Rigidity results for stationary spacetimes

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Introduction

Equilibrium configurations play a fundamental role to understand any physical theory.

- This is particularly important in Gravity because of the geometric nature of the field.
- The aim of this course is to present some of the main results in this area of research.
- The topic is vast and cannot be covered in a few hours.
  - I will restrict myself to the case of pure gravitation, i.e. vacuum spacetimes.
  - Even with this restriction, the presentation will by no means be exhaustive $\Longrightarrow$ Many interesting results will be left out.

The hope is to give an idea of what the subject is about, what are the main techniques, the difficulties and main results.

- In many physical theories there is a canonical notion of time.
- Equilibrium there simply means: independent of time.

In General Relativity, there is no canonical notion of time.

- The definition of "equilibrium state" needs to be formulated in geometric terms.
- Require the metric to remain invariant under a suitable one-parameter isometry group.

Recall that an isometry of a (pseudo-)Riemannian manifold $(\mathcal{M}, g^{\mathcal{M}})$ is a diffeomorphism

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M} \qquad \text{satisfying} \qquad \Phi^\star(g^{\mathcal{M}}) = g^{\mathcal{M}}.$$

- A one-parameter isometry group is a differentiable map

$$\begin{aligned} \Psi : \mathbb{R} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (s,p) &\longrightarrow \Psi(s,p) := \Psi_s(p) \qquad \text{satisfying} \end{aligned}$$

$\forall s, t \in \mathbb{R}$: (i) Ψ_s is a diffeomorphism , (ii) $\Psi_s \circ \Psi_t = \Psi_{s+t}$ (iii) Ψ_s is an isometry .

(i) and (ii) define a one-parameter group of transformations: Generator $\xi(p) = \left.\frac{d\Psi_s(p)}{ds}\right|_{s=0}$

If $\{\Psi_s\}$ is a one parameter isometry group then

$$\Psi_s^\star(g^{\mathcal{M}}) = g^{\mathcal{M}} \quad \Longrightarrow \quad \frac{d\Psi_s^\star(g^{\mathcal{M}})}{ds} = 0 \quad \Longleftrightarrow \quad \mathcal{L}_\xi g^{\mathcal{M}} = 0.$$

Definition (Killing vector)

A Killing field is a vector field ξ satisfying $\mathcal{L}_\xi g^{\mathcal{M}} = 0$.

- Given a complete vector field ξ the one-parameter group of transformations can be reconstructed.

Recall: A vector field $\xi \in \mathfrak{X}(\mathcal{M})$ is complete iff all their integral curves

$$\gamma : I = (a,b) \subset \mathbb{R} \longrightarrow \mathcal{M}, \qquad \dot{\gamma}(s) = \xi(\gamma(s)) \quad \text{have maximal domain } I = \mathbb{R}$$

A non-complete ξ generates a local one-parameter group of transformations.

Theorem (Local one-parameter group of transformations)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a (pseudo-)Riemannian manifold and ξ any vector field. For any point $p \in \mathcal{M}$, there exists an open set U_p and a map

$$\Psi : (a,b) \times U_p \longrightarrow \mathcal{M}$$

such that (i) $\Psi_s := \Psi(s,\cdot)$ is a diffeomorphism onto the image, (ii) $\left.\frac{d\Psi_s(q)}{ds}\right|_{s=0} = \xi(q)$ and (iii) $\Psi_s \circ \Psi_t = \Psi_{s+t}$ for all points and values of s,t where this expression is defined. If, moreover, ξ is a Killing vector then $\Psi_s^\star(g^{\mathcal{M}}) = 0$. (Local isometry group).

To talk about "equilibrium states" one still needs some relation of ξ with "time".

Definition (Spacetime)

A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is a smooth n-dimensional connected manifold ($n \geq 4$) with a smooth metric $g^{\mathcal{M}}$ of Lorentzian signature $\{-,+,\cdots,+\}$ and a time orientation.

- Time orientation: Existence of a timelike vector field u, declared to be future.
- Write $g^{\mathcal{M}}(u,v)$ also by $\langle u, v \rangle$.
  - A causal vector v is future if $\langle v, u \rangle \leq 0$. v is past if $-v$ is future.

The weakest possible notion of "equilibrium state" in General Relativity is

Definition (Time-independent)

A spacetime is $(\mathcal{M}, g^{\mathcal{M}})$ time-independent if it admits a Killing vector ξ which is timelike somewhere.

Aim: classify time-independent spacetimes satisfying suitable field equations: e.g.. vacuum, electrovacuum, Yang-Mills, scalar field, etc.

- For many results, stronger notions of "equilibrium state" will be necessary.
- Theory mostly developed in dimension $n = 4$. Less clear that any reasonable classification exists in higher dimensions.

Setup and notation

Given a spacetime $(\mathcal{M}, g^{\mathcal{M}})$ denote by ∇ the Levi-Civita covariant derivative.

- Curvature operator of ∇: $\text{Riem}^{\mathcal{M}}(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z$.
- Riemann tensor: $\text{Riem}^{\mathcal{M}}(X,Y,Z,W) = \langle X, \text{Riem}^{\mathcal{M}}(Z,W)Y \rangle$

Ricci tensor, curvature scalar and Einstein tensor denoted: $\text{Ric}^{\mathcal{M}}$, $\text{Scal}^{\mathcal{M}}$, $\text{Ein}^{\mathcal{M}}$.

- Einstein field equations:

$$\text{Ein}^{\mathcal{M}} + \Lambda g^{\mathcal{M}} = \chi T.$$

Λ : cosmological constant, $\chi = \frac{8\pi G}{c^4} = 8\pi$, T : energy-momentum tensor,

- Vacuum: $T = 0$,
- Scalar field: $T_{\text{sc}} = d\Phi \otimes d\Phi - \frac{1}{2}|d\Phi|^2_{g^{\mathcal{M}}} g^{\mathcal{M}}$.
- Perfect fluid: $T_{\text{pf}} = (\rho + p)\, u \otimes u + p\, g^{\mathcal{M}}$, $\langle u, u \rangle = -1$

Spacetime indices: $\alpha, \beta, \gamma, \cdots = 0, , \cdots n-1$.

- Electromagnetic field: $(T_{\text{EM}})_{\alpha\beta} = F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mathcal{M}}_{\alpha\beta}$, $\quad F_{\alpha\beta}$ two-form.

$+$ Equations for matter fields. E.g. in electrovacuum: $dF = 0$, $\qquad \text{div}_{g^{\mathcal{M}}} F = 0$.

We will be mostly concerned with vacuum spacetimes $\text{Ric}^{\mathcal{M}} = 0$.

- $\mathcal{G} = \{\xi \in \mathfrak{X}(\mathcal{M}), \xi$ Killing vector field$\}$ endowed with $[\,,\,]$ is a Lie algebra.

A Killing vector satisfies the Killing equations:

$$\mathcal{L}_\xi g^{\mathcal{M}} = 0 \qquad \Longleftrightarrow \qquad \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0.$$

Imply for any $\xi \in \mathcal{G}$:

(i) $F\xi \in \mathcal{G}$ and $F \in \mathcal{F}(\mathcal{M}) \iff F = \text{const}.$

(ii) $\nabla_\beta \nabla_\mu \xi_\nu = \text{Riem}^{\mathcal{M}}{}_{\alpha\beta\mu\nu}\xi^\alpha.$

Consequences of (ii):

- Fix $p \in \mathcal{M}$ and define $\boldsymbol{\xi} = g^{\mathcal{M}}(\xi, \cdot)$. The linear map

$$\begin{aligned} \varphi : \mathcal{G} &\longrightarrow T_q\mathcal{M} \times \Lambda^2(T_p\mathcal{M}) \\ \xi &\longrightarrow (v = \xi|_p, \boldsymbol{\omega} = d\boldsymbol{\xi}|_p) \end{aligned}$$

  is an isomorphism onto its image $\mathcal{H}_p := \varphi(\mathcal{G})$.

- dim $\mathcal{G} \leq \frac{n(n+1)}{2}$ and equality if and only if $(\mathcal{M}, g^{\mathcal{M}})$ is of constant curvature k, i.e.

$$\text{Riem}^{\mathcal{M}}(X,Y)Z = k\left(\langle Y, Z\rangle X - \langle X, Z\rangle Y\right), \qquad k \in \mathbb{R}.$$

Exercises:

(i) Prove that φ is an isomorphism (onto the image)

(ii) Prove that a spacetime is maximally symetric ($\dim(\mathcal{G}) = \frac{n(n+1)}{2}$) if and only if has constant curvature.

Covariantly constant vectors

- A particular case of Killing vector is a covariantly constant vector field.

Definition (Covariantly constant vector field)

A covariantly constant vector field is a vector field satisfying $\nabla_\alpha \xi_\beta = 0$.

Theorem

The dimension of the vector space $\mathcal{C} = \{\xi \in \mathfrak{X}(\mathcal{M}) \quad \xi \text{ covariantly constant } \}$ in any (pseudo)-Riemannian space of dimension n is $\dim(\mathcal{C}) \leq n$.
Moreover $\dim(\mathcal{C}) = n$ if and only if $(\mathcal{M}, g^{\mathcal{M}})$ is locally flat ($Riem^{\mathcal{M}} = 0$).

Exercises:
(i) Prove the theorem.
(ii) Prove: If $\dim(\mathcal{C}) = n - 1$, then $\dim(\mathcal{C}) = n$.

Definition (Integrable Killing vector)

A vector field $\xi \in \mathfrak{X}(\mathcal{M})$ is integrable iff $\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0$ ($\boldsymbol{\xi} := g^{\mathcal{M}}(\xi, \cdot)$).

By Fröbenius: the set of points $\mathcal{M} \setminus \{\xi \neq 0\}$ is foliated by maximal, injectively immersed, codimension-one submanifolds $\{\Sigma_{\mathfrak{a}}\}$ satisfying $\boldsymbol{\xi}|_{T_p\Sigma_{\mathfrak{a}}} = 0$ for all $\Sigma_{\mathfrak{a}}$ and $p \in \Sigma_{\mathfrak{a}}$.

- Integrable (also called hypersurface orthogonal) Killing vectors play a very important role in the classification of stationary spacetimes.

Definition (Time-invariant spacetime)

$(\mathcal{M}, g^{\mathcal{M}})$ is time-invariant if it is time-independent and the corresponding Killing vector ξ is integrable.

Name is motivated by the following result (exercise):

- Let $(\mathcal{M}, g^{\mathcal{M}})$ be time-invariant. For any $p \in \mathcal{M}$ with $\xi|_p$ timelike $\exists$ a neighbourhood $U_p = (-a, a) \times \Sigma$ of p with coordinates (t, x^i) $(i, j = 1, \cdots, n)$ such that

$$g^{\mathcal{M}} = -N^2(x)dt^2 + h_{ij}(x)dx^i dx^j, \qquad \xi = \partial_t, \qquad t(p) = 0.$$

Note: The transformation $(t, x) \longrightarrow (-t, x)$ is an isometry of U_p leaving p invariant.

Examples of time-independent spacetimes

Most fundamental example: Minkowski spacetime.

$$(\mathcal{M} = \mathbb{R}^n, \eta), \qquad \eta = -dt^2 + dx_1^2 + \cdots + dx_{n-1}^2$$

- Has maximal number of Killings vectors and of covariantly constant fields.
- $\zeta = \partial_t$: globally timelike, integrable, Killing vector
- Boost $\xi := x_1 \partial_t + t \partial_{x_1}$ is also a Killing vector.

  Timelike in $|x_1| > |t|$, spacelike in $|x_1| < |t|$, null in $|x_1| = |t|$.

- In the region $x_1 > |t|$, the coordinate change

$$t = X \sinh T, \quad x_1 = X \cosh T, \qquad (X, T) \in \mathbb{R}^+ \times \mathbb{R}$$

transforms the metric into static flat Kasner: $\mathcal{M} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^2$,

$$g^{\mathcal{M}} = -X^2 dT^2 + dX^2 + dx_2^2 + \cdots dx_n^2, \qquad \xi = \partial_T$$

- This spacetime is extendible.

Static flat Kasner belongs to a larger family of, generarically inextendible, vacuum, time-independent spacetimes: Static Kasner spacetimes:

$$\mathcal{M} = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^{n-2}, \qquad g^{\mathcal{M}} = -X^{2\alpha_1} dT^2 + dX^2 + X^{2\alpha_2} dx_2^2 + \cdots + X^{2\alpha_n} dx_{n-1}^2$$

$$\sum_{i=1}^{n-1} \alpha_i = 1, \qquad \sum_{i=1}^{n-1} \alpha_i^2 = 1$$

One-parameter groups of isometries is a particular case of groups of isometries.

Definition (Group of transformations and isometries)

A Lie group G acts as a group of transformations on $\mathcal{M}$ iff there is smooth map $\Phi : G \times \mathcal{M} \longrightarrow \mathcal{M}$ satisfying, for all $h, h_1, h_2 \in G$:

$$\Phi_h(\cdot) := \Phi(h, \cdot) : \mathcal{M} \longrightarrow \mathcal{M} \text{ is a diffeomorhism,} \qquad \Phi_{h_1 \cdot h_2} = \Phi_{h_1} \circ \Phi_{h_2}$$

The group is of isometries iff, in addition, Φ_h is isometry for all $h \in G$.

- For $p \in \mathcal{M}$, the orbit $\mathcal{O}_p$ is the set $\{\Phi_h(p) : h \in G\} \subset \mathcal{M}$.

Definition (Spherical symmetry)

$(\mathcal{M}^n, g^{\mathcal{M}})$ spherically symmetric if $SO(n-1)$ acts a group of isometries with spacelike, codimension-two orbits (or points).

- Area radius function $r : \mathcal{M} \longrightarrow \mathbb{R}^+$:

$$r = \left(\frac{|\mathcal{O}_p|}{\omega_{n-2}} \right)^{\frac{1}{n-2}}, \qquad \omega_n \text{ area of the unit } \mathbb{S}^{n-2} \text{ sphere.}$$

- r is smooth even at points where $r = 0$ (i.e. $\mathcal{O}_p$ is a point).

- Fundamental result in General Relativity: spherically symmetric vacuum spacetimes are classified by a real parameter.

Theorem (Schwarzschild, Birkhoff, Tangherlini)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be spherically symmetric and vacuum. Then $\mathcal{X} := \{p \in \mathcal{M}; |\nabla r|^2 = 0\}$ has empty interior and $\exists m \in \mathbb{R}$ and local coordinates on $\mathcal{M} \setminus \mathcal{X}$ such that

$$g = -\left(1 - \frac{2m}{r^{n-3}}\right)^2 dt^2 + \frac{dr^2}{1 - \frac{2m}{r^{n-3}}} + r^2 g_{\mathbb{S}^{n-2}},$$

- Analog Newtonian result: the only spherical gravitational potential is $\Phi = -\frac{Gm}{r}$.

Besides spherically symmetric, the metric is time independent.

- The spacetime is time-invariant: $\xi = \partial_t$ is hypersurface orthogonal.

Theorem (Kruskal)

Given $m \in \mathbb{R}$ there is a unique maximal spherically symmetric smooth vacuum spacetime of mass m.

- Maximal: cannot be extended to a larger spacetime with the same properties.

This spacetime is called Kruskal spacetime, denoted by $(M_{\mathrm{Kr}}, g_{\mathrm{Kr}})$.

Properties of the Kruskal spacetime

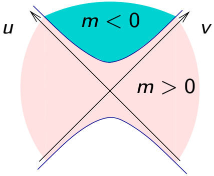
Same qualitative properties in all dimensions $n \geq 4$.

- Explicit form depends on n. Restrict to $n = 4$ for definiteness:

- For $m \neq 0$:
  $M_{\mathrm{Kr}} = (U := \text{c.c.}\{(u,v) \in \mathbb{R}^2; \operatorname{sign}(m) uv < 1\}) \times \mathbb{S}^2$

$$g_{\mathrm{Kr}} = -\frac{32 m^3}{r} e^{-\frac{r}{2m}} du\, dv + r^2 g_{\mathbb{S}^2}, \qquad u, v \in$$

$$r : U \longrightarrow \mathbb{R}^+, \qquad uv = e^{\frac{r}{2m}} \left(1 - \frac{r}{2m}\right).$$



- Singularity at $uv = 1$ (a curvature singularity).
- Approaches Minkowski for large r. The spacetime is asymptotically flat.
- The time-independent Killing vector is

$$\xi = \frac{1}{4m} \left(-u \partial_u + v \partial_v \right).$$

Global properties depend strongly of the sign of m.

- $m = 0$: Maximal extension is Minkowski.
- $m < 0$:
  - (i) Singularity at $r = 0$ visible to any observer: naked singularity.
  - (ii) Schwarzschild coordinates cover the whole manifold, i.e, $\mathcal{X} = \emptyset$.

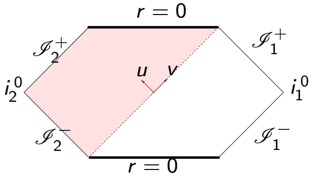
(i) The set $\mathcal{X} = \{|\nabla r|^2 = 0\} \neq \emptyset$ and in fact $\mathcal{X} = \{r = 2m\}$.

(ii) The Killing vector ξ satisfies $\langle \xi, \xi \rangle = \frac{2m}{r}\left(1 - \frac{r}{2m}\right) = \frac{2m}{r} e^{-\frac{r}{2m}} uv$.

- Timelike on $\{uv < 0\}$, spacelike on $\{uv > 0\}$ and null on $\{uv = 0\}$.
- Vanishes on $\{u = v = 0\}$: codimension-two spacelike surface: Bifurcation surface
- $\{u = 0, v > 0\}$ is a null hypersurface where $\xi \neq 0$, null and tangent: Killing horizon
  - Similarly for $\{u = 0, v < 0\}$, $\{u > 0, v = 0\}$, $\{u < 0, v = 0\}$:

Penrose diagram for $m > 0$:

- There are two asymptotic regions.
- Causal curves starting at $u > 0$ cannot reach $\mathscr{I}_1^+$
- Exist points causally disconnected from infinity.



