Uniqueness of the Characteristic Initial Value Problem in General Relativity

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Outlook

Contents

Standard Cauchy Problem (Choquet-Bruhat, Geroch,...)

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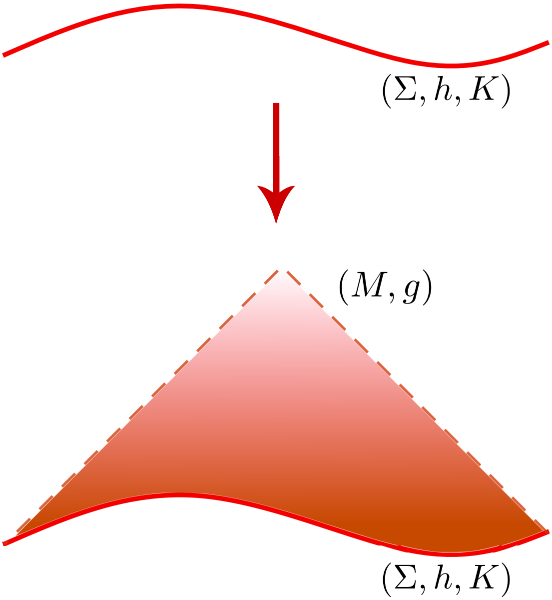
- Then there exists a **Ricci-flat ambient spacetime** in which (Σ, h, K) is embedded as a spacelike hypersurface, with h and K being the first and second fundamental forms, respectively.


(Σ, h, K)



(Σ, h, K)
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ABSTRACT INITIAL DATA

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EINSTEIN SPACETIME IN WHICH THE ABSTRACT DATA IS EMBEDDED

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- It is natural to ask **under which conditions** two initial data sets (Σ_1, h_1, K_1) and (Σ_2, h_2, K_2) **give rise to the same spacetime** (i.e. isometric spacetimes).

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Theorem

Isometric initial data give rise to isometric spacetimes.

ABSTRACT NOTION OF INITIAL DATA

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GEOMETRIC EXISTENCE THEOREM

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GEOMETRIC UNIQUENESS THEOREM

A natural question

IS IT POSSIBLE TO DO THE SAME WITH THE CHARACTERISTIC CAUCHY PROBLEM IN GR?

Characteristic Cauchy Problem (Rendall, Luk,...)

- The initial data is posed on **two null and transverse hypersurfaces** $\mathcal{H}$ and $\underline{\mathcal{H}}$.

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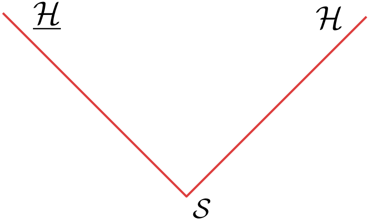

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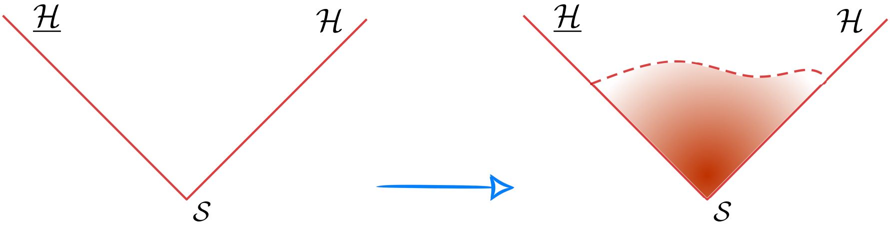

	"Standard" Cauchy Problem	"Characteristic" Cauchy Problem
Abstract Initial Data	(Σ, h, K)	
(Abstract) Existence Theorem		
(Abstract) Uniqueness Theorem		

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3. Prove **existence** and **uniqueness** theorems in this abstract framework

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- A particular case is **"Null Hypersurface Data"**. It consists of a set $\mathcal{D} = \left\{ \mathcal{H}, \gamma, \ell, \ell^{(2)}, Y \right\}$ consisting of a smooth manifold $\mathcal{H}$, a 2-covariant, symmetric tensor field γ, a one-form ℓ, a scalar $\ell^{(2)}$ and a 2-covariant, symmetric tensor field Y, satisfying:

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  2. $\mathrm{Rad}(\gamma) \neq \emptyset$ and γ is **semi-positive definite**.
- This definition is **abstract**: it does not require the presence of any ambient spacetime.

