Gluing constructions for Lorentzian length spaces

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- Introduction to gluing
- Main results: Reshetnyak gluing theorem and causal inheritance
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General introduction

- The theory of Lorentzian length spaces (LLS) can be described as a synthetic version of Lorentzian geometry.
- Inspired by the relationship between metric geometry and Riemannian geometry.
- LLS are a comparatively young approach, hence some elementary concepts are not fully developed yet.
- In the metric picture, gluing is of fundamental importance. It is expected that this is the same case on the Lorentzian side.

Definition (Lorentzian pre-length space).

A tuple $(X, d, \ll, \leq, \tau)$ is called a *Lorentzian pre-length space* (LpLS) if it satisfies the following:

(i) $(X, \ll, \leq)$ is a causal space, i.e., $\leq$ is a reflexive and transitive relation on X and $\ll$ is a transitive relation on X contained in $\leq$ $(x \ll y \Rightarrow x \leq y)$.

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(ii) $\tau : X \times X \to [0, \infty]$ is lower semi-continuous w.r.t. the metric d, i.e., $\liminf \tau(x_n, y_n) \geq \tau(x, y)$.

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(ii) $\tau : X \times X \to [0, \infty]$ is lower semi-continuous w.r.t. the metric d, i.e., $\liminf \tau(x_n, y_n) \geq \tau(x, y)$.

(iii) τ respects the causal structure in the following way: if $x \leq y \leq z$ then $\tau(x, z) \geq \tau(x, y) + \tau(y, z)$ and $\tau(x, y) > 0 \iff x \ll y$.

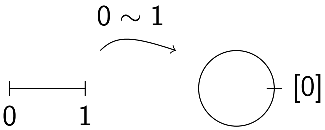
Lorentzian pre-length spaces: basics II

- If X is *intrinsic*, i.e., τ is given by the length of curves, and some technical assumptions hold additionally, then X is a *Lorentzian length space* (LLS).
- "LpLS $\leftrightsquigarrow$ LLS" $=$ "metric space $\leftrightsquigarrow$ length space".
- Any smooth spacetime is a LpLS. Any smooth strongly causal spacetime is a LLS.

- In semi-Riemannian manifolds, sectional curvature bounds are equivalent to *triangle comparison* (using the *signed distance* $\mathrm{sgn}(\gamma)\sqrt{|\langle\gamma'(0),\gamma'(0)\rangle|}$ for a geodesic γ).
- *Geodesics* and *Geodesic triangles* can also be defined in metric spaces and LpLS, without relying on metric tensor or manifold structure.
- Thus, can also define triangle comparison as substitute for sectional curvature in the abstract setting of metric spaces or LpLS.
- In summary, a LpLS has *timelike curvature bounded above (or below) by* K if triangles are slimmer (fatter) than in the Lorentzian model space of constant curvature K.

Gluing: introduction/motivation

- Visually, one can think of points that are identified via an equivalence relation as being glued together, e.g., $[0,1]/\{0,1\} \cong \mathbb{S}^1$.



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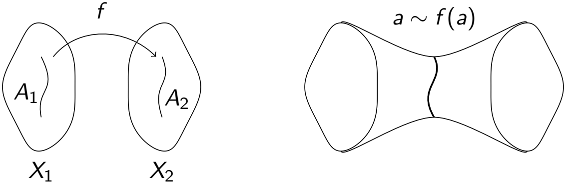
- Topological gluing: form disjoint union of spaces, then form quotient space where the equivalence relation is given by the identification of certain subsets via a map.



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- Metric gluing: do this process with metric spaces and equip the resulting space with a suitable metric.
- Gluing demonstrates one of the advantages of length spaces compared to Riemannian manifolds: easy to construct new spaces out of old ones.

Lorentzian gluing: disjoint union

"Lorentzian disjoint union" is easy to construct.

Definition (Lorentzian disjoint union).

Let $(X_1, d_1, \ll_1, \leq_1, \tau_1)$ and $(X_2, d_2, \ll_2, \leq_2, \tau_2)$ be LpLS. Set $X := X_1 \sqcup X_2$ and define $\ll := \ll_1 \sqcup \ll_2$, i.e., $x \ll y :\Leftrightarrow \exists i \in \{1, 2\} : x, y \in X_i \wedge x \ll_i y$, and $\leq := \leq_1 \sqcup \leq_2$. Define

$$d(x, y) := \begin{cases} d_i(x, y) & x, y \in X_i \\ \infty & \text{else}, \end{cases}$$

and

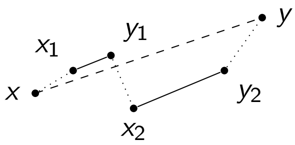
$$\tau(x, y) := \begin{cases} \tau_i(x, y) & x, y \in X_i \\ 0 & \text{else}. \end{cases}$$

Then $(X, d, \ll, \leq, \tau)$ is called the *Lorentzian disjoint union* of X_1 and X_2 (this is always a LpLS).

Lorentzian gluing: quotient structure

Let $(X, d, \ll, \leq, \tau)$ be LpLS and $\sim$ equivalence relation. *Quotient semi-metric* $\widetilde{d}$ is already known:

$$\widetilde{d}([x],[y]) := \inf\{\sum_{i=1}^{n} d(x_i, y_i) \mid x \sim x_1, y \sim y_n, y_i \sim x_{i+1}\}.$$



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Definition (Quotient Lorentzian structure).

