Asymptotically flat manifolds

Lecture 2

Asymptotically flat manifolds

- Asymptotically flat manifolds are NOT the same as compact, without boundary manifolds.
- The fundamental difference is that the AF manifolds have `quantities at infinity'.
- In particular, if the solution to the Einstein equations is asymptotically flat in spacelike directions, then there are two objects at infinity that are (usually) finite.

Asymptotically flat manifolds

- If we choose an asymptotically 3-cartesian coordinate system, then we have

$$E = 1/16\pi \oint_{\infty} (g_{ij,j} - g_{jj,i}) ds^i$$

$$P^i = -1/8\pi \oint_{\infty} \pi^{ij} ds_i$$

and

$$M^i = -1/8\pi \oint_{\infty} \pi^{ij} \xi_j ds_i$$

- Where ξ^i = (-y, +x, 0), or another of the rotational Killing vectors. We could write the linear momentum with one of the translational Killing vectors.

Asymptotically flat spacetimes

- Something `miraculous' happens to these expressions. It looks as if the metric should be $g_{ij} = \delta_{ij} + O(1/r)$ and $\pi^{ij} = O(1/r^2)$ to get finite linear momentum and $\pi^{ij} = O(1/r^3)$ for finite angular momentum. We can get by with significantly slower falloff. In particular, we only need $g_{ij} = \delta_{ij} + O(1/r^{1/2})$, and $\pi^{ij} = O(1/r^{3/2})$ to get finite energy and linear momentum. The situation with finite angular momentum is slightly more complicated.

Asymptotically flat spacetimes

- The key point is that one should not think of these expressions as surface integrals, but rather as volume integrals, and use the constraints to get rid of the terms which appear to be divergent.
- The calculation for the momentum/angular momentum is slightly easier than for the energy.

Asymptotically flat

- Let us put in a translational Killing vector at infinity. The linear momentum now becomes

$$P^{(i)} = -1/8\pi \oint_{\infty} \pi^{ij} \xi_j ds_i$$

- We need not assume that we make any particular coordinate choice at infinity and we do not make any assumptions about how we continue the asymptotic Killing vector into the interior. Let us turn the surface integral into a volume integral.

Asymptotically flat spacetimes

- We get $P^{(i)} = -1/8\pi \oint_{\infty} \pi^{ij}\xi_j ds_i = -1/8\pi \int_V (\nabla_i \pi^{ij}\xi_j + \pi^{ij}\nabla_i \xi_j) dv$
- The vacuum momentum constraint gets rid of the first volume term (if we have sources, we get $\int_V \sqrt{g} J^a \xi_a dv$, but we will ignore this).
- Therefore we have $P^{(i)} = -1/8\pi \int_V \pi^{ab} \nabla_a \overset{(i)}{\xi_b} dv$
- For this to be finite we only need that π^{ab} fall off faster than $r^{-3/2}$ and that $\nabla_a \overset{(i)}{\xi_b}$ also fall off faster than $r^{-3/2}$.

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- If we had a true translational Killing vector such that $\nabla_a \xi_b^{(i)} - \nabla_b \xi_a^{(i)} = 0$ then the volume integral would vanish. In general, however, we get that $\nabla_a \xi_b^{(i)} = \xi_{a,b} - \Gamma^c_{ab} \xi_c$. Near infinity, if we pick $\xi_a = (1, 0, 0)$, then $\xi_{a,b} = 0$, and $\Gamma^c{}_{ab} = 0(r^{-3/2})$ if the metric falls off like $r^{-1/2}$. We also have that $\pi^{ab} = 0(r^{-3/2})$. Therefore the volume integral is finite. Again, it looks as if the metric should fall off like 1/r so as to give finite energy

Asymptotically flat

- However, we get that the metric need only fall off like $O(1/r^{1/2})$ to give finite energy. The constraints play an interesting role here.
- If we assume that $g_{ij} = \delta_{ij} + O(1/r^{1/2})$ and π^{ij} (or K^{ij}) falls off like $O(1/r^{3/2})$ then it appears that the surface integrals diverge. However, something `magical' happens. If we look at either $\oint_{\infty} (g_{ij,j} - g_{jj,i}) n^i ds$ or $\oint_{\infty} \pi^{ab} \xi_a n_b ds$ in

Asymptotically flat spacetimes

- isolation, it is clear that we need $g_{ij} = \delta_{ij}$ +
- O(1/r) and π^{ab} = O(1/r^2) to have finite surface integrals. However, the constraints play a major role in the calculation. It turns out that ($g_{ij,ij} - g_{jj,ii}$) is what we get when we write the energy as a volume integral. This is the `linear' part of the scalar curvature and if we use the Hamiltonian constraint to replace the linear part of the scalar curvature with `$g_{ij,k}g_{ij,k}$' (we actually get many terms but all of them are `first-derivative squared' terms) and `$K^{ab}K_{ab} - K^2$' terms, we see that we only need $r^{-1/2}$ falloff in g_{ij} and $r^{-3/2}$ falloff in K^{ab}.

