Globalization of Curvature Bounds in Lorentzian Pre-length Spaces

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Outline

1. Motivation and Metric Results
2. Alexandrov's Patchwork for Lorentzian Length Spaces
2.1. Recap of Definitions
2.2. Geodesic Fan and Finite Cover
2.3. Triangulation and Gluing Lemma
3. Comments and Outlook
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Synthetic Lorentzian Framework

- This morning we have heard about Lorentzian length space framework of [Kunzinger, Sämann 2018] which aims to be to smooth Lorentzian geometry, what metric length spaces are to Riemannian geometry.
- Used to extend the scope of results on smooth structures to lower regularity ones.
- Lower regularity metrics are useful in the study of physically relevant space-times i.e. cosmic strings, gravitational waves, quantum foam.

Spaces with Curvature Bound

- In the metric case, one tames the properties of spaces by assuming global curvature bound.
- Such spaces have been used to provide results in a wide range of fields — group theory, PDE theory, algebraic topology.
- We want to translate this wider toolkit to the Lorentzian framework, starting with conditions on when a space with local upper curvature bound has a global upper bound.

Alexandrov's Patchwork - Metric Version

Theorem (Alexandrov 1957)

Let X be a metric length space with local curvature bounded above by $k \in \mathbb{R}$ and assume that there exists a unique geodesic joining each pair of points in X which are less than $\mathrm{diam}(M_k)$ apart. If these geodesics vary continuously with their endpoints, then X has global curvature bounded above by k (i.e. X is a $CAT(k)$ space).

A geodesic varies continuously with its endpoints when $x_n \to x$, $y_n \to y$ implies $\gamma_{x_n y_n} \to \gamma_{xy}$ uniformly.

Recap – Lorentzian Pre-length Space

Let (X, d) be a metric space equipped with relations $\leq$, $\ll$ and (time-separation) function $\tau : X \times X \to [0, \infty]$ satisfying

(i) $\leq$ is reflexive and transitive, $\ll$ is transitive and contained in $\leq$

(ii) τ is lower semi-continuous w.r.t d

(iii) $\tau(x,z) \geq \tau(x,y) + \tau(y,z)$ for $x \leq y \leq z$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$

then $(X, d, \ll, \leq, \tau)$ is called a Lorentzian pre-length space.

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The timelike diamond governed by points x, $z \in X$ is given by $I(x, z) := \{y \in X \,|\, x \ll y \ll z\}$.

Recap – Types of Curve in X

- Locally Lipschitz curves $\gamma : [a, b] \to X$ are called future-directed, timelike curves if $\gamma(s) \ll \gamma(t)$ for all parameter values $s < t$. (Analogously for past-directed/ causal)
- Call a causal curve γ_{xy} from x to y a geodesic if it maximises its τ-length, i.e. $L_\tau(\gamma) = \tau(x, y)$.

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- Call a causal curve γ_{xy} from x to y a geodesic if it maximises its τ-length, i.e. $L_\tau(\gamma) = \tau(x,y)$.
- X is called (uniquely) geodesic if there exists a (unique) geodesic between each pair of causally related points in X.
- X is called regular if all geodesics γ_{xy} connecting $x \ll y$ are timelike.

Recap – Triangle Comparison

- M_k denotes the Lorentzian model space of constant curvature k.
- A timelike triangle $\Delta(x, y, z)$ in X consists of three points $x \ll y \ll z$ and three geodesics γ_{xy}, γ_{yz} γ_{xz} between them.

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- A timelike triangle $\Delta(x, y, z)$ in X consists of three points $x \ll y \ll z$ and three geodesics γ_{xy}, γ_{yz} γ_{xz} between them.
- A comparison triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ is a timelike triangle in M_k whose sides are the same τ-length as $\Delta(x, y, z)$ in X.
- A comparison point for $p \in \gamma_{xy}$ (similarly $\gamma_{y,z}$, γ_{xz}) is the unique point $\bar{p} \in \gamma_{\bar{x},\bar{y}}$ satisfying

$$\tau(x, p) = \tau_k(\bar{x}, \bar{p}) \text{ and } \tau(p, y) = \tau_k(\bar{p}, \bar{y})$$

Recap – (Local) Timelike Curvature Bounds

An open set $U \subseteq X$ is a timelike $\leq k$ comparison neighbourhood if

(i) τ is finite and continuous on $U \times U$

(ii) There exists a geodesic contained in U between all $x \ll y$ in U

(iii) For all p, q on the sides of timelike triangles $\Delta(x, y, z)$ and comparison points $\bar{p}$, $\bar{q}$ on $\Delta(\bar{x}, \bar{y}, \bar{z})$ in M_k, one has

$$\tau(p, q) \geq \tau_k(\bar{p}, \bar{q})$$

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X has (local) timelike curvature bounded above if it is covered by such U.

