Geometric invariants on black hole initial data

Robert Sansom

Queen Mary University of London

Interdisciplinary junior scientist workshop: Mathematical General Relativity

March 3rd, 2023

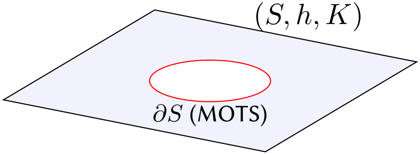
Black hole initial data

Black hole initial data is initial data for the vacuum Einstein field equations that contains a marginally outer trapped surface (MOTS), also called an apparent horizon.

Black hole initial data

Black hole initial data is initial data for the vacuum Einstein field equations that contains a marginally outer trapped surface (MOTS), also called an apparent horizon.

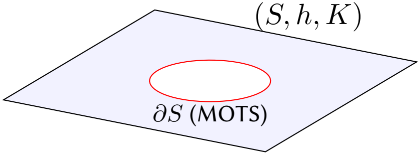
- The MOTS is the **boundary** of the black hole in the initial data.



Black hole initial data

Black hole initial data is initial data for the vacuum Einstein field equations that contains a marginally outer trapped surface (MOTS), also called an apparent horizon.

- The MOTS is the **boundary** of the black hole in the initial data.



For example, in the Schwarzschild spacetime the event horizon coincides with the apparent horizon.

The Schwarzschild spacetime is also stationary - i.e. there exists a time-like Killing vector field.

Question

How do we characterise the **stationarity** (or lack thereof) of an initial data set (S, h, K) of the vacuum Einstein field equations?

Question

How do we characterise the **stationarity** (or lack thereof) of an initial data set (S, h, K) of the vacuum Einstein field equations?

Motivated by:

PRL **93,** 231101 (2004) PHYSICAL REVIEW LETTERS week ending 3 DECEMBER 2004

A New Geometric Invariant on Initial Data for the Einstein Equations

Sergio Dain
Albert-Einstein-Institut, am Mühlenberg 1, D-14476, Golm, Germany
(Received 14 July 2004; published 1 December 2004)

For a given asymptotically flat initial data set for Einstein equations a new geometric invariant is constructed. This invariant measures the departure of the data set from the stationary regime; it vanishes if and only if the data are stationary. In vacuum, it can be interpreted as a measure of the total amount of radiation contained in the data.

DOI: 10.1103/PhysRevLett.93.231101 PACS numbers: 04.20.Ex, 04.20.Cv, 04.20.Fy, 04.20.Ha

The approximate Killing equation

The constraint map

Given $h_{ab} \in \mathscr{M}_2$ space of Riemannian metrics, $K_{ab} \in \mathscr{S}_2$ space of symmetric 2-tensors.

Let $\mathscr{C}$ and $\mathscr{X}$ denote the spaces of scalars and vectors.

The constraint map

Given $h_{ab} \in \mathscr{M}_2$ space of Riemannian metrics, $K_{ab} \in \mathscr{S}_2$ space of symmetric 2-tensors.

Let $\mathscr{C}$ and $\mathscr{X}$ denote the spaces of scalars and vectors.

The **constraint map** is the map $\Phi : \mathscr{M} \times \mathscr{S}_2 \to \mathscr{C} \times \mathscr{X}$ with

$$\Phi \begin{pmatrix} h_{ij} \\ K_{ij} \end{pmatrix} \equiv \begin{pmatrix} r + K^2 - K_{ij}K^{ij} \\ -D^j K_{ij} + D_i K \end{pmatrix}$$

The Linearisation and its adjoint

Linearisation of the constraint map at (h_{ij}, K_{ij}):
$D\Phi : \mathscr{S}_2 \times \mathscr{S}_2 \to \mathscr{C} \times \mathscr{X}$

$$D\Phi \begin{pmatrix} \gamma_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} D^i D^j \gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_h \gamma + H \\ -D^j Q_{ij} + D_i Q - F_i \end{pmatrix}.$$

