X)$ denotes the set of finite point measures on X. $\mathfrak{X}'$ is the collection of all $\xi \in \mathfrak{X}$ with $\xi \neq o$. If $R > 0$ then $\mathfrak{X}'_R = \mathfrak{X}' \cap \mathscr{M}^{\cdot}_R$.

Probability laws P on $\mathscr{M}, \mathscr{M}^{\cdot\cdot}, \mathscr{M}^{\cdot}$ or $\mathfrak{X}$ are called random measures, (point) processes, simple and finite processes in X respectively. $\mathscr{M}^{\cdot\cdot}_n$ is the collection of all n-particle configurations $\mu \in \mathscr{M}^{\cdot\cdot}$ with $|\mu| = n$, where $|\mu| = \mu(X)$ denotes the number of particles in μ. Δ_o denotes the Dirac measure on $\mathscr{M}$ in o.

Function spaces. F or $F(X)$ is the space of $[0, +\infty]$-valued measurable functions on X. U denotes the subspace of non-negative, bounded functions with bounded support and $\mathscr{K}$ the collection of continuous functions with compact support; $\mathscr{K}_+$ the subset of non-negative functions. The decomposition of a function f into positive and negative part is $f = f^+ - f^-$.

Integration with respect Radon measures. If $f \in F$ then the following notations are used for the integral of f with respect to $\mu \in \mathscr{M}$: $\zeta_f(\mu) = \mu(f)$. If $\mu \in \mathscr{M}^{\cdot\cdot}$ we also write $\mu(f) = \sum_{x \in \mu^*} f(x)\mu(x)$. Here μ^* is the support of μ.

Configurations and subconfigurations. A point measure $\mu \in \mathscr{M}^{\cdot\cdot}(X)$ is called a configuration of particles in the space X. μ can be represented by means of a sequence $(x_j)_{j \in J}$ of particles $x_j \in X$ as

$$\mu = \sum_{j \in J} \varepsilon_{x_j}.$$ [5]

Here $J = [n(\mu)] := \{1, \ldots, n(\mu)\}$ is a subset of the natural numbers augmented by $+\infty$ and $n(\cdot)$ is measurable; also the x_j are measurable maps of μ. This representation of

[4] Usually Radon measures are considered on locally compact and second countable Hausdorff topological spaces.

[5] This can be realized by the $1-1$ correspondence $\mu \leftrightarrow \kappa = \sum_{x \in \mu^*} \sum_{i=1}^{\mu\{x\}} \varepsilon_{(x,i)}$ between $\mu \in \mathscr{M}^{\cdot\cdot}(X)$ and $\kappa \in \mathscr{M}^{\cdot}(X \times \mathbb{N})$.

μ, called measurable indexing relation, is unique up to the ordering of the particles. (Cf. [18], Section 5.1.5., or also [10], Lemma 1.6) A (finite) subconfiguration ν of μ corresponds to a finite subset $I \subseteq J$ by $\nu = \sum_{i\in I} \varepsilon_{x_i}$. The empty configuration o corresponds to $I = \emptyset$. We then write $\nu \preceq \mu$ and $y \in \mu$ if $\varepsilon_y \preceq \mu$. For $\xi \in \mathfrak{X}$ and $x \in X$ we define $\xi_x = \xi - \varepsilon_x$ if $x \in \xi$ and $\xi_x = \xi$ else.

Potentials, Mayer functions, energy etc. A measurable and symmetric function $\Phi : X \times X \to]-\infty, +\infty]$ is a pair potential in X. $\overline{\Phi}$ denotes the truncated potential which equals 1 on $\{\Phi = +\infty\}$ and coincides with Φ outside.

$\omega(x,y) = \mathrm{e}^{-\Phi(x,y)} - 1$ is the associated Mayer function and

$$K(x,o) = 1, \quad K(x,\xi) = \prod_{y\in\xi} \omega(x,y), \qquad x \in X, \xi \in \mathfrak{X}, \xi \neq o.$$

$$\mathbb{E}_\Phi(\xi) = \sum_{j=1}^{n} E(x_j) + \sum_{1\leq i<j\leq n} \Phi(x_i,x_j)$$

denotes the energy of $\xi = \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$. We set $\mathbb{E}_\Phi(o) = 0$. Here the self potential E is a measurable function with values in $(-\infty, +\infty]$. We abbreviate the second term on the right hand side by the symbol for the usual energy:

$$E_\Phi(\xi) = \sum_{1\leq i<j\leq n} \Phi(x_i,x_j), \text{ if } |\xi| \geq 2, \;\; E_\Phi(\xi) = 0, \text{ if } |\xi| \in \{0,1\}.$$

Note that

$$2 \cdot E_\Phi(\xi) = \int_X \int_X \Phi(x,y)\, \xi_x(\mathrm{d}y) \xi(\mathrm{d}x).$$

The conditional energy of x in the environment η is

$$\mathbb{W}_\Phi(x,\eta) = E(x) + \int_X \Phi(x,y)\, \eta(\mathrm{d}y),$$

where the second term on the right hand side is denoted by $W_\Phi(x,\eta)$. The conditional energy of the configuration $\xi = \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$ in the environment η is defined by

$$\mathbb{W}_\Phi(\xi,\eta) = \mathbb{W}_\Phi(x_1,\eta) + \mathbb{W}_\Phi(x_2,\eta + \varepsilon_{x_1}) + \cdots + \mathbb{W}_\Phi(x_n, \eta + \varepsilon_{x_1} + \cdots + \varepsilon_{x_{n-1}}).$$

Note also that the energy of ξ is the conditional energy of ξ in the empty environment, i.e. $\mathbb{E}_\Phi(\xi) = \mathbb{W}_\Phi(\xi, o)$ and $E_\Phi(\xi) = W_\Phi(\xi, o)$. We normalize the conditional energy by $W_\Phi(o,\cdot) \equiv 0$.

Given a pair potential Φ we call

$$\mathfrak{V}_\Phi = \{\xi \in \mathfrak{X} : E_\Phi(\xi) < +\infty\}$$

the collection of all Φ-visible configurations. Particularly all configurations which are singletons are visible. [6]

Reference measure on X. If $\rho \in \mathscr{M}(X)$ we consider the measure ρ lifted to the space of finite configurations defined by

$$\Lambda_\rho\, \varphi = \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \cdots \int_X \varphi(\varepsilon_{x_1} + \cdots + \varepsilon_{x_n})\, \rho(\mathrm{d}x_1) \ldots \rho(\mathrm{d}x_n), \qquad \varphi \in F,$$

defines a locally finite measure on $\mathfrak{X}$. The term for $n = 0$ is $\varphi(o)$.

[6] This notion will play a role in the definition of $\mathscr{P}$-stability and in the proof of Lemma 14.

Local Gibbs processes. If (X,ρ') is a measure space and Φ a pair potential in X, then the Gibbs process in a bounded Borel set A of X is formally defined by

$$Q_A(\mathrm{d}\xi) = \frac{1}{\Xi(A)}\,\mathrm{e}^{-\mathbb{E}_\Phi(\xi)}.\Lambda_{\rho'_A}(\mathrm{d}\xi),$$

provided $\Xi(A) < +\infty$. We make throughout the following *convention:* If there is a non-trivial self energy E then it is included in the reference measure. Thus the local Gibbs measures have the form of classical statistical mechanics for the interaction Φ, namely

$$Q_A(\mathrm{d}\xi) = \frac{1}{\Xi(A)}\,\mathrm{e}^{-E_\Phi(\xi)}.\Lambda_{\rho_A}(\mathrm{d}\xi),$$

where $\rho(\mathrm{d}x) = \mathrm{e}^{-E(x)}.\rho'(\mathrm{d}x)$ and $\rho_A = 1_A.\rho$ the restriction of ρ onto A. Here $\Xi(A)$ is the normalizing factor and 1_A denotes the indicator function of the set A.

Laplace transform and modified Laplace transform. If P is a process in X its Laplace transform is defined by

$$\mathcal{L}_P f = \int_{\mathscr{M}^{\cdot\cdot}(X)} \mathrm{e}^{-\mu(f)}\; P(\mathrm{d}\mu), \qquad f \in F.$$

This concept is well defined for any finite positive measure. For infinite measures the right hand side may become infinite. Such measures, which are even signed, appear below as so called cluster measures L on $\mathfrak{X}$. (Cf. p. 23) If they satisfy the condition

$$\mathscr{K}_{\mathrm{L}} f = \mathrm{L}\left(1-\mathrm{e}^{-\zeta_f}\right) < +\infty, \qquad f \in U,$$

$\mathscr{K}_{\mathrm{L}}$ is called the modified Laplace transform of L. (Cf. [21])

2 Methods

Elementary relations for the Mayer function

For $0 \le \alpha^+ \le +\infty$, $0 \le \alpha^- < +\infty$, $\alpha = \alpha^+ - \alpha^-$ and $|\alpha| = \alpha^+ + \alpha^-$ the following relations hold:

$$(1) \qquad |\mathrm{e}^{-\alpha}-1| = \mathrm{e}^{\alpha^-}(1-\mathrm{e}^{-|\alpha|}) \le \alpha^-\,\mathrm{e}^{\alpha^-} + \left(1-\mathrm{e}^{-\alpha^+}\right) \le |\alpha|\mathrm{e}^{\alpha^-}.$$

Compound factorial measures

Define for a given configuration $\mu \in \mathscr{M}^{\cdot\cdot}$ the following measure on $\mathfrak{X}$.

$$\Lambda'_\mu(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} h(\varepsilon_{x_1} + \cdots + \varepsilon_{x_n})\, \widetilde{\mu}^n(\mathrm{d}x_1 \ldots \mathrm{d}x_n), \quad h \in F, \textit{ where}$$
$$\widetilde{\mu}^n(\mathrm{d}x_1 \ldots \mathrm{d}x_n) = \mu(\mathrm{d}x_1)(\mu-\varepsilon_{x_1})(\mathrm{d}x_2)\cdots(\mu-\varepsilon_{x_1}-\cdots-\varepsilon_{x_{n-1}})(\mathrm{d}x_n).$$

$\widetilde{\mu}^n$ is called the *factorial measure* of μ of order n, and Λ'_μ the *compound factorial measure built on* μ. For $n = 0$ the inner integral is understood as $h(o)$, where $o \in \mathfrak{X}$ denotes the *vacuum*, i.e. the 0-measure. If $n = 1$ we set $\widetilde{\mu}^1 = \mu$. Also $\Lambda'_o(h) = h(o)$. In the case where $\mu \in \mathfrak{X}$ we assume that $\widetilde{\mu}^n = o$ for all $n > |\mu|$.

Integration with respect to Λ'_μ means summation over all (finite) subconfigurations of μ, i.e.

$$\Lambda'_\mu(h) = \sum_{\nu \preceq \mu} h(\nu).$$

Mecke's and Minlos' Formula

Lemma 1. *([23, 20]) For every $\rho \in \mathscr{M}(X)$*

$$\int_{\mathfrak{X}} \int_{\mathfrak{X}} h(\xi, \nu - \xi)\, \Lambda'_{\nu}(\mathrm{d}\xi)\Lambda_{\rho}(\mathrm{d}\nu) = \int_{\mathfrak{X}} \int_{\mathfrak{X}} h(\xi, \nu)\, \Lambda_{\rho}(\mathrm{d}\xi)\Lambda_{\rho}(\mathrm{d}\nu), \qquad h \in F.$$

This formula is also valid for all h which are integrable with respect to the measure on the left hand or the right hand side of this equation.

If h depends only on singletons in the first variable we speak of *Mecke's Formula* otherwise of *Minlos' Formula*.

