>>> The scattering problem for the wave equation with negative cosmological constant

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Interdisciplinary Junior Scientist Workshop:
Mathematical General Relativity

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>>> Outline

1. Negative cosmological constant
2. (Asymptotically) Anti-de Sitter spacetimes
3. The black hole stability problem
4. The scattering problem
5. Outlook

>>> Negative cosmological constant

* The Einstein vacuum equation with cosmological constant Λ reads

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

* Typically, the $\Lambda \geq 0$ case is considered.
* However, the $\Lambda < 0$ setting is mathematically very interesting.

>>> Negative cosmological constant

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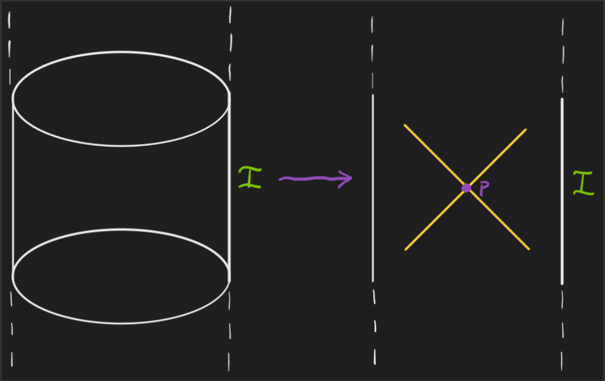
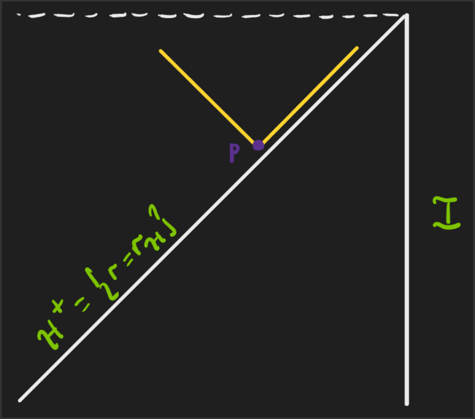
>>> Anti-de Sitter space

* Anti-de Sitter space (AdS) is the simplest solution of the Einstein vacuum equation (EVE) with negative cosmological constant.
* A solid 4-cylinder, with each fixed-time slice a hyperbolic 3-disk.
* Possesses a timelike boundary at infinity.



>>> Schwarzschild-Anti-de Sitter space

* One can consider more interesting spacetimes which are asymptotically AdS.
* Schwarzschild-AdS is a 1-parameter family of black hole spacetimes within the larger 2-parameter family of Kerr-AdS spacetimes



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Why?

>>> Boundary conditions

* Intuition: In certain coordinates, the Einstein vacuum equation becomes a system of nonlinear wave equations. "Fast enough" decay of nonlinear waves $\Box\psi = F$ is key in the proofs of existing stability statements (e.g. $\Lambda = 0$ Schwarzschild).

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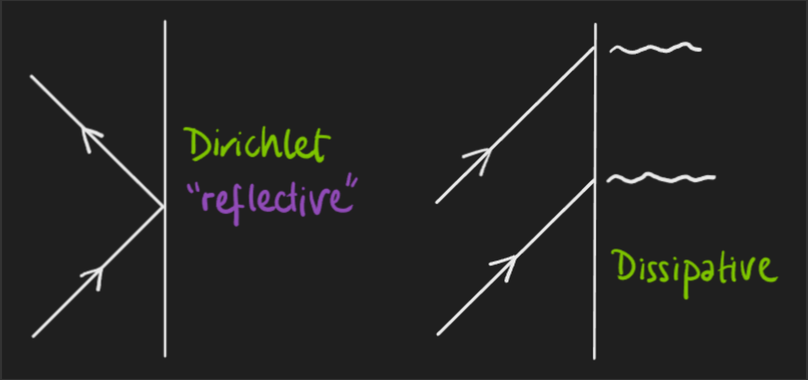
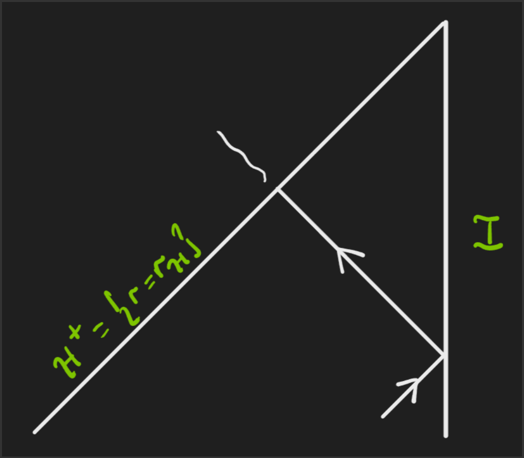


* Dirichlet $\rightarrow$ energy conservation for waves
* Dissipative $\rightarrow$ wave energy escapes to infinity

>>> Is Kerr-AdS stable?