The Linearisation and its adjoint

Linearisation of the constraint map at (h_{ij}, K_{ij}):
$D\Phi : \mathscr{S}_2 \times \mathscr{S}_2 \to \mathscr{C} \times \mathscr{X}$

$$D\Phi \begin{pmatrix} \gamma_{ij} \\ Q_{ij} \end{pmatrix} = \begin{pmatrix} D^i D^j \gamma_{ij} - r_{ij}\gamma^{ij} - \Delta_h \gamma + H \\ -D^j Q_{ij} + D_i Q - F_i \end{pmatrix}.$$

The formal adjoint:

$$D\Phi^* \begin{pmatrix} X \\ X_i \end{pmatrix} = \begin{pmatrix} D_i D_j X - X r_{ij} - \Delta_h X h_{ij} + H_{ij} \\ D_{(i} X_{j)} - D^k X_k h_{ij} + F_{ij} \end{pmatrix}.$$

where H, F_i, H_{ij} and F_{ij} are terms of lower order which vanish under **time symmetry** - $K_{ij} = 0$.

The Killing initial data equations

Surprisingly, the elements of the kernel of $D\Phi^*$ are the **symmetries** of the spacetime determined by the inital data set (S, h, K).

(Moncrief, 1975)

The Killing initial data equations

Surprisingly, the elements of the kernel of $D\Phi^*$ are the **symmetries** of the spacetime determined by the inital data set (S, h, K).
(Moncrief, 1975)

In other words, a solution (N, N^i) to $D\Phi^*(N, N^i) = 0$ also solves the **Killing Initial data (KID) equations**

$$NK_{ij} + D_{(i}N_{j)} = 0,$$
$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j).$$

The Killing initial data equations

Surprisingly, the elements of the kernel of $D\Phi^*$ are the **symmetries** of the spacetime determined by the inital data set (S, h, K).
(Moncrief, 1975)

In other words, a solution (N, N^i) to $D\Phi^*(N, N^i) = 0$ also solves the **Killing Initial data (KID) equations**

$$NK_{ij} + D_{(i}N_{j)} = 0,$$
$$N^k D_k K_{ij} + D_i N^k K_{kj} + D_j N^k K_{ik} + D_i D_j N = N(r_{ij} + KK_{ij} - 2K_{ik}K^k{}_j).$$

The spacetime evolving from the initial data will have a killing vector with **lapse** N and **shift** N^i. (Beig - Chrusciel, 1996)

Bartnik operators

We will make use of the following **Bartnik operators** related to the linearisation of the constraint map and its formal adjoint:

$$\mathscr{P}\begin{pmatrix}\gamma_{ij}\\ q_{kij}\end{pmatrix} \equiv D\Phi\begin{pmatrix}\gamma_{ij}\\ -D^k q_{kij}\end{pmatrix},$$

$$\mathscr{P}^*\begin{pmatrix}X\\ X_i\end{pmatrix} \equiv \begin{pmatrix}1 & 0\\ 0 & D_k\end{pmatrix}\cdot D\Phi^*\begin{pmatrix}X\\ X_i\end{pmatrix}.$$

The approximate Killing equation

The **approximate Killing operator** : $\mathscr{P} \circ \mathscr{P}^* : \mathscr{C} \times \mathscr{X} \to \mathscr{C} \times \mathscr{X}$

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \equiv \begin{pmatrix} 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + 2r \Delta_h X + \text{l.o.t} \\ \\ D^j \Delta_h D_{(i} X_{j)} + D_i \Delta_h D^k X_k + \text{l.o.t} \end{pmatrix}.$$

The approximate Killing equation

The **approximate Killing operator** : $\mathscr{P} \circ \mathscr{P}^* : \mathscr{C} \times \mathscr{X} \to \mathscr{C} \times \mathscr{X}$

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} \equiv \begin{pmatrix} 2\Delta_h \Delta_h X - r^{ij} D_i D_j X + 2r\Delta_h X + \text{l.o.t} \\ \\ D^j \Delta_h D_{(i} X_{j)} + D_i \Delta_h D^k X_k + \text{l.o.t} \end{pmatrix}.$$

Then the **approximate Killing equation** is

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = 0. \qquad \text{(AKE)}$$

Properties of (AKE)

The **approximate Killing equation** is

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = 0. \tag{AKE}$$

- A solution to the KID equations also solves (AKE),
- The approximate Killing operator is self-adjoint, fourth order and elliptic,
- (AKE) is the Euler-Lagrange equation of

$$\int_S \mathscr{P}^* (X, X_i) \cdot \mathscr{P}^* (X, X_i) \, \mathrm{d}\mu.$$

Properties of (AKE)

