The calculation of the asymptotic charges on the critical sets of null infinity

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Motivation

Asymptotic symmetries are linked to:

- **Observation:** The action of asymptotic symmetries can be directly measured by the *gravitational memory effect*. e.g. Gravitational wave detectors [Paul D. Lasky *et al.* (2016), Oliver M. Boersma *et al.* (2020)]
- **Theory:** (Soft theorems) asymptotic symmetry of gravitational scattering: *antipodal* subgroup of $\mathrm{BMS}^{+} \times \mathrm{BMS}^{-}$ [A. Strominger (2014)], *super-rotations* [G. Barnich & C.Troessaert (2010)], [D. Kapec *et al.* (2014)] & [F. Cachazo & A. Strominger (2014)].

Notations

- Signature: $(+---)$.
- Spinors: $v^a \to v^{AA'}$.
- Indices:
  - $a, b, c, \ldots \to$ abstract tensor indices.
  - $A, B, C, \ldots \to$ abstract spinor indices.
  - $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots \to$ frame tensor indices, $\boldsymbol{a} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$.
  - $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots \to$ frame spinor indices, $\boldsymbol{A} \in \{\mathbf{0}, \mathbf{1}\}$.

Conformal methods in GR

- Make use of *conformal transformations* $g_{ab} = \Xi^2 \tilde{g}_{ab}$. where Ξ is a C^∞ function known as the *conformal factor*.

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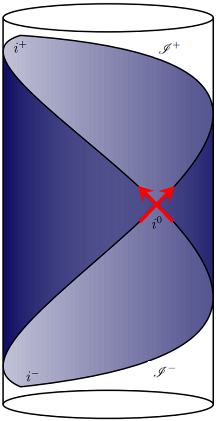
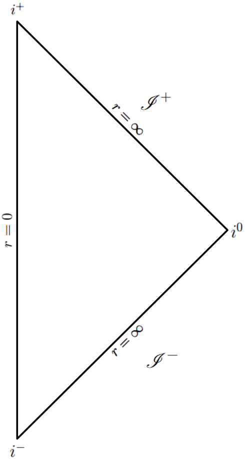
- Make use of *conformal transformations* $g_{ab} = \Xi^2 \tilde{g}_{ab}$. where Ξ is a C^∞ function known as the *conformal factor*.
- This transformation implies transformations laws of other fields $\tilde{R}_{ab} \rightarrow R_{ab}, \tilde{L}_{ab} \rightarrow L_{ab} \ldots$ etc.
- This allows us to write an equivalent of the Einstein field equations on the conformal manifold.

Standard conformal compactification of Minkowski

- $\eta_{ab} = \Xi^2 \tilde{\eta}_{ab}$.





Asymptotic symmetries

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- Matching problem: Relation between the asymptotic charges at past and future null infinity.

Asymptotic symmetries

- Asymptotic symmetries can be studied at: null infinity or spatial infinity.
- Matching problem: Relation between the asymptotic charges at past and future null infinity.
- **The problem of spatial infinity:**
  - Solution: Friedrich's cylinder at spatial infinity, Ashtekar's hyperboloid at spatial infinity.
  - Comparison: [M.M.A.Mohamed & J.A.V.Kroon (2021)].

Goal

Use Friedrich's formulation to evaluate the asymptotic charges at the critical sets of null infinity.

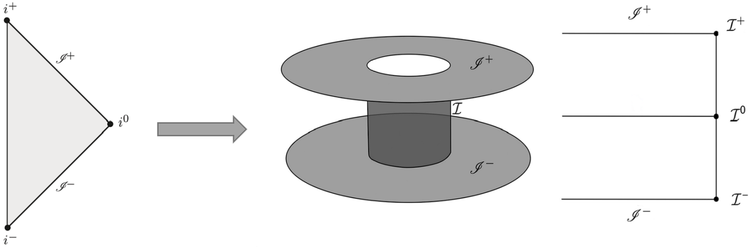
1. The asymptotic charges for the spin-2 field.