Campbell measures and Gibbs measures

Let P be a process in X and $n \geq 1$. The *Campbell measure of P of order n* is the measure on $X^n \times \mathscr{M}^{\cdot\cdot}(X)$ defined by

$$C_P^n(h) = \int_{\mathscr{M}^{\cdot\cdot}(X)} \int_{X^n} h(x, \mu)\, \mu^{\otimes n}(\mathrm{d}x) P(\mathrm{d}\mu), \qquad h \in F.$$

If $n = 1$ we write C_P and call it the *Campbell measure of P*. We'll also use the notion of Campbell measure of higher order for locally finite signed measures instead processes. They are defined analogously.

Let Φ be a pair potential for which the conditional energy $W_{\Phi}(x, \mu)$ is well defined for all x and infinitely extended configurations $\mu \in \mathscr{M}^{\cdot\cdot}$. A process P in X is called a *Gibbs process for* (Φ, ρ), we then write $P \in \mathscr{G}(\Phi, \rho)$, if P is *of first order*, i.e. $\nu_P^1(f) < +\infty$ for all $f \in \mathscr{K}(X)$, and P is a solution of the equation

$$(\Sigma_{\rho}) \qquad C_P(h) = \int_{\mathscr{M}^{\cdot\cdot}} \int_X h(x, \mu + \varepsilon_x)\, \mathrm{e}^{-W_{\Phi}(x,\mu)}\, \rho(\mathrm{d}x) P(\mathrm{d}\mu), \qquad h \in F.$$

Here ν_P^1 denotes the first moment measure of P defined below. An equivalent *compound version of* (Σ_{ρ}) is: For all $h \in F$

$$\int_{\mathscr{M}^{\cdot\cdot}} \int_{\mathfrak{X}} h(\xi, \mu)\, \Lambda'_{\mu}(\mathrm{d}\xi) P(\mathrm{d}\mu) = \int_{\mathscr{M}^{\cdot\cdot}} \int_{\mathfrak{X}} h(\xi, \mu + \xi)\, \mathrm{e}^{-W_{\Phi}(\xi,\mu)}\, \Lambda_{\rho}(\mathrm{d}\xi) P(\mathrm{d}\mu).$$

It is well known (cf. Theorem 2 in [28]) that this is equivalent to saying that P satisfies the DLR-equations as well as Ruelle's equilibrium equation (cf. Equations (5.12) in [34]).

The method of moments

If we use this method in the sequel we assume throughout that the underlying space X is a locally compact, second countable Hausdorff topological space, a lcsc-space for short.

The *moment measure of P of order k* is the measure on X^k defined by

$$\nu_P^k f = \int_{\mathscr{M}^{\cdot\cdot}} \mu^{\otimes k}(f)\, P(\mathrm{d}\mu), \qquad f \in \mathscr{K}(X^k),$$

whereas the *correlation measure*[7] *of P of order k* is the measure given by

$$\widetilde{\nu}_P^k(f) = \int_{\mathscr{M}^{\cdot\cdot}} \widetilde{\mu}^k(f)\, P(\mathrm{d}\mu), \qquad f \in \mathscr{K}(X^k).$$

[7] also called factorial moment measure

The Campbell measure determines all moment measures and all factorial moment measures.

If $\widetilde{\nu}_P^k$ has a density r_P^k with respect to some product measure $\rho^{\otimes k}$, where ρ is a Radon measure on X, then we say that r_P^k is a *correlation function of P of kth order* with respect to ρ. Note that correlation measures and thereby correlation functions are symmetric and thus functions on $\mathfrak{X}$. The process P is called *of order k* if ν_P^k is a Radon measure. P is called *of infinite order* if it is of order k for every k.

If $P \in \mathscr{G}(\Phi,\rho)$ then the compound version of the Gibbs property immediately implies that P has a correlation function given by

$$r_P(\xi) = \int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-W_\Phi(\xi,\mu)}\, P(\mathrm{d}\mu), \qquad \xi \in \mathfrak{X},$$

where

$$W_\Phi(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n},\mu) = W_\Phi(x_1,\mu)+W_\Phi(x_2,\mu+\varepsilon_{x_1})+\cdots+W_\Phi(x_n,\mu+\varepsilon_{x_1}+\cdots+\varepsilon_{x_{n-1}}).$$

This representation is due to Nguyen X.X. and Zessin. (Cf. [28], Formula (4.3)) It exhibits the correlation function of an infinitely extended Gibbs process as a function of *all* finite configurations.

The following basic lemma reduces the study of moment measures to the study of correlation measures.

Lemma 2. *(Krickeberg's decomposition, cf. [12]; [27], Theorem 4.1; and [26], Theorem 4.1.1) Let P be a point process in X of order k. Then the collection $\widetilde{\nu}_P^\ell, \ell = 1,\ldots,k$, of correlation measures of P is the unique family of symmetric Radon measures, decomposing ν_P^k in the following way:*

$$\nu_P^k(f_1\otimes\cdots\otimes f_k) = \sum_{\mathscr{J}\in\wp[k]} \widetilde{\nu}_P^{|\mathscr{J}|}(\otimes_{J\in\mathscr{J}} f_J), \qquad f_1,\ldots,f_k \in \mathscr{K}(X).$$

Here $|\mathscr{J}|$ is the cardinality of $\mathscr{J}$, the sum is taken over all partitions of $[k] := \{1,\ldots,k\}$ into non-empty subsets J, and $f_J = \prod_{j\in J} f_j$.

The next result is a continuity theorem for moment measures which can be found in [39]. (Cf. Theorem 2.2.)

Lemma 3. *Let $(P_n)_n$ be a sequence of processes in X of infinite order satisfying the following conditions*

for each k the limits

$\nu^k(f) = \lim_{n\to\infty} \nu_{P_n}^k(f), \quad f \in \mathscr{K}(X^k)$, exist and satisfy

$$\sum_{k=1}^\infty \nu^k(A^k)^{-\frac{1}{2k}} = +\infty, \qquad A \in \mathscr{B}_0(X).$$

Then there exists one and only one process P in X of infinite order such that

$$P_n \Rightarrow P \text{ and } \nu_P^k = \nu^k \text{ for each } k \in \mathbb{N}.$$

Here "$\Rightarrow$" means the weak convergence.

Algebraic method

We need a method which goes back to Ruelle [33]. If $\xi \in \mathfrak{X}$, integration with respect to Λ'_ξ gives rise to some commutative algebra which plays an important role in the analysis of quantities like the Boltzmann factor or the Ursell function.

Let $\mathscr{A}$ be the set of all measurable, complex valued functions φ on $\mathfrak{X}$ and define a $\star$-multiplication of two functions by

$$\varphi \star \psi(\xi) = \int_{\mathfrak{X}} \varphi(\nu)\psi(\xi - \nu)\Lambda'_\xi(\mathrm{d}\nu), \qquad \xi \in \mathfrak{X}, \varphi, \psi \in \mathscr{A}.$$

In this way $\mathscr{A}$ becomes a commutative algebra with unit $\mathbf{1}$, defined by $\mathbf{1}(\xi) = 1$ if $\xi = o$ and $\mathbf{1}(\xi) = 0$ else. In general, if $\varphi_1, \ldots, \varphi_n \in \mathscr{A}$,

$$(\varphi_1 \star \cdots \star \varphi_n)(\xi) = \sum_{\substack{(\xi_1,\ldots,\xi_n):\xi_1,\ldots,\xi_n \preceq \xi \\ \xi_1+\cdots+\xi_n=\xi}} \varphi_1(\xi_1)\cdots\varphi_n(\xi_n), \quad \xi \in \mathfrak{X}.$$

Let $\mathscr{A}_o = \{\varphi \in \mathscr{A} \mid \varphi(o) = 0\}$. The *algebraic exponential* is defined as the mapping $\Gamma : \mathscr{A}_o \to \mathbf{1} + \mathscr{A}_o$ by

$$\Gamma\varphi = \mathbf{1} + \sum_{k=1}^{\infty} \frac{1}{k!}\varphi^{\star k}, \qquad \varphi \in \mathscr{A}_o.$$

We introduce the operation of *differentiation* in $\mathscr{A}$ by

$$D_x\varphi(\xi) = \varphi(\xi + \varepsilon_x), \qquad x \in X, \xi \in \mathfrak{X}.$$

It satisfies

$$D_x(\Gamma\varphi) = D_x\varphi \star \Gamma\varphi, \qquad \varphi \in \mathscr{A}_o.$$

More generally $D_\xi\varphi = D_{x_1}\cdots D_{x_n}\varphi$ if $\xi = \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$. By Minlos formula

$$\Lambda_\rho(\varphi_1 \star \varphi_2) = \Lambda_\rho(\varphi_1)\cdot\Lambda_\rho(\varphi_2), \qquad \varphi_1, \varphi_2 \in \mathscr{A} \cap L^1(\Lambda_\rho).$$

3 Conditions on the underlying potential

Consider the underlying space $(X, \mathscr{B}, \mathscr{B}_0, \rho)$, where $\rho \in \mathscr{M}(X)$, and a pair potential Φ in X. Throughout the paper we assume stability, regularity and integrability conditions on Φ. These properties may vary depending on the problem considered. We consider three different *stability conditions*.

(A1) *(stability)* There exists a non-negative, measurable function $\mathrm{c} : X \to [0, +\infty)$, such that

$$E_\Phi(\xi) \geq -\xi(\mathrm{c}), \qquad \xi \in \mathfrak{X}.$$

Recall that $\xi(\mathrm{c}) = \sum_{x\in\xi} \mathrm{c}(x)$.

(A2) *(weak $\mathscr{P}$- or w$\mathscr{P}$-stability)* There exists a measurable function $\mathrm{c} : X \to [0, +\infty)$ such that for every $\xi \in \mathfrak{X} \setminus \{o\}$ there exists $x \in \xi$ with

$$W_\Phi(x, \xi_x) \geq -\,\mathrm{c}(x).$$

(A3) *($\mathscr{P}$-stability[8])* There exists a measurable function $\mathrm{c}: X \to [0,+\infty)$ such that for every $x \in X$

$$W_\Phi(x,\xi) \geq -\mathrm{c}(x), \qquad \xi \in \mathfrak{V}_\Phi.$$

If c is given, the two *regularity conditions* are

(B1) *(c-regularity)* There exists a non-negative, measurable function $\mathrm{a}: X \to [0,+\infty)$ such that

$$\int_X |\omega_x|(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\rho(\mathrm{d}y) \leq \mathrm{a}(x), \qquad x \in X.$$

(B2) (*modified* c-*regularity*) There exists a non-negative, measurable function $\mathrm{a}: X \to [0,+\infty)$ such that

$$\int_X |\overline{\Phi}_x|(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\mathrm{e}^{\Phi_x^-(y)}\,\rho(\mathrm{d}y) \leq \mathrm{a}(x), \qquad x \in X.$$

(B2) implies (B1) in view of the elementary equalities (1).

Finally, we need integrability conditions: If (c,a) are given, the conditions are

(C1) (*weak local* (c,a)-*integrability*)

$$\mathrm{e}^{(\mathrm{c}+\mathrm{a})}\,\mathrm{e}^{-\Phi_x^+}.\rho \in \mathscr{M}(X), \qquad x \in X.$$

(C2) (*local* (c,a)-*integrability*)

$$\mathrm{e}^{(\mathrm{c}+\mathrm{a})}.\rho \in \mathscr{M}(X).$$

(C3) (*strong local* (c,a)-*integrability*)

$$\mathrm{e}^{(\mathrm{c}+\mathrm{a})}\,\mathrm{e}^{\Phi_x^-}.\rho \in \mathscr{M}(X).$$

Obviously (C3) $\Longrightarrow$ (C2) $\Longrightarrow$ (C1) .

