Yamabe constant (continued)

Lecture 3

Yamabe constant of AF manifolds

- Let us have an AF manifold which satisfies R = 0. Further, we assume that it is conformally flat outside a region of compact support. This is NOT important, it is purely to simplify things. We now conformally transform the manifold so that it is flat outside some coordinate radius r_0. We know there exists a positive function v satisfying $8\nabla^2 v - Rv = 0, v \to 1$.

Yamabe constant

- This is the conformal factor that transforms us back to the zero scalar curvature manifold. Asymptotically, we know
- $$v = 1 + M/2r + \dots.$$
- with M positive, because we know that the mass of $g' = v^4 g$ (with $g_{ab}' = v^4\delta_{ab}$ near infinity) is positive. We neither know nor care about the interior. The positive energy theorem tells us that R in general is negative, but we do not care about it.

Yamabe constant

- We want to get an upper bound for the Yamabe constant. We use the following test function
- $\xi = \beta v$, $r < r'$, $\xi = \dfrac{\alpha^{1/2}}{(\alpha^2 + r^2)^{1/2}}$, $r' < r < \infty$
- where $r' > r_0$.
- We match $1 + M/2r$ with $\alpha^{1/2}/(\alpha^2 + r^2)^{1/2}$ to get
- $\alpha^2 = 2r'^3/M$

Yamabe constant

- and $\beta = \alpha^{-1/2}(1 + M/2r')^{-3/2}$. We use the conformal factor in the interior and the flat-space Sobolev function in the exterior.
- We break the integral above the line into two parts $\int_{r'}^{\infty} = \int_{0}^{r'} + \int_{r'}^{\infty}$ and we have
- $\int_{0}^{r'} [(\nabla\xi)^2 + 1/8R\xi^2]dv = \beta^2 \int_{0}^{r'} [(\nabla v)^2 + 1/8Rv^2]dv =$

$$\beta^2 \oint_{r'} v\nabla v.dS + \beta^2 \int_{0}^{r'} v(1/8Rv - \nabla^2 v)\,dv$$

Yamabe constant

- But we know that $1/8Rv - \nabla^2 v = 0$. Therefore the integral reduces to

$$\int_0^{r'} = \beta^2 \oint_{r'} v \nabla v . dS = \oint_{r'} \xi \nabla \xi . dS$$

- The other integral is $\int_{r'}^{\infty} [(\nabla \xi)^2 + 1/8R\xi^2] dV = \int_{r'}^{\infty} (\nabla u)^2 d^3x$

$$= \int_{r'}^{\infty} [\nabla.(u\nabla u) - u\nabla^2 u] d^3x = -\oint_{r'} u\nabla u . dS + 3\int_{r'}^{\infty} u^6 dv$$

- where we use that u minimizes the flat Sobolev integral.

Yamabe constant

- When I add the two functions together the surface integrals at r = r' cancel because we matched the functions and first derivatives at r = r'. Therefore we get

$$\int_0^\infty [(\nabla\xi)^2 + 1/8R\xi^2]dv = 3\int_{r'}^\infty u^6 dv$$

- I can explicitly perform the final integration to get

$$\int_{r'}^\infty u^6 dv \approx \pi^2/4 - 4\pi/3(r'/\alpha)^3 = \pi^2/4 - 4\pi/3[M/2r']^{3/2}$$

Yamabe constant

- (This is in the limit where we pick r’ >> M, i.e., α >> r’. In this case we have u(r < r’) $\approx$ $\alpha^{-1/2}$,a constant. Therefore we get $\int_0^{r'} u^6 dv \approx \alpha^{-3}[(4\pi/3)r'^3]$.
- exactly the correction above.)
- The denominator we simply estimate by

$$\int_0^{\infty} \xi^6\, dv > \int_{r'}^{\infty} \xi^6 dv = \int_{r'}^{\infty} u^6 dv.$$

- Thus we get

Yamabe constant

- $$Y < 3[\int_{r'}^{\infty} u^6 dv]^{2/3} = 3[\pi^2/4 - (4\pi/3)[M/2r']^{3/2}]^{2/3}.$$
- We immediately see that Y < $3\pi^2/4$ as long as $M > 0$, and the positive energy theorem gives us M = 0 if and only if the 3-space is flat.
- This is not quite accurate. We cannot exactly match u and v at any given finite radius. In particular, we expect v to have dipole and higher order corrections. However, these become less and less important as we let r' become large. We can make this precise.

