The mean curvature flow on certain generalized Robertson-Walker space-times with non compact space-like slice

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Mathematical General Relativity
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Motivations, i.e. why MCF?

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Remark

In this talk I will always assume the ambient to be (some special kind of) Lorentzian manifold and the submanifold to be an hypersurface. Important to point out that MCF has been analyzed also in other setting. Semi-Riemannian (cf. [LiSa10]). But also in a Kähler(-Einstein) ambient, preserving the Lagrangian property (cf. [Smo12]).

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Be aware of the sign convention, I am using the **positive** *Laplacian.*

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Remark

In the Riemannian setting with M compact, and perhaps $\mathcal{H} = 0$, one can not expect long time existence.

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Note: global hyperbolicity plays a crucial role. Allows to express space-like hypersurfaces as graphs over the space-like slice.

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2. $(M, \widetilde{g})$ satisfies the Omori-Yau maximum principle: that is for every bounded $u \in C^\infty(M)$ there exist sequences $(p_k)_k$ and $(p'_k)_k$ in M so that

$$(i)\ u(p_k) > \sup_M u - 1/k; \quad -\Delta u(p_k) < 1/k$$
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3. $(M, \widetilde{g})$ has bounded geometry.

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PGMCF

A family of functions $u : M \times I \to \mathbb{R}$ is (gives rise to a prescribed mean curvature flow) with initial embedding $F(p,0) = (u_0(p), p)$ if it satisfies:

$$(\partial_t + \Delta)\, u = \frac{f'(u)}{f(u)} \left(m + \frac{|\widetilde{\nabla} u|^2_{\widetilde{g}}}{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}} \right) + \mathcal{H} \frac{f(u)}{\sqrt{f(u)^2 - |\widetilde{\nabla} u|^2_{\widetilde{g}}}}, \qquad (1)$$

$$u(\_, 0) = u_0.$$

Short time existence

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Theorem (G.,Vertman)

Let $u_0 \in C^{2,\alpha}(M)$ *and* $\mathcal{H} \in C^{\ell,\alpha}(M)$ *with* $\ell \geq 2$. *There exists* $T > 0$ *small enough and* $u \in C^{2,\alpha}(M) \cap C^{\ell+2,\alpha}(M \times [\sigma, T])$ *for every* $\sigma > 0$ *solution to* (1).

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Remark

More regularity on u_0 allows for more regularity on u, e.g. $u_0 \in C^{4,\alpha}(M)$ and $\mathcal{H} \in C^{2,\alpha}(M)$ then $u \in C^{4,\alpha}(M \times [0, T])$.

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i) $\mathrm{H}(t = 0) - \mathcal{H} \geq \delta > 0$.
ii) $\mathrm{Ric}^N(X, X) > 0$ *for every time-like vector field* $X \in \Gamma(TN)$ *(TCC).*

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Then there exists $u \in C^{4,\alpha}(M) \cap C^{\ell+2,\alpha}(M \times [\sigma,\infty))$ *for every* $\sigma > 0$ *with uniformly bounded Hölder norm. Moreover* $\|\partial_t u\|_\infty$ *is exponentially decreasing.*

Convergence

Theorem (G., Vertman)

If M satisfies the same assumptions as for the long time existence and it is the interior of a compact manifold with boundary $\overline{M}$ then there exists $u^* \in L^\infty(M)$ so that a solution u to (1) exists and $u \to u^*$ as $t \to \infty$. Moreover $u^* \in C^{\ell+2,\alpha'}(M)$ with $\alpha' < \alpha$ in the interior with curvature $\mathrm{H}(u^*) = \mathcal{H}$.

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Thank you for your attention

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- Bootstrapping (use Krylov-Safonov estimates and bounded geometry).

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Assume a finite maximal time existence $T_{\max} < \infty$. The idea is to prove that $u(T_{\max})$ satisfies the same properties as the initial function u_0. If so one could restart the flow at $u(T_{\max})$ thus making $T_{\max}$ not maximal.

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- C^0-estimates.
- C^1-estimates.
- C^2-estimates.
- Hölder regularity.
- Bootstrapping.

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$$(\partial_t+\Delta)(\mathrm{H}-\mathcal{H})\geq -c(\mathrm{H}-\mathcal{H}).$$
$$(\partial_t+\Delta)\|\mathrm{h}\|^2\leq -a^2\|\mathrm{h}\|^4+b^2.$$
$$(\partial_t+\Delta)(\mathrm{H}-\mathcal{H})^2\leq -\delta'(\mathrm{H}-\mathcal{H})^2.$$

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