Lemma 4. *If Φ is* c*-regular for some* a *then* (C1) *implies* (C3). *In this case the three local integrability conditions are equivalent.*

Proof. We show that $\mathrm{e}^{(\mathrm{c}+\mathrm{a})}\,\mathrm{e}^{\Phi_x^-}.\rho$ is a Radon measure and thereby also $\mathrm{e}^{(\mathrm{c}+\mathrm{a})}\rho$. Indeed, if $g \in U$,

$$\begin{aligned}
&\int_X g(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\mathrm{e}^{\Phi_x^-(y)}\,\rho(\mathrm{d}y)\\
&\qquad\leq \int_X g(y)|\omega_x|(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\rho(\mathrm{d}y) + \int_X g(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\mathrm{e}^{-\Phi_x^+(y)}\,\rho(\mathrm{d}y)\\
&\qquad< +\infty. \qquad \square
\end{aligned}$$

If Φ is non-negative then it is $\mathscr{P}$-stable. Moreover, $\mathscr{P}$-stable potentials for c are $w\mathscr{P}$-stable potentials for c, whereas $w\mathscr{P}$-stable potential for c are c-stable. We need to show that $w\mathscr{P}$-stability for c implies stability for c. Indeed, given $\xi \in \mathfrak{X}$, $|\xi| = n \geq 1$, weak $\mathscr{P}$-stability implies that there exists $x_1 \in \xi$ with $W_\Phi(x_1,\xi_{x_1}) \geq -\mathrm{c}(x_1)$. Iterating this step we obtain a numbering $\xi = \varepsilon_{x_1} + \varepsilon_{x_2} + \cdots + \varepsilon_{x_n}$ such that for all $j = 1,\ldots,n$

$$W_\Phi(x_j,\xi_{x_1x_2\ldots x_j}) \geq -\mathrm{c}(x_j).$$

[8] This notion goes back to Oliver Penrose [29]. (Cf. also Charles B. Morrey [25] and J. Groeneveld [7])

Here $\xi_{x_1x_2\ldots x_j}=\xi-(\varepsilon_{x_1}+\cdots+\varepsilon_{x_j})$. On the other hand

$$E_\Phi(\xi)=\sum_{j=1}^{n}W_\Phi(x_j,\xi_{x_1x_2\ldots x_j})\geq-\xi(\mathrm{c}).$$

Finally, stability for c implies $w\mathscr{P}$-stability for $2\mathrm{c}$.

The following estimate will be used below: If Φ is $w\mathscr{P}$-stable for c then $\Phi(x,y)\geq-\mathrm{c}(x)\wedge\mathrm{c}(y),x,y\in X$, and thereby

$$\Phi^-(x,y)\leq\mathrm{c}(x)\wedge\mathrm{c}(y),\qquad x,y\in X. \tag{2}$$

This inequality is equally valid for all $\mathscr{P}$-stable potentials.

c-regularity (which is an extension of a criterion of Kotecký and Preiss [11]) and the condition of local integrability are due to Ueltschi [36]. The concept of modified regularity, used here with the additional factor $\mathrm{e}^{\Phi_x^-}$, is due to Poghosyan and Ueltschi [30] and was inspired by Procacci [32].

Algebraic method (continued)

Consider the *Boltzmann factor* $B=\mathrm{e}^{-E_\Phi}$. B is an element of the algebra $\mathscr{A}$ with $B(o)=1$. It has an inverse within $\mathscr{A}$, denoted by $B_\star^{-1}$. Another important element $\varkappa$ of the algebra $\mathscr{A}$ is called the *Ursell function* and given by $\varkappa(o)=0,\varkappa(\varepsilon_x)=1$ and for $n\geq2$

$$\varkappa(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n})=\sum_{\gamma\in\mathscr{C}_n}\prod_{\{i,j\}\in\gamma}\omega(x_i,x_j),$$

where $\mathscr{C}_n$ denotes the set of all simple, unoriented, connected graphs γ with n vertices, and the product is taken over all edges in γ. We remark that $\kappa\in\mathscr{A}_o$ and $B=\Gamma\kappa$.

We also need the *improved tree-graph estimate* of Ueltschi [37]:

Lemma 5. *If Φ is* c*-stable then*

$$|\kappa|(\xi)\leq\mathrm{e}^{\xi(\mathrm{c})}\sum_{\gamma\in\mathscr{T}(\xi)}\prod_{\{x,y\}\in\gamma}\Big(1-\mathrm{e}^{-|\Phi|(x,y)}\Big),\quad\xi\in\mathfrak{X}.$$

Here $\mathscr{T}(\xi)$ denotes the family of trees with vertex set ξ.

4 Examples

Classical interactions

Example 1. *(Stable pair potentials in Euclidean space) Let (X,λ_z) be the d-dimensional Euclidean space $E=\mathbb{R}^d$ with measure $\lambda_z=z\ell,z>0$. Here $\ell(\mathrm{d}u)=\mathrm{d}u$ is the Lebesgue measure on E. φ denotes a pair potential in E of the type $\varphi(u,v)=\psi(u-v)$, where ψ is an even measurable function on E, stable with some constant stability function. The* c*-regularity and modified* c*-regularity conditions for any constant* $\mathrm{c}\geq0$ *with* $\mathrm{a}\equiv1$ *take the form*

$$z\cdot\int_E|\mathrm{e}^{-\psi(v)}-1|\;\mathrm{d}v\leq\mathrm{e}^{-(\mathrm{c}+1)},\quad z\cdot\int_E|\overline{\psi}|(v)\,\mathrm{e}^{\psi^-(v)}\;\mathrm{d}v\leq\mathrm{e}^{-(\mathrm{c}+1)}.$$

These conditions are satisfied if the integrals are finite and z is chosen small enough. For concrete examples we refer to the book of Ruelle [33].

Example 2. *($\mathscr{P}$-potentials in Euclidean space) As before the underlying space is $E = \mathbb{R}^d$ with Lebesgue measure $z\ell$. $0 < R$ is a constant and φ a hard-core potential of the following type: If $|u-v| \leq R$ then $\varphi(u,v) = +\infty$; φ is real-valued outside the hard-core, and if $|u-v| > R$ then*

$$|\varphi|(u,v) \leq \psi(|u-v|),$$

where ψ is some positive, decreasing function on $]R,+\infty]$ such that

$$\int_R^\infty \psi(r) \cdot r^{d-1} \, \mathrm{d}r < +\infty.$$

φ is called a Penrose potential or $\mathscr{P}$-potential. (Cf. [29]; see also [25, 7])
It is well known ([25, 29]) that a $\mathscr{P}$-potential is $\mathscr{P}$-stable, and thereby w$\mathscr{P}$-stable, for some stability constant B depending on ψ and R.

We discuss next the modified regularity condition for a $\mathscr{P}$-potential φ for a constant stability function $B > 0$ and a constant function $\mathrm{a} \equiv A > 0$:

$$z \int_E |\overline{\varphi}_u|(v) \, \mathrm{e}^{B+A} \, \mathrm{e}^{\varphi_u^-(v)} \, \mathrm{d}v \leq z \mathrm{e}^{B+A} \left(\tau_d(R) + \mathrm{e}^B \int_R^\infty \psi(t) t^{d-1} \, \mathrm{d}t \right).$$

Here $\tau_d(R)$ denotes the volume of the d-dimensional ball of radius R. Since the right hand side is finite, modified B-regularity holds true if z is small enough.

Examples of ψ are $\psi(t) \equiv 0$, $\psi(t) = \delta \, \mathrm{e}^{-\gamma t}$, and $\psi(t) = \frac{\delta}{t^{d+\varepsilon}}$, where $\varepsilon, \delta, \gamma$ are positive constants. The first leads to purely hard-core, the second to Kac-type (cf. [9]) and the last one to Penrose-type potentials.

Quantum interactions

Let φ be a pair potential in Euclidean space $E = \mathbb{R}^d$ with Lebesgue measure $z\,\mathrm{d}x, z > 0$. Assume that φ is *weakly positive definite*, i.e. $y^{\otimes 2} \varphi \geq 0$ for all $y \in \mathfrak{X}$. Particularly $\varphi(u,u) \geq 0$ for all u. Since

$$y^{\otimes 2} \varphi = \sum_{u,v \in y} \varphi(u,v) = 2E_\varphi(y) + \sum_{u \in x} \varphi(u,u) \geq 0$$

we obtain that φ is stable with stability function $\mathrm{B}(u) = \frac{1}{2}\varphi(u,u)$. φ induces a pair potential in $\mathfrak{X} = \mathfrak{X}(E)$ by

$$\Phi(x,y) = (x \otimes y) \, \varphi, \; x, y \in \mathfrak{X}.$$

Φ is weakly positive definite. Indeed, for all finite configurations κ in the space $\mathfrak{X}$, which we denote by $\mathbb{X}$,

$$\kappa^{\otimes 2} \Phi = \int \int \varphi(a,b) \, (\chi\kappa)(\mathrm{d}a)(\chi\kappa)(\mathrm{d}b) \geq 0,$$

because φ is weakly positive definite. The elements of $\kappa \in \mathbb{X}$ have the form $\kappa = \varepsilon_{y_1} + \cdots + \varepsilon_{y_n}$, $y_j \in \mathfrak{X}$ and $\chi\kappa = y_1 + \cdots + y_n$ is the cluster dissolution of κ. In view of

$$\kappa^{\otimes 2} \Phi = 2E_\Phi(\kappa) + \sum_{y \in \kappa} \Phi(y,y) = 2 \left(E_\Phi(\kappa) + \sum_{y \in \kappa} (\zeta_\mathrm{B}(y) + E_\varphi(y)) \right)$$

this implies that Φ is stable with stability function $\mathrm{c} = \zeta_\mathrm{B} + E_\varphi$, $0 \leq \mathrm{c} < +\infty$. Note that here the self energy $\mathbb{E}_\Phi(\varepsilon_x) = E_\varphi(x), x \in \mathfrak{X}$, is not trivial.

On the other hand if the classical potential φ is $\mathscr{P}$-stable with a constant B then Φ is $\mathscr{P}$-stable with $c = \zeta_B$. Indeed, for all $x \in \mathfrak{X}'_R$ and $\kappa \in \mathfrak{V}_\Phi = \{\kappa | \chi\kappa \in \mathfrak{X}'_R\}$

$$W_\Phi(x,\kappa) = \int_{\mathfrak{X}} \Phi(x,y)\ \kappa(\mathrm{d}y) = \int_{\mathfrak{X}} \int_X \varphi(a,b)\ \chi\kappa(\mathrm{d}b)x(\mathrm{d}a) \geq -B\cdot|x|.$$

The following examples are inspired by the work of Ginibre [4, 6] and describe classes of interactions for quantum gases, i.e. systems of clusters in Euclidean space interacting via the potential Φ.

Below we choose as the basic space the space $\mathfrak{F}' = \mathfrak{F}'(E)$ of all nonempty closed subsets in $E = \mathbb{R}^d$. This set is topologized by means of the Fell topology which is generated by the maps

$$F \to \mathrm{d}(u,F), \qquad u \in E,$$

where $\mathrm{d}(u,F) = \inf_{v\in F} d(u,v)$ and d is the Euclidean metric. $\mathfrak{F}'$ then is a lcsc Hausdorff topological space, in which $\mathfrak{X}'$, the collection of nonempty finite sets in E, is dense. (Cf. [17] Theorem 1-2-1 and Corollary 2 of Theorem 1-2-2)

The collection $\mathscr{B}_0(\mathfrak{F}')$ of bounded Borel sets in $\mathfrak{F}'$ is generated by the collection of all

$$\mathfrak{F}_B = \{F \in \mathfrak{F}' | F \cap B \neq \emptyset\}, \qquad B \in \mathscr{B}_0(E),$$

In the following two examples $\Phi(x,y)$ is the interaction potential between clusters x and y. The energy of a finite configuration κ of clusters is then given by

$$\mathbb{E}_\Phi(\kappa) = \sum_{x\in\kappa} E_\varphi(x) + E_\Phi(\kappa);$$

and we consider the following locally finite measures $\lambda_z = \mathrm{e}^{-E_\varphi}.\lambda'_z$ on $\mathfrak{F}'$. Here

$$\lambda'_z(f) = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_E \int_{E^{m-1}} f(\varepsilon_u + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m})\ \mathrm{B}^u_m(\mathrm{d}u_2 \ldots \mathrm{d}u_m)\,\mathrm{d}u, \quad f \in F,$$

where the first term of the series is $z\int_E f(u)\,\mathrm{d}u$;

$$\mathrm{B}^u_m(\mathrm{d}u_2 \ldots \mathrm{d}u_m) = \gamma(u,u_2)\cdots\gamma(u_m,u)\,\mathrm{d}u_2\ldots\mathrm{d}u_m, \quad m \geq 2,$$

and

$$\gamma(u,v) = \frac{1}{(2\pi\beta)^{d/2}} \exp\left(-\frac{|u-v|^2}{2\beta}\right)$$

is a Gaussian kernel with positive parameter β. Note that λ_z is supported by $\mathfrak{X}'$. Thus quantum models are given by pair potentials Φ on the space $(\mathfrak{X}', \lambda_z)$.