Black hole: Set of events that cannot be joined to $\mathscr{I}^+$ by causal curves.

Event horizon $\mathcal{H}^+$: Boundary of the region causally disconnected from infinity.

Note: Killing vector ξ tangent to the event horizon (invariant under the isometry group).

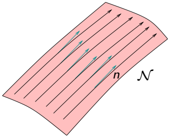
Null hypersurfaces

Definition (Null hypersurface)

A null hypersurface of $(\mathcal{M}^m, g^{\mathcal{M}})$ is an injectively immersed hypersurface $\Psi : \mathcal{N} \longrightarrow M$ such that $h = \Psi^\star(g^{\mathcal{M}})$ is degenerate everywhere.

- For all $p \in \mathcal{N}$, exists $n \neq 0 \in T_p\mathcal{N}$ such that $h(n, \cdot) = 0$ (degeneration vector).

General properties (identify $\mathcal{N}$ and $\Phi(\mathcal{N})$ for local facts):



(i) $h(X, X) \geq 0$ for all $X \in T_p\mathcal{N}$.

(ii) The degeneration direction at each point is unique.

(iii) $\Psi_\star(n)$ is a non-zero normal to the hypersurface.

(iv) For any $p \in \mathcal{N}$, the spacetime geodesic at p with tangent vector $n|_p$ lies in $\mathcal{N}$.

Definition (section of a null hypersurface)

A section of a null hypersurface $\mathcal{N}$ is an injectively immersed, codimension-two, spacelike surface $\Phi : \Sigma \longrightarrow \mathcal{M}$ such that

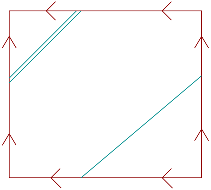
(a) $\Phi(\Sigma) \subset \mathcal{N}$

(b) $\Phi(\Sigma)$ intersects each null generator of $\mathcal{N}$ exactly once.

Definition (Killing prehorizon)

Let $(\mathcal{M}, g^{\mathcal{M}})$ with a Killing vector ξ. A Killing prehorizon of ξ is a connected, null, injectively immersed hypersurface $\mathcal{H}_\xi$ such that $\forall p \in \mathcal{H}_\xi$, $\xi|_p \neq 0$, null and tangent..

- Locally, a Killing prehorizon is embedded. Globally it may fail to be.
- The surface gravity κ_ξ of a Killing prehorizon $\mathcal{H}_\xi$ is defined by $\nabla_\xi \xi \stackrel{\mathcal{H}_\xi}{=} \kappa_\xi \xi$



- A Killing prehorizon is degenerate of $\kappa_\xi = 0$ and non-degenerate if $\kappa_\xi \neq 0$.

Definition (Killing horizon)

A Killing horizon is an embedded Killing prehorizon

In the Kruskal spacetime there are four Killing horizons.

- Each one with surface gravity: $|\kappa_\xi| = \frac{1}{4m}$ (sign depends on the horizon).

- Non-degenerate Killing prehorizons are automatically Killing horizons (Exercise: prove this).
- Killing prehorizons diffeomorphic to $\Sigma \times I$, ξ tangent to $I \subset \mathbb{R}$ and Σ compact are Killing horizons.

Lemma (Raćz & Wald)

Let $\mathcal{H}_\xi$ be a Killing prehorizon for an integrable Killing vector. Then κ_ξ is constant on each arc-connected component of $\mathcal{H}_\xi$.

In the non-integrable case, a similar result holds under energy conditions.

- A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ satisfies the Dominant energy condition if

$$\mathrm{Ein}^{\mathcal{M}}(u,v) \geq 0, \qquad \text{for all } u, v \in T_p\mathcal{M} \text{ future directed and causal.}$$

Lemma (e.g. Wald)

A Killing prehorizon on a spacetime satisfying the dominant energy condition has constant surface gravity on each arc-connected component.

Both are consequence of the following identity:

$$X(\kappa_\xi) = -\mathrm{Ric}^{\mathcal{M}}(X, \xi), \qquad \forall\, X \in \mathfrak{X}(\mathcal{H}_\xi)$$

Exercise: Show that this identity implies the previous results.

The Kruskal spacetime is a particular case of a fundamental class of vacuum spacetimes: the Kerr family.

- Each element identified by two real numbers m and a. Global properties depend on their values.
- When $a \neq 0$, no global chart exists. Useful to restrict to suitable open subsets.

Restrict to spacetime dimension four:

- Exterior Kerr spacetime: Boyer-Lindquist coordinates.

$$\text{Let} \begin{cases} r_+ := \max\{0, m\} & \text{if } |a| \geq m \\ r_+ := m + \sqrt{m^2 - a^2} & \text{if } |a| \leq m \end{cases}, \quad \mathcal{M}_{BL} = \underbrace{\mathbb{R}}_{t} \times \underbrace{(r_+, \infty)}_{r} \times \underbrace{\mathbb{S}^2}_{\{\theta,\phi\}}$$

$$g^{\mathcal{M}} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2 + a^2)d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\rho^2 := r^2 + a^2\cos^2\theta, \quad \Delta := a^2 - 2mr + r^2,$$

- $a = 0$ corresponds to the Schwarzschild metric.
- $m = 0$ corresponds to the Minkowski metric.

Admits generalization to higher dimensions: Myers-Perry spacetime.

The exterior Kerr spacetime is not maximal.

- The maximal extension is somewhat complicated.
- Suffices to consider a partial extension:

Definition (Advanced extension of the Kerr spacetime)

For $m, a \in \mathbb{R}$ let $\mathcal{M}_a = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \{x^2 + y^2 \leq a^2, z = 0\}\right)$, with (x, y, z) Cartesian coordinates in $\mathbb{R}^3$. The advanced Kerr spacetime of mass m and specific angular momentum a is the spacetime $(\mathcal{M}_a, g^{\mathcal{M}}_{m,a})$ with

$$g^{\mathcal{M}}_{m,a} = \underbrace{-dt^2 + dx^2 + dy^2 + dz^2}_{\eta} + \frac{2mr^3}{r^4 + a^2 z^2} \boldsymbol{\ell} \otimes \boldsymbol{\ell},$$

where $r : \mathcal{M}_a \longrightarrow \mathbb{R}^+$ is defined by $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$ and

$$\boldsymbol{\ell} = dt + \frac{r}{r^2 + a^2}(xdx + ydy) + \frac{a}{r^2 + a^2}(ydx - xdy) + \frac{zdz}{r}.$$

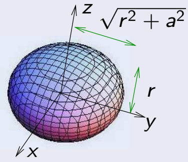


Figure : $\mathbb{S}^2$ factor.

- The ring $z = 0$, $x^2 + y^2 = a^2$ is a curvature singularity.
- The metric extends smoothly to $(z = 0, x^2 + y^2 < a^2)$.
  - The spacetime can still be extended across the disk.
- $m = 0$ is explicitly the Minkowski metric.

- The spacetime is time-independent, with Killing vector $\xi = \partial_t$.
- In addition, it is axially symmetric.

Definition (Axial symmetry)

A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is axially symmetric if

(i) $SO(2)$ acts as a group of isometries $\Psi : SO(2) \times \mathcal{M} \longrightarrow \mathcal{M}$.

(ii) The set of fixed points $\Psi(\alpha, p) = p, \quad \forall \alpha \in SO(2)$ is a codimension-two timelike surface (axis of symmetry).

- The generator of $SO(2)$ in Kerr is $\eta = x\partial_y - y\partial_x$. Fixed points: $\{x = y = 0\}$.

Assume $m \geq 0$.

- Spacetime admits Killing horizons iff $|a| \leq m \neq 0$.
- There are two Killing horizons located at the hypersurfaces $\mathcal{H}_{r_+} = \{r = r_+\}$ and $\mathcal{H}_{r_-} = \{r = r_-\}$ where

$$r_\pm := m \pm \sqrt{m^2 - a^2}.$$

- Topology: $\mathbb{R} \times \mathbb{S}^2$, Killing vector generators: $\xi_\pm = \xi + \frac{a}{2mr_\pm}\eta$. Surface gravity:

$$\kappa_\pm = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}.$$

Degenerate when $|a| = m$: Extreme Kerr

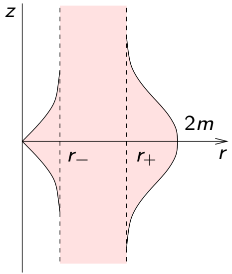
Ergoregions

- The Killing vector $\xi = \partial_t$ has norm $\langle \xi, \xi \rangle = -1 + \frac{2mr^3}{r^4 + a^2 z^2}$

Assume that $0 < |a| \leq m$

- ξ is spacelike on (r_-, r_+) and in the regions

$$|z| < \frac{2mr^3}{a^2} \left(1 - \frac{r}{2m} \right), \qquad r \in (0, r_-] \cup [r_+, 2m)$$



- The exterior Kerr spacetime corresponds to the domain $r > r_+$
- The time-independent Killing vector ξ is not timelike everywhere in this region.

The subset of $\{r > r_+\}$ where ξ is spacelike or null is called ergoregion.

- In particular: ξ spacelike everywhere on each Killing horizon, except on the intersection of the Killing horizon and the axis of symmetry where it is null.

General properties of degenerate Killing horizons

- Degenerate Killing prehorizons satisfy strong restrictions.

Assume that $\mathcal{H}_\xi$ is a degenerate Killing horizon in a spacetime $(\mathcal{M}^n, g^{\mathcal{M}})$ with cross section S (i.e. $\mathcal{H}_\xi = S \times I$, with ξ tangent to $I \subset \mathbb{R}$).

Lemma (Moncrief & Isenberg ($n = 4$), Lewandowski, Pawlowski ($n \geq 3$))

Let q be the (positive definite) induced metric of S. Then

$$Ric^q(X, Y) = Ric^{g^{\mathcal{M}}}(X, Y) + 2(\nabla^q_X s_\xi)(Y) + 2 s_\xi(X) s_\xi(Y), \qquad X, Y \in \mathfrak{X}(S)$$

where $s_\xi(X) := -\frac{1}{2}\langle \ell, \nabla_X \xi \rangle$ and ℓ is the null normal to S satisfying $\langle \ell, \xi \rangle = -2$.

- If ξ is integrable, then s_ξ is closed:

$$\{\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0\} + \{\xi \text{ non-zero }\} \quad \Longrightarrow \quad d\boldsymbol{\xi} = \mathbf{V} \wedge \boldsymbol{\xi}$$

Taking exterior derivative

$$d\mathbf{V} \wedge \boldsymbol{\xi} = 0 \quad \Longrightarrow \quad d\mathbf{V} = \mathbf{A} \wedge \boldsymbol{\xi}$$

- If Φ is the embeddding of S: $s_\xi = \Phi^\star(\mathbf{V})$, so

$$ds_\xi = d\Phi^\star(\mathbf{V}) = \Phi^\star(d\mathbf{V}) = \Phi^\star(\mathbf{A} \wedge \boldsymbol{\xi}) = 0. \qquad \square$$

- Assuming that $(\mathcal{M}, g^{\mathcal{M}})$ is vacuum, the geometry of S satisfies

$$\mathrm{Ric}^q(X,Y) = 2(\nabla^q_X s_\xi)(Y) + 2 s_\xi(X) s_\xi(Y), \qquad X, Y \in \mathfrak{X}(S) \qquad (\star)$$

It is an interesting problem to classify all compact Riemanian manifolds with this property.

- In the static case, the cassification is complete when the section is compact.

Theorem (Chruściel, Reall, Tod)

Assume that s_ξ is closed and (S,q) compact and satisfying $(\star)$. Then $s_\xi = 0$ and q is Ricci flat.

Sketch of proof: For simplicity assume s_ξ is exact: $s_\xi = dF$. Set $\Psi = e^F$. Assume S connected.

$$\Psi \mathrm{Ric}^q = 2\, \mathrm{Hess}_q \Psi.$$

- For later use, let us consider the more general case:

$$\Psi\, \mathrm{Ric}^q = a\, \mathrm{Hess}_q \Psi, \quad 0 \neq a \in \mathbb{R}$$

- Take divergence, use $\Psi\, \mathrm{Scal}^q = a \Delta_q \Psi$ and Bianchi $\mathrm{div}_q(\mathrm{Ric}^q) = \frac{1}{2} d(\mathrm{Scal}^q)$

$$\nabla^q \left(a\,(a-1)\, |d\Psi|^2_q + \Psi^2 Scal^q \right) = 0 \quad \Longrightarrow \quad a\,(a-1)\,|d\Psi|^2_q + \Psi^2 \mathrm{Scal}^q = C$$

Since $\mathrm{Scal}^q = a\Psi^{-1}\Delta_q\Psi$, it follows

$$\Delta_q\left(\Psi^a\right) = C\Psi^{a-2}.$$

- (S, q) compact and Riemannian $\Longrightarrow$ $C = 0$ and $\Psi = \text{const}$.
- From the definitions: $s_\xi = 0$ and hence $\mathrm{Ric}^q = 0$. □

Exercise: Fill in the details of the proof.

Corollary (Non existence of generate vacuum horizons in $n = 4$)

There exists no degenerate integrable Killing horizons with cross sections of spherical topology in vacuum four-dimensional spacetimes.

Corollary

If (Σ^3, h) (h positive definite) is compact and admits a positive function N satisfying

$$N Ric^h = Hess_h N$$

then h is locally flat (and hence its universal covering is $(\mathbb{R}^3, g_E)$) and N is constant.

Degenerate axially symmetric horizons

- Axially symmetric, vacuum, degenerate horizons in four-dimensions with compact cross-sections are rigid.
- A degenerate Killing horizon $\mathcal{H}_\xi$ is cyclically symmetric if $(\mathcal{M}, g^{\mathcal{M}})$ admits a Killing vector η with closed orbits, tangent to $\mathcal{H}_\xi$ and spacelike away from zeroes.

Theorem (Hajicek, Lewandowski & Pawlowski)

Let $\mathcal{H}_\xi$ be a degenerate, cyclically symmetric Killing horizon with a compact cross-section S. If $(\mathcal{M}, g^{\mathcal{M}})$ is four-dimensional and vacuum on $\mathcal{H}_\xi$ then

(i) *S is topologically $\mathbb{S}^2$.*

(ii) *There exists a constant $J > 0$ and spherical-type coordinates $\{\theta, \phi\}$ on S such that induced metric h and connection one-form s_ξ read:*

$$h = J\left((1+\cos^2\theta)^2 d\theta^2 + \frac{4\sin^2\theta}{1+\cos^2\theta} d\phi^2\right), \qquad s_\xi = -\frac{\cos\theta\sin\theta}{1+\cos^2\theta} d\theta + \frac{2\sin^2\theta}{1+\cos^2\theta} d\phi$$

- This geometry corresponds to the geometry of the event horizon of the extreme Kerr spacetime with $m = a = \sqrt{J}$.

The idea of the proof is to use the isometry to reduce the problem to an ODE and solve it.

- This theorem can be extended to include a cosmological constant.

Fixed points of Killing vectors I

Besides horizons, zeros of Killing vectors are also important to understand isometries.

Definition

A fixed point of a Killing vector is a point $p \in \mathcal{M}$ where $\xi|_p = 0$.

- For a non-trivial Killing vector ξ: if p is a fixed point then $\boldsymbol{\omega} := d\boldsymbol{\xi}|_p$ is non-zero.

Theorem (Structure of fixed points)

At a fixed point p of a Killing vector ξ let $W_p := \{v \in T_p\mathcal{M}; \boldsymbol{\omega}(v, \cdot) = 0\}$. Then p lies in a smooth, totally geodesic embedded submanifold S_p such that $T_pS_p = W_p$.

Holds for $(\mathcal{M}, g^{\mathcal{M}})$ of arbitrary dimension and signature.

- Dimension of S_p agrees with dimension of W_p. Causal character of S_p agrees with causal character of W_p.
- If dim $(W_p) = 0$ then $S_p = \{p\}$: isolated fixed point.
- Codimension of W_p can only be even (and $\neq 0$)

Example 1:

- In 3 dimensions, at any fixed point, W_p is necessarily one-dimensional.
- The geodesic through p with tangent vector $z \in W_p$ is a curve of fixed points: axis of symmetry.

Fixed points of Killing vectors (II)

Example 2:

- Bifurcation surface in Kruskal $m > 0$: $\mathfrak{S} = \{u = 0, v = 0\}$. Let $p \in \mathfrak{S}$.
- Killing vector: $\xi = \frac{1}{4m}(-u\partial_u + v\partial_v)$ $\quad \boldsymbol{\omega}|_p = d\boldsymbol{\xi}|_p = \frac{4m}{e} du \wedge dv$.
- $W_p = \{X \in T_p\mathcal{M}, \text{tangent to the } \mathbb{S}^2 \text{ factor}\}$: spacelike and codimension-two.
- The vector $X = \partial_u$ or $\partial_v \in T_p\mathcal{M}$ satisfy

$$\boldsymbol{\omega}(X, \cdot) = -2\lambda g^{\mathcal{M}}(X, \cdot), \qquad \lambda = \frac{1}{4m}$$

- The null geodesics $\gamma(s)$ starting at $\mathfrak{S}$ with tangent X generate a Killing horizon $\mathcal{H}$ and $\xi(\gamma(s)) = \lambda s X(\gamma(s))$

This is generally true:

Theorem

Let p be a fixed point of a Killing vector ξ. Let $u \in T_p\mathcal{M}$ be an eigenvector of $\boldsymbol{\omega} := d\boldsymbol{\xi}|_p$ with eigenvalue $\lambda \neq 0$ (i.e. satisfies $\boldsymbol{\omega}(u) = -2\lambda g^{\mathcal{M}}(u, \cdot)$).
The affinely parametrized geodesic $\gamma_u(s)$ starting at p with tangent vector u is such that:

- *$\xi|_{\gamma_u(s)}$ is tangent to $\gamma_u(s)$ and in fact $\xi|_{\gamma_u(s)} = \lambda s\, u(s)$.*
- *If S_p is spacelike and u is null, then γ_u is a generator of a Killing horizon with surface gravity $\kappa_\xi = \lambda$.*

Further examples of time independent vacuum spacetimes

A large class of axially symmetric vacumm spacetimes admitting a timelike Killing vector is the Weyl class:

$$\mathcal{M} = \underbrace{\mathbb{R}}_{t} \times \underbrace{\mathbb{S}^1}_{\phi} \times \underbrace{\mathcal{U} \subset (\mathbb{R}^+ \times \mathbb{R})}_{\rho,z}, \quad g^{\mathcal{M}} = -e^{2U} dt^2 + e^{-2U} \left(\rho^2 d\phi^2 + e^{2k} \left(d\rho^2 + dz^2 \right) \right)$$

- Field equations:

$$\Delta_\delta U = 0, \quad \text{with } \delta = d\rho^2 + dz^2 + \rho^2 d\phi^2 \text{ on } \mathcal{U} \qquad \partial_\phi U = 0$$

$$dk = \rho \left(U_\rho^2 + U_z \right) d\rho + 2\rho U_\rho U_z dz$$

Properties

- Time-invariant: $\xi = \partial_t$ is timelike and integrable.
- Includes the Schwarzschild spacetime $M > 0$

  $U_M =$ Newtonian potential of a uniform rod of length $2M$ and total mass M

$$U_M = \frac{1}{2} \log \left(\frac{L(\rho,z) - M}{L(\rho,z) + M} \right), \qquad L(\rho,z) := \frac{1}{2} \left(\sqrt{\rho^2 + (z+M)^2} + \sqrt{\rho^2 + (z-M)^2} \right)$$

- $\rho = 0$, $|z| > M$ is a regular axis of symmetry.
- $U \to -\infty$ on the rod: $\rho = 0$ $|z| \leq M$ is a coordinate singularty.
- Corresponds to the event horizon.