X has global timelike curvature bounded above if X is such a neighbourhood - big triangles satisfy (iii).

Translating the Metric Assumptions

Given local upper curvature bounds, which constraints then imply global upper curvature bounds?

- As in metric case, we assume existence (ii) and uniqueness of geodesics between $x \ll y$ in X.
- We also want to 'continuously vary geodesic endpoints' along other geodesics.

Translating the Metric Assumptions

Given local upper curvature bounds, which constraints then imply global upper curvature bounds?

- As in metric case, we assume existence (ii) and uniqueness of geodesics between $x \ll y$ in X.
- We also want to 'continuously vary geodesic endpoints' along other geodesics.
- Can only do so if the second geodesic is a timelike curve — varying the endpoint along a null piece causes issues.
- To fix this, also assume that X is regular.

Additional Assumptions

- A Lorentzian pre-length space X is called strongly causal if $\{I(x,y) \,|\, x,y \in X\}$ is a subbase for the topology induced by d.
- A Lorentzian pre-length space X is called non-timelike locally isolating if $\forall x \in X$ and all neighbourhoods $U \subseteq X$ of x, there exists x_-, $x_+ \in U$ such that $x_- \ll x \ll x_+$.

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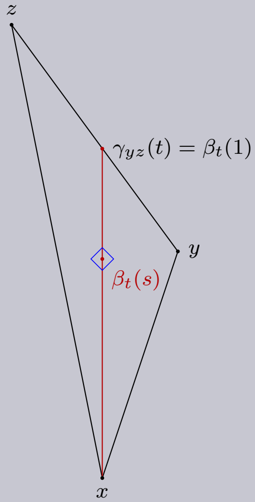
- Metric – can describe neighbourhoods using metric balls. Lorentzian – want to describe neighbourhoods in terms of τ.
- Assuming the above, if X has local curvature bound, then there is a neighbourhood basis of diamonds which are comparison neighbourhoods at each $x \in X$.

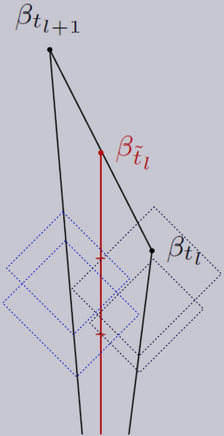
Geodesic Fan

- Parametrise geodesics by $[0,1]$.
- Can construct a (unique) timelike geodesic β_t from x to any point $\gamma_{yz}(t)$ on γ_{yz}.
- β varies continuously with t via $\gamma_{yz}(t)$.
- Any point $\beta_t(s)$ on β_t has a comparison neighbourhood which is a timelike diamond.
- Can choose governing points to be on β_t by continuity of β_t in s (for $s \in (0,1)$).



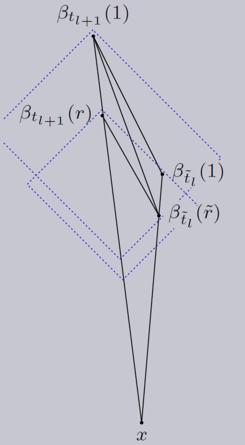
Finite Covering of Diamonds

- Filled triangle is compact (β continuous on compact set $[0,1] \times [0,1]$) so can extract finite subcover of diamonds.
- In particular, carefully covering finitely many β_t also covers the triangle.
- Can do this in such a way that the diamonds overlap and the overlap completely contains a geodesic from x to some $\gamma_{yz}(\tilde{t})$.