The **approximate Killing equation** is

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X_i \end{pmatrix} = 0. \tag{AKE}$$

- A solution to the KID equations also solves (AKE),
- The approximate Killing operator is self-adjoint, fourth order and elliptic,
- (AKE) is the Euler-Lagrange equation of

$$\int_S \mathscr{P}^* (X, X_i) \cdot \mathscr{P}^* (X, X_i) \, \mathrm{d}\mu.$$

Note: We cannot construct (AKE) using $D\Phi^*$ because the functional

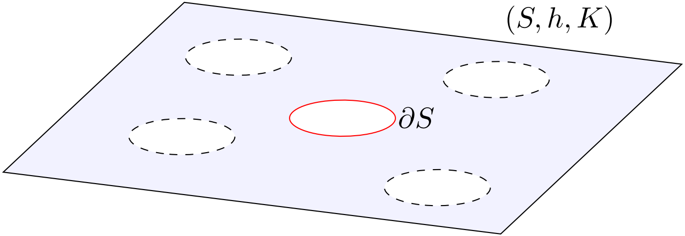
$$\int_S D\Phi^* (X, X_i) \cdot D\Phi^* (X, X_i) \, \mathrm{d}\mu$$

contains terms of inconsistent physical dimension.

Solvability of (AKE)

The setup

- Focus on the symmetry corresponding to time translation,
- One **asymptotic end**,
- One (or several) inner bdrys that are **marginally outer trapped surfaces (MOTS)**.



Asymptotic Conditions

Assume that the initial data is **asymptotically flat**

$$h_{ab} - \delta_{ab} \in H^{\infty}_{-\frac{1}{2}} \qquad (= o_{\infty}(|x|^{-\frac{1}{2}}))$$
$$K_{ab} \in H^{\infty}_{-\frac{3}{2}}$$

Asymptotic Conditions

Assume that the initial data is **asymptotically flat**

$$h_{ab} - \delta_{ab} \in H^{\infty}_{-\frac{1}{2}} \qquad (= o_{\infty}(|x|^{-\frac{1}{2}}))$$
$$K_{ab} \in H^{\infty}_{-\frac{3}{2}}$$

The initial data is called **stationary** if there exists $(N, N^i) \in H^2_{1/2}$ such that $\mathscr{P}^*(N, N^i) = 0$.

A stationary lemma

Lemma

Assume on the bdry ∂S that one has

$$\begin{cases} N|_{\partial S} = 0 \\ \Delta_h N|_{\partial S} = 0 \\ N^i|_{\partial S} = 0 \\ \not{D} N^i|_{\partial S} = 0 \end{cases}$$

then there exists a solution $(N, N^i) \in H^\infty_{1/2}$ to (AKE) if and only if the data is stationary.

Where $\not{D}$ is the covariant derivative along the normal direction to ∂S.

Main existence theorem

Theorem (S, Valiente Kroon '22)

Given smooth functions f, g, f^i, and h^i on ∂S, then the BVP

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0 \qquad \text{on } S,$$

$$\begin{cases} X|_{\partial S} = f, \\ \Delta_h X|_{\partial S} = g, \\ X^i|_{\partial S} = f^i, \\ \not{D} X^i|_{\partial S} = h^i, \end{cases}$$

has a unique solution such that

$$X = \lambda|x| + \vartheta, \qquad \vartheta \in H^\infty_{\frac{1}{2}}$$

$$X^i \in H^\infty_{\frac{1}{2}}.$$

Futhermore, λ vanishes **if** the initial data is stationary.

The number λ is **Dain's invariant**.

Dain's invariant

- Under time symmetry, $K_{ij} = 0$, Dain's invariant can be written as the boundary integral

$$\lambda = -\frac{1}{8\pi} \oint_{S_\infty} n^a D_a \Delta_h X \mathrm{d}S,$$

Dain's invariant

- Under time symmetry, $K_{ij} = 0$, Dain's invariant can be written as the boundary integral

$$\lambda = -\frac{1}{8\pi} \oint_{S_\infty} n^a D_a \Delta_h X \mathrm{d}S,$$

- It is the **obstruction to stationarity** at the asymptotic end,

Dain's invariant

- Under time symmetry, $K_{ij} = 0$, Dain's invariant can be written as the boundary integral

$$\lambda = -\frac{1}{8\pi} \oint_{S_\infty} n^a D_a \Delta_h X \mathrm{d}S,$$

- It is the **obstruction to stationarity** at the asymptotic end,
- It can be used to measure the deviation from stationarity at the asymptotic end.