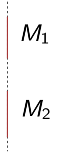
Weyl class (II)

The Weyl class admits a superposition principle:

- If U_1, U_2 are solutions, then $U_1 + U_2$ is a solution.

Are there other solutions besides Schwarzschild that admit smooth extensions?

- E.g., consider two disjoint rods of mass M_i and length $2M_i$.
- It turns out that the metric at $\rho = 0$ between the rods is not regular.



- If η is the generator of an $SO(2)$ and $\mathcal{Z}$ is the symmetry axis, then it must hold:

$$\text{Elementary flatness} \qquad \lim_{p \to \mathcal{Z}} \frac{\langle dX, dX \rangle}{4X} = 1, \qquad X := \langle \eta, \eta \rangle.$$

- Fails to be true for any superposition of Schwarschild rods [Weyl 1917].
- Deficit angle interpreted as a strut that exerts a force to keep the back holes appart.

Natural to ask about uniqueness of Schwarzschild within the Weyl class.

Theorem (Müller zum Haagen & Seifert 1973, Gibbons 1974)

Under suitable technical assumptions, the Schwarzschild solution is the only spacetime within the Weyl class that admitts a regular axis and a smooth extension across the singularities of the potential U.

Quotient formalism

Quotient space

- Let $(\mathcal{M}, g^{\mathcal{M}})$ admit a Killing vector ξ.
- Define an equivalence relation
  $p \sim q \iff \{$ exists an integral curve of ξ connecting them$\}$.
- At any $p \in \mathcal{M}$ with $\xi|_p \neq 0$, exists $\mathcal{M} \supset \mathcal{U}_p \ni p$ such that

$$Q_p := \mathcal{U}_p / \sim \text{ smooth manifold} \quad \text{and} \quad \begin{array}{cccc} \pi : & \mathcal{U}_p & \longrightarrow & Q_p \\ & q & \longrightarrow & \overline{q} \end{array} \text{ is a submersion.}$$

- $\ker(\pi^\star|_q) = \{a\,\xi|_q, a \in \mathbb{R}\}$.

Assume that $\xi|_p$ is not null. Restricting $\mathcal{U}_p$, ξ non-null on $\mathcal{U}_p$.

- Define the spaces

$$\mathcal{F}_\xi(\mathcal{U}_p) := \{f \in \mathcal{F}(\mathcal{U}_p) \text{ with } \mathcal{L}_\xi f = 0\}$$
$$\mathfrak{X}_\xi(\mathcal{U}_p) := \{X \in \mathfrak{X}(\mathcal{U}_p) \text{ with } \mathcal{L}_\xi X = 0 \text{ and } \langle \xi, X \rangle = 0\}$$

- Elements of $\mathfrak{X}_\xi$ can be multiplied by elements in $\mathcal{F}_\xi$ without leaving the space.
- $\mathfrak{X}_\xi$ is a module over $\mathcal{F}_\xi$.

Proposition

The following properties hold:

(i) *The map* $\pi_\star : \mathcal{F}_\xi(\mathcal{U}_p) \longrightarrow \mathcal{F}(Q_p)$ *defined by* $\pi_\star(f)(\overline{q}) = f(q)$ *is well-defined (i.e. independent of* $q \in \overline{q}$*) and an isomorphism.*

(ii) *The map* $\pi_\star : \mathfrak{X}_\xi(\mathcal{U}_p) \longrightarrow \mathfrak{X}(Q_p)$ *defined by*

$$\pi_\star(X)|_{\overline{q}} = \pi_\star|_q(X|_q)$$

is well-defined and also an isomorphism (in fact a module-isomorphism).

Vector fields Lie-constant along ξ and orthogonal to ξ can be transferred to the quotient.

- The spacetime metric $g^{\mathcal{M}}$ is Lie-constant along ξ:

Proposition (Quotient metric)

There exists a metric h *on* Q_p *such that*

$$g^{\mathcal{M}}(X,Y) = h(\pi_\star(X), \pi_\star(Y)) \qquad \forall X, Y \in \mathfrak{X}_\xi(\mathcal{U}_p)$$

- If ξ_p timelike, then h is positive definite.

- The idea is to write down the field equations in $(\mathcal{M}, g^{\mathcal{M}})$ in terms of objects defined in the quotient.
- Denote by D the covariant derivative w.r.t. h.
- Let q be the projector orthogonal to ξ. In index notation:

$$q^{\alpha}_{\beta} := \delta^{\alpha}_{\beta} + \frac{1}{\lambda}\xi^{\alpha}\xi_{\beta}, \qquad \lambda := -\langle \xi, \xi \rangle. \qquad \textit{Properties}: \quad q(\xi) = 0, \quad q \circ q = q$$

Needed: relate derivatives with respect to D with diferential operations in the ambient space.

Proposition

Let $T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$ be a tensor on $(\mathcal{M}, g^{\mathcal{M}})$ completeley orthogonal to ξ and Lie-constant along ξ. Then

$$(DT)^{\gamma_1 \cdots \gamma_p}_{\nu \delta_1 \cdots \delta_q} := q^{\mu}_{\nu} q^{\gamma_1}_{\alpha_1} \cdots q^{\gamma_p}_{\alpha_q} q^{\beta_1}_{\delta_1} \cdots q^{\beta_q}_{\delta_q} \nabla_{\mu} T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

has the same properties and

$$D\left(\pi_{\star}(T)\right) = \pi_{\star}\left(DT\right).$$

Exercise: Prove this.

- We can transfer equations in the ambient space to equations in the quotient.

- The scalar $\lambda = -\langle \xi, \xi \rangle$ satisfies $L_\xi \lambda = 0$. Descends to the quotient.
- Define the tensor

$$\hat{\omega}_{\alpha\beta} = q_\alpha^\nu q_\beta^\nu \nabla_\mu \xi_\nu.$$

- $\hat{\omega}$ descends to a a two-form in the quotient.

The following decomposition holds

$$\nabla_\alpha \xi_\beta = \hat{\omega}_{\alpha\beta} - \frac{1}{2\lambda} \xi_\alpha \nabla_\beta \lambda + \frac{1}{2\lambda} \xi_\beta \nabla_\alpha \lambda.$$

- Combining with $\nabla_\alpha \nabla_\beta \xi_\mu = \mathrm{Riem}^{\mathcal{M}}{}_{\rho\alpha\beta\mu} \xi^\rho$ yields:

Proposition

Let ξ be a Killing vector nowhere null on $\mathcal{U}_p$. Then the following identities hold:

$$q_\mu^\beta q_\nu^\alpha \nabla_\alpha \nabla_\beta \lambda = -2 \hat{\omega}_\mu{}^\rho \hat{\omega}_{\nu\rho} + \frac{1}{2\lambda} \nabla_\mu \lambda \nabla_\nu \lambda + 2 Riem^{\mathcal{M}}{}_{\mu\rho\nu\sigma} \xi^\rho \xi^\sigma,$$

$$q^{\alpha\beta} q_\gamma^\mu \nabla_\alpha \hat{\omega}_{\beta\mu} = -\frac{1}{2\lambda} \hat{\omega}_{\alpha\gamma} \nabla^\alpha \lambda - Ric^{\mathcal{M}}{}_{\delta\sigma} \xi^\delta q_\gamma^\sigma,$$

$$\nabla_\alpha \left(\frac{1}{\lambda} \hat{\omega}_{\beta\mu} \right) q_{[\rho}^\alpha q_\sigma^\beta q_{\delta]}^\mu = 0$$

- Define alert $\mathrm{Riem}^{\mathcal{M}}{}_{\xi} := \mathrm{Riem}^{\mathcal{M}}(\cdot, \xi, \cdot, \xi)$. Symmetric, completely orthogonal to ξ and Lie constant along ξ, hence descends to the quotient.
- Define $\mathrm{Ric}^{\mathcal{M}}{}_{\xi}{}^{\perp}{}_{\gamma} := \mathrm{Ric}^{\mathcal{M}}{}_{\beta\sigma}\xi^{\beta} q^{\sigma}_{\gamma}$, orthogonal to ξ and Lie constant. It also descends.

Theorem

Let $V := +\sqrt{|\lambda|}$. The following equations hold

$$
\begin{aligned}
D_a D_b \lambda &= -2\hat{\omega}_a{}^c \hat{\omega}_{bc} + \frac{1}{2\lambda} D_a\lambda D_b\lambda + 2\pi_\star (Riem^{\mathcal{M}}{}_{\xi})_{ab} \\
D_a \left(V \hat{\omega}^a{}_b \right) &= -V \pi_\star (Ric^{\mathcal{M}}{}_{\xi}{}^{\perp})_b \\
d\left(\frac{1}{\lambda} \hat{\boldsymbol{\omega}} \right) &= 0.
\end{aligned}
$$

- Second equation involves tangential-normal component of Ricci.
- First equation invovles tangential-normal-tangential-normal component of Riemann.

Taking trace (in h) of the hessian equation:

$$\Delta_h \lambda = -2|\hat{\omega}|^2_h + \frac{1}{2\lambda}|D\lambda|^2_h + 2\pi_\star(\mathrm{Ric}^{\mathcal{M}}(\xi,\xi))$$

- Relates tangential-tangential components of Ricci with objects in the quotient.
- What about the remaining components of Riemann (and Ricci)?

They are related to the Riemann curvature of h.

Proposition

The following identity holds

$$Riem^h_{abcd} + \frac{2}{\lambda}\hat{\omega}_{ab}\hat{\omega}_{cd} + \frac{1}{\lambda}\left(\hat{\omega}_{ac}\hat{\omega}_{bd} - \hat{\omega}_{ad}\hat{\omega}_{bc}\right) = \pi_\star(Riem^{\mathcal{M}\,\perp})_{abcd}$$

where $\qquad Riem^{\mathcal{M}\,\perp}{}_{\mu\nu\gamma\rho} := q^\alpha_\mu q^\beta_\nu q^\gamma_\rho q^\delta_\sigma Riem^{\mathcal{M}}_{\alpha\beta\gamma\delta}$

Proof: Use de Ricci identity in the quotient and relate with projected derivarives in the ambient.

Exercise: Do this.

- When $(\mathcal{M}, g^{\mathcal{M}})$ is orientable then the quotient is also orientable.
- Volume forms can be related by $\pi_\star$.

Proposition

Let $(\mathcal{M}, g^{\mathcal{M}})$ be oriented with volume form $\boldsymbol{\eta}$. The $(n-1)-$form $(\boldsymbol{\eta}_\xi)_{\alpha_2 \cdots \alpha_n} := \boldsymbol{\eta}_{\alpha_1 \cdots \alpha_n} \xi^{\alpha_1}$ descends to the quotient and satisfies

$$\pi_\star(\eta_\xi) = V \boldsymbol{\eta}^h \qquad \text{where } \eta^h \text{ is the volume form of } h.$$

All previous equations hold in any dimension, signature and sign of λ.

- When $n = 4$, $\hat{\omega}_{\alpha\beta}$ can be replaced by a one-form called twist one-form.

Definition

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be four-dimensional with volume form $\boldsymbol{\eta}_{\alpha\beta\mu\nu}$. The twist one-form of a Killing vector is

$$\omega_\alpha := \eta_{\alpha\beta\mu\nu} \xi^\beta \nabla^\mu \xi^\nu.$$

Properties:

(i) ω_α is orthogonal to ξ and Lie constant along ξ.

(ii) Denote $\pi_\star(\boldsymbol{\omega})$ by the same symbol. Then

$$\omega_a = -\boldsymbol{\eta}^{\mathbf{h}}{}_{abc}\omega^{bc}$$

Proposition

In spacetime dimension four, the following itentities in the quotient hold

$$D_a D_b \lambda = \frac{\epsilon}{2\lambda}\left(|D\boldsymbol{\omega}|^2_h h_{ab} - \omega_a\omega_b\right) + \frac{1}{2\lambda} D_a\lambda D_b\lambda + 2\pi_\star(Riem^{\mathcal{M}}{}_\xi)_{ab}$$

$$D_a\omega_b - D_b\omega_a = -2V\boldsymbol{\eta}^{\mathbf{h}}{}_{abc}\pi_\star(Ric^{\mathcal{M}}{}_\xi{}^\perp)^c$$

$$D_a\left(\frac{1}{V^3}\omega^a\right) = 0$$

$$Ric^h_{ab} - \frac{1}{2\lambda}D_aD_b\lambda + \frac{1}{4\lambda^2}D_a\lambda D_b\lambda - \frac{\epsilon}{2\lambda^2}\left(|D\boldsymbol{\omega}|^2_h h_{ab} - \omega_a\omega_b\right) = \pi_\star(Ric^{\mathcal{M}\perp})_{ab}$$

where $Ric^{\mathcal{M}}{}^\perp_{\alpha\beta} = q^\mu_\alpha q^\nu_\beta Ric^{\mathcal{M}}{}_{\mu\nu}$ *and* $\epsilon = sign(\det(g^{\mathcal{M}}))$

- When $\mathrm{Ric}^{\mathcal{M}}{}^\perp_\xi = 0$, then the twist one-form is closed, hence locally exact. The local potential is called twist potential.

Conformal metric in the quotient

- In terms of the metric h, the equation for Ric^h involves the Hessian of λ.
- It is useful to use a conformally rescaled metric $\gamma := V^2 h$.
- Using the transformation law for ∇ and Riem under conformal rescalling:

Proposition

In spacetime dimesion $n = 4$, the following equations in the quotient hold

$$\overline{D}_a \overline{D}_b \lambda = \frac{1}{2\lambda} \left[\left(|\overline{D}\lambda|^2_\gamma + \epsilon |\overline{D}\boldsymbol{\omega}|^2_\gamma \right) \gamma_{ab} - \left(\overline{D}_a \lambda \overline{D}_b \lambda + \epsilon\, \omega_a \omega_b \right) \right] + 2\pi_\star (Riem^{\mathcal{M}}{}_\xi)_{ab}$$

$$\overline{D}_a \omega_b - \overline{D}_b \omega_a = -2 (\boldsymbol{\eta}^\gamma)_{abc} \pi_\star (Ric^{\mathcal{M}}{}_\xi{}^\perp)^c$$

$$\overline{D}_a \left(\frac{1}{\lambda^2} \omega^a \right) = 0$$

$$Ric^\gamma_{ab} - \frac{1}{2\lambda^2} \left(\overline{D}_a \lambda \overline{D}_b \lambda - \epsilon\, \omega_a \omega_b \right) = \pi_\star (Ric^{\mathcal{M}\perp})_{ab} - \frac{sign(\lambda)}{\lambda^2} \gamma_{ab}\, \pi_\star (Ric^{\mathcal{M}}(\xi, \xi))$$

where $\epsilon = sign(\det(g^{\mathcal{M}}))$ and $\overline{D}$ is the covariant derivative of $\gamma := V^2 h$.

- The trace of the Hessian equation involves only the spacetime Ricci

$$\Delta_\gamma \lambda = \frac{1}{\lambda} \left(|\overline{D}\lambda|^2_\gamma + \epsilon |\overline{D}\omega|^2_\gamma \right) + \frac{2}{V^2} \pi_\star (\mathrm{Ric}^{\mathcal{M}}(\xi, \xi)), \qquad V^2 = |\lambda|.$$

Harmonic map formulation

- The Einstein vacuum field equations in the quotient have an interesting harmonic map (wave map) structure.

Definition (harmonic map)

A harmonic (wave) map is a smoth map $\Psi : (N, \gamma) \longrightarrow (M, \overline{g})$ satisfying the Euler-Lagrange equations of the energy functional

$$E(\Psi) = \int_M \frac{1}{2} \gamma^{ab}(x) \overline{g}_{ij}(\Psi(x)) \frac{\partial \Psi^i(x)}{\partial x^a} \frac{\partial \Psi^j(x)}{\partial x^b} \boldsymbol{\eta}^{\gamma}$$

- The field equations are: $\Delta_\gamma \Psi^i + \Gamma^{\overline{g}\,i}{}_{jk}|_{\Psi(x)} \frac{\partial \Psi^j}{\partial x^a} \frac{\partial \Psi^k}{\partial x^b} \gamma^{ab}(x) = 0.$

Definition

A harmonic (wave) map $\Psi : (N, \gamma) \longrightarrow (M, \overline{g})$ is called coupled to gravity if it satisfies

$$\mathrm{Ric}^{\gamma}_{ab}(x) = \frac{1}{2} \overline{g}_{ij}(\Psi(x)) \frac{\partial \Psi^i}{\partial x^a} \frac{\partial \Psi^j}{\partial x^b}$$

Exercise: Find a Lagrangian for the equations of a harmonic map coupled to gravity.