Example 3. *(Positive potentials in spaces of finite clusters) Let φ be of the type $\varphi(u,v) = \psi(u-v)$. Assume that ψ is non-negative and integrable. φ thus is $w\mathscr{P}$-stable with stability constant $B \equiv 0$. The associated pair potential Φ is also non-negative and stable for* $\mathrm{c} = 0$. *We discuss the condition of* c*-modified regularity and thereby of* c*-regularity of Φ, which amounts to*

$$\int_{\mathfrak{X}'} \Phi(x,y)\,\mathrm{e}^{\mathrm{a}(y)}\ \lambda_z(\mathrm{d}y) \leq \mathrm{a}(x), \qquad x \in \mathfrak{X}'.$$

It had been remarked in [30] for functions $\mathrm{a}(x) = A\cdot|x|, A > 0$, *that this is true if*

$$\frac{\|\psi\|_1}{(2\pi\beta)^{d/2}} \sum_{m=1}^{\infty} \frac{z^m\,\mathrm{e}^{Am}}{m^{d/2}} \leq A.$$

This holds for z sufficiently small. ($\|\psi\|_1$ denotes the L^1-norm.) (Cf. Section 4.3 of [36].)

Unfortunately we are not able to show regularity or modified regularity in the case when the potential is stable and has an attractive part. The next example shows that this is the case if we consider $\mathscr{P}$-potentials.

Example 4. *($\mathscr{P}$-potentials in spaces of finite clusters) Let φ be a $\mathscr{P}$-potential in $E = \mathbb{R}^d$ for some stability constant B from Example 2. This implies that Φ is $\mathscr{P}$-stable on $\mathfrak{X}'_R$ with stability function $\mathrm{c} = \zeta_B$.*

We next show along the lines of [4, 30] that Φ is strongly modified c-regular for the parameter $0 < p < 1$ and the function $\mathrm{a}(y) = |\mathbb{S}_{R/2}(y)| + |y|, y \in \mathfrak{X}'_R$, if z is small enough.[9] Here $\mathbb{S}_{R/2}(y) = \bigcup_{u \in y} B_{R/2}(u)$ denotes the sausage around the cluster y. The range of a is contained in the interval $[1, +\infty)$.

If $y \in \mathfrak{X}'_R$ then $\mathrm{a}(y) = (\tau_d(R/2) + 1) \cdot |y|$, because $\ell(\mathbb{S}_{R/2}(y)) = \tau_d(R/2) \cdot |y|$. Note also that λ_z is supported by $\mathfrak{X}'_R$.

We shall show below that for z small enough and $p < 1$

$$(\star) \qquad \int_{\mathfrak{X}'_R} |\overline{\Phi}_x|(y)\, \mathrm{e}^{(\zeta_B + \mathrm{a})(y)}\, \mathrm{e}^{\Phi_x^-(y)}\, \lambda_z(\mathrm{d}y) \le p \cdot \mathrm{a}(x), \qquad x \in \mathfrak{X}'_R.$$

Since $E_\varphi(y) \ge -B|y|, y \in \mathfrak{X}'_R$, and $\Phi_x^-(y) \le \mathrm{c}(y)$ for every x, the integral on the left hand side can be estimated from above by

$$\sum_{m=1}^{\infty} \frac{(z\,\mathrm{e}^{3B+\tau_d(R/2)+1})^m}{m} \int_E \int_{E^{m-1}} |\overline{\Phi}|(x, \varepsilon_u + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m})\, \mathrm{B}_m^u(\mathrm{d}u_2 \ldots \mathrm{d}u_m)\, \mathrm{d}u$$

Using the device, that for all $f \in F$

$$\int f(u, u_2, \ldots, u_m)\, \mathrm{B}_m^u(\mathrm{d}u_2 \ldots \mathrm{d}u_m) = \int f(u, u_2 + u, \ldots, u_m + u)\, \mathrm{B}_m^0(\mathrm{d}u_2 \ldots \mathrm{d}u_m),$$

we obtain

$$\sum_{m=1}^{\infty} \frac{(z\,\mathrm{e}^{3B+\tau_d(R/2)+1})^m}{m} \int_{E^{m-1}} \left[\int_E |\overline{\Phi}|(x, \varepsilon_u + \varepsilon_{u_2+u} + \cdots + \varepsilon_{u_m+u})\, \mathrm{d}u \right] \mathrm{B}_m^0(\mathrm{d}u_2 \ldots \mathrm{d}u_m)$$

The inner integration with respect to $\mathrm{d}u$ is now decomposed into the region $\mathbb{S}(x, \varepsilon_0 + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m})$ of all $u \in E$ where $|\overline{\Phi}|(x, \varepsilon_u + \varepsilon_{u_2+u} + \cdots + \varepsilon_{u_m+u}) \equiv 1$ and its complement. One obtains for the inner integral

$$\ell(\mathbb{S}(x, \varepsilon_0 + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m})) + m \cdot |x| \cdot \|\psi\|_1.$$

where $\|\psi\|_1 = \int_{|u|>R} \psi(u)\, \mathrm{d}u$. It is easy to see that

$$\mathbb{S}(x, \varepsilon_0 + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m}) = \mathbb{S}_R(x) \cup \bigcup_{j=2}^{m} \mathbb{S}_R(x - u_j),$$

so that

$$\ell(\mathbb{S}(x, \varepsilon_0 + \varepsilon_{u_2} + \cdots + \varepsilon_{u_m})) \le m 2^d \cdot \ell(\mathbb{S}_{R/2}(x)).$$

Here $x - u_j$ denotes the translation of x by $-u_j$. Thus condition $(\star)$ is valid if

$$\frac{1}{(2\pi\beta)^{d/2}} \sum_{m=1}^{\infty} \frac{(z\,\mathrm{e}^{3B+\tau_d(R/2)+1})^m}{m^{d/2}} \le \frac{p}{2^d \vee \|\psi\|_1}.$$

This inequality holds true if z is small enough. Local (c, a)-integrability is obvious.

[9] The definition of this regularity condition can be found at the beginning of Section 10.

Part II

Construction of a limiting process by the method of moments

In this part we present elements of the method of cluster expansions, construct the limiting Gibbs process and exhibit its Gibbsian character.

5 Tools from cluster expansion

Lemma 6. *The function $G(\xi,\cdot)$, defined by $G(\xi,\cdot) = B_\star^{-1} \star D_\xi B, \xi \in \mathfrak{X}$, is an element of the algebra $\mathscr{A}$ and, as a function of both variables, the unique solution of the non-integrated Kirkwood-Salsburg equation: For each choice of x in ξ*

$$\begin{cases} G(o,\eta) = \delta_{o,\eta}, \\ G(\xi,\eta) = \mathrm{e}^{-W_\Phi(x,\xi_x)} \int_{\mathfrak{X}} K(x,\nu) \cdot G(\xi_x + \nu, \eta - \nu) \Lambda'_\eta(\mathrm{d}\nu) \quad \xi \neq o. \end{cases}$$

Particularly, the right hand side of this equation does not depend on the choice of $x \in \xi$.

This lemma is well known [33, 24, 30]. G is called the (classical) *Ursell kernel*. (Cf. [24]) It is well defined inductively by this equation and thereby uniquely determined. Moreover, $G(o,o) = G(\varepsilon_x, o) = 1$.

We next estimate the function G by means of another function H_c that satisfies a similar equation and is accessible to detailed analysis. If we assume that Φ is a $w\mathscr{P}$-stable pair potential in X with stability function c then the Ursell kernel can then be estimated from above by

$$|G|(\xi,\eta) \leq \mathrm{e}^{\mathrm{c}(x)} \cdot \int_{\mathfrak{X}} |K|(x,\nu) \cdot |G|(\xi_x + \nu, \eta - \nu) \Lambda'_\eta(\mathrm{d}\nu) \tag{3}$$

for all $x \in \xi$ with $W_\Phi(x,\xi_x) \geq -\mathrm{c}(x)$.

Observation (4) is taken to define $H_\mathrm{c}(\xi,\eta)$ as the unique solution H of the equation

$$\begin{cases} H(o,\eta) = \delta_{o,\eta}, \\ H(\xi,\eta) = \mathrm{e}^{\mathrm{c}(x)} \int_{\mathfrak{X}} |K|(x,\nu) \cdot H(\xi_x + \nu, \eta - \nu) \Lambda'_\eta(\mathrm{d}\nu), \quad \xi \neq o. \end{cases} \tag{4}$$

Under the assumption of weak c-stability of Φ

$$|G|(\xi,\eta) \leq H_\mathrm{c}(\xi,\eta), \qquad \xi,\eta \in \mathfrak{X}. \tag{5}$$

Our next step is to write the function H_c explicitly in terms of rooted forests. An unoriented simple graph is called *rooted forest* if its connected components are rooted trees. Here a *rooted tree* is a tree in which one vertex is specified as a root. $\mathscr{F}(\xi,\eta)$ denotes the collection of forests with the set of vertices $\xi + \eta$ and roots ξ.

Lemma 7. *(Minlos, Poghosyan [24]) For $\xi \neq o$ the function $H_\mathrm{c}(\xi,\eta)$ is given by*

$$H_\mathrm{c}(\xi,\eta) = \mathrm{e}^{(\xi+\eta)(\mathrm{c})} \sum_{\gamma \in \mathscr{F}(\xi,\eta)} \prod_{\{x,y\} \in \gamma} |\omega|(x,y). \tag{6}$$

Since H_c is uniquely determined as a solution of the equation (4) it is enough to check that this ansatz of H_c satisfies the equation (4). (For details see [24], Lemma 1 and Lemma 2 as well as Lemma 6.2 and Lemma 6.3 in [30].)

Lemma 8. *If* $\xi = \varepsilon_{x_1} + \cdots + \varepsilon_{x_n}$*, then for all* $\eta \in \mathfrak{X}$

$$H_{\mathrm{c}}(\xi,\eta) = H_{\mathrm{c}}(x_1,\cdot) \star \cdots \star H_{\mathrm{c}}(x_n,\cdot)\,(\eta). \tag{7}$$

Proof. Consider the relation

$$\mathscr{F}(\xi,\eta) \equiv \mathscr{T}(x_1,\eta_1) \times \cdots \times \mathscr{T}(x_x,\eta_n),$$

where on the right hand side one has an n-fold Cartesian product of sets of trees. This establishes a $1-1$ correspondence between forests γ rooted in ξ with vertex set $\xi+\eta$ and n-tuples of disjoint trees $\tau_j \in \mathscr{T}(x_j,\eta_j), j=1,\ldots,n$, rooted in x_j with vertex set $\varepsilon_{x_j}+\eta_j$, where $\eta_1,\ldots,\eta_n \preceq \eta$ with $\eta_1+\cdots+\eta_n=\eta$. Therefore the right hand side of (6) can be written as

$$\sum_{\substack{(\eta_1,\ldots,\eta_n):\eta_1,\ldots,\eta_n\preceq\eta\\ \eta_1+\cdots+\eta_n=\eta}} H_{\mathrm{c}}(x_1,\eta_1)\cdots H_{\mathrm{c}}(x_n,\eta_n),$$

which equals the right hand side of (7) by definition of the $\star$-product. □

The Ruelle estimate

Ruelle's estimate of the (correlation) function r is in the center of all classical statistical mechanics. $r(\xi), \xi \in \mathfrak{X}$*, is estimated by means of the kernel* H_{c} *from Lemma 7. In the case of singletons* $|\xi| = 1$ *we obtain Ruelle's estimate in terms of the Ursell function under a weaker condition.*

Below we use the following version of equation (2.9) of Theorem 2.1 in [30], which provides important upper bounds for the integrals appearing in the lemma.

