The coupled Einstein constraint equations on non-compact manifolds

Rodrigo Avalos

Institut für Mathematik, Potsdam University, Germany.
E-mail: rdravalos@gmail.com

1. R. Avalos and J.H. Lira, *Einstein type elliptic systems*, Annales Henri Poincaré 23, 32213264 (2022).
2. R. Avalos, J.H. Lira and N. Marque, *Einstein Type Systems on Complete Manifolds*, arXiv:2201.08347v1 (2022).

March 7, 2023

Layout of the talk

- Introduction - The ECE and the conformal method
- Coupled charged fluid system on AE manifolds
- Coupled system on complete manifolds

The constraint equations I

From previous talks we know that the Gauss-Codazzi equations impose constraints on the admissible initial data for the Cauchy problem in GR. Also, we know that:

- An initial data set is given by a tuple (M^n, g, K), where (g, K) are subject to the Einstein constraint equations:

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 2\epsilon \\ \mathrm{div_g} K - d\mathrm{tr}_g K &= J, \end{aligned} \tag{1}$$

  where $\epsilon \doteq T(n,n)|_{t=0}$ and $J = -T(n,\cdot)$ denote, respectively, the induced energy and momentum by the physical sources modelled in space-time by some energy-momentum tensor field T.

- The system (1) is not only a necessary but also a sufficient condition for the initial data to admit (short-time) evolution.

One is interested in:

1. Producing solutions, ideally with freedom to model interesting situations;
2. Parametrize the space of solutions;

The constraint equations II

We intent to comment on the analysis of the existence of solutions to (1). Let us fist make a few remarks:

- The space-time equations associated with matter fields can impose further constraints;
- This is the case for the electromagnetic field equations (and more generally, for Yang-Mills fields coupled to the Einstein equations);
- In the case of a charged fluid, the full system of constraint equations reads as follows:

$$\begin{aligned} R(g) - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 2\epsilon \\ \mathrm{div}_g K - d\mathrm{tr}_g K &= J, \\ \mathrm{div}_g E &= \tilde{q}, \\ dF &= 0, \end{aligned} \tag{2}$$

where E is a vector field on M and $F \in \Omega^2(M)$ representing the electric and magnetic fields induced in the initial data set. Thus, in realistic situations, the ECE will typically couple with further equations.

Conformal method I

The constraint equations appear as a highly underdetermined system. It is therefore natural to split the initial data into arbitrarily prescribed data and unknowns. The proposal of the conformal method is to consider the following:

$$
\begin{aligned}
g &= \phi^{\frac{4}{n-2}}\gamma \\
K &= \phi^{-2}\tilde{K} + \frac{\tau}{n}g, \\
\tilde{K} &= \pounds_{\gamma,conf}X + U,
\end{aligned}
\tag{3}
$$

where we have denoted by $\tau \doteq \mathrm{tr}_g K$ and defined the conformal killing operator

$$
\pounds_{\gamma,conf}X \doteq \pounds_X\gamma - \frac{2}{n}\mathrm{div}_\gamma X\,\gamma.
$$

In the above decomposition, the symmetric traceless $(0,2)$-tensor field U is supposed to be *transverse*, *i.e*, $\mathrm{div}_\gamma U = 0$. Using the above conformal splitting, the Gauss-Codazzi constraints are rewritten as follows:

Conformal method II

$$
\begin{aligned}
&\Delta_\gamma \phi - c_n R_\gamma \phi + c_n |\tilde{K}|^2_\gamma \phi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \phi^{\frac{n+2}{n-2}} = 0, \\
&\Delta_{\gamma, conf} X - \left(\frac{n-1}{n} d\tau + J \right) \phi^{\frac{2n}{n-2}} = 0,
\end{aligned} \tag{4}
$$

where we have introduced the Conformal Killing Laplacian, defined by $\Delta_{\gamma,conf} X \doteq \mathrm{div}_\gamma (\pounds_{\gamma,conf} X)$.