Asymptotically flat spacetimes

- The angular momentum is slightly more complicated. We get that we need that the metric falls off like $1/r$, while the extrinsic curvature falls off like $1/r^2$. The $1/r$ falloff of the metric is critical in the sense that if the metric falls off faster than $1/r$, we get that the energy is zero, and if the 3-manifold is complete and a vacuum solution to the constraints, the space (and the spacetime) must be flat.

Asymptotically flat manifolds

- The momentum and angular momentum are much more flexible. If the extrinsic curvature falls off faster than $1/r^2$ we will have zero linear momentum, while if the extrinsic curvature falls off faster than $1/r^3$ the angular momentum is zero. If the linear momentum is zero we immediately get that the energy is the mass. In general we get that

$Y = \lambda[\int u^6 dv]^{2/3}$

- $$M = (E^2 - P^2)^{1/2},$$
- the standard formula for special relativity.

Yamabe constant

- We can define the Yamabe constant for an asymptotically flat manifold, just as we did for a compact, without boundary, one. It is now defined as

$$Y(g) = \frac{\inf}{\theta \in C_0^\infty} \frac{\int [(\nabla\theta)^2 + 1/8R\theta^2]dv}{[\int \theta^6\, dv]^{1/3}}$$

- The only difference is that we now evaluate it over functions of compact support, rather than smooth functions.

Yamabe constant

- The Yamabe constant is conformally invariant. Further, manifolds which are compact, without boundary, can be conformally rescaled to AF manifolds, with a metric which goes flat like 1/r. The Yamabe constants of both manifolds are the same. The AF Yamabe constant is closely related to the Sobolev constant,

$$S(g) = \frac{\inf}{\theta \in C_0^\infty} \frac{\int [(\nabla\theta)^2] dv}{[\int \theta^6 dv]^{1/3}}$$

Sobolev constant

- A major difference is that the Sobolev constant is always positive while the Yamabe constant can be negative, if the metric is far away from flat space. We need to keep in mind the distinction between AF spaces and compact, without boundary, spaces. In the compact, without boundary case, if the scalar curvature is negative, the Yamabe constant is negative. This is NOT true in the AF case.

Yamabe constant

- The function which minimizes the Yamabe functional satisfies $-\nabla^2\mu + 1/8R\mu = \lambda\mu^5$
- where λ is a constant. The relationship between λ and Y is $Y = \lambda[\int \mu^6 dv]^{2/3}$
- If we are dealing with a compact manifold everything is straightforward. The function μ which minimizes the Yamabe functional is everywhere positive. If we have that the manifold is AF, λ is constant, but μ goes to zero at infinity.

Yamabe constant

- The Yamabe constant is a conformal invariant. Let us conformally transform a metric g on a manifold M by some positive function ϕ, i.e., $g'_{ij} = \phi^4 g_{ij}$. Given that the scalar curvature R transforms as $R' = \phi^{-4} R - 8\phi^{-5}\nabla^2\phi$
- it is easy to show that (with $\theta' = \theta/\phi$)

$$\int [(\nabla\theta')^2 + 1/8 R\theta'^2] dv' = \int [(\nabla\theta)^2 + 1/8 R\theta^2] dv$$

- and $$\int \theta'^6 \, dv' = \int \theta^6 dv$$

Yamabe constant

- Thus it immediately follows that
- $$Y(M, g') = Y(M, g)$$
- The metric $g'_{ab} = \mu^4 g_{ab}$ satisfies $R' = 8\lambda$, which is obviously constant. If we are given a manifold with $R = R_0$, a constant, then the minimizing equation is clearly satisfied by μ = constant. In turn, we get $Y = 1/8 R_0 [\int dv]^{2/3} = 1/8 R_0 V^{2/3}$
- In particular the sign of the Yamabe constant is determined by the sign of the scalar curvature one can transform to.