Positive energy

- If we are given an asymptotically flat metric, with positive Yamabe constant, we can solve

$$8\nabla^2\phi - R\phi = 0, \phi > 0, \phi \to 1 at \infty$$

- This is the Lichnerowicz equation. We can rewrite it as $8\nabla^2\theta - R\theta = R, \theta \to 0 at \infty$
- This is where we write $\phi = 1 + \theta$.

Brill Waves

- In 1959, essentially as his PhD thesis, Dieter Brill published the first positive energy proof. It was for axi-symmetric, moment-of-time-symmetry data. He started off with a base metric $dS^2 = e^{2Aq}(d\rho^2 + dz^2) + \rho^2 d\theta^2$, where $q = q(\rho, z)$ is a function which decays faster than $1/r$ at infinity, $q = q_{,\rho} = 0$ along the z axis, and A is a (positive) constant. We look for a positive conformal factor ϕ.

Brill waves

- This goes to 1 at infinity. It must satisfy

$$8\nabla^2\phi - R\phi = 0$$

- This requires that A be `small'. This is what guarantees that the Yamabe constant of the manifold is positive. We can work out that
- $R = -2Ae^{-2Aq} (q_{,\rho\rho} + q_{,zz})$.
- We also have that
- $g = \rho^2 e^{4Aq}$.

Brill waves

$$\int R\,dv = \int \sqrt{g}\,R\,d^3x = -4A\pi \int \rho(q,_{\rho\rho} + q,_{zz})\,d\rho\,dz$$

$$= -4\pi A \int [(\rho q,_{\rho} - q),_{\rho} + (\rho q,_{z}),_{z}]\,d\rho\,dz$$

$$= 0$$

$$2\pi M = -\oint_{\infty} \nabla^a \phi\, ds_a = -\oint_{\infty} (\nabla^a \phi / \phi)\, ds_a$$

$$= \int [-\nabla^2 \phi / \phi + (\nabla\phi)^2 / \phi^2]\,dv = \int [-R/8 + (\nabla\phi)^2 / \phi^2]\,dv = \int [(\nabla\phi)^2 / \phi^2]\,dv > 0$$

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- What choices of q (or Aq) allow us to have an everywhere positive ϕ? It has to be `small'. More precisely, it must be small enough so that the Yamabe constant of the metric must be positive.
- In particular, we can prove the following:

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- Cantor and Brill:
- There exists a positive ϕ going to 1 at infinity such that $g'_{ab} = \phi^4 g_{ab}$ has $R' = 0$ if and only if

$$8\nabla^2\phi - R\phi = 0$$

$$\int [8(\nabla f)^2 + Rf^2]dv > 0$$

- For all f of compact support (with f not identically zero).
- It turns out that we can find a condition for this using the Sobolev constant.

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- We can show this using the Sobolev constant

$$\int (\nabla f)^2 dv > S(g)[\int f^6 dv]^{1/3}$$

- and the Holder inequality

$$\int R(g) f^2 dv < [\int |R(g)|^{3/2} dv]^{2/3} [\int f^6 dv]^{1/3}$$

- If $[\int |R(g)|^{3/2} dv]^{2/3} < 8S(g)$ we can show that the integral $\int [8(\nabla f)^2 + Rf^2] dv$ is positive for every f and so regular solutions to $8\nabla^2 \phi - R\phi = 0$ exist.

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- The next thing that is interesting about Brill waves is that if A is large (either positive or negative) for any fixed q, a regular solution to
- $$8\nabla^2\phi - R\phi = 0$$
- cannot exist.
- Let us define a positive function f_+ which has support only where R is positive. We then have

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$$\int (\nabla f_+)^2 dv = 2\pi \int [(f_{+},_{\rho})^2 + (f_{+},_{z})^2]\rho d\rho dz$$

$$\int R(g)(f_+)^2 dv = 4\pi A \int -(q,_{\rho\rho} + q,_{zz}) f_+^2 \rho d\rho dz$$

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- Let me define $|A_-| = \dfrac{\int [(f_+,_\rho)^2 + (f_+,_z)^2]\rho\, d\rho\, dz}{2\int -(q,_{\rho\rho} + q,_{zz}) f_+^2 \rho\, d\rho\, dz}$
- If A < -|A$_-$|, we cannot get a regular solution to the Hamiltonian constraint. In a similar way, we can define an A$_+$ and again we get no regular solution.
- This is not just a mathematical game. We re trying to map out `conformal superspace',

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- More precisely, we are trying to map out `axially symmetric conformal superspace'. We have q as defining a `direction' in conformal superspace, with A as the magnitude. If A is either large and positive or large and negative we emerge from the allowed region of conformal superspace, and we no longer can solve the conformal constraints.