The Future (in)Stability of Relativistic Perfect Fluids

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Joint work with Florian Beyer (Otago) and Todd Oliynyk (Monash)

Outline

1. Intro to Cosmology

2. Stability of Cosmological Solutions
   - Introduction to Cosmological Stability
   - Einstein-Euler in Gowdy Symmetry

Cosmology

Cosmology is concerned with the study of the large-scale behaviour of our universe.

Key questions for relativists include:

- Which solution of the field eqns corresponds to our universe? (Or at least a close approximation of it)
- What sort of behaviour do these models produce for the universe in the past or future?
- Other interesting mathematical/physical properties of these solutions?

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Initial Assumptions

As a first step towards answering these question we use a mix of observational data and (mainly) assumptions about the structure of our universe.

1. Spatial Isotropy - There is no geometrically preferred direction, i.e. the universe 'looks the same' in any direction.
2. Spatial Homogeneity - The universe looks the same from every point, i.e. it is isotropic at every point.

Current evidence suggests our universe, on large scales, is approximately isotropic and homogeneous.

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FLRW Metric

The standard spatially isotropic and homogeneous solution is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$g = -\mathrm{d}t^2 + a(t)^2 h_{IJ} \mathrm{d}x^I \mathrm{d}x^J, \tag{1}$$

where $a(t)$ is the scale factor which measures the *relative motion* of isotropic observers and h_{IJ} is the (time-independent) metric on spatial hypersurfaces.

Cosmological Fluids

- Only matter compatible with homogeneity and isotropy is a perfect fluid.
- Stress-energy tensor $\rightarrow$ $T^{ij} = (p+\rho)v^i v^j + pg^{ij}$.
- Need an equation of state to close system $\rightarrow$ $p = K\rho$ (linear).
  - $K = 0$ (Dust universe)
  - $K = 1/3$ (Radiation Fluid)
  - $K = 1$ (Stiff fluid)

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Non-Linear Stability Definition

Consider a solution $X_0 = (M, g, \phi)$ to the EFE which is future causally geodesically complete and generated from initial data $x_0 = (g_0, \phi_0) = (g, \phi)|_{t_0}$.

For initial data suitably close to x_0, if the corresponding maximal Cauchy development is future causally geodesically complete then we say X_0 is future stable.

Physical relevance - Are FLRW solutions stable to the future?

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Previous Results on Non-Linear Stability

FLRW fluid solutions with linear equation of state $p = K\rho$:

- *Euler between dust and radiation*: Speck '12, Rodnianski and Speck '13.
- *Radiation fluid*: Lübbe and Valiente '13.
- *Dust*: Hadzic and Speck '13.

Other stability results have been obtained Friedrich, LeFloch, Oliynyk, Beyer, Ringström, and Wei (Amongst many others)

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Current Picture

	$0 \leq K < 1/3$	$K = 1/3$	$1/3 < K < 1$
Isotropic	Stable	Stable	Unstable? **
Non-Isotropic		Stable	Stable*

*Only for relativistic Euler equations on fixed FLRW background [Oliynyk, 2021, Marshall and Oliynyk, 2022]

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Relativisitic Euler for K>1/3

Two key results:

1. For each $K \in (1/3, 1)$ the numerical solutions of the relativistic Euler equations display ODE behaviour at late times.
2. For each $K \in (1/3, 1)$ the density contrast $(\frac{\partial_I \rho}{\rho})$ of the fluid develops steep gradients near a finite number of spatial points where it becomes unbounded as $t \searrow 0$.

NB: Only holds for suitably small perturbations of FLRW solution

Einstein-Euler System

Is the behaviour of the density contrast the same when the fluid is coupled to the gravitational field?

The simplest way to test this numerically is by considering the Einstein-Euler equations in Gowdy symmetry.

Numerical Implementation - Gowdy Metric

Compactified $\mathbb{T}^3$-Gowdy metric:

$$g = \frac{1}{\tau^2}\left(e^{2(\eta-U)}(-e^{2\alpha}\mathrm{d}\tau^2 + \mathrm{d}\theta^2) + e^{2U}(dy + Adz)^2 + e^{-2U}\mathrm{d}z^2\right). \tag{2}$$

- The functions η, U, α, and A depend only on $(\tau, \theta) \in (0, 1] \times \mathbb{T}$.
- We take θ to be a periodic coordinate on the 1-torus $\mathbb{T}$ obtained by identifying the ends of the interval $[0, 2\pi]$.
- 1+1 problem with periodic boundary condtions
- Finite time domain to evolve over

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Numerical Implementation - Discretisation

Our numerical scheme is straightforward:

- 2nd order central finite difference stencils to discretise spatial derivatives.
- 2nd order Runge-Kutta (RK2) method to evolve in time.

