Graphs and Discrete Dirichlet Spaces

Wu Edition[1]

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[1]This text has the same enumeration as the published version of the book.

Contents

Preface

The present book deals with the spectral geometry of infinite graphs. This topic involves the interplay of three different subjects: geometry, the spectral theory of Laplacians and the heat flow of the underlying graph. These three subjects are brought together under the unifying perspective of Dirichlet forms.

The spectral geometry of manifolds is a well-established field of mathematics. On manifolds, the focus is on how Riemannian geometry, the spectral theory of the Laplace–Beltrami operator, Brownian motion and heat evolution interact. In the last twenty years large parts of this theory have been subsumed within the framework of strongly local Dirichlet forms. Indeed, this point of view has proven extremely fruitful.

The spectral geometry of graphs concerns discrete objects. For graphs, geometry is encoded in combinatorial notions, the Laplacian is a difference operator and the heat evolution is given by a Markov jump process. Developments in this area often come about as a discrete analogue to the situation on manifolds. In particular, the spectral geometry of graphs appears in approximation procedures. However, it can also be studied without any reference to manifolds.

Our point of view is fundamentally different: our perspective is that of Dirichlet forms. In this context, manifolds and graphs are treated on an equal footing. Specifically, manifolds provide the prototype for local Dirichlet forms and graphs provide the prototype for non-local Dirichlet forms. Therefore, conceptually, the similarities result from the common context of Dirichlet forms and the differences are a consequence of the local as opposed to non-local character.

Beyond the conceptual beauty of this approach, it also offers various practical advantages. As far as results are concerned, there is no need for restrictive boundedness assumptions on the geometry of the underlying graph. At the same time, the reader is offered a very accessible introduction to the powerful theory of Dirichlet forms. In fact, our approach enables the reader to quickly reach cutting-edge research topics with a minimum number of prerequisites.

The structure of this book. We now discuss the structure of this book. The Prelude, Chapter 0, presents many of the relevant ideas in the context of finite graphs. In this part of the book, the reader will already encounter the main players and concepts in a finite-dimensional context. This chapter will be accessible to undergraduate students who have seen basic linear algebra, analysis, some probability theory and ordinary differential equations. It can be used for a one-semester topics course at this level. Alternatively, individual sections can be used to motivate topics studied in full generality in later chapters, or Chapter 0 can be skipped altogether.

The actual discussion of infinite graphs starts after the Prelude with Chapters 1 and 2. Here, Chapter 1 covers all of the basic notions and concepts needed for a discussion of infinite graphs. This material is used virtually everywhere in the book. Chapter 2, on the other hand, covers some more advanced tools as well as additional aspects needed in specific places only. So, the reader may skip Chapter 2 at first reading and only come back to the relevant parts of it as needed.

In fact, the first section of Chapter 1 already presents the setting of infinite graphs and the connection to Dirichlet forms and the Laplacian. Besides introducing basic quantities, this section also has the character of a summary as it collects all essential definitions. Therefore, having read the first section of Chapter 1, the reader can jump into any of the following chapters as they should now be familiar with the basic concepts and notations. In fact, the reader is invited to explore the topics of the book by browsing through the chapters. Each chapter starts with a summary and care has been taken to ensure that the chapters can be used independently of each other. Thus, depending on the interest of the reader, any of the numerous topics can be pursued, offering substantial flexibility.

In this way, Chapter 1 together with a choice of subsequent chapters can serve as a basis for a one-year graduate course. This course will only require some basic topics from functional analysis. The more advanced tools which are necessary to deal with the material are provided in a series of appendices at the end of the book. In this way, the spectral geometry of graphs gives a wonderful opportunity to learn abstract operator-theoretic concepts "on the job."

Following Chapter 0, the book is divided into three parts. Part 1, which consists of Chapters 1 to 7, deals with "Foundations and fundamental topics." As discussed already, the first chapter discusses all of the basic objects needed for the theory. A core theme of the remaining chapters is how various quantities and concepts of interest can be investigated via generalized solutions.

Part 2, Chapters 8 to 10, deals with "Classes of graphs." In this part, we study graphs with a uniformly positive measure, graphs with a spherical symmetry and graphs with suitable sparseness properties. Taken together, these are the most common models encountered in the study of infinite graphs.

Part 3, Chapters 11 to 14, deals with "Geometry and intrinsic metrics." In this part, the geometry of graphs is approached via the recently developed tool of intrinsic metrics for graphs.

Each part of the book starts with a brief synopsis and each chapter begins with a summary giving an overview of the contents. As mentioned previously, the Prelude, Chapter 0 is an independent portion of the book which can be used for an undergraduate course. As such, Chapter 0 has an extended nontechnical introduction to give a general mathematical and scientific context for our viewpoint on graphs. Finally, the book concludes with a series of appendices summarizing the required background from spectral theory and the theory of Dirichlet forms required for parts of the book starting with Chapter 1.