2. The asymptotic charges in full gravity.

Friedrich's cylinder at spatial infinity on Minkowski spacetime

$$\eta_{ab} = \Theta^2 \tilde{\eta}_{ab}, \quad \Theta = \rho(1-\tau^2),$$

$$\eta_{ab} = \boldsymbol{d}\tau \otimes \boldsymbol{d}\tau + \frac{\tau}{\rho}(\boldsymbol{d}\tau \otimes \boldsymbol{d}\rho + \boldsymbol{d}\rho \otimes \boldsymbol{d}\tau) - \frac{1-\tau^2}{\rho^2}\boldsymbol{d}\rho \otimes \boldsymbol{d}\rho - \boldsymbol{\sigma}.$$



Define the following sets of the conformal boundary ($\Theta = 0$):

$$\mathcal{I} \equiv \{p \in \mathcal{M} |\, |\tau(p)| < 1, \rho(p) = 0\} \;\; \text{cylinder at spatial infinity}$$

$$\mathcal{I}^{\pm} \equiv \{p \in \mathcal{M} |\, \tau(p) = \pm 1, \rho(p) = 0\} \;\; \text{critical sets of null infinity}$$

The supertranslation charges

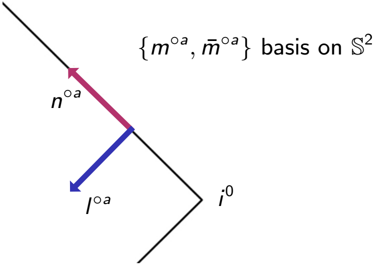
The spin-2 charges

$$\mathscr{P} = \int_{\mathcal{C}} \lambda \mathcal{W}^{\circ}_{abcd} l^{\circ a} n^{\circ b} m^{\circ c} \bar{m}^{\circ d} \mathrm{d}S.$$

where

$$\mathcal{W}^{\circ}_{abcd} \quad \text{Weyl-like tensor}$$
$$\{l^{\circ a}, n^{\circ a}, m^{\circ a}, \bar{m}^{\circ a}\} \quad \text{NP-null tetrad}$$

On $\mathscr{I}^+$,



Spinor expressions for the charges

- In spinors

$$\mathcal{W}^{\circ}_{abcd} \rightarrow W^{\circ}_{AA'BB'CC'DD'} = -\psi^{\circ}_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\psi}^{\circ}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}.$$

- Components of the spin-2 spinor

$$\psi^{\circ}_{0} \equiv \psi^{\circ}_{\mathbf{0000}}, \quad \psi^{\circ}_{1} \equiv \psi^{\circ}_{\mathbf{0001}}, \quad \psi^{\circ}_{2} \equiv \psi^{\circ}_{\mathbf{0011}},$$

$$\psi^{\circ}_{3} \equiv \psi^{\circ}_{\mathbf{0111}}, \quad \psi^{\circ}_{4} \equiv \psi^{\circ}_{\mathbf{1111}}.$$

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- Final expression for the spin-2 charges

$$\mathscr{P} \equiv \mathscr{P}(\lambda, \psi^{\circ}_{2}).$$

The spin-2 field on Minkowski

- The spin-2 field:

$$\Box \psi_{ABCD} = 0.$$

- Introduce F-gauge frames (adapted to conformal geodesics): $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$.
- NP-gauge to F-gauge: $\psi_2^{\circ} = \psi_2 \rightarrow \mathscr{P}(\lambda, \psi_2^{\circ}) = \mathscr{P}(\lambda, \psi_2)$.
- Expand near $\rho = 0$

$$\psi_2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{2;l,m}(\tau)\, {}_0Y_{l,m} + o_1(\rho),$$

The spin-2 field on Minkowski

- For ψ_2,

$$(1-\tau^2)\ddot{a}_{2;l,m} - 2\tau\dot{a}_{2;l,m} + l(l+1)a_{2;l,m} = 0,$$

- Given initial data

$$\psi_2(0) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{2;l,m}(0)\,{}_0Y_{l,m} + o_1(\rho),$$

the solution is given by

$$a_{2;l,m} = A_{l,m}P_l(\tau) + B_{l,m}Q_l(\tau)$$

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- The asymptotic charges are generally not well-defined at the critical sets. restrict the free initial data set.
- Initial data that satisfy *regularity condition* + assume $\lambda = Y_{l,m}$.

The supertranslation charges for the spin-2 field

Theorem

Given initial conditions for the spin-2 field equations satisfying certain regularity conditions, the asymptotic charges on $\mathcal{I}^+$ for the spin-2 field are given by

$$\mathscr{P}|_{\mathcal{I}^+} = \begin{cases} 2(l+1)Q_{l+1}(0)(a_2)_* & \text{for even } l \geq 0, \\ \sqrt{l(l+1)}Q_l(0)\left((a_1)_* - (a_3)_*\right) & \text{for odd } l. \end{cases}$$

If the regularity conditions are not satisfied then the supertranslation charges are not well defined at $\mathcal{I}^+$.