Main existence theorem

Theorem (S, Valiente Kroon '22)

Given smooth functions f, g, f^i, and h^i on ∂S, then the BVP

$$\mathscr{P} \circ \mathscr{P}^* \begin{pmatrix} X \\ X^i \end{pmatrix} = 0 \qquad \text{on } S,$$

$$\begin{cases} X|_{\partial S} = f, \\ \Delta_h X|_{\partial S} = g, \\ X^i|_{\partial S} = f^i, \\ \not{D} X^i|_{\partial S} = h^i, \end{cases}$$

has a unique solution such that

$$X = \lambda |x| + \vartheta, \qquad \vartheta \in H^\infty_{\frac{1}{2}}$$

$$X^i \in H^\infty_{\frac{1}{2}}.$$

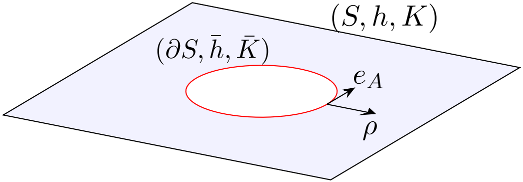
Futhermore, λ vanishes **if** the initial data is stationary.

Specifying the boundary data

How do we turn the **if** in the previous theorem into an **if and only if**?

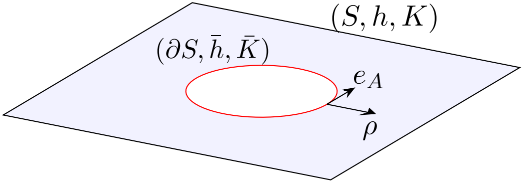
‘How much’ of the KID equations can be solved on ∂S?

To do this, perform a **2+1 decomposition** of the KID equations.



‘How much’ of the KID equations can be solved on ∂S?

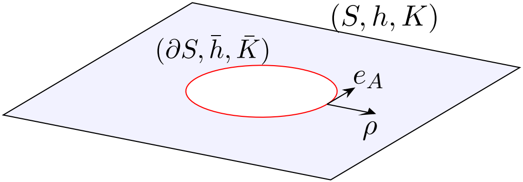
To do this, perform a **2+1 decomposition** of the KID equations.



Now, **assume time symmetry**, $K_{ij} = 0$. This assumption sets $X^i = 0$ on S.

‘How much’ of the KID equations can be solved on ∂S?

To do this, perform a **2+1 decomposition** of the KID equations.



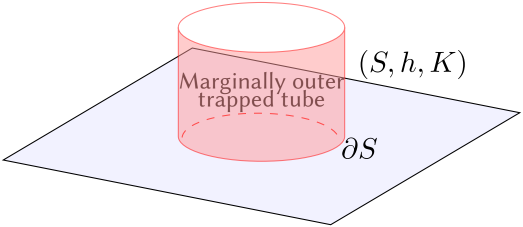
Now, **assume time symmetry**, $K_{ij} = 0$. This assumption sets $X^i = 0$ on S.

Projecting the KID equations with the 2-metric: $\bar{h}_{ij} = h_{ij} - \rho_i \rho_j$ onto ∂S obtains the intrinsic equation on ∂S:

$$\Delta_{\bar{h}} X - \frac{1}{2}(\bar{r} + |\bar{K}|^2) X = 0$$

where $|\bar{K}|^2 = \bar{K}_{AB} \bar{K}^{AB}$.

