Generating spacetimes with singularities

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Interdisciplinary junior scientist workshop:
Mathematical General Relativity

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Outline

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- Preliminary definitions
- Penrose's 1965 singularity theorem
- Singularities in the cosmological setting
- Singularities in the Gannon-Lee setting

In this talk singularity is synonymous with an incomplete null geodesic.

Preliminary definitions

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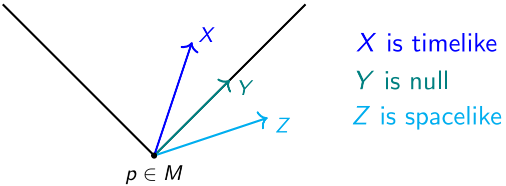
Definitions

- A spacetime (M, g) is a time-oriented Lorentzian manifold. For nonzero vectors $X \in T_pM$, we have a decomposition:

  1. $\langle X, X \rangle < 0$ iff X is timelike,
  2. $\langle X, X \rangle = 0$ iff X is null,
  3. $\langle X, X \rangle > 0$ iff X is spacelike.

  If X is timelike or null, then X is a causal vector.



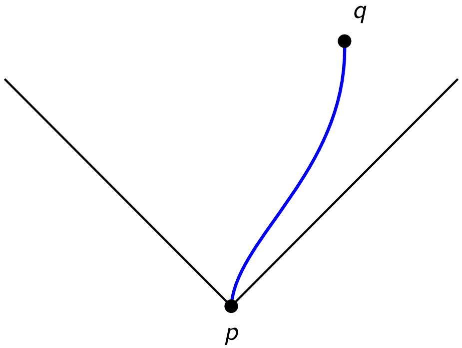
- Time-oriented means there is a timelike vector field X on M. If Y is causal, then we say Y is future or past if $\langle X, Y \rangle$ is negative or positive, respectively.

Preliminary definitions

- The causal future of a point p is the set

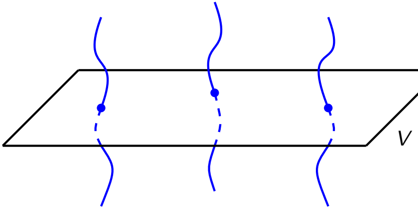
$$J^+(p) = \{q \mid \exists \text{ a future-directed causal curve from } p \text{ to } q\}$$

Similar definitions for $J^+(S)$ for any $S \subset M$ and the causal past J^-.



Preliminary definitions

- A Cauchy surface V is a subset of M such that every inextendible future directed timelike curve intersects V exactly once.



Remarks.

- Cauchy surfaces are automatically topological hypersurfaces.
- If M has a Cauchy surface V, then M is topologically $\mathbb{R} \times V$.
- In the 2000's Bernal and Sánchez improved the above result to a diffeomorphism and an orthogonal metric splitting.
- The existence of a Cauchy surface is equivalent to global hyperbolicity of M. In Lorentzian geometry, globally hyperbolic spacetimes often play the role of complete Riemannian manifolds.

Penrose's 1965 Singularity Theorem

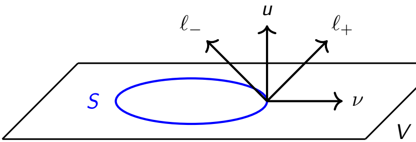
Penrose's 1965 Singularity Theorem

Trapped surfaces

- A surface S in a spacetime M (i.e. a codimension 2 submanifold) is future trapped if its mean curvature vector is past directed timelike.

Initial data perspective

Suppose V is a spacelike Cauchy surface and $S \subset V$ is two-sided.



$\ell_\pm = u \pm \nu$

- Null 2nd fundamental forms: $\chi_\pm(X, Y) = \langle \nabla_X \ell_\pm, Y \rangle$.
- Null expansion scalars: $\theta_\pm = \mathrm{tr}_S \chi_\pm = \mathrm{tr}_S K \pm H$
  - K is the 2nd fundamental form of V in M
  - H is the mean curvature of S in V.

- S is future trapped if $\theta_+ < 0$ and $\theta_- < 0$.

Penrose's 1965 Singularity Theorem

Theorem (Penrose (1965))

Suppose V is a noncompact Cauchy surface in a spacetime M satisfying the null energy condition, i.e. $\mathrm{Ric}(X, X) \geq 0$ for all null X. If M contains a future trapped compact surface S, then there is an incomplete future directed null geodesic emanating from S.

Sketch of proof.

$$\begin{aligned} \text{Geodesic completeness} \quad &\Longrightarrow \quad \partial J^+(S) \text{ is compact,} \\ &\Longrightarrow \quad \partial J^+(S) \approx V, \\ &\Longrightarrow \quad V \text{ is compact.} \\ &\rightarrow\leftarrow \end{aligned}$$

□

Topology and singularities in cosmological spacetimes

Cosmology in brief

- In the 1920's Hubble observed that the universe is expanding by measuring the redshift of distant galaxies.