Triangulation Process

- $\beta_{\tilde{t}_l}$ and $\beta_{t_{l+1}}$ are both covered by the same diamonds and form slim triangle.
- Choose a point $\beta_{\tilde{t}_l}(\tilde{r})$ in the intersection of the top two diamonds.
- Also choose a point $\beta_{t_{l+1}}(r)$ which is in the second diamond and timelike after $\beta_{\tilde{t}_l}(\tilde{r})$.



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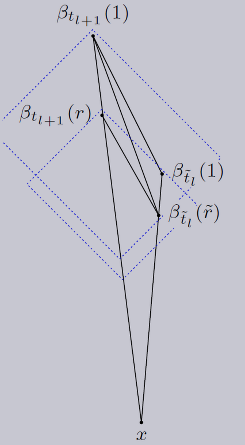
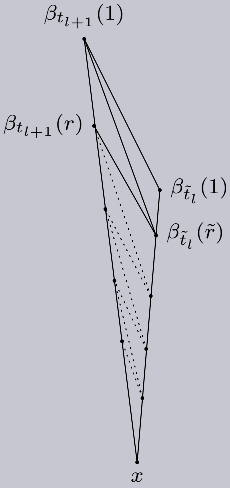
- Also choose a point $\beta_{t_{l+1}}(r)$ which is in the second diamond and timelike after $\beta_{\tilde{t}_l}(\tilde{r})$.
- Join these points by a geodesic contained in the intersection of the last two diamonds.
- Split the quadrilateral into two triangles which are contained in the final diamond.



Triangulation Process

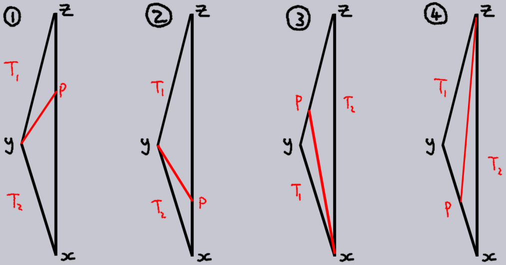
- Use subsequent pairs of timelike diamonds to repeat this for the rest of the thin triangle.
- Each of the smaller triangles now lives in a comparison neighbourhood given by the timelike diamond.
- As we have local curvature bounds, these satisfy curvature comparison.
- Also works on the triangles of form $\Delta(x, \beta_{t_l}(1), \beta_{\tilde{t}_l})$ as $\beta_{\tilde{t}_l}$ also lies in the β_{t_l} cover.



Gluing Lemma for Timelike Triangles

Theorem (Beran, Rott 2022)

Let X be a Lorentzian pre-length space and $U \subseteq X$ be an open subset satisfying (i) and (ii) for timelike curvature bounds. If a timelike triangle $\Delta(x, y, z)$ in U can be split in any of the below ways, such that T_1 and T_2 satisfy (iii) of curvature bounds for some k, then $\Delta(x, y, z)$ also satisfies (iii) for k.



Globalization of Finiteness and Continuity

- We require X to globally satisfy (i) to apply Gluing Lemma.
- Can prove (not today) that continuity globalizes with the same assumptions as (iii) — do not need to assume it.

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- We require X to globally satisfy (i) to apply Gluing Lemma.
- Can prove (not today) that continuity globalizes with the same assumptions as (iii) — do not need to assume it.
- Metric theory only considers points in X which are less than $\mathrm{diam}(M_k)$ apart in spaces with larger diameter.
- To bring this to Lorentzian pre-length spaces, we need to modify the definition of curvature bounds to consider only such curves.

Lorentzian Alexandrov's Patchwork

Theorem (Beran, N., Rott 2023)

Let X be a strongly causal, non-timelike locally isolating, and regular Lorentzian pre-length space with local curvature bounded above by k. Assume that there exists a unique geodesic between each pair of points $x \ll y$ in X and that geodesics vary continuously with their endpoints. Then X has global curvature bounded above by k.

Outlook

- Work so far suggests definition of curvature bounds needs minor technical adjustment to only consider 'not too long' curves.
- Metric length spaces also have globalization theorem for curvature bounded below [Toponogov 1959, Burago et al 1992]
- Lorentzian case does not have this yet, but we do have a bound on finite diameter for spaces with global lower bound.
- Locally finite, connected metric graphs have local upper curvature bound [Burago et al 2001] — Lorentzian analogue seems to be causal sets. Do they play well with our framework?

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