Remarks:

- The specific form of the energy-momentum tensor imposes conformal scaling for the sources. One says J is York-scaled if $J = \phi^{-\frac{2n}{n-2}} \tilde{J}$, where $\tilde{J}$ is constructed with freely prescribed data.
- In this case, under a CMC assumption, the constraints read as follows.

$$
\begin{aligned}
&\Delta_\gamma \phi - c_n R_\gamma \phi + c_n |\tilde{K}|^2_\gamma \phi^{-\frac{3n-2}{n-2}} + c_n \left(\frac{1-n}{n} \tau^2 + 2\epsilon \right) \phi^{\frac{n+2}{n-2}} = 0, \\
&\Delta_{\gamma, conf} X = \tilde{J}.
\end{aligned} \tag{5}
$$

Conformal method III

- Above we are treating these equations as equations for (ϕ, X), with free parameters $(\gamma, \tau, U, \tilde{\epsilon}, \tilde{J})$, and the only coupling between the two equations in (5) is through $\tilde{K}(X)$.
- Thus, if the momentum constraint is solvable for some prescribed $\tilde{J}$, then the equations decouple, and **we are left with the study of the Lichnerowicz equation.** For vacuum, CMC data on closed manifolds, one has the following classical classification:

	$\tau = 0$ $U = 0$	$\tau = 0$ $U \neq 0$	$\tau \neq 0$ $U = 0$	$\tau \neq 0$ $U \neq 0$
$\mathcal{Y}([\gamma]) > 0$	No	Yes	No	Yes
$\mathcal{Y}([\gamma]) = 0$	Yes	No	No	Yes
$\mathcal{Y}([\gamma]) < 0$	No	No	Yes	Yes

The above classification relies on two big steps:

1. Monotone iteration scheme for equations of the form

$$\Delta_\gamma \phi = f(\cdot, \phi) \doteq \sum_I a_I \phi^I, \quad a_I \in L^p \tag{6}$$

2. Construction of barriers for the monotone iteration.

The coupled system

Let us consider now the conformally formulated ECE for a charged (dust) fluid

$$a_n\Delta_\gamma\phi - R_\gamma\phi + |\tilde{K}(X)|^2_\gamma\phi^{-\frac{3n-2}{n-2}} + \left(2\epsilon_1 - \frac{n-1}{n}\tau^2\right)\phi^{\frac{n+2}{n-2}} + 2\epsilon_2(f)\phi^{-3} + 2c_n\epsilon_3\phi^{\frac{n-6}{n-2}} = 0,$$

$$\Delta_{\gamma,conf}X - \frac{n-1}{n}d\tau\phi^{\frac{2n}{n-2}} - \omega_1\phi^{2\frac{n+1}{n-2}} + \omega_2(f) = 0,$$

$$\Delta_\gamma f = \tilde{q}\phi^{\frac{2n}{n-2}}$$

where, setting $\tilde{E} = \nabla f + \vartheta \in \Gamma(T^*M)$,

$$\epsilon_1 = \mu\left(1 + |\tilde{u}|^2_\gamma\right),\ \epsilon_2 = \frac{1}{2}|\tilde{E}|^2_\gamma,\ \epsilon_3 = \frac{1}{4}|\tilde{F}|^2_\gamma,$$

$$\omega_{1k} = \mu\left(1 + |\tilde{u}|^2_\gamma\right)^{\frac{1}{2}}\tilde{u}_k,\ \omega_{2k} = \tilde{F}_{ik}\tilde{E}^i,\ \tilde{q} = q(1 + |\tilde{u}|^2_\gamma)^{\frac{1}{2}}.$$

New difficulties:

- The system is fully coupled (even under a CMC condition);
- An iteration/fixed point scheme will have to depend on the existence of uniform barriers (because the coefficients are changing along the iteration!);

Analysis on AE manifolds I

Definition (Weighted Sobolev spaces)