- For the isotropic FLRW models of cosmology,

$$g = -dt^2 + a(t)^2 h,$$

  an expansion means $\dot{a}(t_0) > 0$ for our current cosmic time t_0.

- An expanding universe implies that the second fundamental form K is positive definite since for FLRW cosmology

$$K = \frac{\dot{a}(t)}{a(t)} h.$$

Definition. A spacelike Cauchy surface V is expanding in all directions if its second fundamental form K is positive definite.

Topology and singularities in cosmology

Theorem (Galloway and L. (2017))

Suppose V is a 3-dimensional compact spacelike Cauchy surface for M. Assume V is expanding in all directions and the NEC holds. If V is not a spherical space, then M is past null geodesically incomplete.

V is a spherical space if it's a quotient of the three-sphere.

Remarks.

- The assumption that V is expanding in all directions is stronger than the positive mean curvature assumption in Hawking's cosmological singularity theorem.
- However, we more than make up for it since we only assume the NEC whereas Hawking assumes the SEC. Therefore our theorem applies to spacetimes with $\Lambda > 0$ and inflationary models.
- De Sitter space and its quotients are examples of spacetimes with spherical space Cauchy surfaces expanding in all directions but are nevertheless complete.

Topology and singularities in cosmology

Proof of the theorem:

If there exists an embedded minimal surface $S \subset V$, then

$$\begin{aligned} \theta_\pm &= \mathrm{tr}_S K \pm H \\ &= \mathrm{tr}_S K \\ &> 0 \quad \text{(since } V \text{ is expanding in all directions).} \end{aligned}$$

Then S would be past trapped.

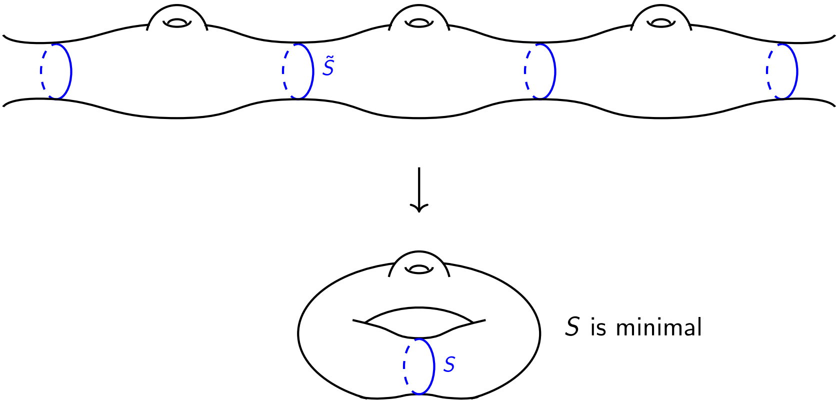
Goal: Find a minimal $S \subset V$ and a covering $p\colon \tilde{V} \to V$ such that $\tilde{V}$ is noncompact and $\tilde{S}$, the lift of S, contains an isometric copy of S.

This induces a canonical spacetime covering $P\colon \tilde{M} \to M$ with a Cauchy surface $\tilde{V}$ in $\tilde{M}$. Apply Penrose's theorem in $\tilde{M}$ to obtain a past incomplete null geodesic in $\tilde{M}$. This projects down to an incomplete null geodesic in M.

Lemma: If $H_2(V, \mathbb{Z}) \neq 0$, then the goal can be achieved.

Visual proof of Lemma

$H_2(V,\mathbb{Z}) \neq 0 \implies$ there is an oriented, minimal, embedded $S \subset V$ which is nonseparating.



S is minimal

Topology and singularities in cosmology

Prime decomposition:

$$V \text{ is orientable} \quad \Longrightarrow \quad V = V_1 \# \cdots \# V_n.$$

Two cases:

(1) $\pi_1(V_i) < \infty \implies V_i$ is spherical (elliptization conjecture).

(2) $\pi_1(V_i) = \infty$. Then either:

  (i) $V_i = S^1 \times S^2$, or

  (ii) V_i is irreducible.

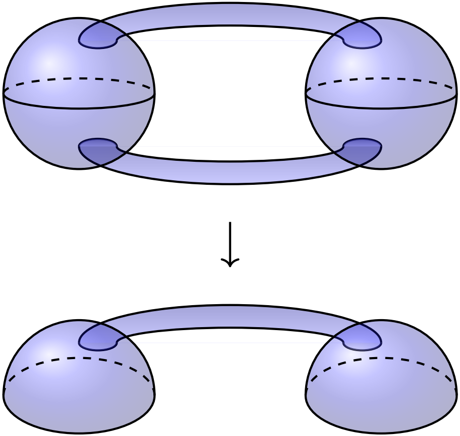
In either case (i) or (ii), the goal can be achieved:

- Case (i) $\implies H_2(V, \mathbb{Z}) \neq 0 \implies$ Goal is achieved.
- In case (ii), the positive resolution of the surface subgroup conjecture along with classical results from Schoen-Yau imply that there is a minimal immersion $f\colon S_g \to V$ for some genus $g \geq 1$ surface such that the induced homomorphism f_* is injective. Consider the covering $p\colon \tilde{V} \to V$ such that $p_*\pi_1(\tilde{V}) = f_*\pi_1(S_g)$. Then $\tilde{V}$ is noncompact and S_g is minimal and immersed in $\tilde{V}$ via the map lifting criterion. Goal is achieved.