The *quotient time separation function* $\widetilde{\tau}$ is defined as

$$\widetilde{\tau}([x],[y]) := \sup\{\sum_{i=1}^{n} \tau(x_i, y_i) \mid x \sim x_1, y \sim y_n, y_i \sim x_{i+1}, x_i \leq y_i\}.$$

Moreover, define $[x] \mathrel{\widetilde{\ll}} [y] :\Leftrightarrow \widetilde{\tau}([x],[y]) > 0$ and $[x] \mathrel{\widetilde{\leq}} [y] :\Leftrightarrow \{\sum_{i=1}^{n} \tau(x_i, y_i) \mid \ldots\} \neq \emptyset$.

Lorentzian gluing: amalgamation I

To glue two (or more) LpLS, put these two concepts together: Form Lorentzian disjoint union $X_1 \sqcup X_2$ and choose closed subsets $A_i \subseteq X_i$ together with a map f which "preserves structure": need $f : A_1 \to A_2$ to be

- τ-preserving ($\tau_1(a, b) = \tau_2(f(a), f(b))$)
- $\leq$-preserving ($a \leq_1 b \iff f(a) \leq_2 f(b)$)
- locally bi-Lipschitz homeomorphism (ensures $\widetilde{d}$ is definite).

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Then apply quotient process to the disjoint union with respect to the equivalence relation generated by $a \sim f(a)$.

The resulting space is almost a LpLS: it may happen that $\widetilde{\tau}$ is not lower semi-continuous. The following condition on the glued sets solves this issue:

Definition (Non-timelike local isolation).

A subset $A \subseteq X$ of a LpLS is called *non-timelike locally isolating* if $\forall a \in A$ and all nbhds U of a $\exists b_-, b_+ \in U \cap A : b_- \ll a \ll b_+$.

Lorentzian gluing: amalgamation II

Definition (Lorentzian amalgamation).

Let X_1 and X_2 be two LpLS, $A_i \subseteq X_i$ closed and non-timelike locally isolating subsets and $f : A_1 \to A_2$ a τ- and $\leq$-preserving locally bi-Lipschitz homeomorphism. Consider the Lorentzian disjoint union $X_1 \sqcup X_2$ and let $\sim$ be the equivalence relation on $X_1 \sqcup X_2$ generated by $a \sim f(a)$. Then $((X_1 \sqcup X_2)/\sim, \widetilde{d}, \widetilde{\ll}, \widetilde{\leq}, \widetilde{\tau})$ is called *Lorentzian amalgamation* of X_1 and X_2 and denoted by $X_1 \sqcup_A X_2$.

$X_1 \sqcup_A X_2$ is always a LpLS.

Lorentzian gluing theorem

Reshetnyak gluing theorem: gluing of metric spaces is compatible with upper curvature bounds (X_1, X_2 CAT$(k) \Rightarrow X_1 \sqcup_A X_2$ CAT(k)).

Due to the missing concept of spacelike distance in LpLS, a Lorentzian version is currently only possible for spacetimes.

Theorem (Beran, R., '22).

Let X_1 and X_2 be two smooth strongly causal spacetimes. Let $A_i \subseteq X_i$ be closed non-timelike locally isolating and $f : A_1 \to A_2$ a τ- and $\leq$-preserving locally bi-Lipschitz homeomorphism which locally preserves the signed distance. If A_1 and A_2 are *convex* ("$\forall x, y \in A_i : \gamma_{xy} \subseteq A_i$") and X_1 and X_2 have sectional curvature bounded above by $K \in \mathbb{R}$, then the Lorentzian amalgamation $X_1 \sqcup_A X_2$ is a Lorentzian pre-length space with timelike curvature bounded above by K.

Gluing of LLS and the causal ladder I

- Investigate the compatibility of gluing and the causal ladder, as well as other elementary properties of LpLS.
- For example: if X_1 and X_2 are strongly causal or causally path-connected, what about $X_1 \sqcup_A X_2$?
- Most steps of the causal ladder appear to interact well with gluing.

Gluing of LLS and the causal ladder II

Theorem (R., '22).

Let X_1 and X_2 be two LpLS and X the Lorentzian amalgamation.

(i) If X_i are strongly causal and locally compact LLS, then X is a LLS.

(ii) If X_i are chronological/causal/strongly causal, then so is X.

(iii) If X_i are globally hyperbolic LLS with A_i time observing ($\forall x, y \in X_i \; \exists a, b \in A_i : J(x, y) \cap A_i \subseteq J(a, b) \cap A_i$), then X is globally hyperbolic (causal + $J(x, y)$ cpt.).

Outlook to Lorentzian gluing

Possible applications and further work in this direction include:

- Globalization/Alexandrov's Patchwork (done!).
- Generalize Reshetnyak via spacelike distance for LpLS.
- Matching of spacetimes.
- Gluing of spaces with lower curvature bounds along boundary.
- Gluing of spaces with synthetic Ricci curvature bounds.

References

S. B. Alexander and R. L. Bishop.
Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds.
Comm. Anal. Geom., 16(2):251–282, 2008.

T. Beran, L. Napper, and F. Rott.
Alexandrov's Patchwork and the Bonnet-Myers theorem for Lorentzian length spaces, 2023.
https://arxiv.org/abs/2302.11615.

T. Beran and F. Rott.
Gluing constructions for Lorentzian length spaces., 2022.
https://arxiv.org/abs/2201.09695.

F. Cavalletti and A. Mondino.
Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications., 2020.
https://arxiv.org/abs/2004.08934.

M. Kunzinger and C. Sämann.
Lorentzian length spaces.
Ann. Glob. Anal. Geom., 54(3):399–447, 2018.

E. Minguzzi.
Further observations on the definition of global hyperbolicity under low regularity, 2023.
https://arxiv.org/abs/2302.09284.

F. Rott.
Gluing of Lorentzian length spaces and the causal ladder., 2022.
https://arxiv.org/abs/2209.06894.