* Black hole horizons provide an escape route for reflected energy of waves, so one might think this is enough for stability.



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Conjecture (Holzegel, Smulevici '13)

Kerr-AdS is asymptotically <u>unstable</u> as a solution of the Einstein vacuum equation with Dirichlet boundary conditions.

>>> Main result

Theorem (GH, in preparation)

There exist a class of exponentially decaying waves on the Kerr-AdS black hole exterior with Dirichlet boundary conditions.

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Theorem (GH, in preparation)

Let $h_{\mathcal{H}} : [0, \infty) \times S^2 \to \mathbb{R}$ be scattering data on $\mathcal{H}$ satisfying

$$\overline{D_{k\mathcal{H}}} := \sum_{0 \le m \le k} \int_0^\infty \int_{S^2} \exp(\alpha(k) \cdot t^*) |\partial^m h_{\mathcal{H}}| \mathrm{d}t^* \mathrm{d}\omega < \infty$$

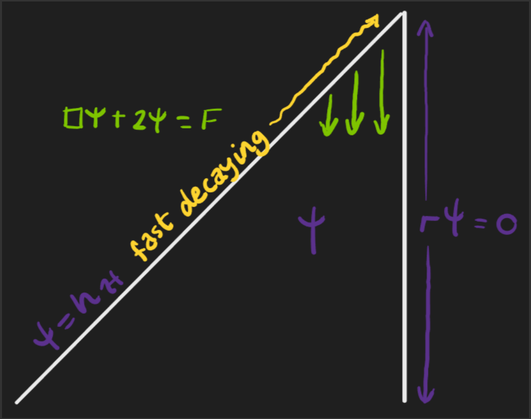
for $k \ge 3$, $\alpha(k) > 0$ sufficiently large. Then, given t_0^* sufficiently large, there exists a unique solution $\psi : [t_0^*, \infty) \times [r_{\mathcal{H}}, \infty) \times S^2 \to \mathbb{R}$ of

$$\begin{cases} \Box\psi + 2\psi = F(\psi, \partial\psi), \\ (\psi, \nabla_{t^*}\psi)|_{\mathcal{H}} = (h_{\mathcal{H}}, \nabla_{t^*} h_{\mathcal{H}}), \quad r\psi|_{\mathcal{I}} = 0 \end{cases}$$

such that $|\psi(\tau)| \lesssim \frac{\overline{D_{3\mathcal{H}}}}{r^{\frac{3}{2}}} \exp(-B \cdot \tau)$ for some $B > 0$, $\tau \in [t_0^*, \infty)$.

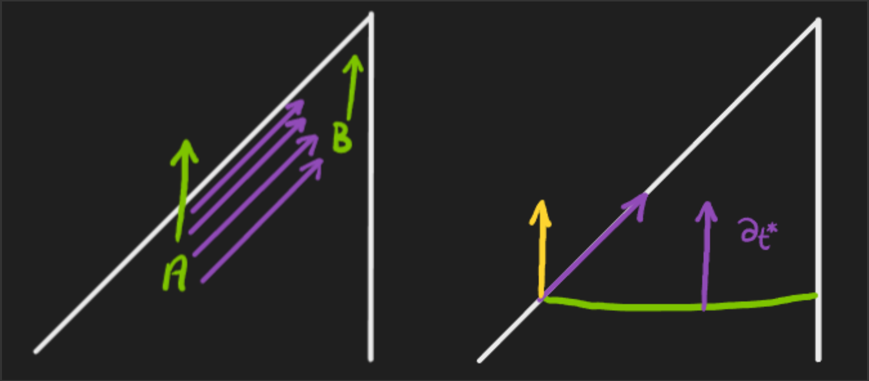
>>> Nonlinear waves from exponentially decaying data

* Idea: construct a non-generic class of nonlinear wave equation solutions which decay as fast as we like
* Take exponentially decaying data on the horizon
* Dirichlet data at the boundary
* Solve towards the past



>>> Key difficulty: gravitational redshift

* The redshift effect helps one deduce decay in forwards evolution
* In backwards evolution this becomes a blueshift, a geometric obstruction to proving decay



>>> Outlook

* A scattering construction for the Einstein vacuum equation with asymptotically-AdS data
* Open question: do there exist slower decaying solutions?
* Is Kerr-AdS, indeed, unstable?

Thank you for your attention!

>>> Idea of the proof

* Approximate the target global solution by a sequence of solutions of finite problems.
* Prove estimates on solutions in the sequence and their derivatives.
* Derive a convergent subsequence whose limit is a global solution.