Gauge group

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- Given $\mathcal{D} = \{\mathcal{H}, \gamma, \ell, \ell^{(2)}, Y\}$ and $(z, \zeta) \in \mathcal{F}^{\star}(\mathcal{H}) \times \Gamma(T\mathcal{H})$ we define $\mathcal{D}' = \{\mathcal{H}, \gamma', \ell', \ell^{(2)\prime}, Y'\}$ by means of

$$\begin{aligned}
\gamma' &:= \gamma, \\
\ell' &:= z\left(\ell + \gamma(\zeta, \cdot)\right), \\
\ell^{(2)\prime} &:= z^2\left(\ell^{(2)} + 2\ell(\zeta) + \gamma(\zeta, \zeta)\right), \\
Y' &:= zY + \ell \otimes_s \mathrm{d}z + \frac{1}{2}\mathscr{L}_{z\zeta}\gamma.
\end{aligned}$$

Relation with a "standard" null hypersurface

Let $\mathcal{D} = \left\{\mathcal{H}, \gamma, \ell, \ell^{(2)}, Y\right\}$ be NHD.

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- The gauge freedom is associated to the freedom in the choice of a transverse vector to the hypersurface (the so-called **rigging vector** ξ).

Abstract constraint tensor

- It is possible to write down **all** tangential components of the ambient Ricci tensor in the embedded case in terms of the abstract data

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- The explicit form of f_{ab} is

$$\begin{aligned} f_{ab} = &\left(\gamma_{af}\overline{R}^{f}{}_{cbd} + 2\overline{\nabla}_{[d}\left(K_{b]c}\ell_a\right) + 2\ell^{(2)}K_{c[b}K_{d]a}\right)P^{cd} \\ &- \left(\ell_d\overline{R}^{d}{}_{bac} + 2\ell^{(2)}\overline{\nabla}_{[c}K_{a]b} + K_{b[a}\overline{\nabla}_{c]}\ell^{(2)} + \ell_d\overline{R}^{d}{}_{abc}\right. \\ &\left. + 2\ell^{(2)}\overline{\nabla}_{[c}K_{b]a} + K_{a[b}\overline{\nabla}_{c]}\ell^{(2)}\right)n^c. \end{aligned}$$

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- K, P, n and $\overline{\nabla}$ are abstract (they are defined from $\left\{\gamma, \ell, \ell^{(2)}, Y\right\}$).

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- This tensor allows us to define the constraint equations **abstractly** by

$$\mathcal{R}_{ab} = \lambda \gamma_{ab}.$$

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\mathcal{R}(n,X) &= -\big(\widetilde{\nabla}_n \tau\big)(X) + 2\nabla^h_X \omega - (\chi \cdot \tau)(X) - \tau(X) \operatorname{tr}_h \chi + \operatorname{div}(\chi)(X) \\
&\quad - \nabla^h_X \operatorname{tr}_h \chi - 2\omega \nabla^h_X \log|\lambda| - (\chi \cdot \mathrm{d}(\log|\lambda|))(X), \\
\mathcal{R}_{AB} &= R^h_{AB} - 2\widetilde{\nabla}_n \Upsilon_{AB} + 4\omega \Upsilon_{AB} - \nabla^h_A \eta_B - \nabla^h_B \eta_A - 2\eta_A \eta_B \\
&\quad - \chi_{AB} \operatorname{tr}_h \Upsilon - \Upsilon_{AB} \operatorname{tr}_h \chi, \\
H &= n\left(\operatorname{tr}_h \Upsilon\right) + |\eta|^2 + \operatorname{div}(\eta) - \frac{1}{2} R^h + \left(\operatorname{tr}_h \chi - 2\omega\right) \operatorname{tr}_h \Upsilon,
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- Then, $\mathcal{R}$ **generalizes them** to the **abstract setting** and to **any gauge**.

Contents

- Given NHD $\mathcal{D} = \{\mathcal{H}, \gamma, \ell, \ell^{(2)}, Y\}$ and a non-degenerate submanifold $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{H}$ (i.e. $i^{\star}\gamma =: h$ is a metric), we would like to define its two second fundamental forms and torsion one-forms.

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$$\mathcal{K}^{\mathfrak{t}}(X, Z) := \frac{1}{2}\left(\mathscr{L}_T\gamma\right)(X, Z) + CY(X, Z) + \frac{1}{2}\left(X(C)\ell(Z) + Z(C)\ell(X)\right),$$

$$\daleth_{ij}(X) := \left(\ell(T_i) + C_i\ell^{(2)}\right)\left(X(C_j) - K(X, T_j)\right) + C_i\ell\left(\overline{\nabla}_X T_j\right) + \gamma(T_i, \overline{\nabla}_X T_j) + \frac{1}{2}C_iC_jX\left(\ell^{(2)}\right) + C_j\Pi(X, T_i)$$

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- When the data is embedded, $t := \Phi_\star T + C\xi$ is **normal** to $\mathcal{S}$.
- Given a basis of NPs $\{\mathfrak{t}_1, \mathfrak{t}_2\}$ one can define two 2-covariant tensors $\mathcal{K}^{\mathfrak{t}_1}$, $\mathcal{K}^{\mathfrak{t}_2}$ together with a one-form $\daleth[\mathfrak{t}_1, \mathfrak{t}_2]$.
- When $\mathcal{D}$ happens to be embedded, $\mathcal{K}^{\mathfrak{t}_1}$, $\mathcal{K}^{\mathfrak{t}_2}$ and $\daleth[\mathfrak{t}_1, \mathfrak{t}_2]$ coincide with the two **second fundamental forms** and the **torsion one-form** in the basis $\{t_1 = T_1 + C_1\xi, t_2 = T_2 + C_2\xi\}$.