Lemma 9. *Suppose that* Φ *is* c*-regular for some function* a*. Then*

$$\int_{\mathfrak{X}} \mathrm{e}^{\eta(\mathrm{c})}\cdot \sum_{\gamma\in\mathscr{T}(x,\eta)} \prod_{\{y,y'\}\in\gamma} |\omega|(y,y')\,\Lambda_\rho(\mathrm{d}\eta) \le \mathrm{e}^{\mathrm{a}(x)}, \qquad x\in X. \tag{8}$$

If Φ *is* c*-stable and satisfies the condition*

$$\int_X (1-\mathrm{e}^{-|\Phi(x,y)|})\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\;\rho(\mathrm{d}y) \le \mathrm{a}(x), \tag{9}$$

which is weaker than c*-regularity, then*

$$\int_{\mathfrak{X}} |\kappa|(\xi+\varepsilon_x)\,\Lambda_\rho(\mathrm{d}\xi) \le \mathrm{e}^{(\mathrm{a}+\mathrm{c})(x)}, \qquad x\in X. \tag{10}$$

The proof of Lemma 9 follows the same lines as in [30]. The reasoning for the estimate (10) uses the tree-graph estimate in Lemma 5. Lemma 9 is an instance of the following estimate of Ruelle.

Corollary 1. *Let* Φ *be a* w$\mathscr{P}$*-stable pair potential with stability function* c *and* c*-regular for some* a*. Then the function*

$$r(\xi) = \int_{\mathfrak{X}} G(\xi,\eta)\,\Lambda_\rho(\mathrm{d}\eta), \qquad \xi\in\mathfrak{X}.$$

is a well defined element of $\mathbf{1}+\mathscr{A}_o$ *and satisfies the Ruelle estimate*

$$|r|(\xi) \le \mathrm{e}^{\xi(\mathrm{c}+\mathrm{a})}, \qquad \xi\in\mathfrak{X}. \tag{$\aleph_{\mathrm{c}}^{\mathrm{a}}$}$$

Proof. Inequality (5) combined with Lemma 7 and Lemma 8 imply

$$\begin{aligned}\int_{\mathfrak{X}} |G|(\xi,\eta)\,\Lambda_\rho(\mathrm{d}\eta) &\leq \int_{\mathfrak{X}} H_{\mathrm{c}}(\xi,\eta)\,\Lambda_\rho(\mathrm{d}\eta)\\ &= \prod_{x\in\xi}\int_{\mathfrak{X}} H_{\mathrm{c}}(x,\eta)\,\Lambda_\rho(\mathrm{d}\eta)\\ &\leq \mathrm{e}^{\xi(\mathrm{c}+\mathrm{a})}.\end{aligned}$$

□

We remark here that for the proof of the estimate for $r(x)$ that

$$G(x,\cdot) = B_\star^{-1} \star D_x(\Gamma\kappa) = D_x\kappa.$$

Thus, using (9) and (10), we obtain $(\aleph_{\mathrm{c}}^{\mathrm{a}})$ under weaker conditions. To summarize:

Theorem 1. *If Φ is $w\mathscr{P}$-stable for* c *and* c*-regular for some* a *then the function $r(\xi)$, $|\xi| \geq 2$ satisfies the Ruelle estimate $(\aleph_{\mathrm{c}}^{\mathrm{a}})$. The same estimate for $r(x)$ is valid under the weaker conditions of* c*-stability combined with the regularity condition (9).*

6 Ursell kernel expansion of correlation functions

In the sequel we assume that Φ satisfies the following conditions called *standard conditions*.

$(\mathfrak{W}_\rho)$ Φ is $w\mathscr{P}$-stable with stability function c, c-regular for some a and weak locally (c,a)-integrable.

Given $A \in \mathscr{B}_0(X)$ define the Gibbs process in A with empty boundary conditions as the probability Q_A on $\mathfrak{X}(A)$ which is absolutely continuous with respect to Λ_{ρ_A} and given by

$$Q_A(\mathrm{d}\xi) = \frac{1}{\Xi(A)}\,\mathrm{e}^{-E_\Phi(\xi)}\cdot\Lambda_{\rho_A}(\mathrm{d}\xi).$$

This process is well defined for all c-stable potentials Φ, which are locally c-integrable, particularly for $w\mathscr{P}$- and $\mathscr{P}$-stable potentials.

Lemma 10. *The correlation function of Q_A is*

$$r_A(\xi) = \frac{1}{\Xi(A)}\int_{\mathfrak{X}(A)} \mathrm{e}^{-E_\Phi(\xi+\eta)}\,\Lambda_\rho(\mathrm{d}\eta), \quad \xi\in\mathfrak{X}(A). \tag{11}$$

This follows immediately from the calculation of correlation measures by means of Mecke's formula. Obviously r_A is supported by the visibility set $\mathfrak{V}_\Phi$.

Proposition 1. *The correlation function of Q_A has the following expansion in terms of the Ursell kernel:*

$$r_A(\xi) = \int_{\mathfrak{X}(A)} G(\xi,\eta)\,\Lambda_\rho(\mathrm{d}\eta), \qquad \xi\in\mathfrak{X}(A). \tag{12}$$

Proof. Note that the integrand in Formula (11) is $D_\xi B$. Minlos' Formula implies

$$\Lambda_{\rho_A}(D_\xi B) = \Lambda_{\rho_A}(B\star B_\star^{-1}\star D_\xi B) = \Xi(A)\Lambda_{\rho_A}(G(\xi,\cdot)).$$

This is due to the integrability of B and $B_\star^{-1}\star D_\xi B$ with respect to Λ_{ρ_A}, where the latter follows from the corresponding Ruelle estimate. □

We remark that in formulas (11) and (12) $r_A(o) = 1$. A classical result of Minlos [22] and Ruelle [33] shows that r_A converges pointwise to the socalled limiting correlation function r if $A \uparrow X$ along some increasing sequence of bounded Borel sets exhausting X. It gives also an estimate of the rate of convergence. We obtain a weaker version of this result directly from Theorem 1 by means of Lebesgue's dominated convergence theorem.

Corollary 2. *Under standard conditions* $(\mathfrak{W}_\rho)$ *the function r is the limiting correlation function in the sense of pointwise convergence, i.e.*

$$r(\xi) = \lim_{A \uparrow X} r_A(\xi), \qquad \xi \in \mathfrak{X},$$

and satisfies the Ruelle bound $(\aleph_c^a)$. *Furthermore, r is supported by the collection of visible configurations* $\mathfrak{V}_\Phi$.

Proof. Given $\xi \in \mathfrak{X}$, we note that $1_{\mathfrak{X}(A_n)}(\cdot)|G|(\xi,\cdot), n \geq 1$, is a sequence in $L^1(\Lambda_\rho)$ majorized by $|G|(\xi,\cdot) \in L^1(\Lambda_\rho)$ in view of $(\aleph_c^a)$. Thus Lebesgue's dominated convergence theorem implies

$$1_{\mathfrak{X}(A_n)}(\cdot)G(\xi,\cdot) \to G(\xi,\cdot) \quad \text{in } L^1(\Lambda_\rho),$$

when $A_n \uparrow X$. This implies the assertions. □

7 Construction of a limiting Gibbs process with correlation function r

We now show that r is not only a limiting correlation function. We use the method of moments to show that there exists a unique process P having r as its correlation function.[10] *Recall that from now on the underlying space X is lcsc.*

Lemma 11. *(cf. [39], Corollary 2.2.) Let* $(P_n)_n$ *be a sequence of point processes* P_n *in X of infinite order satisfying the conditions*

(13) *for each k the limits*

$$\widetilde{\nu}^k(f) = \lim_{n\to\infty} \widetilde{\nu}^k_{P_n}(f), \quad f \in \mathscr{K}(X^k), \quad \text{exist and satisfy}$$

(14)
$$\sum_{k=1}^{\infty} \nu^k(A^k)^{-\frac{1}{2k}} = +\infty \quad \text{for each } A \in \mathscr{B}_0(X),$$

where

$$\nu^k(A^k) = \sum_{\mathscr{J}} \widetilde{\nu}^{|\mathscr{J}|}(A^{|\mathscr{J}|}).$$

Then there exists one and only one point process P in X of infinite order such that $P_n \Rightarrow P$ *and* $\widetilde{\nu}^k_P = \widetilde{\nu}^k$ *for each k.*

Lemma 11 follows from Krickeberg's decomposition Lemma 2 combined with Lemma 3.

[10] Thus we realize Minlos' program on limiting Gibbs processes from [22] by means of methods from point process theory which had been developed later.

For the construction of the limiting process we work under the standard conditions on Φ. Let $(K_n)_n$ be a sequence of bounded Borel sets in X exhausting X and consider the sequence $P_n = Q_{K_n}$ of Gibbs processes in X. By Proposition 1 the associated correlation measures are given for n large enough by

$$\widetilde{\nu}^k_{P_n} f = \int_{X^k} f(x_1,\dots,x_k) \int_{\mathfrak{X}(K_n)} G(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}, \eta) \Lambda_\rho(\mathrm{d}\eta)\rho(\mathrm{d}x_1)\dots\rho(\mathrm{d}x_k),$$

where $f \in \mathscr{K}(X^k)$. In view of local integrability, the Ruelle bound implies that each P_n is of infinite order. Moreover, it is clear from Corollary 2 that Condition (13) holds true with limits

$$\widetilde{\nu}^k(f) = \int_{X^k} f(x_1,\dots,x_k) \int_{\mathfrak{X}} G(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}, \eta)\,\Lambda_\rho(\mathrm{d}\eta)\rho(\mathrm{d}x_1)\dots\rho(\mathrm{d}x_k).$$

Thus it remains to prove (14). Let $A \in \mathscr{B}_0(X)$. Denoting by $S(k,m)$ the Stirling number of the second kind, Krickeberg's decomposition Lemma 2 implies under $(\mathfrak{W}_\rho)$

$$\begin{aligned} \nu^k(A^k) &= \sum_{m=1}^{k} S(k,m)\widetilde{\nu}^m(A^m) \\ &\leq \sum_{m=1}^{k} S(k,m) \left[\rho_A(e^{\mathrm{c}+\mathrm{a}})\right]^m \\ &< +\infty. \end{aligned}$$

Here we used once more $(\aleph^{\mathrm{a}}_{\mathrm{c}})$. Recalling Stirling's Formula (cf. [35], Formula (24 a)), i.e.

$$S(k,m) = \frac{1}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j} j^k,$$

we conclude that there is a positive constant C

$$\nu^k(A^k) \leq \mathrm{e}^C (2k)^{2k}$$

so that for some positive constant C'

$$\nu^k(A^k)^{\frac{1}{2k}} \leq C' \cdot k.$$

This implies (14). To summarize:

Theorem 2. *Under the standard conditions on Φ there exists a unique process* G *of infinite order with correlation function r, which is the limiting Gibbs process with empty boundary conditions (in the sense of weak convergence) of the sequence Q_{K_n} of local Gibbs processes.*

Theorem 1 and Theorem 2 are applicable to all potentials of Examples 1 - 4.