Vacuum field equations as a harmonic (wave) map coupled to gravity

- Consider the vacuum field equations with a non-null Killing vector.
  - The twist potential ω is closed.
  - Locally there exists a twist potential: $\omega_a = \overline{D}_a \omega$.
- The field equations are

$$\Delta_\gamma \lambda = \frac{1}{\lambda} \left(|\overline{D}\lambda|^2_\gamma + \epsilon |\overline{D}\omega|^2_\gamma \right), \qquad \overline{D}_a \left(\frac{1}{\lambda^2} \omega^a \right) = 0,$$
$$\mathrm{Ric}^\gamma_{ab} = \frac{1}{2\lambda^2} \left(\overline{D}_a \lambda \overline{D}_b \lambda - \epsilon \overline{D}_a \omega \overline{D}_b \omega \right)$$

- The map:

$$\Psi : (Q_p, \gamma) \longrightarrow \left(\mathbb{R}^+ \times \mathbb{R}, \overline{g} = \frac{d\lambda^2 - \epsilon d\omega^2}{\lambda^2} \right)$$
$$x \longrightarrow (|\lambda|, \omega)$$

  is a harmonic (wave) map coupled to gravity.
- For $(M, g^{\mathcal{M}})$ of Lorentzian signature $\epsilon = -1$. The target space if $\mathbb{H}^2$.
- The result holds both for ξ timelike (harmonic map) or ξ spacelike (wave map).

Axially symmetric, time independent spacetimes

Recall: Axially symmetry is an $SO(2)$ isometry action

$$\Psi : SO(2) \times \mathcal{M} \longrightarrow \mathcal{M}$$

with two-dimensional and timelike axis of symmetry: $\mathcal{A}$

- The Killing generator ζ of $SO(2)$ has $\mathbb{S}^1$ orbits.
- There exists a neighbourhood $\mathcal{U}$ of $\mathcal{A}$ such that ζ is spacelike in $\mathcal{U} \setminus \mathcal{A}$.
- Elementary flatness holds at the axis:

$$\lim_{p \to \mathcal{A}} \frac{\langle dX, dX \rangle}{4X} = 1, \qquad X = \langle \zeta, \zeta \rangle$$

Important result concerning axially symmetric and time-independent spacetimes:

Theorem (Carter, 1970)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a axially symmetric with generator ζ. Assume that $(\mathcal{M}, g^{\mathcal{M}})$ admits a Killing vector ξ which is timelike somewhere on $\mathcal{A}$. Then

$$[\xi, \zeta] = 0$$

- Admits generalization to arbitrary dimension.

Orthogonal transitivity

- Time-independent, axially symmetric spacetimes admit an orthogonal splitting.

Theorem (Kundt, Trümper (1966), Papapetrou (1966))

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be admit two commuting Killing vectors ξ and ζ. Then

$$\nabla_\mu \left(\eta_{\alpha\beta\gamma\delta} \xi^\alpha \zeta^\beta \nabla^\gamma \xi^\delta \right) = 2\eta_{\alpha\beta\gamma\mu} (Ric^{\mathcal{M}}{}_\xi)^\alpha \xi^\beta \zeta^\gamma$$

$$\nabla_\mu \left(\eta_{\alpha\beta\gamma\delta} \zeta^\alpha \xi^\beta \nabla^\gamma \zeta^\delta \right) = 2\eta_{\alpha\beta\gamma\mu} (Ric^{\mathcal{M}}{}_\zeta)^\alpha \zeta^\beta \xi^\gamma$$

where $Ric^{\mathcal{M}}{}_\xi := Ric^{\mathcal{M}}(\xi, \cdot)$ and $Ric^{\mathcal{M}}{}_\zeta := Ric^{\mathcal{M}}(\zeta, \cdot)$.

- A matter model is said to satisfy the circularity condition if

$$\mathrm{Ric}^{\mathcal{M}}{}_\xi, \mathrm{Ric}^{\mathcal{M}}{}_\zeta \in \mathrm{span}\{\xi, \zeta\}$$

Examples:

- Vacuum.
- Perfect fluid with velocity vector $u \in \mathrm{span}\{\xi, \zeta\}$.
- Electromagnetic field satisfying $\mathcal{L}_\xi F = \mathcal{L}_\zeta F = 0$ and electric current $J^\alpha := \nabla_\beta F^{\alpha\beta}$:

$$J = \mathrm{span}\{\xi, \zeta\} \qquad \text{in particular, electrovacuum}$$

Time-independent axially symmetric spacetimes

Axially symmetric, time-independent spacetimes satisfying

(i) The circularity condition.

(ii) The time-independent Killing ξ is timelike somewhere on $\mathcal{A}$

have the properties:

(a) The time-independent and axial Killings commute $[\xi, \zeta] = 0$.

(b) $\boldsymbol{\xi} \wedge \boldsymbol{\zeta} \wedge d\boldsymbol{\xi} = 0$ and $\boldsymbol{\xi} \wedge \boldsymbol{\zeta} \wedge d\boldsymbol{\zeta} = 0$

- Define $\mathcal{O}_p := \text{span}\,\{\xi|_p, \zeta|_p\}$, $\mathcal{O}_p^\perp :=$ orthogonal space to $\mathcal{O}_p$.
- Consider the subset $\widetilde{\mathcal{M}} := \{p \in \mathcal{M}; \mathcal{O}_p$ two-dimensional and timelike$\}$.

The distributions $\{\mathcal{O}\}$ and $\{\mathcal{O}^\perp\}$ are both integrable on $\widetilde{\mathcal{M}}$. By Fröbenius:

- For all $p \in \widetilde{\mathcal{M}}$, $\exists$ two injectively immersed, maximal, codim-two surfaces $\mathcal{T}_p$, S_p:
  - $\mathcal{T}_p$ is timelike and S_p is spacelike and orthogonal to $\mathcal{T}_p$.

The spacetime Einstein tensor can be fully written in terms of the geometry of S_p.

Weyl coordinates (I)

- Define on $\widetilde{\mathcal{M}}$ $\rho := \sqrt{\langle \xi, \zeta \rangle^2 - \langle \xi, \xi \langle \rangle \zeta, \zeta \rangle}$.
- Write $\eta_1 = \xi$ and $\eta_2 = \zeta$. The following equations holds:

$$\Delta_q \rho = \frac{1}{\rho} \mathrm{Ric}^{\mathcal{M}}(\zeta_A, \zeta_B) \langle \zeta_C, \zeta_D \rangle \epsilon^{AC} \epsilon^{BD} \qquad \epsilon^{AB} \text{ Levi-Civita symbol.}$$

- Assume vacuum and $d\rho \neq 0$ everywhere on a simply connnected domain $\hat{S}_p$.
- Equation $\Delta_q \rho = 0$ can be rewritten as

$$d(\star_q d\rho) = 0.$$

- Define $dz = \star_q d\rho$. Since $dz \neq 0$: $\{\rho, z\}$ and global coordinates on $\hat{S}_p$.

The coordinates $\{\rho, z\}$ are called Weyl coordinates. The metric q on $\hat{S}_p$ is necessarily

$$q = e^{2k} \left(d\rho^2 + dz^2 \right), \qquad k(\rho, z)$$

- There exists a spacetime neighbourhood $\mathcal{U}_p$ of p of the form $\mathcal{U}_p = I \times \mathbb{S}^1 \times \hat{S}_p$ and coordinates $\{t, \phi, \rho, z\}$ such that

$$\xi = \partial_t \qquad \zeta = \partial_\phi$$

Choosing metric coefficients depends on the causal character of ξ and/or ζ.

Weyl coordinates (II)

- If $\lambda > 0$, it is natural to write the metric as

$$g^{\mathcal{M}} = -\lambda \left(dt + Ad\phi\right)^2 + \frac{\rho^2}{\lambda} d\phi^2 + \frac{e^{2k}}{\lambda}\left(d\rho^2 + dz^2\right), \qquad \lambda(\rho, z) > 0 \quad \text{and} \quad A(\rho, z).$$

This choice breaks down at ergoregions. Unsuitable choice.

- If the spacetime is causal (admits no closed causal curves) the orbits of ζ are spacelike away from the axis.
- Hence $X := \langle \zeta, \zeta \rangle > 0$ on S_p.
- The spacetime metric in $\mathcal{U}_p$ can be written as

$$g^{\mathcal{M}} = X \left(d\phi + Adt\right)^2 - \frac{\rho^2}{X} dt^2 + \frac{e^{2k}}{X}\left(d\rho^2 + dz^2\right), \qquad X(\rho, z) > 0 \quad \text{and} \quad A(\rho, z).$$

- The twist one-form $\boldsymbol{\omega}$ of ζ is

$$\omega_\rho = -\frac{X^2}{\rho}\partial_z A, \qquad \omega_z = \frac{X^2}{\rho}\partial_\rho A$$

- In vacuum $\boldsymbol{\omega} = d\omega$. Knowledge of ω determines A except for an (irrelevant) additive constant.

Field equations of time-independent, axially symmetric vacuum spacetimes

- The vacuum field equations for X, ω are:

$$\left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}\right) X = \frac{1}{X}\left((\partial_\rho X)^2 + (\partial_z X)^2 - (\partial_\rho \omega)^2 - (\partial_z\ \omega)^2 \right)$$
$$\left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}\right) \omega = \frac{1}{X}\left(\partial_\rho X \partial_\rho \omega + \partial_z X \partial_z \omega \right)$$

- k can be obtained by a line-integral once $\{X, \omega\}$ are known.
- The Laplacian in the flat metric $g_E = d\rho^2 + dz^2 + \rho^2 d\phi^2$ acting on functions $F(\rho, z)$ is

$$\Delta_{g_E} F = \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz}\right) F$$

- The equations for (X, ω) can be written as a map $\Psi : \hat{S}_p \to \mathbb{R}^+ \times \mathbb{R}$
- It is in fact a harmonic map, in the following sense:

Theorem (Ernst)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a time-independent, axially symmetric vacuum spacetime, with corresponding Killing vectors ξ and η.

Assume $(\mathcal{M}, g^{\mathcal{M}})$ to be causal and let $p \in \mathcal{M}$ be any point where $\mathrm{span}\{\xi|_p, \zeta|_p\}$ is two-dimensional and timelike. Let $\rho := \sqrt{-\langle \xi, \xi \rangle \langle \zeta, \zeta \rangle + \langle \xi, \zeta \rangle^2}$ satisfy $d\rho|_p \neq 0$.

Then $\exists$ a neighbourhood $\mathcal{U}_p \simeq I \times \mathbb{S}^1 \times \hat{S}_p$ of p, where $\hat{S}_p$ is a domain of $\mathbb{R}^+ \times \mathbb{R}$ coordinated by (ρ, z).

Let $X := \langle \zeta, \zeta \rangle$ and ω the twist potential of ζ.

The vacuum field equations are equivalent to the map

$$\Psi : \left(\hat{S}_p \times \mathbb{S}^1, g_E = d\rho^2 + dz^2 + \rho^2 d\phi^2 \right) \longrightarrow \left(\mathbb{H}^2, \frac{dX^2 + d\omega^2}{X^2} \right)$$

with Ψ being harmonic and satisfying $\partial_\phi \Psi = 0$.

- An important issue is under which conditions is this harmonic map formulation is global.

Properties of harmonic maps

- The uniqueness theorem for stationary, axially symmetric black holes is based on the harmonic map formulation of the problem.

Strategy:

- Show that a black hole with these properties is defined by a PDE with boundary data that agrees (in a suitable sense) to the data of the Kerr spacetime.
- Use PDE techniques to prove that this elliptic boundary value problem has a unique solution.

Core of the proof: the distance function in hyperbolic space has a positive semi-definite hessian.

- In turn, this relies on the fact that $\mathbb{H}^2$ is negatively curved.

Start with general properties of harmonic maps

- Let $\Psi : (\mathcal{N}, h) \longrightarrow (\mathcal{K}, \overline{g})$ be a C^2 map.
- Denote the corresponding covariant derivatives by ∇^h and $\nabla^{\overline{g}}$.

The Hessian of Ψ is defined as

$$(\text{Hess}\Psi)^A_{ij} = \nabla^h_i \nabla^h_j \Psi^A + \Gamma^{\overline{g}}{}^A_{BC}|_\Psi \frac{\partial \Psi^B}{\partial x^i} \frac{\partial \Psi^C}{\partial x^j}$$

- The Hessian is a tensor both in $\mathcal{N}$ and in $\mathcal{K}$ in the sense that, for any $X, Y \in \mathfrak{X}(\mathcal{N})$,

$$\text{Hess}(X(p), Y(p)) \in T_{\Psi(p)}\mathcal{K}, \qquad \text{for all } p \in \mathcal{N}.$$

- A map $\Psi : (\mathcal{N}, h) \longrightarrow (\mathcal{K}, \overline{g})$ is a harmonic map if

$$\Delta\Psi = 0, \qquad \text{where} \qquad \Delta\Psi = \text{tr}_h(\text{Hess}\Psi).$$

Consider a function $f : \mathcal{K} \to \mathbb{R}$.

- It is useful to determine the Hessian and Laplacian of the composed function $f \circ \Psi : \mathcal{N} \to \mathbb{R}$

Lemma

The following identity holds for any C^2 $\Psi : (\mathcal{N}, h) \longrightarrow (\mathcal{K}, \overline{g})$ and $f : \mathcal{K} \to \mathbb{R}$

$$Hess_h\left(f \circ \Psi\right)_{ij} = \left(Hess_{\overline{g}}\,\Psi\right)^A_{ij} \partial_A f + \left(\nabla^{\overline{g}}_A \nabla^{\overline{g}}_B f\right) \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j}$$

Exercise: Prove this.

- Taking trace on $\mathcal{N}$ and assuming Ψ harmonic

$$\Delta_h\left(f \circ \Psi\right) = h^{ij}\left(\nabla^{\overline{g}}_A \nabla^{\overline{g}}_B f\right) \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j} \geq 0 \qquad \text{if } \text{Hess}_{\overline{g}} f \text{ positive semi-definite}$$

- The idea to show uniqueness of the harmonic map problem is to show that the distance between the two solutions vanishes.
- Given two harmonic maps Ψ_1 and Ψ_2. The distance between the maps is $\mathrm{dist}(\Psi_1, \Psi_2) = \sup_{p \in \mathcal{N}}(\mathrm{dist}_{\overline{g}}(\Psi_1(p), \Psi_2(p)))$.
- Necessary to know properties of the distance function in the target.

The distance between two points is the infimum of the lengths of all smooth curves joining the two points.

- In general the distance function is not smooth everywhere.
  - Cartan-Hadamard Theorem: The distance function $\mathrm{dist}(p, q)$ is smooth at $p \neq q$ for complete, simply connected manifolds with non-positive curvature.
  - Morever:

Proposition

Let $(\mathcal{K}, \overline{g})$ be a Riemannian manifold with non-positive curvature. Let $dist_p^{\overline{g}} := dist_{\overline{g}}(p, q)$ and assume that $dist_p^{\overline{g}}$ is C^2 on $B_p(R) \setminus p$ (ball of radius R from p). Then

$$Hess_{\overline{g}}(dist_p^{\overline{g}}) \geq 0.$$

- Hyperbolic space is complete, simply connected and with constant negative curvature. Hence $\mathrm{Hess}_{\mathbb{H}^2}(\mathrm{dist}^{\mathbb{H}^2}) \geq 0$ and

$$\Delta_{g^{\mathbb{H}^2}} \mathrm{dist}^{\mathbb{H}^2}(\Psi_1(x), \Psi_2(x)) \geq 0 \qquad \Psi_i : (\mathcal{N}, h) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}^2}\right)$$

- A uniqueness theorem for harmonic maps follows immediately

Theorem

Let (N, h) be a compact Riemannian manifold with smooth boundary. If two C^2 harmonic maps $\Psi_i : (\mathcal{N}, h) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}^2}\right)$ satisfy

$$\Psi_1|_{\partial\mathcal{N}} = \Psi_1|_{\partial\mathcal{N}} \qquad \textit{then} \qquad \Psi_1 = \Psi_2.$$

Proof. The function $F(x) := \mathrm{dist}^2_{\mathbb{H}^2}(\Psi_1(x), \Psi_2(x))$ is smooth everywhere, non-negative and satisfies $\Delta_{g^{\mathbb{H}^2}} F \geq 0$.

- The maximum principle implies that F cannot have an interior maximum unless $F = \mathrm{const}$.

Since $F|_{\partial\mathcal{N}} = 0$ and it is everywhere non-negative $\Longrightarrow F = 0$ everywhere and $\Psi_1(x) = \Psi_2(x)$ for all $x \in \mathcal{N}$.

- The uniqueness problem for black holes is much more difficult because $(\mathcal{N} = \mathbb{R}^3 \setminus \mathcal{Z}, g_E)$ and Ψ diverges on the z-axis $\mathcal{Z}$.
- This could be addressed by making sure that the behaviour near the axis (and at infinity) of the harmonic maps is sufficiently similar.
  - Unclear whether the behaviour near of the axis of any black hole solution is a priori sufficiently similar to the behaviour of the Kerr near the axis.

This difficulty has been solved by using a stronger version of the maximum principle.

Proposition (Chruściel, Li, Weinstein)

Let $f \in C^0(\mathbb{R}^3 \setminus \mathcal{Z})$ satisfy $\Delta_{g_E} f \geq 0$ in $\mathbb{R}^3 \setminus \mathcal{Z}$ in the distributional sense.

$$\text{If} \quad \lim_{x \in \mathbb{R}^3 \setminus \mathcal{Z}, |x| \to \infty} f(x) = 0 \quad \text{and} \quad sup_{x \in \mathbb{R} \setminus \mathcal{Z}} f \leq \infty \quad \text{then } f \equiv 0 \text{ on } \mathbb{R}^3 \setminus \mathcal{Z}.$$

Corollary (Uniqueness of Harmonic Map)

Let $\Psi_A : \left(\mathbb{R}^3 \setminus \mathcal{Z}, g_E = d\rho^2 + dz^2 + \rho^2 d\varphi^2\right) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}} = \frac{dX^2 + d\omega^2}{X^2}\right)$ $(A = 1, 2)$ be two harmonic maps satisfying

$$\lim_{x \in \mathbb{R}^3 \setminus \mathcal{Z}, |x| \to \infty} f(x) = 0 \quad \text{and} \quad sup_{x \in \mathbb{R} \setminus \mathcal{Z}} f \leq \infty$$

where $f(x) = dist_{\mathbb{H}^2}(\Psi_1(x), \Psi_2(x))$. Then $\Psi_1 = \Psi_2$.