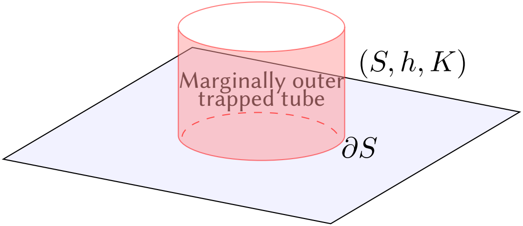
Stability of MOTS



A MOTS evolves into a marginally outer trapped tube if it is **stable** - the lowest eigenvalue of the MOTS stability operator, $\mathscr{L}$, is positive

$$\mathscr{L} \equiv -\Delta_{\bar{h}} + \frac{1}{2}(\bar{r} - |\bar{K}|^2)$$

Stability of MOTS



A MOTS evolves into a marginally outer trapped tube if it is **stable** - the lowest eigenvalue of the MOTS stability operator, $\mathscr{L}$, is positive

$$\mathscr{L} \equiv -\Delta_{\bar{h}} + \frac{1}{2}(\bar{r} - |\bar{K}|^2)$$

KID equation on ∂S: $\Delta_{\bar{h}} X - \frac{1}{2}(\bar{r} + |\bar{K}|^2)X = 0$

Existence of solutions to intrinsic KID equation

Lemma

If the MOTS is stable then the only solution to the intrinsic KID equation

$$\Delta_{\bar{h}} X - \frac{1}{2}(\bar{r} + |\bar{K}|^2) X = 0$$

is the trivial one: $X = 0$.

Existence of solutions to intrinsic KID equation

Lemma

If the MOTS is stable then the only solution to the intrinsic KID equation

$$\Delta_{\bar{h}} X - \frac{1}{2}(\bar{r} + |\bar{K}|^2)X = 0$$

is the trivial one: $X = 0$.

Setting $X = 0$ in the KID equations leaves only

$$D_i D_j X = 0$$

on ∂S. This suggests setting $\Delta_h X = 0$ as the boundary condition for (AKE).

A new invariant

The other components of the decomposition of the KID equations (normal-normal and normal-tangential etc.) leads to the following geometric invariant.

Lemma

Let $X = \Delta_h X = 0$ on ∂S. Then the KID equations are satisfied at ∂S if and only if $\omega = 0$ where

$$\omega = \oint_{\partial S} |\bar{K}|^2 |\not{D} X|^2 \mathrm{d}S.$$

A new invariant

The other components of the decomposition of the KID equations (normal-normal and normal-tangential etc.) leads to the following geometric invariant.

Lemma

Let $X = \Delta_h X = 0$ on ∂S. Then the KID equations are satisfied at ∂S if and only if $\omega = 0$ where

$$\omega = \oint_{\partial S} |\bar{K}|^2 |\not{D} X|^2 \mathrm{d}S.$$

Thus, in the time symmetric setting, The boundary conditions $X|_{\partial S} = \Delta_h X|_{\partial S} = 0$ with $\omega = 0$ is enough to ensure that $\lambda = 0$ **if and only if the data is stationary**.

Conversely, if $\omega \neq 0$ then the data is not stationary.

Conclusions

- **The non-time symmetric case?** Much harder to see how to specify boundary data because data must also be specified for X^i. For $\vec{X} = (X, X^A)$ the intrinsic part of the KID equations can be written as an elliptic system

$$\Delta_{\bar{h}}\vec{X} + T \cdot \bar{D}\vec{X} + C \cdot \vec{X} = \vec{F}.$$

The structure of this equation needs to be understood,

Conclusions

- **The non-time symmetric case?** Much harder to see how to specify boundary data because data must also be specified for X^i. For $\vec{X} = (X, X^A)$ the intrinsic part of the KID equations can be written as an elliptic system

$$\Delta_{\bar{h}}\vec{X} + T \cdot \bar{D}\vec{X} + C \cdot \vec{X} = \vec{F}.$$

  The structure of this equation needs to be understood,
- Unique continuation of the KID equations from ∂S to S,

Conclusions

- **The non-time symmetric case?** Much harder to see how to specify boundary data because data must also be specified for X^i. For $\vec{X} = (X, X^A)$ the intrinsic part of the KID equations can be written as an elliptic system

$$\Delta_{\bar{h}} \vec{X} + T \cdot \bar{D} \vec{X} + C \cdot \vec{X} = \vec{F}.$$

  The structure of this equation needs to be understood,
- Unique continuation of the KID equations from ∂S to S,
- Make connection with uniqueness of black holes using approximate symmetries,
- Evolution of Dain's invariant.

Thank you!