Support of limiting Gibbs processes. A general exclusion principle

Let Φ be an arbitrary pair potential in X. Consider the collection of all *infinitely extended visible configurations*, i.e. configurations in $\mathscr{M}^{\cdot\cdot}(X)$ of finite local energy, defined by

$$\mathscr{M}^{\cdot\cdot}_{\upsilon} = \{\mu = \sum_{j=1}^{\infty} \varepsilon_{x_j} \in \mathscr{M}^{\cdot\cdot}(X) | (i \neq j \Rightarrow \Phi(x_i,x_j) < +\infty)\}.$$

Here we suppose that $o, \varepsilon_x \in \mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$ for all x. Note that $\mathfrak{V}_\Phi = \mathfrak{X} \cap \mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$. The aim is to characterize processes P which are supported by $\mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$. Let

$$D_\infty = \{(x,y) \in X^2 | \Phi(x,y) = +\infty\}.$$

We first characterize $\mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$: Let $\mu \in \mathscr{M}^{\cdot\cdot}$. Then $\mu \in \mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$ if and only if $\widetilde{\mu}^2(D_\infty) = 0$. The proof is immediate and left to the reader. This implies

Proposition 2. *(Exclusion principle) Let P be a process in X. Then P is supported by $\mathscr{M}_{\mathfrak{V}}^{\cdot\cdot}$ in the sense that $P\,\mathscr{M}_{\mathfrak{V}}^{\cdot\cdot} = 1$, if and only if*

$$\widetilde{\nu}_P^2(D_\infty) = 0.$$

In this case we say that P is a *visible* process for Φ. Denote by $D = \{(x,x) : x \in X\}$ the diagonal in X^2. An immediate implication of the exclusion principle is

Corollary 3. *Let Φ be an arbitrary pair potential in X satisfying $\Phi(x,y) = +\infty$ iff $x = y$. Then P is a simple process, i.e. supported by the collection of simple point configurations $\mathscr{M}^{\cdot}(X)$, if and only if $\widetilde{\nu}_P^2\, D = 0$.*

This is Theorem 1 in [40]. Another corollary of Proposition 2 is

Corollary 4. *If P has a correlation function r_P, which is supported by the Φ-visible configurations $\mathfrak{V}_\Phi$, then P is visible.*

In view of Corollary 2 we obtain

Corollary 5. *The limiting Gibbs process* G *from Theorem 2 is visible.*

Corollary 5 may be applied to all Examples 1- 4. Particularly, we have

Example 5. *Consider Example 2: Let φ be a $\mathscr{P}$-potential in E with parameter R. The collection of hard-core configurations $\mathscr{M}_R^{\cdot}(E)$ are the visible configurations. Thus Corollary 4 implies that the limiting Gibbs process* G *for φ is a hard-core process. The same holds true for the hard-core model of Example 4. One has to work now with the pseudometric*

$$D(x,y) = \min_{u \in x, v \in y} |u - v|, \qquad x, y \in \mathfrak{X}'.$$

8 The Gibbsian character of the limiting Gibbs process

We next identify the limiting Gibbs process G *as a Gibbs process in the DLR-sense under slightly stronger than the standard conditions:*

Assume the conditions

$(\mathfrak{W}_\rho^\sharp)$ Φ is $w\mathscr{P}$-stable with stability function c and modified c-regular for some function a. Moreover, Φ is weak locally (c,a)-integrable.

Note that $(\mathfrak{W}_\rho^\sharp)$ implies the standard conditions $(\mathfrak{W}_\rho)$.

Theorem 3. *Let ρ be a Radon measure on X and Φ a pair potential in X satisfying $(\mathfrak{W}_\rho^\sharp)$. Then* G *is a visible Gibbs process in X for Φ and ρ.*

Theorem 2 guarantees the existence of a limiting Gibbs process G of infinite order. To show its Gibbsianess it suffices to show that G is a solution of equation (Σ_ρ) for functions h of the form $f \otimes \mathrm{e}^{-\zeta_g}$, where $f, g \in U$. (Cf. [21], proof of Theorem 10) Let $(K_n)_n$ be an increasing sequence in $\mathscr{B}_0(X)$ exhausting X. Since by construction $\mathrm{G}_n := \mathrm{G}_{K_n} \Rightarrow \mathrm{G}$ as $n \to \infty$, we know from the generalized Palm-Khinchin theorem ([18], Proposition 10.1.5 or also Corollary 3.3.2 in [26]), that the left hand side converges, i.e. $C_{\mathrm{G}_n} h \to C_{\mathrm{G}} h$.

Since the G_n are Gibbs processes in K_n, they solve equation $(\Sigma_{\rho_{K_n}})$. (Cf. Lemma 1 in [41]) To show convergence of the right hand side of $(\Sigma_{\rho_{K_n}})$ consider the sequence of integrands on the space (X, ρ) given by

$$\iota_n(x) = f(x)\,\mathrm{e}^{-g(x)} \cdot \int_{\mathfrak{X}(K_n)} \mathrm{e}^{-\mu(g)}\,\mathrm{e}^{-W_\Phi(x,\mu)}\ \mathrm{G}_n(\mathrm{d}\mu) \cdot 1_{K_n}(x).$$

The idea is to use Lebesgue's dominated convergence theorem. Observe first that

$$\int_{\mathfrak{X}(K_n)} \mathrm{e}^{-\mu(g)}\,\mathrm{e}^{-W_\Phi(x,\mu)}\ \mathrm{G}_n(\mathrm{d}\mu) = \mathcal{L}_{\mathrm{G}_n}(g + \Phi_x).^{11}$$

On the other hand

$$\int_{\mathfrak{X}(K_n)} \mathrm{e}^{-W_\Phi(x,\mu)}\ \mathrm{G}_n(\mathrm{d}\mu) = r_{\mathrm{G}_n}(x),$$

which follows directly from equation $(\Sigma_{\rho_{K_n}})$. This implies that the ι_n are dominated by some ρ-integrable function. Indeed,

$$\iota_n(x) \le f(x) r_{\mathrm{G}_n}(x) \le f(x)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(x)}.$$

Here we used Theorem 1 combined with the bound $(\aleph^{\mathrm{a}}_{\mathrm{c}})$. The function on the right hand side is ρ-integrable by local (c, a)-integrability.

Thus the problem is to show pointwise convergence

$$\mathcal{L}_{\mathrm{G}_n}(g + \Phi_x) \to_n \mathcal{L}_{\mathrm{G}}(g + \Phi_x), \qquad x \in X$$

This difficult question has been answered by Benjamin Nehring [26] for the case of a classical gas. For the convenience of the reader we recall his proof where we replace his Lemma 5.2.5 by an abstract version which allows a proof in our different setting.

Proof. (We proceed in steps labeled from A to F.)

A. Consider the socalled cluster measure $\mathrm{L} = \varkappa.\Lambda_\rho$. (Cf. [31]) This is an infinite and signed measure on $\mathfrak{X}$ of first order. The latter follows from Mecke's formula combined with the estimate (10) of Lemma 9 and local integrability of the potential: For $f \in U$

$$\begin{aligned}
\nu^1_{|\mathrm{L}|} f &= \int_{\mathfrak{X}} \int_X f(x) |\varkappa|(\xi)\, \xi(\mathrm{d}x)\ \Lambda_\rho(\mathrm{d}\xi) \\
&= \int_X f(x) \int_{\mathfrak{X}} |\varkappa|(\xi + \varepsilon_x)\ \Lambda_\rho(\mathrm{d}\xi) \rho(\mathrm{d}x) \\
&\le \int_X f(x)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(x)}\ \rho(\mathrm{d}x) \\
&< +\infty.
\end{aligned}$$

This implies that $\nu^1_{\mathrm{G}} f \le \nu^1_{|\mathrm{L}|} f$, $f \in F$.

[11] Although the function $g + \Phi_x$ is not an element of U we speak, in an abuse of language, of the Laplace transform evaluated in $g + \Phi_x$.

B. The observation that L is of first order allows to represent the limiting Gibbs process G in another way: Proposition 3 or Proposition 4 in [31] characterize G as the unique process $\mathfrak{I}_{\mathrm{L}}$ in X with Laplace transform

$$\mathcal{L}_{\mathfrak{I}_{\mathrm{L}}} f = \exp\left(-\mathscr{K}_{\mathrm{L}}(f)\right), \qquad f \in U.$$

Recall that $\mathscr{K}_{\mathrm{L}}(f) = \mathrm{L}(1 - \mathrm{e}^{-\zeta_f})$ is the modified Laplace transform. Furthermore, the local Gibbs processes G_n satisfy

$$\mathcal{L}_{\mathrm{G}_n} f = \exp\left(-\mathrm{L}_n\left(1 - \mathrm{e}^{-\zeta_f}\right)\right) \to_n \mathcal{L}_{\mathrm{G}} f, \quad f \in U.$$

Here $\mathrm{L}_n = 1_{\mathfrak{X}(K_n)}.\mathrm{L}$. We now use this representation for the solution of our problem, i.e. we show the convergence of the modified Laplace transform

$$(a) \qquad \mathscr{K}_{\mathrm{L}_n}(g + \Phi_x) \to_n \mathscr{K}_{\mathrm{L}}(g + \Phi_x),$$

and the equalities

$$(b) \qquad \mathcal{L}_{\mathrm{G}_n}(g + \Phi_x) = \exp\left(-\mathscr{K}_{\mathrm{L}_n}(g + \Phi_x)\right),$$

$$(c) \qquad \mathcal{L}_{\mathrm{G}}(g + \Phi_x) = \exp\left(-\mathscr{K}_{\mathrm{L}}(g + \Phi_x)\right).$$

C. The main lemma to obtain these assertions is the following: Fix some x and set $\mathrm{k} = g + \Phi_x$. Consider the function

$$\daleth(\xi) = \xi(g + |\overline{\Phi}_x|)\,\mathrm{e}^{\xi(\Phi_x^-)}, \qquad \xi \in \mathfrak{X}.$$

Lemma 12. *$\daleth$ is integrable with respect to $|\mathrm{L}|$ and satisfies*

$$|1 - \mathrm{e}^{-\zeta_{\mathrm{k}}}| \leq \daleth.$$

Proof. $|1 - \mathrm{e}^{-\zeta_{\mathrm{k}}}|$ is seen to be bounded from above by $\daleth$ if one takes into account the elementary inequalities (1). Integrability of $\daleth$ is implied by the following reasoning:

$$\begin{aligned}
&\int_{\mathfrak{X}} \int_X (g(y) + |\overline{\Phi}_x|(y)\,\mathrm{e}^{\xi(\Phi_x^-)}\,|\varkappa|(\xi)\;\xi(\mathrm{d}y)\Lambda_\rho(\mathrm{d}\xi) \\
&= \int_{\mathfrak{X}} (g(y) + |\overline{\Phi}_x|(y)\,\mathrm{e}^{\Phi_x^-(y)} \int_X |\varkappa|(\xi + \varepsilon_y)\,\mathrm{e}^{\xi(\Phi_x^-)}\;\Lambda_\rho(\mathrm{d}\xi)\rho(\mathrm{d}y) \\
&\leq \int_X (g(y) + |\overline{\Phi}_x|(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\mathrm{e}^{\Phi_x^-(y)}.\rho(\mathrm{d}y)
\end{aligned}$$

In view of the second part of Lemma 9 the inner intergral on the right hand side of the equality can be estimated from above by $\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}$. Finally, the last integral is a sum of two integrals which are both finite. □

An immediate corollary is assertion (a): Indeed,

$$|(\mathrm{L} - \mathrm{L}_n)(1 - \mathrm{e}^{-\zeta_{\mathrm{k}}})| \leq |\mathrm{L}|\left(|1 - \mathrm{e}^{-\zeta_{\mathrm{k}}}|(1 - 1_{\mathfrak{X}(K_n)})\right) \searrow_n 0.$$

Since $\mathrm{e}^{\zeta_{\Phi_x^-}} - 1 \leq \daleth$ we also have

Corollary 6. *The function $\left(\mathrm{e}^{\zeta_{\Phi_x^-}} - 1\right)$ is integrable with respect to $|\mathrm{L}|$.*

D. It remains to show (b) and (c). It suffices to do it for (c). Recall that $\mathrm{G} = \mathfrak{I}_{\mathrm{L}}$. We observe first that monotone convergence implies

$$\int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-\mu(\mathrm{k}_+)} \sum_{n=o}^{\infty} \frac{\mu(\mathrm{k}_-)^n}{n!}\ \mathrm{G}(\mathrm{d}\mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-\mu(\mathrm{k}_+)}\, \mu(\mathrm{k}_-)^n\ \mathrm{G}(\mathrm{d}\mu)$$
$$= \mathcal{L}_{\mathrm{G}}(\mathrm{k}_+) + \sum_{n=1}^{\infty} \frac{1}{n!} C^n_{\mathrm{G}}\left(\mathrm{k}_-^{\otimes n} \otimes \mathrm{e}^{-\zeta_{\mathrm{k}_+}}\right).$$

Here C^n_{G} is the Campbell measure of order n of G. Note that the left hand side equals the Laplace transform $\mathcal{L}_{\mathrm{G}}(\mathrm{k})$ in the generalized sense. Our aim is to show that both terms on the right hand side are finite and sum up to $\exp(-\mathscr{K}_{\mathrm{L}}(\mathrm{k}))$.