Topology and singularities in cosmology

The remaining case is when V is the connected sum of two or more spherical spaces. In this case, there is a covering $\tilde{V}$ with $H_2(\tilde{V}, \mathbb{Z}) \neq 0$:

S^1 x S^2



$RP^3 \# RP^3$

Topology and singularities in cosmology

Generating examples:

Let V be any compact 3-manifold. There is a metric h on V such that R_h is constant (Yamabe problem).

The vacuum Einstein constraint equations with Λ:

$$R_h - |K|_h^2 + (tr_h K)^2 = 2\Lambda$$
$$D_i K^i{}_j - D_j K^i{}_i = 0 \qquad (D \text{ is the } h\text{-covariant derivative}).$$

Choose $K = h$. Then

- K is positive definite (expanding in all directions),
- $DK = 0$ so the second constraint is satisfied,
- The LHS of the first constraint is a constant.

Pick Λ so that the first constraint is satisfied. Then

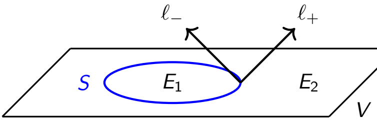
$$V \text{ is not spherical} \quad \Longrightarrow \quad \text{MGHD is past null incomplete.}$$

Singularities in the Gannon-Lee setting

Singularities in the Gannon-Lee setting

Setting:

$S \approx S^2$ separates a Cauchy surface V into E_1 and E_2:



$\ell_\pm = u \pm \nu$

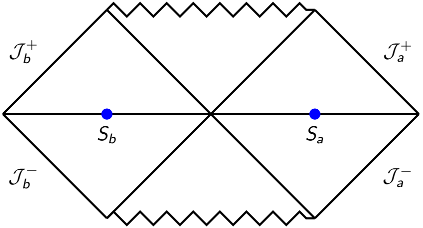
Theorem (Gannon-Lee)

Assume

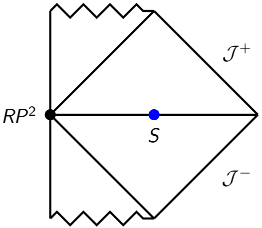
- *the null energy condition holds,*
- *S is future inner trapped (i.e. $\theta_- < 0$),*
- *E_2 is noncompact.*

If $\pi_1(E_1)$ is nontrivial, then M is future null geodesically incomplete.

Singularities in the Gannon-Lee setting



The Schwarzschild RP^3 geon:



Singularities in the Gannon-Lee setting

Generating examples:

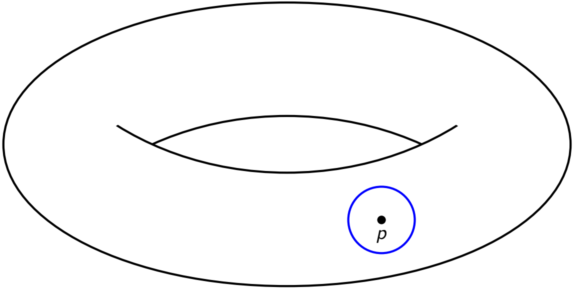
- Let $\mathcal{V}$ be a Riemannian manifold with positive scalar curvature.
- Fix $p \in \mathcal{V}$. The Green's function at p for the conformal Laplacian on $\mathcal{V}$ exists and is strictly positive.
- Consequently, $V := \mathcal{V} \setminus \{p\}$ admits a metric h with $R_h = 0$ and is asymptotically flat with $p \in \mathcal{V}$ representing infinity.
- Let (M, g) denote the MGHD of $(V, h, K = 0)$ for the vacuum Einstein equations.

Then

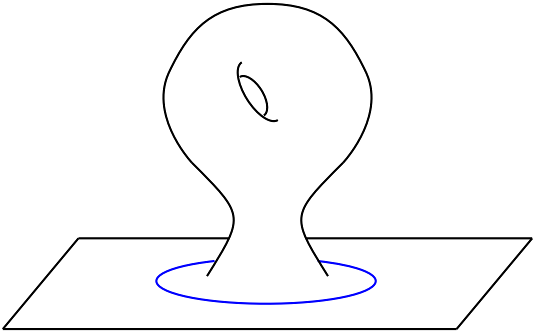
$$\pi_1(\mathcal{V}) \text{ is nontrivial} \quad \Longrightarrow \quad (M, g) \text{ is future null incomplete.}$$

Singularities in the Gannon-Lee setting

$\mathcal{V} = S^1 \times S^2$:



Singularities in the Gannon-Lee setting



Thank you!