Yamabe constant

- One case where the Yamabe constant is easily evaluated is for a 2-sphere with constant scalar curvature. In this case we get $Y = 3(\pi^2/4)^{2/3}$. This is a special number. It is the Sobolev constant of flat space. Rick Schoen's completion of the Yamabe theorem consisted of showing that the Yamabe constant was strictly less than $3(\pi^2/4)^{2/3,}$ except when the space is conformally flat. In this case it equals $3(\pi^2/4)^{2/3}$.

Yamabe constant

- There are a number of things we can say about the sign of the Yamabe constant for compact manifolds. If we have a manifold with non-positive scalar curvature, we can use $\theta = 1$ and immediately get $Y \leq V^{-1/3}\int R\,dv < 0$. The only situation we need be careful is when R = 0 everywhere. On the other hand if R is everywhere non-negative we get

Yamabe constant

- Then we get $\int [(\nabla\theta)^2 + 1/8 R\theta^2] dv > 0$ for every test-function. This implies that Y is greater than or equal to zero. Thus, on a compact manifold, the global sign of the scalar curvature is a conformal invariant. We know that

$$R' = \phi^{-4} R - 8\phi^{-5}\nabla^2\phi$$

- Multiply by ϕ^5, and integrate, which gives

$$\int \phi^5 R' dv' = \int [\phi R - 8\nabla^2\phi] dv = \int \phi R\, dv$$

Yamabe constant

- We cannot have R’ positive and R negative (or vice versa). This argument does NOT work for AF manifolds. In the AF case, it only works one way, we can have a manifold with everywhere negative scalar curvature, but with positive Yamabe constant, but a manifold with everywhere positive scalar curvature must have positive Yamabe constant. In particular, an AF manifold with zero scalar curvature must have positive Yamabe constant.

Yamabe constant

- If we have an AF manifold with everywhere small but negative scalar curvature, it will have positive Yamabe constant and we can conformally transform it into an AF manifold with positive scalar curvature. We must have the scalar curvature significantly negative to prevent this. If we can solve $8\nabla^2\xi - R\xi = 0, \xi \to 1$
- with $\xi > 0$, we will get an AF manifold with $g'_{ab} = \xi^4 g_{ab}$ which has $R' = 0$.

Yamabe constant

- This obviously works if $R > 0$, but even if $R < 0$ (and R small) it still holds. This manifold has $Y > 0$, even though we can set it, on an asymptotically flat manifold, with $R = 0$. Therefore, $R = 0$ on an AF manifold has positive Yamabe constant, while $R = 0$ on a compact manifold has zero Yamabe constant.

Yamabe constant

- Flat space, where the Sobolev constant equals $3(\pi^2/4)^{2/3}$ and compact without boundary conformally flat space, which has $Y = 3(\pi^2/4)^{2/3}$ are intimately related. The conformal transformation $g_{ab} = u^4\,\delta_{ab}$, with $u = \frac{\alpha^{1/2}}{(\alpha^2 + r^2)^{1/2}}$
- Where α is any constant, transforms flat space to a compact, without boundary sphere. The function u satisfies $\nabla^2 u + 3u^5 = 0$ and is the minimizing function for the Sobolev constant.

Yamabe function

- Rick Schoen considered a compact manifold with positive scalar curvature and looked at the Green function of the operator $8\nabla^2 - R$, i.e., a solution ζ to $8\nabla^2\zeta - R\zeta = \delta(x - x_0)$. It is easy to show that $\zeta > 0$. Let us assume the opposite. Let us assume that $\zeta < 0$ on some subset M' of M. We will assume that M' does not contain x_0.

Yamabe constant

- We assume $\zeta = 0$ on δM’. We multiply the equation above by ζ and integrate over M’. We get $\int_{M'} (8\zeta\nabla^2\zeta - R\zeta^2)dv = 0$. We know that we get

$$\oint_{\delta M'} \zeta \nabla \zeta . dS = 0$$

- Because ζ vanishes on δ M’ and $\nabla\zeta$ is regular.

Yamabe constant

- Therefore we get that $\int_{M'} [8(\nabla\zeta)^2 + R\zeta^2] dv = 0$. This makes no sense. Therefore we can assume that $\zeta > 0$. This means that we can use ζ as a conformal factor and (M, g') = (M, ζ^4g) can be regarded as an asymptotically flat manifold. Further, since $8\nabla^2\zeta - R\zeta = 0$ everywhere except at x_0, the `point at infinity', (M, g') is an asymptotically flat manifold with R = 0.

Yamabe constant

- This, of course, shows that an asymptotically flat manifold with $R = 0$ has strictly positive Yamabe constant.