Global Properties of stationary spacetimes

Stationary and static spacetime

- So far, most of the considerations have been purely local.
- For global results, stronger conditions are required.

Definition

A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is stationary if it admits a Killing vector with complete integral curves which are timelike on a subset $\emptyset \neq \mathcal{U} \subset \mathcal{M}$.

"Stationary" is clearly stronger than "time-independent".

- $(\mathcal{M}, g^{\mathcal{M}})$ strictly stationary: stationary and $\mathcal{U} = \mathcal{M}$.
- $(\mathcal{M}, g^{\mathcal{M}})$ is static: stationary $+$ associated Killing vector ξ is integrable
- $(\mathcal{M}, g^{\mathcal{M}})$ strictly static: static and $\mathcal{U} = \mathcal{M}$.

Theorem (Global structure theorem for strictly stationary $(\mathcal{M}, g)$ [Harris, 1992])

A strictly stationary chronological spacetime is of the form $\mathcal{M} = \mathbb{R} \times S$ with ξ translation along the $\mathbb{R}$ factor.

- Chronological: there are no closed timelike curves.
- The quotient $\mathcal{N} = \mathcal{M}/\sim$ is a smooth manifold.

Conformal compactification of Euclidean space

- Consider Euclidean space $(\mathbb{R}^3 \setminus \{0\}, g_E)$ and define the conformal metric

$$\hat{h} = \Omega^2 g_E, \quad \Omega = \frac{1}{|x|^2_{g_E}} \text{ on } \qquad \mathbb{R}^3 \setminus \{0\}.$$

- A sequence of points diverging at infinity in $\mathbb{R}^3 \setminus \{0\}$ is a Cauchy sequence in $(\mathbb{R}^3 \setminus \{0\}, \hat{h})$
- Infinity is compactified to a point Λ
- The coordinate transformation $\hat{x} = \frac{x}{|x|^2_{g_E}}$ makes:
    - $\Omega = |\hat{x}|^2_{\hat{h}}$
    - $\hat{h} = \delta_{ij} d\hat{x}^i d\hat{x}^j$
- The conformal factor Ω satisfies

$$\Omega|_\Lambda = 0, \qquad d\Omega|_\Lambda = 0, \qquad \text{Hess}_{\hat{h}} \Omega|_\Lambda = 2\hat{h}$$

- The idea is to define asymptotic flatness using a conformal compatification to a point.

Asymptotically flat four-end

Definition (Stationary asymptotically flat 4-end)

A vacuum chronological stationary spacetime $(\mathcal{M}, g^{\mathcal{M}})$ with stationary Killing ξ has a stationary asymptotically flat four-end iff

(i) There is a connected component $\mathcal{M}^\infty$ of the subset $\mathcal{M}^+ := \{p \in \mathcal{M}, \lambda(p) > 0\}$ such that the quotient $\mathcal{N}^\infty = \mathcal{M}^\infty / \sim$ is diffemorphic to $\mathbb{R}^3 \setminus \overline{B}(a)$.

(ii) There exists a manifold $\overline{\mathcal{N}}^\infty \simeq B(b)$ and a diffeomorphism

$$\Psi : \mathcal{N}^\infty \longrightarrow \overline{\mathcal{N}}^\infty \setminus \{\Lambda\}, \qquad \text{where } \Lambda \text{ is the center of the ball } \overline{B}(b).$$

(iii) $\overline{\mathcal{N}}^\infty$ admits a Riemannian metric $\hat{h}$ and a function $\Omega \in C^\infty(\overline{\mathcal{N}}^\infty, \mathbb{R}^+)$ satisfying,

$$\Omega|_\Lambda = 0, \qquad d\Omega|_\Lambda = 0, \qquad \mathrm{Hess}_{\hat{h}}\Omega|_\Lambda = 2\hat{h}$$

such that $\Psi : (\mathcal{N}^\infty, h) \longrightarrow (\overline{\mathcal{N}}^\infty \setminus \{\Lambda\}, \Omega^{-2}\hat{h})$ is an isometry.

(iv) There exists smooth functions $\tilde{\lambda}, \tilde{\omega}$ in $\overline{\mathcal{N}}^\infty$ such that

$$\lambda - 1 = \Psi^\star(\sqrt{\Omega}\,(\tilde{\lambda} - 1)), \qquad \omega = \Psi^\star(\sqrt{\Omega}\,\tilde{\omega}).$$

- Global structure theorem: $\mathcal{M}^\infty = \mathbb{R} \times \mathcal{N}^\infty$ with ξ translation along the $\mathbb{R}$.
- $\mathcal{M}^\infty$ admits spacelike cross section Σ^∞.

A Riemannian manifold $(\mathcal{N}^{\infty}, h)$ satisfying (ii) and (iii) is called Asymptotically Euclidean in the sense of conformal compatification.

- The manifold $\mathcal{N}^{\infty}$ is simply connected: twist potential ω exists globally.
- The definition requires, in particular $\lambda \to 1$ and $\omega \to 0$ at infinity.

Loosely speaking. Asymptotically flat four-end states that the metric h in Cartesian coordinates $\{x\}$ of $\mathcal{N}^{\infty} \simeq \mathbb{R}^3 \setminus B(a)$ is, with $r := |x|$:

$$h_{ij} = \delta_{ij} + \frac{1}{r} h^{(1)}_{ij} + \frac{1}{r^2} h^{(2)}_{ij} + \cdots \qquad \Omega = \frac{1}{r^2} + O(\frac{1}{r^3})$$

$$\lambda = 1 + \frac{1}{r}\lambda^{(1)} + \frac{1}{r^2}\lambda^{(2)} + \cdots, \qquad \omega = \frac{1}{r}\omega^{(1)} + \frac{1}{r^2}\omega^{(2)} + \cdots.$$

- Write this as $\omega = O_{\infty}\left(\frac{1}{r}\right)$ and $\lambda = 1 + O_{\infty}\left(\frac{1}{r}\right)$.

An analogous definition exists in higher dimensions. We will refer to asymptotically flat n-end in this case.

Rigidity of strictly stationary spacetimes

- Asymptotic flatness is motivated by the physical expectation that the gravitational field far away from the sources should approach a state of gravitational field where effect of sources and of propagating degrees of freedom is negligigle.
- Generally taken for granted that Minkowski is the only spacetime with these properties.
- As noticed by M. Anderson, this relies on the expectation

The only vacuum spacetime wich is stationary everwhere and complete is the Minkowski spacetime.

- Stationary everywhere: No gravitational waves or black holes.
- Complete: No singularities

Is such a result true?

- Lichnerowicz (1955) proved this result assuming asymptotic flatness.

Theorem (Lichnerowicz, 1955)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ is strictly stationary, chronological, complete, vacuum spacetime. Assume

(i) $(\mathcal{M}, g^{\mathcal{M}})$ *has an asymptotically flat four-end* $(\mathcal{M}^\infty, g^{\mathcal{M}})$.

(ii) $(\mathcal{M} \setminus \mathcal{M}^\infty)/\sim$ *is compact.*

Then $(\mathcal{M}, g^{\mathcal{M}})$ *is isometric to the Minkowski spacetime.*

- "Vacuum and complete" encode the property that there are no sources.
- "Chronological" needed to apply the global structure theorem.
- Conditions (i) and (ii) mean that that the quotient $\mathcal{N} = \mathcal{M}/\sim$ is $\mathcal{N} = \mathcal{C} \cup \mathbb{R}^3$.
    - Outside a compact set, the spacetime approaches the Minkowski spacetime.

Theorem (Anderson, 2000)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a geodesically complete, chronological, strictly stationary, vacuum spacetime. Then $(\mathcal{M}, g^{\mathcal{M}})$ is either the Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$ or a quotient of this spacetime by a discrete group of isometries commuting with a time translation $t \longrightarrow t + c$, $c \in \mathbb{R}$.

- In physical terms one can state this theorem as

  There are no gravitational solitons, i.e. non-trivial gravitational field with no sources and in equilibrium.

- Other non-linear physical theories do admit such states, so this is a highly non-trivial statement about General Relativity.
- Justifies the condition that far away from the sources, the field approaches Minkowski.

This theorem is specific to four-dimensions. Many parts of the argument are only valid in this dimension.

Theorem: Sketch pf proof (I)

- $(\mathcal{M}, g^{\mathcal{M}})$ geodesically complete implies $(\mathcal{N}, h)$ is complete.
- Passing to the universal cover, assume Σ simply connected $\Longrightarrow \boldsymbol{\omega} = d\omega$.
- In the conformal metric $\gamma = \lambda h$ we have a harmonic map coupled to gravity

$$\Psi : (\mathcal{N}, \gamma) \longrightarrow \left(\mathbb{H}^2, \frac{d\lambda^2 + d\omega^2}{\lambda} \right) \qquad \text{Ernst map}$$

For any harmonic map $\Psi : (\mathcal{N}, \gamma) \to (\Sigma, g)$ the Bochner identity holds

- Define the energy density

$$e(\Psi) := \frac{1}{2} \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j} \gamma^{ij}(x) g_{AB}(\Psi(x))$$

- If Ψ is a harmonic map:

$$\Delta_\gamma e(\psi) = ||\text{Hess}\,\Psi||^2 + \text{Ric}_\gamma^{ij} \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j} g_{AB}(\Psi) - \text{Riem}^g_{ABCD} \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j} \frac{\partial \Psi^C}{\partial x^k} \frac{\partial \Psi^D}{\partial x^l} \gamma^{ik} \gamma^{jl}$$

Theorem: Sketch of proof (II)

For the Ernst map:

- $g = g^{\mathbb{H}}$, metric in the hyperbolic plane.
- The Ricci tensor of the domain is (because we have a harmonic map coupled to gravity):

$$\mathrm{Ric}^{\gamma}_{ij} = \frac{1}{2} g^{\mathbb{H}}_{AB}(\Psi(x)) \frac{\partial \Psi^A}{\partial x^i} \frac{\partial \Psi^B}{\partial x^j}$$

- $g^{\mathbb{H}}$ has constant curvature $= -1$:

$$\mathrm{Riem}^{g^{\mathbb{H}}}_{ABCD} = -\left(g^{\mathbb{H}}_{AC} g^{\mathbb{H}}_{BD} - g^{\mathbb{H}}_{AD} g^{\mathbb{H}}_{BC} \right)$$

The Bochner indentity gives:

$$\Delta_{\gamma} e(\Psi) = ||\mathrm{Hess}\,\Psi||^2 + 2|\mathrm{Ric}^{\gamma}|^2_{\gamma} + \gamma^{ik}\gamma^{jl} \left(A_{ik} A_{jl} - A_{il} A_{jk} \right) \geq 0 \qquad (\star) \qquad (1)$$

$$\textit{where} \qquad A_{ij} := \langle V_i, V_j \rangle_{g^{\mathbb{H}}}, \qquad V^A_i = \frac{\partial \Psi^A}{\partial x^i}$$

Exercise: Show that the last term is non-negative.

- Thus: $e(\Psi)$ is subharmonic.

We are assuming $(\mathcal{N}, h)$ is complete

- $\not\Longrightarrow (\mathcal{N}, \gamma)$ complete as λ may approach zero. Argument splits in two cases:
  - (a) $(\mathcal{N}, \gamma)$ complete
  - (b) $(\mathcal{N}, \gamma)$ non-complete.

We only discuss case (a) with the additional condition $|\mathrm{Ric}^{\gamma}|$ bounded.

- We have $\mathrm{Scal}^{\gamma} \leq C$ and $\mathrm{Scal}^{\gamma} = e(\Psi) \geq 0$.
- Take a maximizing sequence $\{x_n\}$ such that $\mathrm{Scal}^{\gamma}(x_n) \to \sup_{\mathcal{N}} \mathrm{Scal}^{\gamma} \leq C$.
- Scal^{γ} approaches its suppremum and $(\mathcal{N}, \gamma)$ is complete + Ric^{γ} bounded:
  - $\Delta_{\gamma}(\mathrm{Scal}^{\gamma})|_{x_n}$ must approach zero.

However $\Delta_{\gamma} e(\Psi)$ is a sum of non-negative terms, among which there is $|\mathrm{Ric}^{\gamma}|^2$.

$$|Ric^{\gamma}(x_n)|_{\gamma} \longrightarrow 0$$

Hence also $Scal^{\gamma}(x_n) \longrightarrow 0$. But we approached the suppremum. It follows

$$\mathrm{Scal}^{\gamma} = 0 \qquad \text{contradiction.} \qquad \square$$

Rigidity of stationary black hole spacetimes

Black hole region

- Black holes can be defined in non-stationary, asymptotically flat spacetimes.
- We adapt the definition to the stationary case.

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a stationary spacetime with stationary Killing ξ.

- Since the orbits of ξ are complete, there exist a one-parameter group of isometries:

$$\begin{aligned} \Psi : \mathbb{R} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (t_0, p) &\longrightarrow \Psi(t_0, p) := \Psi_{t_0}(p), \qquad \Psi^{\star}_{t_0}(g^{\mathcal{M}}) = g^{\mathcal{M}} \end{aligned}$$

- Recall the definition of timelike future of a point

Definition (Cronological future of a point)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a spacetime and $p \in \mathcal{M}$. The future of p is

$$I^{+}(p) := \{q \in \mathcal{M}; \text{exists a future directed timelike curve from } p \text{ to } q\}$$

- $I^{+}(p)$ is open.
- For $\Omega \subset \mathcal{M}$: $I^{+}(\Omega) = \cup_{p \in \Omega} I^{+}(p)$.
- $I^{-}(p)$ and $I^{-}(\Omega)$ are defined similarly.

Black hole region and domain of outer communications

Definition (Black hole region)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be a stationary spacetime with an asymptotically flat n-end $(\mathcal{M}^\infty, g^{\mathcal{M}})$. The black hole region of $\mathcal{M}^\infty$ is the closed set

$$\mathcal{B}^+ = \mathcal{M} \setminus I^-(\mathcal{M}^\infty)$$

- The white hole region defined analogously.

Definition (Domain of outer communications)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be a stationary spacetime with an asymptotically flat n-end $(\mathcal{M}^\infty, g^{\mathcal{M}})$. The domain of outer communications (D.O.C) of the asymptotic end $\mathcal{M}^\infty$ is

$$\langle\langle \mathcal{M}^\infty \rangle\rangle := I^+(\mathcal{M}^\infty) \cap I^-(\mathcal{M}^\infty)$$

- Obviously $\langle\langle \mathcal{M}^\infty \rangle\rangle = \mathcal{M} \setminus (\mathcal{B}^+ \cup \mathcal{B}^-)$
- The domain of outer communication in the Kerr spacetime is
  - If $|a| \leq m$: $\{r \geq r_+\}$.
  - If $|a| > m$: The whole spacetime.

- The D.O.C is the set of events that can send signals to the asymptotic region and also receive signals from infinity.
- The black hole region is the set of points that cannot send signals to infinity.

Definition (Stationary black hole spacetime)

A stationary black hole spacetime is a stationary spacetime with an asymptotically flat n-end and $\mathcal{B}^+ \neq \emptyset$.

- The future event horizon of a black hole spacetime is $\mathcal{H}^+ = \partial \mathcal{B}^+$.
- This notion can be defined in more general setups than stationary spacetimes.

Properties:

- In a general setup, $\mathcal{H}^+$ is a Lipschitz null hypersurface ruled by null geodesics (not smooth in general).
- A future null geodesic γ which is tangent to $\mathcal{H}$ at one point $p = \gamma(s_0)$ remains fully contained in $\mathcal{H}^+$ for $s \geq s_0$ (not true in general for $s < s_0$).

Stationary black holes have been studied under additional global restrictions

- Ongoing effort to lift as many such global assumptions as possible

I^+ regularity

- A convenient set of global assumptions is encoded in the following definition [Chruściel & Lopes Costa]

Definition (I^+ regular stationary black hole)

A stationary black hole spacetime is I^+-regular is the following three-conditions are satisfies

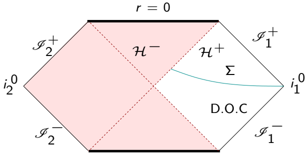
(i) The domain of outer communications $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is globally hyperbolic.

(ii) $\langle\langle \mathcal{M}^\infty \rangle\rangle$ contains a spacelike acausal hypersurface Σ containing an asymptotically flat end Σ^∞ where the stationary Killing vector is timelike.

(iii) The closure $\overline{\Sigma}$ satisfies:

- (iii.a) $\overline{\Sigma} \setminus \Sigma^\infty$ is compact.
- (iii.b) $\partial\Sigma$ is a topological manifold satisfying $\partial\Sigma \subset \partial\langle\langle \mathcal{M}^\infty \rangle\rangle \cap I^+(\mathcal{M}^\infty)$ and intersecting each generator of $\langle\langle \mathcal{M}^\infty \rangle\rangle \cap I^+(\mathcal{M}^\infty)$ precisely once.

- Several notions still need definition.
- Pictorically, the definition means:



Definition (Globally hyperbolic)

A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is globally hyperbolic if it admits a Cauchy hypersurface, i.e. a topological hypersurface Σ such that any inextendible future directed curve in $(\mathcal{M}, g^{\mathcal{M}})$ intersects Σ exactly once.

- By a result of Bernal & Sánchez, Σ can be chosen spacelike and smooth.
- A globally hyperbolic spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is diffemorphic to $\Sigma \times \mathbb{R}$.