Let $E \to \mathbb{R}^n$ be vector bundle over $\mathbb{R}^n$. The weighted Sobolev space $W^{k,p}_{\delta}$, with k a non-negative integer, $1 < p < \infty$ and $\delta \in \mathbb{R}$, of sections u of E, is defined as the subset of $W^{k,p}_{loc}$ for which the norm

$$\|u\|_{W^{k,p}_{\delta}(\mathbb{R}^n)} \doteq \sum_{|\alpha| \leq k} \|\sigma^{-\delta - \frac{n}{p} + |\alpha|} \partial^{\alpha} u\|_{L^p(\mathbb{R}^n)} \tag{7}$$

is finite, where $\sigma(x) \doteq (1 + |x|^2)^{\frac{1}{2}}$ and α denotes an arbitrary multi-index.

Remark: Using a partition of unity one extends the definition of $W^{k,p}_{\delta}$-spaces to an arbitrary AE manifold.

Definition ($W^{k,p}_{-\tau}$-AE manifolds)

Let (M^n, g) be a connected, n-dimensional Riemannian manifold and let $\tau > 0$. We say that (M, g) is an Asymptotically Euclidean (AE) manifold of class $W^{k,p}_{-\tau}$ if:

1. $g \in W^{k,p}_{\mathrm{loc}}(M)$ where $p > \frac{n}{k}$ (and consequently g is continuous).

Analysis on AE manifolds II

2. There exists a compact set $K \subset M$ and a diffeomorphism $\Phi : M \backslash K \mapsto \mathbb{R}^n \backslash \overline{B_1(0)}$;
3. For each $1 \leq i, j \leq n$ $\left((\Phi^{-1})^{*} g\right)_{ij} - \delta_{ij} \in W^{k,p}_{-\tau}(\mathbb{R}^n \backslash \overline{B_1(0)})$.

With the above definitions in mind, one proceeds as follows:

- First, analyse the behaviour of the linear parts involved in the above system between appropriate (weighted) spaces;
- For non-compact manifolds one needs to be careful with the behaviour of ϕ at infinity;
- Since ϕ should not decay to zero at infinity (but to some constant value) one splits $\phi = \omega + \varphi$ where ω is a harmonic function which captures the behaviour of ϕ at infinity;
- Denoting the linear operator appearing in the left-hand side by

Analysis on AE manifolds III

$$\mathcal{P} : W^{2,p}_{\delta}(M;E) \mapsto L^{p}_{\delta-2}(M;E),$$
$$(\varphi, f, X) \mapsto (\Delta_{\gamma}\varphi, \Delta_{\gamma} f, \Delta_{\gamma,\mathrm{conf}}X)$$

we rewrite the above system more compactly as

$$\mathcal{P}(\psi) = \mathbf{F}(\psi), \tag{8}$$

The idea is to solve the above problem by solving a sequence of linear problems: Given $\psi_0 \in W^{2,p}_{\delta}(M;E)$, if we get a unique solution $\psi_1 = \mathcal{P}^{-1}\mathbf{F}(\psi_0)$, we can begin an iteration scheme. If we find a fixed point $\bar{\psi}$ in this iteration, then such fixed point solves

$$\mathcal{P}(\bar{\psi}) = \mathbf{F}(\bar{\psi}),$$

which is equivalent to solving the original system. If, furthermore, we get that $\phi > 0$, then such solution actually solves the conformal problem associated to a charged fluid. In order to satisfy this last condition, we will need to produce barriers ϕ_- and ϕ_+, and make sure that the iteration stays within $[\phi_-, \phi_+]$.

Obtaining solution via the above method then requires

Analysis on AE manifolds IV

1. An iteration scheme replacing the one for Lichnerowicz type equations, adapted to systems;
2. Construction of (strong) global barriers for the particular system.
3. Both of the above can be done and one can obtain the following type of results:

Theorem (Yamabe positive existence - free τ)

Let (M^n,γ) be a $W^{2,p}_{\delta}$-Yamabe positive AE manifold, $p>n$, $n\geq 3$ and $2-n<\delta<0$. Consider the system associated to a charged fluid with conformal data $\tau, U, \tilde{F}, \vartheta \in W^{1,p}_{\delta-1}$, $\mu \in W^{1,p}_{2(\delta-1)}, \tilde{u} \in W^{1,p}_{\delta-1}, \tilde{q} \in L^{p}_{\delta-2}$. If $U, \tilde{F}, \vartheta, \mu, \tilde{q}$ are sufficiently small, then, there is a $W^{2,p}_{\delta}$-solution to the conformal problem.