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- The charges on $\mathcal{I}^-$ are

$$\mathscr{P}|_{\mathcal{I}^-} = (-1)^l \mathscr{P}|_{\mathcal{I}^+}$$

The supertranslation charges in full gravity

- For a cut $\mathcal{C}$ of null infinity, the charges associated with a function $\lambda \in \mathbb{S}^2$ is given by

$$\mathscr{Q} = -\oint_{\mathcal{C}} \varepsilon_2 \lambda (\mathcal{P}^\circ + \tfrac{1}{2}\sigma^{\circ ab} N^\circ_{ab}).$$

where

$$\sigma^{\circ ab} \rightarrow \text{Shear tensor}, \qquad N^\circ_{ab} \rightarrow \text{News tensor}.$$

- The $\mathcal{P}^\circ$ is related to zero-spin component ϕ°_2 of the rescaled Weyl tensor $d^\circ_{abcd} = \Xi^{-1} C^\circ_{abcd}$.
- The final expression

$$\mathscr{Q} \equiv \mathscr{Q}(\lambda, \phi^\circ_2, \sigma^\circ, \gamma^\circ, \mu^\circ),$$

where

$$\sigma^\circ = \Gamma^\circ_{\mathbf{0}\mathbf{1}'}{}^{\mathbf{1}}{}_{\mathbf{0}}, \;\; \gamma^\circ = -\Gamma^\circ_{\mathbf{1}\mathbf{1}'}{}^{\mathbf{0}}{}_{\mathbf{0}}, \;\; \mu^\circ = \Gamma^\circ_{\mathbf{0}\mathbf{1}'}{}^{\mathbf{0}}{}_{\mathbf{1}}.$$

Translation from NP-gauge to F-gauge

- General transformation

$$g^{\circ}_{ab} = \theta^{2} g_{ab},$$

this implies

$$e^{\circ}_{a} = \theta^{-1} \Lambda^{b}{}_{a} e_{b},$$

and

$$\epsilon^{\circ}_{A} = \theta^{-\frac{1}{2}} \Lambda^{B}{}_{A} \epsilon_{B}.$$

- With this, we can write e.g.

$$\sigma^{\circ} \equiv \sigma^{\circ}\left(\theta, \Lambda^{A}{}_{B}, \Lambda_{B}{}^{A}, \Gamma_{AA'}{}^{C}{}_{D}\right),$$

$$\phi^{\circ}_{2} \equiv \phi^{\circ}_{2}\left(\theta, \Lambda^{A}{}_{B}, \epsilon_{A}, d_{AA'BB'CC'DD'}\right) \ldots \text{etc.}$$

- Evaluate for θ and $\Lambda^{b}{}_{a}$ (or $\Lambda^{B}{}_{A}$). *Simple in Minkowski.* In general, depends on solutions of field equations.

The Extended Conformal Field Equations (ECFE)

- $\tilde{R}_{ab} = 0.$
- $g_{ab} = \Xi^2 \tilde{g}_{ab}.$
- Introduce the Weyl connection $\hat{\nabla}_a g_{bc} = -2 f_a g_{bc}.$
- The extended conformal field equations are

$$[\boldsymbol{e_b}, \boldsymbol{e_b}] - \left(\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} - \hat{\Gamma}_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{a}}\right) \boldsymbol{e_c} = 0,$$

$$\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} - \hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}} = 0,$$
$$\hat{\nabla}_{\boldsymbol{c}} \hat{L}_{\boldsymbol{db}} - \hat{\nabla}_{\boldsymbol{d}} \hat{L}_{\boldsymbol{cb}} - d_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0,$$
$$\hat{\nabla}_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} - f_{\boldsymbol{a}} d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}} = 0,$$

for the unknowns $\left(\boldsymbol{e_a}, \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}, \hat{L}_{\boldsymbol{db}}, d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}\right)$. The tensors $\hat{P}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ and $\hat{\rho}^{\boldsymbol{c}}{}_{\boldsymbol{dab}}$ are the geometric and algebraic curvature, respectively.

The Extended Conformal Field Equations (ECFE)

- We require 3 extra supplementary equations to relate solutions of the conformal field equations to Einstein field equations.