Definition (Acausal hypersurface)

A hypersurface Σ in a spacetime $(\mathcal{M}, g^{\mathcal{M}})$ is acausal if no future directed causal curve intersects Σ twice.

Definition (Asymptotically flat end)

An asymptotically flat end is a spacelike hypersurface Σ^{∞} diffeomorphic to $\mathbb{R}^{n-1} \setminus \overline{B}_1$, which, in the Euclidean coordinates x, satisfies

$$g_{ij} - \delta_{ij} = O^2\left(\frac{1}{|x|^{n-3}}\right), \quad K_{ij} = O^2\left(\frac{1}{|x|^{n-2}}\right), \quad \rho, |\mathbf{J}| = O(|x|^{-p}), \quad p > n-2.$$

where g is the induced metric, K the second fundamental form, $\rho := \mathrm{Ein}^{\mathcal{M}}(\nu, \nu)$ and $\mathbf{J} := -\mathrm{Ein}^{\mathcal{M}}(\nu, \cdot)$, where ν is the unit normal to Σ^{∞}.

- The notation $f(x^i) = O^k(r^{\alpha})$, $k \in \mathbb{N}$, means $f(x^i) = \mathrm{O}(r^{\alpha})$, $\partial_j f(x^i) = O(r^{\alpha-1})$ and so on for all derivatives up to and including the kth ones.

Definition

An spacelike hypersurface Σ^{n-1} (possibly with boundary) is asymptotically flat if there is a compact set $\mathcal{K}$ such that $\Sigma \setminus \mathcal{K}$ is a finite union of asymptotically flat ends.

- Condition (i), (ii) and $(iii.a)$ in the definition of I^+ regularity state that
  - $\langle\langle \mathcal{M}^\infty \rangle\rangle$ admits an asymptotically flat hypersurface.
  - The stationary Killing vector is timelike in the asymptotic end.
    - Note that our definition of stationary black hole requires that ξ approaches a time-translation near infinity.
    - This is for presentation purposes only. It can shown under very general circumstances.

Condition $(iii.b)$ is a condition on the future event horizon $\mathcal{E}^+ := \partial \langle\langle \mathcal{M}^\infty \rangle\rangle \cap I^+(\mathcal{M}^\infty)$

- It states that $\mathcal{E}^+$ has compact cross sections.

- The hypersurface Σ is generally not a Cauchy surface of $\langle\langle \mathcal{M}^\infty \rangle\rangle$.
- Cauchy surfaces of $\langle\langle \mathcal{M}^\infty \rangle\rangle$ need not be of the form

$$\Sigma_{\text{Cauchy}} = \mathcal{K} \cup \Sigma^\infty, \qquad \mathcal{K} \text{ compact}$$

- For instance:

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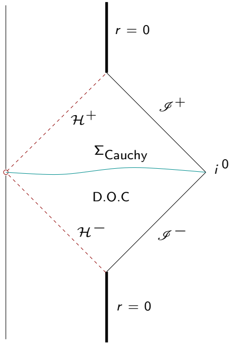


Consequences of I^+ regularity

- The definition of I^+ regularity is useful in order to obtain global properties of the black hole spacetime.
- The proofs make fundamental use of causality theory.
- We make no attempt to explain the proofs. Details can be found in [Chruściel & Lopes Costa] and references therein.

The condition of $\langle\langle \mathcal{M}^\infty \rangle\rangle$ globally hyperbolic guarantees that $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is simply connected under a very mild energy condition.

- A spacetime $(\mathcal{M}, g^{\mathcal{M}})$ satisfies the null energy condition iff

$$\mathrm{Ein}_g(k,k) \geq 0, \qquad \forall p \in \mathcal{M} \quad \text{and} \quad \forall k \in T_p\mathcal{M} \quad \text{null}.$$

Theorem (Galloway, Chruściel & Wald)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a stationary black hole spacetime. If the domain of outer communications is globally hyperbolic and satisfies the null energy condition then it is simply connected.

- This theorem also holds without stationarity, for a suitable definition of domain of outer communications [Galloway 1995].

Corollary (Topology of event horizon)

In I^+ regular, stationary four-dimensional black hole spacetimes satisfying the null energy condition, cross sections of $\mathcal{E}^+$ have spherical topology.

- I^+ regularity is also used to show regularity of the event horizon.

Theorem (Smoothness of the event horizon)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be a stationary black hole spacetime satisfying the regularity condidition I^+ and the null energy condition. Let $\mathcal{E}_0^+$ be a connected component of $\mathcal{E}^+$

(i) *[Chruściel & Lopes Costa] There exists a compact Lipschitz hypersurface S_0 of $\mathcal{E}_0^+$ transverse both to the stationary Killing vector ξ and to the generators of $\mathcal{E}_0^+$ and which meeets every generator of $\mathcal{E}_0^+$ precisely once. In particular, $\cup_{t\in\mathbb{R}}\Phi_t(S_0) = \mathcal{E}_0^+$.*

(ii) *[Chruściel, Delay, Galloway, Howard]. If $(\mathcal{M}, g^{\mathcal{M}})$ is vacuum at large distances in the the asymptotic region then S_0 and $\mathcal{E}_0^+$ are smooth (analytic if the spacetime is).*

Hawking rigidity theorem

- At present, a fundamental step of the classification of stationary black holes is the claim: rotating black holes are axisymmetric, known as Hawking rigidity theorem.
- A rotating black hole spacetime is defined by the property that the stationary Killing vector ξ is not-tangent to the generators of $\mathcal{E}^+$.

The Hawking rigidity theorem procceds in several steps: Local step

Theorem (Hawking rigidity near $\mathcal{H}$ [Hawking $n = 4$, Hollands, Ishibashi, Wald $n \geq 4$)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be an analytic spacetime with an analytic null hypersurface $\mathcal{H}$ such that

(i) *$\mathcal{M}$ admits a complete Killing vector ξ tangent to $\mathcal{H}$.*

(ii) *$\mathcal{H}$ admits a compact cross section S transverse to ξ.*
Define $u : \mathcal{H} \to \mathbb{R}$ by $\xi(u) = 1$ and $u|_S = 0$.

(iii) *The average surface gravity $\langle \kappa_\xi \rangle = \frac{-1}{2|S|} \int_S \langle k, \nabla_\ell \ell \rangle \boldsymbol{\eta}_S$ is nonzero, where ℓ is the null generator of $\mathcal{H}$ satisfying $\ell(u) = 1$ and k is $\perp$ to S, null and with $\langle \ell, k \rangle|_{\mathcal{H}} = -2$.*

Then there is a neighbourhood $\mathcal{U}$ of $\mathcal{H}$ and a Killing vector η on $\mathcal{U}$ which is null, non-zero and tangent to $\mathcal{H}$ (i.e. $\mathcal{H}$ is a Killing horizon for η).
In fact, if ξ is not tangent to the generators of $\mathcal{H}$ then there exists $N \geq 1$ Killing vectors $\zeta_{(A)}$, commuting with each other and with ξ, with 2π orbits, and constants $\Omega_{(A)}$ such that

$$\eta = \xi + \Omega_{(1)}\zeta_{(1)} + \cdots + \Omega_{(N)}\zeta_{(N)}.$$

In spacetime dimension $n = 4$ condition (iii) can be replaced by the condition that S has spherical topology.

- Idea of the proof: ξ, being non-tangent to the generators and transverse to the cross-section, transforms generators into generators which at the same time "advancing" along $\mathcal{H}$.
- Since S is compact, a given generator becomes transformed into another one infinitely close after advancing sufficiently far along the isometry group.
- Since ξ is Killing, it does not change any geometric property.
- This gives enough information to show that the generator η of $\mathcal{H}$ (with a suitable choice of scale) satisfies

$$\mathcal{L}_X \cdots \mathcal{L}_X \mathcal{L}_\xi g^{\mathcal{M}}|_{\mathcal{H}} = 0, \qquad X \in \mathfrak{X}(\mathcal{M})$$

  - Analicity of the spacetime and of $\mathcal{H}$ is then used to show that η is a Killing vector in a neighbourhood of $\mathcal{H}$.

- Passing from the local stament to a global statement in the domain of outer communications was justified by Hawking using analytic continuation.
- However, as noticed by Chruściel, analytic continuation alone is not sufficient to reach this conclusion.

Theorem (Hawking rigidity: global part [Chruściel & Lopes Costa, Hollands *et al*])

Let $(M^n, g^{\mathcal{M}})$ be a stationary black hole spacetime with stationary Killing vector ξ and assume that $(\mathcal{M}, g^{\mathcal{M}})$ satisfies the I^+-regularity condition.
Assume further that on one connected component S_0 of $\overline{\Sigma} \subset \mathcal{E}^+$ we have that either

(i) *The average surface gravity $\langle \kappa_\xi \rangle|_{S_0} \neq 0$, or*

(ii) *the stationary flow is periodic in the space of generators of the connected component $\mathcal{E}^+_0$ of $\mathcal{E}^+$ containing S_0*

and ξ is non-everywhere tangent to the generators of $\mathcal{E}^+_0$.

Then there exist $N \geq 1$ complete, independent, Killing vector $\zeta_{(A)}$ on $\langle\langle \mathcal{M}^\infty \rangle\rangle$ with $2\pi-$periodic orbits, commuting with each other and with ξ and costants $\Omega_{(A)}$ such that

$$\eta := \xi + \Omega_{(1)}\zeta_{(1)} + \cdots + \Omega_{(N)}\zeta_{(N)}.$$

is null, non-zero and tangent to $\mathcal{E}^+_0$

Stationary and axially symmetric black holes

The Hawking rigidity theorem makes the study of stationary and axially symmetric black holes of fundamental interest.

- The Weyl coordinates happen to be global in this case

Theorem (Interior structure theorem)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a stationary and axially symmetric, vacuum I^+-regular black hole. Let ξ the generator of the stationary isometry and ζ the generator of the $SO(2)$ isometry and denote by $\mathcal{A}$ the axis of symmetry. Then the following properties hold:

(i) $\langle\langle \mathcal{M}^\infty \rangle\rangle \setminus \mathcal{A}$ *is diffeomorphic to* $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{R}$.

(ii) *In the coordinates* $\{t, \phi, \rho, z\}$ *defined by this diffeomorfism:* $\xi = \partial_t$ *and* $\zeta = \partial_\phi$.

(iii) *The metric on* $\langle\langle \mathcal{M}^\infty \rangle\rangle \setminus \mathcal{A}$ *takes the* $2+2$ *form*

$$g^{\mathcal{M}} = -\frac{\rho^2}{X} dt^2 + X \left(d\phi + A dt \right)^2 + \frac{e^{2k}}{X} \left(d\rho^2 + dz^2 \right).$$

Detailed proof in [Chruściel & Lopes Costa] building on work of several other authors.

Theorem (Boundary structure theorem, [Chruściel & Lopes Costa])

Assume that all connected components of $\mathcal{E}^+$ have non-zero average surface gravity.

Then the quotient $\mathcal{Q} := \overline{\langle\langle \mathcal{M}^\infty \rangle\rangle}/(\mathbb{R} \times SO(2))$ is a smooth manifold with boundary and:

(i) $\mathcal{Q} \simeq [0,\infty] \times \mathbb{R}$ *with corresponding coordinates* $\{\rho, z\}$.

(ii) *There exists m disjoint compact intervals* $I_i = [c_i^-, c_i^+]$ *such that:*

- *The portion of the axis* $\mathcal{A}$ *in* $\overline{\langle\langle \mathcal{M}^\infty \rangle\rangle}$ *projects onto* $\{0\} \times (\mathbb{R} \setminus \cup_i int(I_i))$.
- *Each connected component* $\mathcal{E}_0^+$ *projects onto an interval* $\{0\} \times I_i$.

(iii) $\log(\frac{X}{\rho^2})$ *and* ω *are smooth functions of* ρ^2 *and* z *on near* $\{0\} \times (\mathbb{R} \setminus \cup_i I_i)$.

(iv) ω *is constant on each connected component of* $\{0\} \times (\mathbb{R} \setminus \cup_i I_i)$.

(v) X *and* ω *are smooth functions of* ρ, z *near* $\{0\} \times (-c_i^+, c_i^+)$, *for all* i.

(vi) *Near each point* $c_i^\pm$: $X = \frac{\rho^2 f_i^\pm(\rho,z)}{z - c_i^\pm \pm \sqrt{(z-c_i^\pm)^2 + \rho^2}}$ *with* $f_i^\pm$ *positive and bounded and*

$$\omega = \omega_i^\pm + \frac{\hat{\omega}(\rho,z)\left(a_i^\pm \rho + \sqrt{2(z - c_i^\pm \pm \sqrt{(z-c_i^\pm)^2 + \rho^2})}\, O((z-c_i^\pm)^2 + \rho^2)\right)^2}{2\left(z - c_i^\pm \pm \sqrt{(z-c_i^\pm)^2 + \rho^2}\right)}$$

where $\omega_i^\pm$ *and* $a_i^\pm > 0$ *are constant and* $\hat{\omega}(\rho,z)$ *is a bounded function.*

- Denote by ω_a $(a = 0, \cdots, m)$ the constant values of ω on each connnected component of $\{0\} \times (\mathbb{R} \setminus \cup_i I_i)$.

Definition (Axis data)

The collection of m disjoint intervals $I_i = [c_i^-, c_i^+]$ $(i = 1, \cdots m)$ and of constants ω_a are called axis data.

- The points $\rho = 0, z = c_i^{\pm}$ correspond to the two points where the connected component $\mathcal{E}_i^+$ of the horizon intersects the axis of symmetry.
- The detailed behaviour of X and ω near those points is necessary to show that two harmonic maps with the same axis data lie at finite distance from each other.

Theorem (Chruściel & Lopes Costa)

Let $\Psi_A : \left(\mathbb{R}^3 \setminus \mathcal{Z}, g_E = d\rho^2 + dz^2 + \rho^2 d\varphi^2\right) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}} = \frac{dX^2 + d\omega^2}{X^2}\right)$ $(A = 1, 2)$ *be two harmonic maps corresponding to two four-dimensional, stationary and axially symetric vacuum* I^+*-regular black holes* $(\mathcal{M}_A, g_A^{\mathcal{M}})$.

If the axis data of both spacetimes agree, then

$$\sup_{p \in \mathbb{R}^3} \left(dist_{\mathbb{H}^2}(\Psi_1(p), \Psi_2(p)) \right) < \infty$$

- The Kerr spacetime of mass m and specific angular momentum $|a| < m$ corresponds to the axis data:

$$I = [-\mu, \mu], \qquad \omega|_{\{0\}\times[\mu,+\infty)} = 8ma \qquad \omega|_{\{0\}\times(-\infty,-\mu]} = 0, \qquad \mu = +\sqrt{m^2 - a^2}.$$

- Note that a global shift of the intervals I_i is irrelevant due to the freedom $z \to z +$ const in the definition of z.
- Also ω is defined up to an additive constant, so the value of ω on one of the components of the axis (say $\{0\} \times [c_m^+, \infty)$) can be specified freely.

An interesting question is whether for any axis data there exists a harmonic map with the behaviour prescribed by the boundary structure theorem.

Theorem (Existence, Weinstein)

Given any axis data there exists a unique harmonic map $\Psi : \left(\mathbb{R}^3 \setminus \mathcal{Z}, g_E = d\rho^2 + dz^2 + \rho^2 d\varphi^2\right) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}} = \frac{dX^2 + d\omega^2}{X^2}\right)$ *fulfulling the axis data conditions and satisfying*

$$dist_{\mathbb{H}^2}(\Psi, \Psi_{m,a}) \longrightarrow 0 \quad for \quad \sqrt{\rho^2 + z^2} \longrightarrow \infty$$

where $\Psi_{m,a}$ *is the harmonic map corresponding to the Kerr spacetime with parameters* $\{m, a\}$ *defined is a specific manner from the axis data.*

- Not known whether these maps lead to stationary and axially symmetric black hole spacetimes.
    - A vacuum, stationary and axially symmetric spacetime does exist.
    - However, not known whether it has singularities at the axis of symmetry.
- The spacetimes corresponding to these harmonic maps are called Weinstein spacetimes.

Combining the various ingredients:

- The Interior structure theorem.
- The Boundary structure theorem.
- The Existence theorem.
- The Uniqueness theorem for harmonic maps

$$\Psi : \left(\mathbb{R}^3 \setminus \mathcal{Z}, g_E\right) \longrightarrow \left(\mathbb{H}^2, g^{\mathbb{H}} = \frac{dX^2 + d\omega^2}{X^2}\right)$$

Theorem (Uniqueness theorem for stationary axially symmetric black holes)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a vacuum, stationary and axially symmetric, I^+-regular black hole spacetime.

If the event horizon is connected and has non-zero average surface gravity then $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is isometric to the exterior of a Kerr black hole spacetime.

If the event horizon has more than one connected component and each component has non-zero average surface gravity, then either the spacetime does not exist or else $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is isometric to one of the Weinstein spacetimes.

- In this form, the theorem has been formulated and proved by Chrúsciel and Lopes Costa, building on work by many authors:

Carter, Robinson, Hawking, Mazur, Bunting, Weinstein, Galloway, etc.

Non-rotating black hole spacetimes

- The Hawking rigidity theorem splits the analysis in rotating and non-rotating black holes (under analiticity).
- Non-rotating black holes can be reduced to static black holes under suitable conditions
    - We consider first when and how the non-rotating case reduces to the static one.
    - Uniqueness of the static black black holes is then discussed (in a more general setting).