Remark:

- The restrictions on the size of the coefficients appear when constructing barriers;

Analysis on non-compact complete manifolds I

Motivations:

- Physical motivation: Analysis of system with bounded initial data, but not necessarily decaying.
- Mathematical motivation: Extending the analysis of Lichnerowicz on complete manifolds to the conformally formulated ECE.

Difficulties:

- The previous theorems relied on global estimates and compactness theorems for the chosen functional spaces, which are no longer available.

One way to overcome these difficulties is:

1. Solve the system along an exhaustion of M by compact sets $\{\Omega_k\}_{k=1}^{\infty}$ under the assumption of existence of uniform barriers;
2. Obtain uniform estimates on interior compacts;
3. Extract a diagonal sequence converging to a solution in $W^{2,p}_{loc}(M)$;
4. Construct uniform barriers.

Analysis on non-compact complete manifolds II

- Steps 1-3 above can be done for general complete manifolds and provide a substitute for the iteration scheme;
- Step 4 can be obtained under the condition of bounded geometry.

Example: Consider the simplified system

$$\Delta_\gamma \phi - c_n R_\gamma \phi + c_n \left|\tilde{K}(X)\right|_\gamma^2 \phi^{-\frac{3n-2}{n-2}} - c_n \frac{n-1}{n}\tau^2 \phi^{\frac{n+2}{n-2}} + 2c_n \epsilon_1 \phi^{\frac{n+2}{n-2}} + 2c_n \epsilon_2 \phi^{-3} + 2c_n \epsilon_3 \phi^{\frac{n-6}{n-2}} = 0, \tag{9}$$

$$\Delta_{\gamma,conf} X - \frac{n-1}{n} d\tau \phi^{\frac{2n}{n-2}} - \omega_1 \phi^{2\frac{n+1}{n-2}} + \omega_2 = 0,$$

The above system corresponds to the electromagnetic constraints, but with $\tilde{q} = 0$. In this case, one can prove results along the lines of the following theorem:

Analysis on non-compact complete manifolds III

Theorem

Let (M^n, γ) be a smooth complete Riemannian manifold of bounded geometry, let $n \geq 3$ be its dimension and $p > n$. We make the following assumptions:

$$R_\gamma, \epsilon_1, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{\mathrm{loc}}(M) \text{ and } \omega_1, \omega_2, d\tau \in L^2(M) \cap L^p(M).$$

$$\lambda_{1,\mathrm{conf}} > 0,$$

$$a \doteq c_n R_\gamma + b_n \tau^2 \in L^\infty(M), \;\; a \geq a_0 > 0.$$

Assume further that:

$$\epsilon_2 + \epsilon_3 > 0 \text{ if } n \leq 6$$
$$\epsilon_2 > 0 \text{ if } n > 6.$$

Then, there exists $C(n, M, \gamma, \lambda_{1,\mathrm{conf}})$ such that if

$$|R_\gamma| + \max\left(\|d\tau\|_{L^2(M)}, \|d\tau\|_{L^p(M)}\right) + \max\left(\|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)}\right) + \max\left(\|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)}\right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau^2,$$

then (9) admits a $W^{2,p}_{\mathrm{loc}}$ solution.

Remarks

- Under different conditions on the geometric parameters and physical sources one can get different versions of the above theorems;
- One can in particular introduce boundary conditions on compact inner boundaries;
- When one deals with controlled asymptotic geometry extensions to larger systems of constraints seem fairly natural;
- In both settings, one can produce solutions in L^2-spaces (which might be better suited for the evolution problem..), but in this case more subtle regularity issues appear.

Thank you for your attention!