Volume singularities in general relativity

Leonardo García-Heveling

Radboud University Nijmegen

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Outline

Singularities in general relativity

Singularities in GR: motivation

- Some spacetimes in GR have singularities
- Not so easy to define mathematically
- Physical significance: Big Bang and black holes

Singularities in GR: definition

Basic idea: spacetime is singular if an observer can "disappear" in finite proper time

Observer at rest: timelike geodesic incompleteness

There exists a timelike **geodesic** $\gamma \colon [0, b) \to M$ with

$$\nexists \lim_{s \to b} \gamma_s, \qquad L_g(\gamma) < \infty$$

Any observer: bounded acceleration incompleteness

There exists a timelike **curve** $\gamma \colon [0, b) \to M$ with

$$g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) < C, \qquad \nexists \lim_{s \to b} \gamma_s, \qquad L_g(\gamma) < \infty$$

Singularities in GR: inextendibility

- Trivial example of incompleteness: take any spacetime and remove some points
- Q: when do we have a “real” singularity?
- A: if spacetime is inextendible through the singularity

Definition

An *extension* of (M, g) is a spacetime $(\tilde{M}, \tilde{g})$ and a non-surjective isometric embedding $(M, g) \hookrightarrow (\tilde{M}, \tilde{g})$

Lemma

If a curvature scalar blows up at the singularity, then the spacetime is C^2-inextendible.

In general, inextendibility results are hard (especially low-regularity). See e.g. works of Sbierski, Galloway–Ling–Sbierski, Graf–Ling...

Singularities in GR: resolution

Problem: what does an observer experience when hitting a singularity?

Perhaps singularities are resolved by quantum effects? Say, when only a Planck time is left?

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44} \text{ s}$$

Still need curvature blow-up so that the energy also reaches the Planck scale...

Volume singularities (new!)

Volume singularities

Recall $I^+(x) := \{y \in M\colon \exists \text{ future timelike curve from } x \text{ to } y\}$

Definition (LGH, upcoming)

A spacetime (M, g) is **volume incomplete** if there exists a causal curve $\gamma\colon [0, b) \to M$ with

$$\mathrm{vol}_g\left(I^\pm(\gamma_s)\right) \to 0$$

Theorem (LGH, upcoming)

A spacetime (M, g) is volume incomplete if and only if

$$\exists p \in M \quad \text{with} \quad \mathrm{vol}_g\left(I^\pm(p)\right) < \infty$$

Satisfied e.g. in Schwarzschild

Big Bang setting

(*all* of spacetime is swallowed by the singularity)

A Hawking-type singularity theorem

Theorem (LGH, upcoming)

Let $\beta > 0$ and $\kappa \geq -(\beta/n)^2$. Suppose

1. (M, g) contains a compact Cauchy surface Σ
2. $\mathrm{Ric}_g(v, v) \geq n\kappa$ for all v with $g(v, v) = -1$
3. Mean curvature $H_\Sigma \geq \beta$

Then for every $p \in M$

$$\mathrm{vol}_g \left(I^-(p) \right) < \infty$$

- Proof based on Treude & Grant (2013), Graf (2016)
- If $\kappa > -(\beta/n)^2$, then also every timelike geodesic is incompete (usual Hawking thm)
- Example of $\kappa = -(\beta/n)^2$ case: $g = -dt^2 + e^{2t}h$

Cosmological time

Definition

$t\colon M \to \mathbb{R}$ is a *time function* if it is continuous and strictly increasing along every future-directed causal curve

Theorem (LGH, upcoming)

If $\mathrm{vol}_g\left(I^-(\gamma_s)\right) \to 0$ for *every* past-inextendible causal curve γ, then

1. (M, g) is globally hyperbolic
2. $p \mapsto \mathrm{vol}_g\left(I^-(p)\right)$ is a time function

Proof based on Dieckmann (1988) and Andersson–Galloway–Howard (1998)

Black hole setting

(only *some* observers enter the singular region)

Weak cosmic censorship

Conjecture (Penrose)

In a physically realistic spacetime, every singularity is hidden behind an event horizon

Proposition (LGH, upcoming)

The set

$$\mathcal{B} := \{p \in M \colon \operatorname{vol}_g(I^+(p)) < \infty\}$$

is a future set, i.e.

$$I^+(p) \subseteq \mathcal{B} \quad \text{for every} \quad p \in \mathcal{B}$$

hence ∂B is an event horizon

Strong cosmic censorship

Definition

A singularity is *locally naked* if there is a point $p \in M$ and a curve γ that hits the singularity, such that

$$\gamma \subset I^-(p)$$

Conjecture (Penrose)

Physically realistic singularities are never locally naked

Proposition (LGH, upcoming)

$$\mathrm{vol}_g\left(I^+(\gamma_s)\right) \to 0 \implies \bigcap_s I^+(\gamma_s) = \emptyset$$

Hence volume singularities are never locally naked

Outlook

So far:

- New notion of incompleteness based on $\mathrm{vol}_g(I^+(p)) < \infty$
- Hawking type theorem (with "$=$" case)
- Cosmic time similar to Andersson–Galloway–Howard
- Cosmic censorship satisfied

Still open:

- Penrose type theorem
- Rigidity in Hawking thm (cf Bartnik splitting conjecture)
- Dynamical formation

Thank you for listening!