Killing Initial data set

- Recall the fundamental definition of initial data set:

Definition (Initial data set)

An initial data set is a triple (Σ^m, g, K), $m \geq 3$, where (Σ, g) is a Riemannian manifold, K a symmetric two-tensor. The matter content is the scalar ρ and one-form $\mathbf{J}$:

$$2\rho := \mathrm{Scal}(g) - |K^2|_g + (\mathrm{tr}_g K)^2, \qquad \mathbf{J} := -\mathrm{div}_g \left(K - (\mathrm{tr}_g K)\, g\right)$$

- (Σ, g, K) is embedded in $(\mathcal{M}, g^{\mathcal{M}})$ with embedding $\Phi : \Sigma \to (\mathcal{M}, g^{\mathcal{M}})$ if $g = \Phi^\star(g^{\mathcal{M}})$ and K is the extrinsic curvature with respect to a unit normal ν.
  - From Gauss and Codazzi: $\rho \stackrel{\Sigma}{=} \mathrm{Ein}^{\mathcal{M}}(\nu, \nu)$, $\qquad \mathbf{J} = -\Phi^\star(\mathrm{Ein}^{\mathcal{M}}(\nu, \cdot))$

Definition (Killing initial data (KID))

$(\Sigma; g, K; N, Y; \mathcal{T})$ is said to be a Killing initial data if (Σ^m, g, K) is initial data and the scalar N, vector field Y and symmetric tensor $\mathcal{T}$ satisfy the Killing initial data equation:

$$\mathcal{L}_Y g = -2NK,$$

$$\mathcal{L}_Y K = -\mathrm{Hess}_g N + N\left(\mathrm{Ric} + \mathrm{tr}_g(K)\, K - 2K \circ K\right) - N\left(\mathcal{T} - \frac{1}{m-1}(\mathrm{tr}_g \mathcal{T} - \rho) g\right).$$

- N (lapse) and Y (shift) correspond to the normal and tangential components of the Killing vector at Σ.
- Concerning $\mathcal{T}$:

Proposition

Let $(\Sigma; g, K; N, Y; \mathcal{T})$ be a KID embedded in a spacetime with embedding Φ and assume that $(\mathcal{M}, g^{\mathcal{M}})$ admits a Killing vector ξ satisfying $\xi \stackrel{\Sigma}{=} N\nu + Y$. Then $\mathcal{T} = \Phi^\star(Ein^{\mathcal{M}})$.

- A KID is vacuum iff $\rho = \mathbf{J} = \mathcal{T} = 0$.

Theorem

Let $(\Sigma; g, K; N, Y; \mathcal{T})$ be vacuum Killing initial data embedded in a spacetime $(\mathcal{M}, g^{\mathcal{M}})$. Then the spacetime admits a Killing vector ξ and

$$\xi \stackrel{\Sigma}{=} N\nu + Y.$$

- Similar results holds for other matter models, e.g. for electrovacuum.

On a Killing initial data we define $\lambda := N^2 - |Y|^2_g$ and call it square norm of the Killing.

Definition (Asymptotically flat KID end)

A Killing initial data $(\Sigma^m; g, K; N, Y; \mathcal{T})$ is a stationary asymptotically flat end if (Σ, g, K) is asymptotically flat and

$$N - N_\infty = O^2\left(\frac{1}{|x|^{m-2}}\right), \qquad Y_i - Y_{\infty\, i} = O^2\left(\frac{1}{|x|^{m-2}}\right),$$

where N_∞, $Y_{\infty\, i}$ are constants satisfying $N_\infty > |Y_\infty|_\delta$.

The definition of asymptotic flatness is analogous to the general case:

Definition (Asymptotically flat KID)

A KID $(\Sigma^m; g, K; N, Y; \mathcal{T})$ (possibly with boundary) is asymptotically flat if there is a compact set $\mathcal{K}$ such that $\Sigma \setminus \mathcal{K}$ is a finite union of stationary asymptotically flat KID ends.

Staticity theorem

- The staticity theorem (i.e. that non-rotating black holes are static) proceeds in two steps:

Step (i): show that the black hole admits an asymptotically flat maximal hypersurface.

- Recall: Maximal means that has vanishing mean curvature ($\mathrm{tr}\, K = 0$).

Step (ii): show that ξ is hypersurface orthogonal and timelike in $\langle\langle \mathcal{M}^{\infty} \rangle\rangle$

To be precise: A black hole is called non-rotating if the stationary Killing ξ is tangent to the null generators of $\mathcal{E}^{+}$ on each one of its connected components.

Theorem (Staticity theorem, step (i), Chruściel & Lopes Costa)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$ be a vacuum, I^{+}-regular, stationary and non-rotating black hole spacetime. Assume that each connected component of $\mathcal{E}^{+}$ is non-degenerate and that $\langle\langle \mathcal{M}^{\infty} \rangle\rangle$ contains no non-embedded Killing prehorizons.

Then, there exists another vacuum stationary black hole spacetime $(\mathcal{M}', g^{\mathcal{M}'}, \xi')$ with $\langle\langle \mathcal{M}'^{\infty}, g^{\mathcal{M}'}, \xi' \rangle\rangle$ isometric to $\langle\langle \mathcal{M}^{\infty}, g^{\mathcal{M}}, \xi \rangle\rangle$ which admits a maximal asymptotically flat Cauchy hypersurface Σ' with smooth boundary $\partial\Sigma'$ satisfying $\xi'|_{\partial\Sigma'} = 0$

Theorem (Staticity theorem, step (ii), Sudarsky & Wald)

Let $(\Sigma^m; g, K; N, Y; \mathcal{T})$ be an asymptotically flat KID with smooth compact boundary $\partial\Sigma$ where $N = Y = 0$. Assume that

(i) $tr_g K = 0$ *("maximal")*

(ii) *The energy flux* $\mathbf{J}$ *vanishes.*

(ii) *The energy-type inequality* $(m-2)\rho + tr_g \mathcal{T} \geq 0$ *holds.*

Then the KID is time-symmetric, $N > 0$ in $int(\Sigma)$ and, after possibly a refefinition of Killing, $Y = 0$.

Thus, the Killing vector is timelike and hypersurface orthogonal outside the horizon $\partial\Sigma$.

- Taking trace to the Hessian equation for N and using the Hamiltonian constraint:

$$\Delta_g N + Y(\mathrm{tr}_g K) = N\left(|K|_g^2 + \frac{1}{m-1}\left((m-2)\rho + \mathrm{tr}_g \mathcal{T}\right)\right)$$

- Under the energy assumption, the equation is $(-\Delta_g + c)\, N = 0$, $c \geq 0$

Lemma (Hopf Maximum principle)

For C^2 solutions of $(-\Delta_g + c)\, N = 0$, $c \geq 0$ N cannot attain a non-positive minimum on Σ unless N is constant.

- Since $N \longrightarrow N_\infty > 0$ at ∞ and $N|_{\partial\Sigma} = 0$, it follows $N > 0$ in $\mathrm{int}(\Sigma)$.
- The momentum constraint is $\mathrm{div}_g\left(K - (\mathrm{tr}_g K) g\right) = \mathrm{div}_g K = 0$.
- Using $\nabla^g_i Y_j + \nabla^g_j Y_i = -2NK_{ij}$:

$$\nabla^g_i \left(K^{ij} Y_i\right) = -N|K|_g^2.$$

- Integrate on Σ and use that $K \to 0$ at infinity and $N = 0$ at $\partial\Sigma$.

$$\int_\Sigma N|K|^2 \boldsymbol{\eta}_g = 0 \qquad \Longrightarrow \qquad K = 0.$$

- $(\Sigma : g, K = 0; N, Y = 0, \rho, \mathbf{J} = 0, \mathcal{T})$ is a Killing initial data with $N > 0$ in $\mathrm{int}(\Sigma)$.

Rigidity of static, asymptotically flat Killing initial data

In the static case it makes sense to study rigidity of stationary black holes by studying rigidity of asymptotically flat Killing initial data with integrable Killing vector.

- Studying rigidity at the Killing initial data level gives much stronger results, as no global in time assumptions on the spacetime are made.

First step: transfer the condition of ξ being hypersurface orthogonal to the initial data level.

- The condition of integrable Killing vector can be transferred to the data:

Definition (Integrable KID)

A KID $(\Sigma; g, K; N, Y; \mathcal{T})$ is integrable iff satisfies the staticity equations

$$N d\mathbf{Y} + 2\mathbf{Y} \wedge Z = 0,$$
$$\mathbf{Y} \wedge d\mathbf{Y} = 0,$$

where $Z := dN + K(Y, \cdot)$ and $\mathbf{Y} := g(Y, \cdot)$.

- The staticity equations imply (recall $\lambda = N^2 - |Y|^2_g$)

$$\lambda d\mathbf{Y} + \mathbf{Y} \wedge d\lambda = 0.$$

- λ does not change sign on any integral manifold of $\mathbf{Y}^{\perp}$

Definition

An asymptotically flat KID is static if $(\Sigma; g, K; N, Y; \mathcal{T})$ is integrable.

Asymptotics in the vacuum case

For vacuum data, asymptotic decay can be improved (staticity not assumed). Define first orbit and conformal metric.

Definition (Orbit and conformal metric)

Consider a KID $(\Sigma; g, K; N, Y; \mathcal{T})$. On any open set $U \subset \Sigma$ where $\lambda > 0$, $h := g + \frac{\mathbf{Y} \otimes \mathbf{Y}}{\lambda}$. is a Riemannian metric, called orbit metric. The conformal metric is $\gamma := \lambda^{\frac{1}{m-2}} h$.

Theorem (Schwarzsch. decay Beig & Simon, Kennefick & Ó Murchadha, Beig & Chruściel**)**

Let $(\Sigma^m; g, K; N, Y; \mathcal{T})$ be a vacuum asymptotically flat KID and let $\lambda_\infty := N_\infty^2 - |Y_\infty|^2$ (necessarily positive). Then, there is a constant M such that

$$\lambda - \lambda_\infty \left(1 + \frac{2M}{|x|^{m-2}}\right) = O^2\left(\frac{1}{|x|^{m-1}}\right), \quad \lambda^{\frac{1}{m-2}} h_{ij} - \delta_{ij} = O^2\left(\frac{1}{|x|^{m-1}}\right).$$

In particular, if $Y_\infty = 0$ and $N_\infty = 1$ then

$$N - 1 + \frac{M}{|x|^{m-2}} = O^2\left(\frac{1}{|x|^{m-1}}\right), \qquad g_{ij} - \left(1 + \frac{2M}{(m-2)|x|^{m-2}}\right)\delta_{ij} = O^2\left(\frac{1}{|x|^{m-1}}\right).$$

- The constant M is precisely the ADM mass of the data.

Bunting and Massood-ul-Alam uniqueness theorem

- Uniqueness for static vacuum black holes was first studied by Israel, by studing the level sets of $V = \sqrt{\lambda}$.
- Besides technical assumptions (removed later by Müller zum Hagen) the method needs connected horizon.
- A major breakthrough was made by Bunting and Masood-ul-Alam, who removed the connectedness condition.
  - Very elegant proof based on a clever application of the positive energy theorem.
  - Works directly at the the initial data level.
  - Originally in dimension $3+1$, but the proof extends to all dimensions for which the positive energy theorem holds

Definition

A manifold Σ^m (without boundary) containing an asymptotic region $\Sigma^\infty \simeq \mathbb{R}^m \setminus \overline{B(1)}$ is of positive energy type if it admits no complete and non-flat metric g with $\mathrm{Scal}^g \geq 0$ and which, on Σ^∞, is asymptotically Euclidean and has vanishing ADM energy.

By the Positive Energy Theorem: manifolds $\Sigma^m = \mathcal{K} \cup \Sigma^\infty$ with $3 \leq m \leq 7$ are of positive energy type.

Notation: A KID $(\Sigma; g, K; N, Y; \mathcal{T})$ is called time symmetric if $Y = K = 0$.

Theorem (Bunting & Masood-al-Alam ($m = 3$))

Let (Σ^m, h, N), $(3 \leq m \leq 7)$ by a time-symmetric, vacuum, asymptotically flat Killing initial data set with compact boundary. Assume:

(i) $N|_{\partial\Sigma} = 0$.

(ii) $N|_{int(\Sigma)} > 0$.

Then there exists $M > 0$ such that (Σ, h, N) is isometric to the $t = 0$ slice of the Schwarzschild spacetime of mass M.

The $t = 0$ slice of Schwarzschild is the time symmetric KID

$$\Sigma_{\mathrm{Sch}} = \mathbb{R}^m \setminus B\left(\left(\frac{M}{2}\right)^{\frac{1}{m-2}}\right), \qquad g_{\mathrm{Sch}} = \left(1 + \frac{M}{2|x|^{m-2}}\right)^{\frac{4}{m-2}} g_E, \quad N_{\mathrm{Sch}} = \frac{1 - \frac{M}{2|x|^{m-2}}}{1 + \frac{M}{2|x|^{m-2}}}.$$

Proof of the Bunting and Masood-al-Alam theorem

- The field equations are, since $K = Y = 0$

$$N\mathrm{Ric}^h = \mathrm{Hess}_h N, \qquad \Delta_h N = 0, \qquad N|_{\partial\Sigma} = 0, \quad N > 0 \text{ on } \mathrm{int}(\Sigma)$$

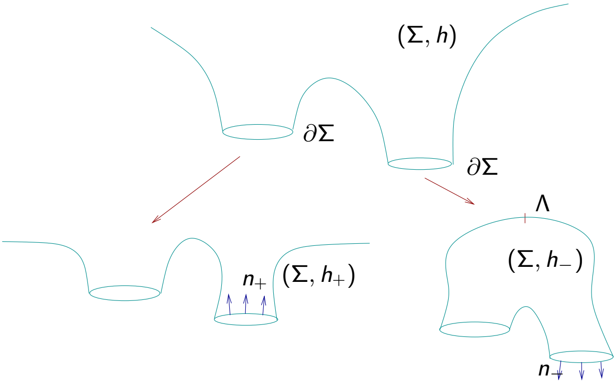
- $|\nabla^h N|^2_h|_{\partial\Sigma}$ is a non-zero constant.
  - Constancy is immediate. If zero, then $N = 0$, $\nabla^h N = 0$ on $\partial\Sigma$, which implies $N \equiv 0$ on $\Sigma \quad \longrightarrow \quad$ Contradiction.
- The boundary $\partial\Sigma$ is totally geodesic:
  - $\nabla^h N$ is a non-zero normal and $\mathrm{Hess}_h N = 0$ on $\partial\Sigma$.
- By the maximum principle: $0 < N < 1$.

Consider the two conformal metrics $h_\pm = \Omega_\pm^{\frac{4}{m-2}} h$ where $\Omega_\pm = \frac{1 \pm N}{2}$.

- The transformation law for curvature scalar and the field equation imply $\mathrm{Scal}_{h_\pm} = 0$.

By the asymptotics of N and h:

- The space (Σ, h_+) is asymptotically flat with vanishing mass.
- The space (Σ, h_-) admits a one point compactification of infinity.



Glue them together across the boundary

- The metrics $h_\pm$ agree on $\partial\Sigma$.
- $n_\pm$: unit normal vectors to $\partial\Sigma$ in $h_\pm$ (n_+ inwards, n_- outwards)

$$n_+(\Omega_+) \stackrel{\partial\Sigma}{=} n_-(\Omega_-) \qquad \Longrightarrow \qquad K^+|_{\partial\Sigma} = K^-_{\partial\Sigma}$$

The manifold $\hat{\Sigma}$ after gluing has $C^{1,1}$ metric $\hat{h}$. The positive energy theorem still applies.

- Asymptotically flat manifold with $\text{Scal}_{\hat{h}} = 0$ and $M_{\text{ADM}} = 0 \Longrightarrow (\hat{\Sigma} = \mathbb{R}^m, \hat{h} = g_E)$.

- The properties of Ric_{h_+} imply

$$\text{Hess}_{g_E}\left(\frac{1+N}{1-N}\right)^{\frac{2}{m-2}} = 2F(N)|\nabla N|^2_{g_E} g_E \qquad F > 0$$

$$N \text{ cannot be constant} \quad \Longrightarrow \quad |\nabla N|^2_{g_E} = \frac{c}{F(N)}, \quad c \in \mathbb{R}^+$$

- Set $M = 2c^{-\frac{m-2}{2}} > 0$. It follows $\frac{1+N}{1-N} = \frac{2}{M}(x_1^2 + \cdots x_m^2)^{\frac{m-2}{2}}$

$$N = \frac{1 - \frac{M}{2|x|^{m-2}}}{1 + \frac{M}{2|x|^{m-2}}} = N_{\text{Sch}},$$

$$h = \Omega_+^{-\frac{4}{m-2}} g_E = \left(1 + \frac{M}{2|x|^{m-2}}\right)^{\frac{4}{m-2}} g_E = g_{\text{Sch}}$$

- This theorem goes a long way of proving uniqueness of static vacuum black holes.
- Not sufficient by itself because of the potential existence of Killing prehorizons.

In a static, KID $(\Sigma^m; g, K; N, Y; \mathcal{T})$, $\mathbf{Y} \wedge d\mathbf{Y} = 0$: By Fröbenius:

- Any $p \in \Sigma$ with $Y|_p \neq 0$ lies in a unique (maximal) integral manifold $\mathcal{N}_p$ of $\{Y^\perp\}$.

Properties:

(i) $\mathcal{N}_p$ is an injectively immersed, arc-connected, codimension-one submanifold.

(ii) Y is normal to $\mathcal{N}_p$

(iii) $\lambda|_{\mathcal{N}_p}$ does not change sign (remains zero if vanishes at one point)

A horizon $\mathcal{H}_{\mathfrak{J}}$ is an integrable manifold $\mathcal{N}_p$ where λ vanishes.

- $\kappa_{\mathfrak{J}} := \frac{1}{2|Y|^2} Y(\lambda)$ is constant on each horizon ($\sim$ surface gravity).
- If $\kappa_{\mathfrak{J}} \neq 0$ then $\mathcal{H}_{\mathfrak{J}}$ is embedded.
    - If $\overline{H}_{\mathfrak{J}}$ contains a fixed point (i.e. a point p where $N|_p = Y|_p = 0$), then $\kappa_{\mathfrak{J}} \neq 0$.
- If $\kappa_{\mathfrak{J}} = 0$, then $\overline{\mathcal{H}}_{\mathfrak{J}}$ may be not embedded ($\sim$ Killing prehorizon).

Consider a static, asymptotically flat KID $(\Sigma^m; g, K; N, Y; \mathcal{T})$. Define

$\Sigma^T :=$ connected component of $\{p \in \Sigma; \lambda(p) > 0\}$ containing the asymptotically flat end.