E. In view of $\mathrm{k}_- \leq \Phi^-_x$ Corollary 6 implies

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathfrak{X}} \xi(\mathrm{k}_-)^n \mathrm{e}^{-\xi(\mathrm{k}_+)}\ |\mathrm{L}|(\mathrm{d}\xi) \leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathfrak{X}} \xi(\Phi^-_+)^n\ |\mathrm{L}|(\mathrm{d}\xi)$$
$$\leq \int_{\mathfrak{X}} \left(\mathrm{e}^{\xi(\Phi^-_x)} - 1\right)\ |\mathrm{L}|(\mathrm{d}\xi)$$
$$< +\infty$$

This argument also shows that

$$(\daleth) \qquad \nu^n_{|\mathrm{L}|}(\mathrm{k}_-^{\otimes n}) \leq \nu^n_{|\mathrm{L}|}((\Phi^-_x)^{\otimes n}) < +\infty, \qquad n \geq 1.$$

F. We now can show $\mathcal{L}_{\mathrm{G}}(\mathrm{k}) = \exp(-\mathscr{K}_{\mathrm{L}}(\mathrm{k}))$. To see this note that

$$\mathscr{K}_{\mathrm{L}}\,\mathrm{k} = \mathrm{L}\left(1 - \mathrm{e}^{-\zeta_{\mathrm{k}_+}}\right) - \int_{\mathfrak{X}} \mathrm{e}^{-\xi(\mathrm{k}_+)} \sum_{n=1}^{\infty} \frac{1}{n!} \xi(\mathrm{k}_-)^n\ \mathrm{L}(\mathrm{d}\xi).$$

This is the difference of two positive terms which are not equal to $+\infty$. For the first terms this is clear by $1 - \mathrm{e}^{-\zeta_{\mathrm{k}_+}} \leq \gimel$; for the second it follows from the convergence of the integral which has been shown under E. Therefore

$$\exp(-\mathscr{K}_{\mathrm{L}}(\mathrm{k})) = \exp(-\mathrm{L}(1 - \mathrm{e}^{-\zeta_{\mathrm{k}_+}})) \cdot \exp\left(\int_{\mathfrak{X}} \mathrm{e}^{-\xi(\mathrm{k}_+)} \sum_{n=1}^{\infty} \frac{1}{n!} \xi(\mathrm{k}_-)^n\ \mathrm{L}(\mathrm{d}\xi)\right)$$

The first factor on the right hand side equals $\mathcal{L}_{\mathrm{G}}(\mathrm{k}_+)$. This is easily seen by dominated convergence, if one approximates k_+ from below by the sequence of functions in U defined by $f_n := (\mathrm{k}_+ \wedge n).1_{K_n}$. For the second factor we use the exponential formula of combinatorial theory (cf. [35], chapter 4, Equation (4.14)):

$$\exp\left(\int_{\mathfrak{X}} \mathrm{e}^{-\xi(\mathrm{k}_+)} \sum_{n=1}^{\infty} \frac{1}{n!} \xi(\mathrm{k}_-)^n\ \mathrm{L}(\mathrm{d}\xi)\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathscr{J} \in \wp(|n|)} \prod_{J \in \mathscr{J}} C^{|J|}_{\mathrm{L}}\left(\mathrm{k}_-^{\otimes |J|} \otimes \mathrm{e}^{-\zeta_{\mathrm{k}_+}}\right)$$

Because the series in the argument of the exponential is absolutely convergent, the series on the right hand side has this property too. In view of ($\daleth$) we can use Theorem 4.3.1 in [26] to obtain

$$\mathcal{L}_{\mathrm{G}}(\mathrm{k}_+) \cdot \sum_{\mathscr{J} \in \wp(|n|)} \prod_{J \in \mathscr{J}} C^{|J|}_{\mathrm{L}}\left(\mathrm{k}_-^{\otimes |J|} \otimes \mathrm{e}^{-\zeta_{\mathrm{k}_+}}\right) = C^n_{\mathrm{G}}\left(\mathrm{k}_-^{\otimes n} \otimes \mathrm{e}^{-\zeta_{\mathrm{k}_+}}\right).$$

Thus

$$\exp(-\mathscr{K}_{\mathrm{L}}(\mathrm{k})) = \mathcal{L}_{\mathrm{G}}(\mathrm{k}_+) + \sum_{n=1}^{\infty} \frac{1}{n!} C_{\mathrm{G}}^n \left(\mathrm{k}_-^{\otimes n} \otimes \mathrm{e}^{-\zeta_{\mathrm{k}_+}} \right),$$

which in view of paragraph D. implies the assertion. (The reasoning for assertion (b) is exactly the same.) ☐

Theorems 1 - 3 may be applied to all Examples 1 - 4. The following observation from the last proof will be important in the next part.

Remark 1. *In view of $\nu_{\mathrm{G}}^1 f \le \nu_{|\mathrm{L}|}^1 f$, $f \in F$, condition $(\urcorner)$ for $n = 1$ yields $\nu_{\mathrm{G}}^1(\Phi_x^-) < +\infty$. This implies $\zeta_{\Phi_x^-} < +\infty$ almost surely with respect to* G *and shows that the conditional energy $W_\Phi(x, \cdot)$ in x is well defined* G*-almost surely. A direct argument which avoids $(\urcorner)$ is: Due to condition $(\mathfrak{W}_\rho^\sharp)$ we know that $\nu_{\mathrm{G}}^1(|\overline{\Phi}_x|) < +\infty$ and thus also $\nu_{\mathrm{G}}^1(\Phi_x^-) < +\infty$.*

Part III
Uniqueness of Gibbs processes

To obtain uniqueness of Gibbs processes we have to restrict the class of potentials again. This is done in two steps. The method for the proof uses the classical method of Kirkwood-Salsburg equations.

9 Kirkwood-Salsburg equation

In this section we assume that Φ is a $w\mathscr{P}$-stable pair potential on X with stability function c, which is *strongly* c*-regular*, i.e. there exist parameters $\varepsilon > 0, 0 < p < 1$ and a measurable function $\mathrm{a} : X \to [\varepsilon, +\infty)$ such that

$$\int_X |\omega_x|(y)\, \mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\ \rho(\mathrm{d}y) \le p\,\mathrm{a}(x), \qquad x \in X.$$

In addition we assume that Φ is weak locally (c, a)-integrable. These assumptions imply $(\mathfrak{W}_\rho)$. Thus the associated limiting Gibbs process G exists; its correlation function is $r(\xi) = \int_{\mathfrak{X}} G(\xi, \eta)\, \Lambda_\rho(\mathrm{d}\eta)$.

Proposition 3. *r satisfies the equations $r(o) = 1$ and*

$$(\mathsf{K}\Sigma_\rho) \qquad r(\xi) = \mathrm{e}^{-W_\Phi(x,\xi_x)} \cdot \int_{\mathfrak{X}} K(x,\eta) r(\xi_x + \eta)\, \Lambda_\rho(\mathrm{d}\eta), \quad x \in \xi \ne o.$$

Particularly, the right hand side of the equation does not depend on the choice of x.

Proof. We obtain from the non-integrated Kirkwood-Salsburg equation of Lemma 6 by means of Minlos' Formula for all $x \in \xi$

$$\begin{aligned} r(\xi) &= \mathrm{e}^{-W_\Phi(x,\xi_x)} \cdot \int_{\mathfrak{X}} \int_{\mathfrak{X}} K(x,\nu) G(\xi_x + \nu, \eta - \nu)\, \Lambda'_\eta(\mathrm{d}\nu) \Lambda_\rho(\mathrm{d}\eta) \\ &= \mathrm{e}^{-W_\Phi(x,\xi_x)} \cdot \int_{\mathfrak{X}} \int_{\mathfrak{X}} K(x,\nu) G(\xi_x + \nu, \eta)\, \Lambda_\rho(\mathrm{d}\nu) \Lambda_\rho(\mathrm{d}\eta) \\ &= \mathrm{e}^{-W_\Phi(x,\xi_x)} \cdot \int_{\mathfrak{X}} K(x,\nu) r(\xi_x + \nu)\, \Lambda_\rho(\mathrm{d}\nu). \end{aligned}$$

To complete the proof we have to justify the use of Minlos' Formula as well as the interchange of the integrations with respect to ν and η. Combining the Ruelle bound $(\aleph_{\mathrm{c}}^{\mathrm{a}})$ with Fubini's theorem yields in view of c-regularity

$$\begin{aligned}
&\int_{\mathfrak{X}}\int_{\mathfrak{X}} |K|(x,\nu)\cdot|G|(\xi_x+\nu,\eta)\,\Lambda_\rho(\mathrm{d}\nu)\Lambda_\rho(\mathrm{d}\eta)\\
&\quad\le \mathrm{e}^{\xi_x(\mathrm{c}+\mathrm{a})}\int_{\mathfrak{X}}|K|(x,\nu)\,\mathrm{e}^{\nu(\mathrm{c}+\mathrm{a})}\,\Lambda_\rho(\mathrm{d}\nu)\\
&\quad= \mathrm{e}^{\xi_x(\mathrm{c}+\mathrm{a})}\exp\left(\int_X|\omega_x|(y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\rho(\mathrm{d}y)\right)\\
&\quad\le \mathrm{e}^{\xi_x(\mathrm{c}+\mathrm{a})}\,\mathrm{e}^{\mathrm{a}(x)}<+\infty.
\end{aligned}$$

□

We next follow the classical reasoning of Ruelle [33], Ginibre [4] and Minlos [22]: Let $\mathscr{E}$, be the Banach space of all complex valued measurable functions $\varphi:\mathfrak{X}'\to\mathbb{C}$ such that

$$\|\varphi\|=\sup_{\xi\in\mathfrak{X}'}\frac{|\varphi|(\xi)}{\mathrm{e}^{\xi(\mathrm{c}+\mathrm{a})}}<+\infty.$$

Recall that $\mathfrak{X}'=\mathfrak{X}\setminus\{o\}$. Since r satisfies the Ruelle bound $(\aleph_{\mathrm{c}}^{\mathrm{a}})$ we have

Lemma 13. *The correlation function r is an element of $\mathscr{E}$ with norm smaller or equal than* 1.