Contents

Idea of the definition

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- The ∂-isometry Ψ is unique given $\mu \in \mathcal{F}^{\star}(S)$ and ϕ.
- We still need to impose some **compatibility conditions** on $\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$:

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- They are **gauge-covariant** and independent on the basis of normal pairs.

Contents

Existence theorem

Theorem (M. Mars, G. Sánchez-Pérez, arXiv:2205.15267)

Let $\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$ be double null data of dimension $m > 1$ satisfying the abstract constraint equations

$$\mathcal{R} = \frac{2\Lambda}{m-1}\gamma \quad \textit{and} \quad \underline{\mathcal{R}} = \frac{2\Lambda}{m-1}\underline{\gamma},$$

where $\Lambda \in \mathbb{R}$. Then there exists a development $(\mathcal{M}, g)$ of $\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$ (possibly restricted if necessary) solution of the Λ-vacuum Einstein equations. Moreover, for any two such developments $(\mathcal{M}, g)$ and $(\widehat{\mathcal{M}}, \widehat{g})$, there exist neighbourhoods of $\mathcal{H} \cup \underline{\mathcal{H}}$, $\mathcal{U} \subseteq \mathcal{M}$ and $\widehat{\mathcal{U}} \subseteq \widehat{\mathcal{M}}$, and a diffeomorphism $\varphi : \mathcal{U} \longrightarrow \widehat{\mathcal{U}}$ such that $\varphi^{\star}\widehat{g} = g$.

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Definition

We say that two double null data $\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$ and $\{\widehat{\mathcal{D}}, \underline{\widehat{\mathcal{D}}}, \widehat{\mu}\}$ are isometric if there exist diffeomorphisms $\psi : \mathcal{H} \longrightarrow \widehat{\mathcal{H}}$ and $\underline{\psi} : \underline{\mathcal{H}} \longrightarrow \underline{\widehat{\mathcal{H}}}$ and gauge parameters (z, ζ) and $(\underline{z}, \underline{\zeta})$ in $\mathcal{D}$ and $\underline{\mathcal{D}}$, respectively, such that the pull-back double null data $\Xi^\star\{\widehat{\mathcal{D}}, \underline{\widehat{\mathcal{D}}}, \widehat{\mu}\}$ satisfies

$$\Xi^\star\{\widehat{\mathcal{D}}, \underline{\widehat{\mathcal{D}}}, \widehat{\mu}\} = \mathcal{G}\left(\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}\right).$$

Uniqueness theorem

Theorem (M. Mars, G. Sánchez-Pérez, arXiv:2301.02722)

Let $\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$, $\{\widehat{\mathcal{D}}, \widehat{\underline{\mathcal{D}}}, \widehat{\mu}\}$ *be two isometric DND satisfying the abstract constraint equations, and let* $(\mathcal{M}, g)$, $(\widehat{\mathcal{M}}, \widehat{g})$ *be respective developments. Then there exist neighbourhoods* $\mathcal{U} \subseteq \mathcal{M}$ *and* $\widehat{\mathcal{U}} \subseteq \widehat{\mathcal{M}}$ *of* $\mathcal{H} \cup \underline{\mathcal{H}}$ *and* $\widehat{\mathcal{H}} \cup \widehat{\underline{\mathcal{H}}}$*, respectively, and a diffeomorphism* $\varphi : \mathcal{U} \longrightarrow \widehat{\mathcal{U}}$ *such that* $\varphi^{\star}\widehat{g} = g$.

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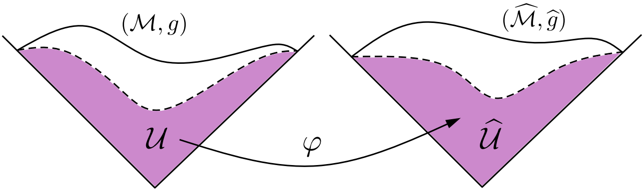

	"Standard" Cauchy Problem	"Characteristic" Cauchy Problem
Abstract Initial Data	(Σ, h, K)	$\{\mathcal{D}, \underline{\mathcal{D}}, \mu\}$
(Abstract) Existence Theorem		
(Abstract) Uniqueness Theorem		

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- It encompasses **all possible initial data** constructed from the metric and its first transverse derivative.
- It allows us to **work in any gauge** (not necessarily in the one in which the null structure equations are written).
- We want to study characteristic **Killing Initial Data** within this formalism (end-of-degree project with a student) and also the conformal field equations in arbitrary dimension.

References

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