Remark

Given a solution to the extended field equations and given that the supplementary equations are satisfied. Then the metric $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ is a solution to the vacuum Einstein field equations on the open set where $\Xi \neq 0$.

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- No equations to fix Ξ and $\hat{\nabla}$ (gauge freedom).

The Extended Conformal Field Equations (ECFE)

Fix gauge freedom using a conformal Gaussian gauge:

- Based on conformal geodesics.
- Fix Ξ and $\hat{\nabla}$ and are known a priori.
- Extended conformal field equations $\rightarrow$ symmetric hyperbolic system.
- Evolution equations $\rightarrow$ Transport system along the conformal geodesics.

Scalarising the equations introduces:

- 45 transport equations (Background fields).
- 5 Bianchi evolution + 3 Bianchi constraints.

The Extended Conformal Field Equations (ECFE)

- Consider initial data $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ [Lan-Hsuan Huang (2010)] with certain asymptotics as $r \to \infty$.
- The initial value problem for the ECFE is in general not regular at spatial infinity.

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- Consider initial data $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{K}})$ [Lan-Hsuan Huang (2010)] with certain asymptotics as $r \to \infty$.
- The initial value problem for the ECFE is in general not regular at spatial infinity.
- Conformal rescaling $\to$ regular conformal initial data in terms of conformal normal coordinates z^i.

$$\Omega = \kappa^{-1}\left(\rho^2 + \frac{1}{6}\Pi_3[\Omega]\rho^3 + O(\rho^4)\right) \quad \text{with } \kappa = O(\rho),$$

$$h_{ij} = \delta_{ij} + O(\rho^3), \quad K_{ij} = O(\rho).$$

From constraint equations,

$$L_{ij} = O(\rho), \quad d_{ij} = O(1), \quad d_{ijk} = O(1).$$

Zero-order solution

- At zero order, the transport system decouples from the Bianchi system.

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- At zero order, the transport system decouples from the Bianchi system.
- For the Bianchi system, we decompose our fields near $\rho = 0$ as follows

$$\phi_n = \sum_{l=2-n}^{\infty} \sum_{m=0}^{2l} \phi_{n;2l,m}(\tau) T_{2l}{}^{m}{}_{l-2+n} + O(\rho),$$

where $T_m{}^j{}_k$ are complex functions associated with representations of $SU(2,\mathbb{C})$.
- Bianchi system transforms to ODEs to solve for the coefficients $\phi_{n;2l,m}(\tau)$. Which component contribute to the charges at zero order?

The asymptotic charges

- Reminder:

$$\mathcal{Q} \equiv \mathcal{Q}(\lambda, \phi_2^\circ, \sigma^\circ, \gamma^\circ, \mu^\circ).$$

- NP-gauge to F-gauge:

e.g. $\sigma^\circ \equiv \sigma^\circ\left(\theta, \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}, \Lambda_{\boldsymbol{B}}{}^{\boldsymbol{A}}, \Gamma_{\boldsymbol{AA}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\right), \quad \phi_2^\circ \equiv \phi_2^\circ\left(\theta, \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}, \epsilon_{\boldsymbol{A}}, d_{\boldsymbol{AA}'\boldsymbol{BB}'\boldsymbol{CC}'\boldsymbol{DD}'}\right),$

- Calculations for $\theta, \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{B}}$ [H. Friedrich & J. Kánnár (2000)] e.g.

$$\theta = 1 + O(\rho),$$

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{1}} = O(\rho^{\frac{1}{2}}), \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{1}} = O(\rho^{\frac{3}{2}}),$$

$$\Lambda^{\boldsymbol{0}}{}_{\boldsymbol{0}} = O(\rho^{\frac{1}{2}}), \quad \Lambda^{\boldsymbol{1}}{}_{\boldsymbol{0}} = O(\rho^{-\frac{1}{2}}).$$

- Given this, we can show that only $\phi_{2;2l,m}(\tau)$ contribute at zero order to the charges.

The asymptotic charges

- Next: The asymptotic behaviour of $\sigma^\circ, \gamma^\circ, \mu^\circ$.
  Contribution at higher order (To be confirmed).
- The asymptotic charges can be written as

$$\mathscr{Q}^{(0)} \equiv \mathscr{Q}^{(0)}(\lambda, \phi_2^{(0)}).$$

  i.e. we only need to consider a solution for $\phi_2^{(0)}$ for the Bianchi system.
- Regularity conditions need to be imposed on the initial data in order to obtain well-defined charges at the critical sets.

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