We define on $\mathscr{E}$ the linear operator K by

$$\begin{aligned}
\mathsf{K}\,\varphi(\varepsilon_y)&=\int_{\mathfrak{X}'}K(y,\eta)\varphi(\eta)\,\Lambda_\rho(\mathrm{d}\eta),\quad y\in X,\\
\mathsf{K}\,\varphi(\xi)&=\mathrm{e}^{-W_\Phi(x,\xi_x)}\cdot\int_{\mathfrak{X}}K(x,\eta)\varphi(\eta+\xi_x)\Lambda_\rho(\mathrm{d}\eta),\quad x\in\xi\neq o.
\end{aligned}$$

Using the operator K we can write the Kirkwood-Salsburg equations as a fixed point equation in the Banach space $\mathscr{E}$, namely

$$r=\mathsf{K}\,r+\mathbf{1}.$$

If the norm of the operator satisfies $\|\,\mathsf{K}\,\|<1$, then this equation has a unique solution given by the Neumann series

$$r=\sum_{n=0}^{\infty}\mathsf{K}^n\,\mathbf{1}.$$

Under the strong regularity condition the operator K is bounded. Indeed, let $\varphi\in\mathscr{E}$ such that $\|\varphi\|\le 1$. Then we obtain as above

$$\begin{aligned}
|(\mathsf{K}\,\varphi)|(\xi)&\le \mathrm{e}^{\mathrm{c}(x)}\int_{\mathfrak{X}}|K|(x,\eta)\,\mathrm{e}^{(\eta+\xi_x)(\mathrm{c}+\mathrm{a})}\,\Lambda_\rho(\mathrm{d}\eta)\\
&\le \mathrm{e}^{\xi(\mathrm{c})+\xi_x(\mathrm{a})}\cdot\exp\left(\rho(|\omega_x|\,\mathrm{e}^{\mathrm{c}+\mathrm{a}})\right)\\
&\le \mathrm{e}^{\xi(\mathrm{c})+\xi_x(\mathrm{a})}\cdot\mathrm{e}^{p\cdot\mathrm{a}(x)}\\
&\le \mathrm{e}^{\xi(\mathrm{c}+\mathrm{a})}\,\mathrm{e}^{-(1-p)\varepsilon}.
\end{aligned}$$

Thus $\|\,\mathsf{K}\,\|<1$ and we have

Theorem 4. *Let Φ be a w$\mathscr{P}$-stable pair potential with stability function* c *which is strongly c-regular for some* a *and weak locally* (c,a)*-integrable. Then the limiting Gibbs process* G *exists, its correlation function r is an element of the Banach space* $\mathscr{E}$ *and the unique solution of the equation* $(\mathsf{K}\Sigma_\rho)$.

Theorems 1 - 4 may be applied to all Examples 1 - 4.

10 Generalities on Gibbs processes

To obtain uniqueness we compare the limiting Gibbs process G with any other Gibbs process P having the same properties as G. Accordingly we work under conditions which imply the assumptions of Theorem 3 as well as Theorem 4 and strengthen again our assumtions:

$(\mathscr{S}^\sharp_\rho)$ Φ is $\mathscr{P}$-stable with stability function c and *strongly modified* c*-regular*. This means that there exist parameters $\varepsilon > 0$ and $0 < p < 1$ and a measurable function $\mathrm{a} : X \to [\varepsilon, +\infty)$ such that

$$\int_X |\overline{\Phi}|(x,y)\,\mathrm{e}^{(\mathrm{c}+\mathrm{a})(y)}\,\mathrm{e}^{\Phi_x^-(y)}\;\rho(\mathrm{d}y) \le p\,\mathrm{a}(x), \qquad x \in X.$$

Finally, Φ is assumed to be weak locally (c,a)-integrable.

We consider processes $P \in \mathscr{G}(\Phi,\rho)$ having the property

$$\nu_P^1(\Phi_x^-) < +\infty, \qquad x \in X. \tag{15}$$

(15) guaranties that the conditional energies $W_\Phi(x,\cdot)$, $x \in X$, are well defined P-almost surely. (Cf. Remark 1) Moreover, P is visible. Indeed,

$$\widetilde{\nu}_P^2(D_\infty) = \int_{\{\Phi=+\infty\}} \int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-W_\Phi(\varepsilon_x+\varepsilon_y,\mu)}\;P(\mathrm{d}\mu)\rho(\mathrm{d}x)\rho(\mathrm{d}y) = 0.$$

Here we use that $W_\Phi(\varepsilon_x+\varepsilon_y,\mu) = W_\Phi(x,\mu) + W_\Phi(y,\mu) + \Phi(x,y)$.

Visible processes $P \in \mathscr{G}(\Phi,\rho)$ satifying (15) are called *tempered Gibbs processes*, we then write $P \in \mathscr{G}_t(\Phi,\rho)$.

Lemma 14. *Every tempered Gibbs process is* c*-extendable in the sense that for every x and P-almost all* μ

$$W_\Phi(x,\mu) \ge -\mathrm{c}(x),$$

Its correlation function is given by

$$r_P(\xi) = \int_{\mathscr{M}^{\cdot\cdot}} \exp(-W_\Phi(\xi,\mu)\;P(\mathrm{d}\mu)), \qquad \xi \in \mathfrak{X},$$

so that r_P *satisfies the Ruelle bound:*

$$r_P(\xi) \le \mathrm{e}^{\xi(\mathrm{c})}, \quad \xi \in \mathfrak{X},$$

and P is of infinite order. Finally, P is uniquely determined by its correlation function and visible, i.e. supported by $\mathscr{M}^{\cdot\cdot}_{\upsilon}$.

Proof. In view of visibility of P the $\mathscr{P}$-stability for c implies that the conditional energies are infinitely c-extendable. This implies

$$\int_{\mathcal{M}^{\cdot\cdot}} \mathrm{e}^{-W_\Phi(\xi,\mu)}\ P(\mathrm{d}\mu) \le \mathrm{e}^{\xi(\mathrm{c})}, \quad \xi \in \mathfrak{X}.$$

On the other hand the left hand side of this inequality is the locally integrable correlation function of P. This follows immediately from the compound version of (Σ_ρ).

With respect to uniqueness note that the first part of the proof shows that for every $f \in \mathscr{K}_+(X)$

$$\sum_{k=1}^{\infty} \frac{1}{k!} \widetilde{\nu}_P^k(f^{\otimes k}) \le \exp \rho(f\,\mathrm{e}^{\mathrm{c}}) < \infty.$$

It is well known that this is a sufficient condition for P to be uniquely determined by its correlation functions. (Cf. [14, 39]) □

Kirkwood-Salsburg equation for Gibbs processes

Proposition 4. *The correlation function r_P of a tempered Gibbs process $P \in \mathscr{G}_t(\Phi,\rho)$ solves the Kirkwood-Salsburg equation $(\mathsf{K}\Sigma_\rho)$.*

Proof. Similarly to the proof of Proposition 3 we can see that the right hand side of $(\mathsf{K}\Sigma_\rho)$ converges absolutely.

(Step 1) The proof of the assertion uses the following two identities: For all $\xi \neq o$, $x \in \xi$ and P-almost all μ

$$W_\Phi(\xi,\mu) = W_\Phi(x,\xi_x) + W_\Phi(x,\mu) + W_\Phi(\xi_x,\mu), \tag{16}$$

$$\mathrm{e}^{-W_\Phi(x,\mu)} = \Lambda'_\mu(K(x,\cdot)) = \sum_{k=0}^{\infty} \frac{1}{k!} \widetilde{\mu}^k(\omega_x \otimes \cdots \otimes \omega_x). \tag{17}$$

(The term $k=0$ of the last series is equal to 1.)

The second identity is the key of the proof and appeared already in Ruelle's proof of Lemma 6 (cf. [33] section 4.4.6) in the case of finite μ. Thus, in view of Lebesgue's dominated convergence theorem, we need to know whether the right hand side of equation (17) is absolutely convergent almost-surely with respect to P. Indeed, Formula (Σ_ρ) in its iterated compound form implies for every x

$$\begin{aligned}
&\int_{\mathcal{M}^{\cdot\cdot}} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} |\omega_x|^{\otimes k}(y_1,\ldots,y_k)\, \widetilde{\mu}^k(\mathrm{d}y_1 \ldots \mathrm{d}y_k) P(\mathrm{d}\mu) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathcal{M}^{\cdot\cdot}} \int_{X^k} \prod_{j=1}^{k} |\omega_x|(y_j)\, \mathrm{e}^{-W_\Phi(\varepsilon_{y_1}+\ldots\varepsilon_{y_k},\mu)}\ \rho(\mathrm{d}y_1)\ldots\rho(\mathrm{d}y_k) P(\mathrm{d}\mu).
\end{aligned}$$

We continue by using extended $\mathscr{P}$-stability and obtain

$$\begin{aligned}
&\le \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} \prod_{j=1}^{k} \left(|\omega_x|(y_j)\, \mathrm{e}^{\mathrm{c}(y_j)} \right) \rho(\mathrm{d}y_1)\ldots\rho(\mathrm{d}y_k) \\
&\le \exp\left(\int_X |\omega_x|(y)\, \mathrm{e}^{\mathrm{c}(y)}\ \rho(\mathrm{d}y) \right) < +\infty.
\end{aligned}$$

(Step 2) We are now in the position to show the assertion.
Let $\xi \in \mathfrak{X}'$ and choose some $x \in \xi$. Using once more the Gibbsian character of P we obtain by formulas (16) and (17)

$$r_P(\xi) = \mathrm{e}^{-W_\Phi(x,\xi_x)} \int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-W_\Phi(x,\mu)}\, \mathrm{e}^{-W_\Phi(\xi_x,\mu)}\, P(\mathrm{d}\mu)$$

$$= \mathrm{e}^{-W_\Phi(x,\xi_x)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathscr{M}^{\cdot\cdot}} \int_{X^k} \prod_{j=1}^{k} \omega_x(y_j)\, \mathrm{e}^{-W_\Phi(\xi_x,\mu)}\, \widetilde{\mu}^k(\mathrm{d}y_1 \dots \mathrm{d}y_k) P(\mathrm{d}\mu)$$

$$= \mathrm{e}^{-W_\Phi(x,\xi_x)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{X^k} K(x, \varepsilon_{y_1} + \cdots + \varepsilon_{y_k})$$

$$\int_{\mathscr{M}^{\cdot\cdot}} \mathrm{e}^{-W_\Phi(\xi_x,\mu+\varepsilon_{y_1}+\cdots+\varepsilon_{y_k}) - W_\Phi(\varepsilon_{y_1}+\cdots+\varepsilon_{y_k},\mu)}\, P(\mathrm{d}\mu)\rho(\mathrm{d}y_1)\dots\rho(\mathrm{d}y_k).$$

Here the interchange of integration and limit building is justified if we take into account $\mathscr{P}$-stability of the potential. Noting that the inner integral is $r_P(\xi_x + \varepsilon_{y_1} + \cdots + \varepsilon_{y_k})$ we see that the correlation function of a Gibbs process satisfies the Kirkwood-Salsburg equation $(\mathsf{K}\Sigma_\rho)$. □

11 Uniqueness of Gibbs processes

We now compare the limiting Gibbs process G, which exists by Theorem 2 and is an element of $\mathscr{G}_t(\Phi,\rho)$ by Theorem 3, with any $P \in \mathscr{G}_t(\Phi,\rho)$ by comparing the correlation function r of G with r_P, the correlation function of P.

We know that r and r_P solve the same Kirkwood-Salsburg equation $(\mathsf{K}\Sigma_\rho)$ and are elements of the Banach space $\mathscr{E}$. Since $(\mathsf{K}\Sigma_\rho)$ has a unique solution by Theorem 4 we obtain

Theorem 5. *Let Φ be a pair potential in X, which satisfies the assumptions $(\wp_\rho^\sharp)$. Then the collection $\mathscr{G}_t(\Phi,\rho)$ of tempered Gibbs processes contains only the limiting Gibbs process* G *for* (Φ,ρ).

Example 6. *Consider the Examples 1 - 4. Theorem 5 covers Example 2. It contains a more-dimensional version of the results of van Hove [38], which is Theorem 5.6.7 in Ruelle's book [33], and of Gallavotti, Miracle-Solé, and Ruelle [2, 3]. If φ is non-negative Theorem 5 can be applied to Example 1. This covers Theorem 2.1 in [8] under slightly stronger conditions. Finally Theorem 5 can be applied to Example 3 and Example 4, i.e. to systems of interacting clusters.*

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