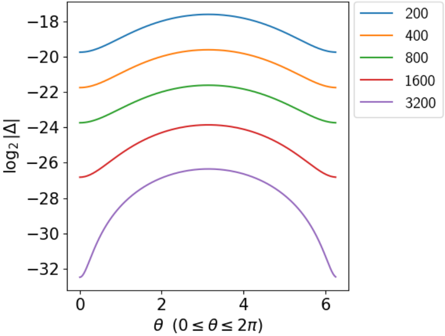

Figure: Convergence plot of $\tilde{\rho}$ at $\tau = 0.599$

Numerical Implementation - Einstein Equations

The Einstein equations are $G_{ab} + \Lambda g_{ab} = T_{ab}$.

For the Gowdy metric we get:

- Three wave equations for A, U, and η,
- A first order evolution equation for α,
- The Hamiltonian and momentum constraints.

First Order System

- Need to introduce first order variables for U, A.
- Use Hamiltonian constraint as evolution equation for η.

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Numerical Implementation - Euler Equations

We contract $\nabla_i T^{ij} = 0$ with v^j and $\delta^j_J - \frac{v_J}{v_0}\delta^j_0$ to get Euler equations

$$B^0 \begin{pmatrix} \tilde{\rho} \\ \tilde{v}_1 \end{pmatrix} + B^1 \begin{pmatrix} \tilde{\rho} \\ \tilde{v}_1 \end{pmatrix} = F_{\tilde{\mathbf{v}}}$$

Euler Variables

- Only two non-zero fluid components, v^0 and v^1
- Eliminate v^0 using normalisation condition
- Introduce new variables $\tilde{\rho}$ and $\tilde{v}_1$ to remove leading-order behaviour in τ

Numerical Implementation - Initial Data

We require that:

- The four-velocity must vanish somewhere on the domain.
- The momentum constraint is satisfied.
- The constraints from the definition of the first order variables are satisfied.
- The initial data is a (small) perturbation of the FLRW solution.

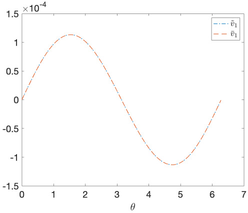
ODE Behaviour

How do we determine whether the system behaves like an ODE?

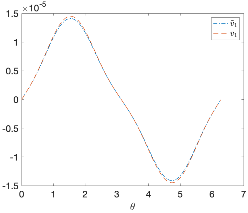
- If the system behaves like an ODE the spatial derivative terms should be negligible.
- Construct asymptotic system from EFEs by setting spatial derivatives to 0.
- Compare full EFEs and asymptotic system

$$\partial_t u + \partial_x u = f(u), \quad \text{Full System (PDE)}$$
$$\partial_t u = f(u), \quad \text{Asymptotic System (ODE)}$$

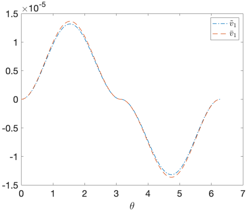
ODE Behaviour



(a) $\tau = 0.001$

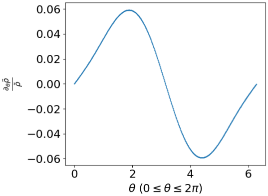


(b) $\tau = 5.55 \times 10^{-6}$



(c) $\tau = 9.79 \times 10^{-10}$

Density Contrast Blowup



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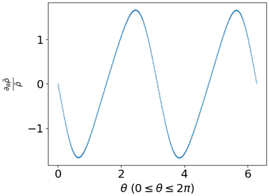

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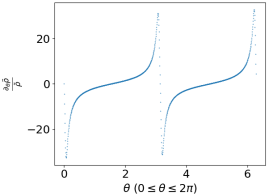

(c) $\tau = 3 \times 10^{-8}$

Future Directions

- ‘Generic’ initial data.
- Full 3+1 code (i.e. no symmetry).
- Stability of non-isotropic solutions when coupled to gravity.

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