- Points in the topological boundary $\partial^t \Sigma$ are either fixed points or lie on horizons.
- Fixed points or non-degenerate horizons are fine.

Theorem (Chruściel)

For any $p \in \partial^t \Sigma^T$ not lying on a degenerate horizon there exists a neighbourhood U_p and a differentiable structure on $\overline{U}_p := \overline{\Sigma_T} \cap U_p$ such that

(i) *$\overline{U}_p$ is a smooth manifold-with-boundary.*

(ii) *$(\overline{U}_p, h, K = 0, N = \sqrt{\lambda}, Y = 0)$ is static KID, where h is the orbit metric.*

- Since $N|_{\partial \overline{U}_p} = 0$ and $N|_{\mathrm{int}(\overline{U}_p)} > 0$, these points are well-adapted to the Bunting and Masood-ul-Alam theorem.

- Degenerate horizons can be excluded in general assuming only a very mild energy condition

Theorem (M., Reiris)

Let $(\Sigma^m; g, K; N, Y; \mathcal{T})$ be an asymptotically flat, static, Killing initial data set satisfying the null energy condition. If Σ has boundary, assume that $\overline{\Sigma^T} \cap \partial\Sigma = \emptyset$.
Then, each arc-connected component of $\partial^t \Sigma^T \setminus \{Y = 0\}$ is an embedded, compact manifold.

- Thus $(\overline{\Sigma^T}, h, K = 0, N = \sqrt{\lambda}, Y = 0)$ is an asymptotically flat initial data set satisfying all hypotheses of the Bunting and Masood-ul-Alam theorem.
- The only requirement to apply the theorem is to make sure that Σ^T does not intersect $\partial\Sigma$.
    - In particular holds for I^+-regular black holes.

Putting the results together:

Theorem (Uniqueness theorem for static black holes)

Let $(\mathcal{M}^n, g^{\mathcal{M}})$, $(4 \leq n \leq 8)$ be a vacuum, static I^+-regular black hole spacetime.

Then $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is isometric to the domain of outer communications of a Schwarzschild spacetime of mass $M > 0$.

- Rigidity can be proved under much weaker global assumptions.

Definition (Weakly outer trapped domain)

Consider an initial data set (Σ, g, K). A domain Σ^+ is called weakly outer trapped if

- $\partial^t \Sigma^+$ is smooth.
- The mean curvature H w.r.t the inward normal satisfies $H + \mathrm{tr}_{\partial^t \Sigma^+} K \leq 0$.

Theorem (M. , Reiris)

Let $(\Sigma^m; g, K; N, Y; \mathcal{T})$ $(m \leq 3 \leq 7)$ be a static, asymptotically flat, vacuum KID. Assume that int(Σ) contains an asymptotically flat weakly outer trapped domain. Then there exists $M > 0$ such that (Σ^T, g, K, N, Y) can be embedded in the domain of outer communications of the Schwarzschild spacetime of mass M.

- This theorem applies to any matter model for which a Bunting and Masood-ul-Alam static uniqueness proof exists.
- Generalizes to the non-time symmetric case and to any dimension a previous theorem by Miao:

Theorem (Miao)

Let (Σ^3, g, N) be a time-symmetric, static, vacuum, asymptotically flat KID. Assume $\partial\Sigma$ is the outermost minimal surface Then there exists $M > 0$ such that (Σ, g, N) is isometric to $(\Sigma_{Sch}, g_{Sch}, N_{Sch})$ of mass M.

Uniqueness of static spacetimes with a photon sphere

- In Schwarzschild all null geodesics starting at $\{r = 3M\}$ with $\dot{r} = 0$ initially stay at $\{r = 3M\}$. Note $N^2 = -\langle \xi, \xi \rangle$ is constant on $\{r = 3M\}$.

Are there other static, asymptotically flat vacuum spacetimes with a photon sphere?

Definition (Photon surface)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a static spacetime of the form $(\mathcal{M} = \mathbb{R} \times \Sigma^m, g^{\mathcal{M}} = -N^2 dt^2 + h)$. An embedded timelike hypersurface $\mathcal{T}$ is an photon surface if and only if any null geodesic initially tangent to $\mathcal{T}$ remains tangent to $\mathcal{T}$.

Definition (Photon sphere)

A photon sphere is a photon surface $\mathcal{T}$ satisfying $N|_{\mathcal{T}} = N_0$ constant.

Photon spheres make static vacuum solutions rigid

Theorem (Cederbaum)

Let $(\mathbb{R} \times \Sigma^3, g^{\mathcal{M}} = -N^2 dt^2 + h)$, where (Σ, h, N) is an asymptotically flat vacuum KID with inner boundary $\partial\Sigma$. If $\mathcal{T} := \mathbb{R} \times \partial\Sigma^m$ is a photon sphere and N has nowhere vanishing gradient, then $(\mathcal{M}, g^{\mathcal{M}})$ is isometric to the region outside $\{r = 3M\}$ of Schwarzshild spacetime of mass $M = \frac{1}{\sqrt{3}H}$, where H is the mean curvature of $\mathcal{T}$.

Let us return to the back hole setting.

Combining:

- The Hawking rigidity theorem,
- The Uniqueness theorem of stationary and axially symmetric black holes.
- The Uniqueness theorem of static black holes.

Theorem (Rigidity of stationary black holes)

Let $(\mathcal{M}^4, g^{\mathcal{M}})$ be a vacuum, analytic, I^+-regular black hole spacetime. Assume that the event horizon $\mathcal{E}^+$ is connected and has non-zero average surface gravity.

Then $\langle\langle \mathcal{M}^\infty \rangle\rangle$ is isometric to a exterior Kerr spacetime.

Black hole rigidity without analyticity

- The classification stationary black holes via the stationary and axially symmetric and static cases relies on Hawking rigidity theorem.
- This requires analyticity.

This assumption is unjustified: Strong a priori restriction on the spacetime

- Important long standing problem: Remove the analyticity assumption.

Interesting progress in the last few years (Ionescu, Klainerman, Alexakis).

Main ingredients of the new approach:

- A tensorial characterization of the Kerr metric among stationary vacuum spacetimes.
- Uniqueness results of ill-posed wave-type equations.

Local characterization of the Kerr metric

- Self-dual Killing two-form $\mathcal{F}_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta + \frac{i}{2}\eta_{\alpha\beta\gamma\mu}\nabla^\gamma \xi^\mu$. Square: $\mathcal{F}^2 := \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$.
- Ernst one-form: $\sigma_\mu = 2\xi^\alpha \mathcal{F}_{\alpha\mu}$
- In vacuum $\sigma_\mu = \partial_\mu \sigma$ locally (Ernst potential). Defined up to an additive constant.
- Self-dual Weyl tensor: $\mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{i}{2}\eta_{\gamma\delta\rho\sigma} C_{\alpha\beta}{}^{\rho\sigma}$.
- The canonical metric: $\mathcal{I}_{\alpha\beta\gamma\delta} \equiv (g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma} + i\, \eta_{\alpha\beta\gamma\delta})/4$

Theorem (M.)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be a vacuum spacetime with a Killing field ξ. If the additive constant in σ can be chosen so that the tensor

$$\mathcal{S}_{\alpha\beta\gamma\delta} \equiv \mathcal{C}_{\alpha\beta\gamma\delta} + \frac{6}{1-\sigma}\left(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} - \frac{1}{3}\mathcal{F}^2 \mathcal{I}_{\alpha\beta\gamma\delta}\right)$$

vanishes identically and $\exists p \in \mathcal{M}$ where $0 \neq \mathcal{F}^2(1-\sigma)^{-4} \in \mathbb{R}$, then $(M, g^{\mathcal{M}})$ is locally isometric to a Kerr spacetime.

- If $(\mathcal{M}, g^{\mathcal{M}})$ is asymptotically flat and ξ is timelike at infinity, the condition on $\mathcal{F}^2(1-\sigma)^{-4}$ is automatically satisfied.
- Theorem inspired on an earlier characterization of Kerr on the quotient manifold near infinity due to Simon.

- Suggests a possible strategy to uniqueness $\longrightarrow$ Show that $\mathcal{S}_{\alpha\beta\gamma\delta} \equiv 0$ in any asymptotically, flat, stationary black hole.
- Accomplished in a particular (but relevant) case.

Theorem (Ionescu, Klainerman, 2007)

Let $(\mathcal{M}, g^{\mathcal{M}})$ be an asymptotically flat vacuum black hole spacetime satisfying

(i) *$(M, g^{\mathcal{M}})$ admits a complete Killing vector ξ which is timelike at infinity.*

(ii) *The black hole and white hole event horizons, $\mathcal{H}^+$ and $\mathcal{H}^-$, intersect on a smooth surface S_0 with $\mathbb{S}^2$ topology.*

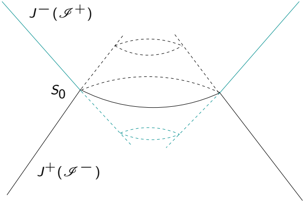
(iii) *$\mathcal{H}^+$ and $\mathcal{H}^-$ are smooth null hypersurfaces in some neighbourhood of S_0. Their null expansions vanish and ξ is not identically zero on S_0.*

(iv) *There exists a real constant M neq0 such that the self-dual two-form $\mathcal{F}_{\mu\nu}$ and the Ernst potential σ associated to ξ satisfy*

$$-4M^2\mathcal{F}^2 \stackrel{S_0}{=} (1-\sigma)^4, \quad Re(2(1-\sigma)^{-1}) > 1 \text{ somewhere on } S_0$$

Then $(\mathcal{M}, g^{\mathcal{M}})$ is isometric to a Kerr spacetime with ADM mass M and angular momentum $|J| < M^2$.

- Conditions (ii) and (iii) mean basically that the horizon is non-degenerate, rotating and connected.



- Condition (iv) is an a priori restriction on the spacetime with no physical justification. Needed for the proof.

Basic steps of the argument:

- Combining the technical assumption $4M^2\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \stackrel{S_0}{=} (1-\sigma)^4$ with the Bianchi identities and the properties of $\mathcal{H}^{\pm}$ $\Longrightarrow$

$$\mathcal{S}_{\alpha\beta\mu\nu}|_{\mathcal{H}^{\pm}} = 0$$

- In one could show that $\mathcal{S}_{\alpha\beta\mu\nu}$ vanishes everywhere, the result would follow.

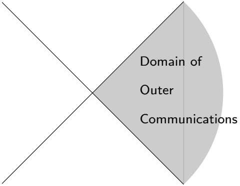
Does the tensor $\mathcal{S}$ satisfy any useful identities?

Ionescu & Klainerman Theorem (II)

- Combining the Bianchi identities, the vacuum field equations and the Killing equations:

$$\begin{aligned} \Delta_{g^{\mathcal{M}}} \mathcal{S}_{\alpha\beta\gamma\delta} &= A_{\alpha\beta\gamma\delta}{}^{\mu\nu\rho\sigma}(\mathcal{S})\mathcal{S}_{\mu\nu\rho\sigma} + B_{\alpha\beta\gamma\delta}{}^{\mu\nu\rho\sigma\kappa}(\mathcal{S})\nabla_\kappa \mathcal{S}_{\mu\nu\sigma\kappa} \\ \mathcal{L}_\xi \mathcal{S}_{\alpha\beta\gamma\delta} &= P_{\alpha\beta\gamma\delta}{}^{\mu\nu\rho\sigma}(\mathcal{S})\mathcal{S}_{\mu\nu\rho\sigma} \qquad \text{(in fact with } P = 0). \end{aligned}$$

- Wave type equations on a domain of the form:
  - Ill posed problem: Existence does not hold
- However, we only need uniqueness!
- The characteristic initial value problem implies $\mathcal{S}_{\alpha\beta\mu\nu} = 0$ in regions I and II.
- Required: a unique continuation result.

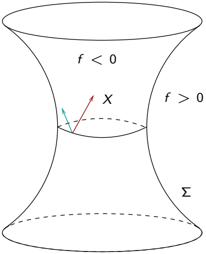


The authors prove this fact using so-called Carleman estimates.

Carleman estimates

- Let Σ be a smooth timelike hypersurface defined by f (i.e. $f = 0$ and $df \neq 0$ on Σ).
- m the unit normal pointing towards $f > 0$.
- Assume Σ is pseudo-convex: The second fundamental form K of Σ along m satisfies

$$K(X,X) < 0, \qquad X \text{ null and tangent to } \Sigma$$



- Carleman estimate: There exists a constant $C > 0$ such that for any ϕ of compact support and $\forall \lambda > C$,

$$\lambda || e^{-\lambda f} \phi ||_{L^2} + || e^{-\lambda f} \nabla \phi ||_{L^2} \leq C \lambda^{-1/2} || e^{-\lambda f} \Delta_{g^{\mathcal{M}}} \phi ||_{L^2}$$

Consequence:

- If $\phi = 0$ on $\{f < 0\}$ and $\Delta_{g^{\mathcal{M}}} \phi = 0$ everywhere $\Longrightarrow$ $\phi = 0$ in a neighbourhood of Σ.

The boundary "$<$" is neither timelike nor smooth, but it can be approximated by such hypersurfaces.

- Exploiting the Carleman estimate, Ionescu and Klainerman show $\mathcal{S}_{\alpha\beta\mu\nu} = 0$ everywhere $\Longrightarrow$ Kerr metric.

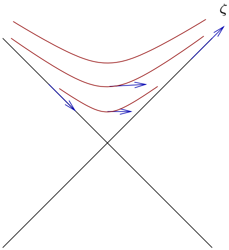
- Alternative approach to prove uniqueness: Show that the axial Killing vector exists without assuming analyticity.

Main idea:

- Hawking argument proves (without assuming analyticity) that there is a vector field ζ on $\mathcal{H}^+ \cup \mathcal{H}^-$ satisfying $\mathcal{L}_\ell \mathcal{L}_\ell \cdots \mathcal{L}_\ell \mathcal{L}_\zeta g^{\mathcal{M}}|_{\mathcal{H}^+ \cup \mathcal{H}^-} = 0$.
- Analyticity is invoked to imply that ζ extends to a Killing everywhere.

Can one still show that ζ extends without assuming analyticity?

- The Killing equations in vacuum imply $\Delta_{g^{\mathcal{M}}} \zeta = 0$ for any Killing.
- The characteristic initial value problem allows to extend ζ to regions *I* and *II*
    - Ill posed PDE in the domain of outer communications.



Alexakis, Ionescu & Klainerman Theorem

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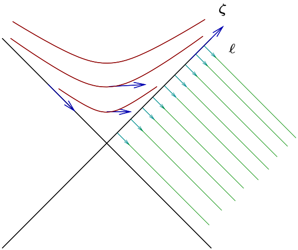
- Extend ζ by Lie dragging along the null geodesics generated by ℓ



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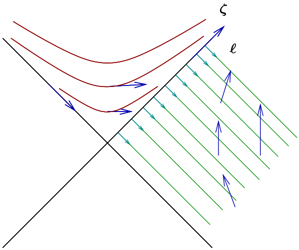
- The Killing equations in vacuum imply $\Delta_{g^{\mathcal{M}}} \zeta = 0$ for any Killing.
- The characteristic initial value problem allows to extend ζ to regions *I* and *II*
  - Ill posed PDE in the domain of outer communications.
- Extend ζ by Lie dragging along the null geodesics generated by ℓ
- Prove (somehow) that ζ is still a Killing...



Alexakis, Ionescu & Klainerman Theorem (II)

- Method: prove a unique continuation theorem for vacuum metrics.

Theorem (Alexakis, 2008)

Let $g^{(4)\prime}$ be a vacuum Lorentzian metric on a neighbourhood $\mathcal{U}$ of the bifurcation surface S_0 in $(\mathcal{M}, g^{\mathcal{M}})$. Assume

- $g^{\mathcal{M}} = g^{(4)\prime}$ *in the regions I and II.*
- *The null geodesics with tangent ℓ are also geodesics in $g^{(4)\prime}$.*

Then $g^{\mathcal{M}} = g^{(4)\prime}$ on a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of S_0

- Proof relies on Carleman estimates.
- Consequence (Alexakis, Ionescu & Klainerman (2008)): ζ is a Killing vector in $\mathcal{U}'$ (apply the theorem to $\mathcal{M}$ and to $\Phi_t^{\star}(g^{\mathcal{M}})$, with $\{\Phi_t\}$ the local group of diffeomorphism of ζ).
- Does ζ extend as a Killing vector to the whole domain of outer communications?
- Not known in general. However, it does if $(\mathcal{M}, g^{\mathcal{M}})$ is a priori close to Kerr.

Theorem (Alexakis, Ionescu & Klainerman, 2008)

With the same assumption as in the previous uniqueness theorem, replace condition (iv) by

(iv)' *The exists a small constant* ϵ *such that* $|(1-\sigma)\mathcal{S}| \leq \epsilon$.

Then $(\mathcal{M}, g^{\mathcal{M}})$ *isometric to a Kerr spacetime with ADM mass* M *and angular momentum* $|J| < M^2$.

- If $(\mathcal{M}, g^{\mathcal{M}})$ close the Kerr a priori ($\mathcal{S}$ is small everywhere) then, it is in fact Kerr (i.e. there are no black holes in a neighbourhood of Kerr).
- Result extended to disconnected (non-degenerate) horizon by Wong and Yu (and extended also to electrovacuum).

Outlook

- I have concentrated in four-dimensional spacetimes and vacuum.
- The main open problems for vacuum, 4-d are:
  - (i) Remove as many global conditions on the black hole as possible.
  - (ii) Remove the analiticity condition.
  - (iii) Settle down the multicomponent black hole case.
  - (iv) Remove the non-degenerate condition on the horizon.

- Many of the results in vacuum have been extended to electrovacuum.
- Other matter models have been studied in detail only in the static case.
  - Uniqueness does not always holds, cf. Bartnik and Mckinnon solutions.
- In static vacuum there has been recent activity trying to:
  - Remove the a priori condition of asymptotic flatness (Reiris)
  - Make no positivity assumption on the lapse N near infinity (Miao, Tam).
- In higher dimensions, stationary spacetimes have much more freedom and much less is known.

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