A word about the * sections. There are a few sections where neither the results nor the notations are necessary to understand the remaining parts of the book. To indicate this, these sections are marked with a * in the title.

A word about the exercises. There are three types of exercises found at the end of each chapter, separated into the categories of Excavation, Example and Extension.

The Excavation Exercises serve the purpose of recalling (and in this sense "excavating") prerequisites from linear algebra, probability and functional analysis and applying them in the context of the book. These exercises can be used in a course to bring students up to speed and to enliven their background knowledge. However, as their purpose is to make the prerequisites transparent, they are only mentioned at the beginning of each section so that they do not interrupt the flow of the presentation.

The Example Exercises let the reader apply the theory to concrete examples. In some cases, the topics of an entire chapter can be worked out for a particular example. These exercises may either serve as a review and summary of the chapter or they may be split up and used to illuminate topics found in specific sections. Thus, they are usually not explicitly mentioned within the text.

Finally, the Extension Exercises consist of material that goes beyond the core of our theory and in this sense "extends" the perspective of the reader. In some cases, these are interesting observations that illuminate a certain aspect and, in other cases, they provide a link to related topics which are not treated exhaustively in the book. These exercises appear as remarks within the text.

A word about the historical notes. Each chapter ends with notes discussing the history of the subject as well as pointing out corresponding references. Furthermore, at the end of the notes of Chapters 0 and 1, we include standard references which intersect with the topics treated in this book.

Acknowledgments

It takes a village to raise a child. It is no different with a book – the present one being brought to life by many who inspired us and generously shared their knowledge and wisdom over the last two decades. Although many of these villagers are found in the references, we take this opportunity to single out a few.

First of all, there are the elders of the village, whose tales we listened to all night by the fire and who shaped our perspective on the world which is presented in this book. Among these elders Józef Dodziuk, Sasha Grigor′yan, Peter Stollmann and Wolfgang Woess are of particular importance to us. Moreover, the stories that Isaac Chavel, Leon Karp, Norbert Peyerimhoff, Yehuda Pinchover and Ivan Veselić shared on specific occasions planted seeds which bore ample fruit in what follows.

Companions are needed when exploring the surroundings of a village, to expand your horizons and to see places you never would have found by yourself. Companions to us have been a number people with whom we wrote articles that grew into chapters of this book. This includes Frank Bauer, Michel Bonnefont, Anne Boutet de Monvel, Rupert Frank, Agelos Georgakopoulos, Sylvain Golénia, Batu Güneysu, Sebastian Haeseler, Bobo Hua, Xueping Huang, Shiping Liu, Jun Masamune, Florentin Münch, Felix Pogorzelski, Hendrik Vogt and Daniel Wingert. Among them, Marcel Schmidt stands out as his influence permeates a substantial part of the content.

If the village is healthy, then one will eventually learn more from the younger generation than what one has taught them. Hence, we are grateful to the students who studied the material with us and helped us to improve it during various lectures, seminars and theses. Many of them gave us feedback on the text, we mention Philipp Bartmann, Siegfried Beckus, Florian Fischer, Philipp Hake, Wiebke Hanl, Matti Richter, Christian Rose, Christian Scholz, Michael Schwarz, Daniel Sell, Aljoscha Sukeylo, Margarete Sydow, Melchior Wirth, Elias Zimmermann and Ian Zimmermann. Here, Simon Puchert deserves a special mention since he went the extra mile more than once in proofreading chapters and spotting various issues that would have later caused embarrassment.

There are of course plenty of other people who have had an important influence in our mathematical life such as Jonathan Breuer, David

Damanik, Dan Mangoubi, Pedro Freitas, Simone Warzel, Joachim Weidmann and Jean-Claude Zambrini, but we certainly cannot mention all of them.

One who should definitely be mentioned is Rémi Lodh from Springer, who was very enthusiastic and supportive right from the beginning of this project and patiently accompanied the extensive procrastination in finishing this book.

Finally, a home in the village is not only a source of refuge and respite but it is also the actual place where most of this book was written. We all enjoyed the hospitality of each other's families on multiple occasions and we are grateful to Yvonne, Elliott, Lumen, Ferris and Nelson; Emil and Felix; Kaylyn; as well our families and friends.

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Part 0

Prelude

Synopsis

A graph is a geometric structure on a set of vertices. At the same time, a graph comes with both a Dirichlet form and a Laplacian defined on the set of functions on its vertices. The interplay between this geometric structure and the spectral theory of the Laplacian is a main focus of this book. Certain unboundedness features of the geometry as well as boundary structures can only occur if the underlying set of vertices is infinite. Still, many phenomena of interest already appear in the case of finite graphs. This is discussed in this part. The material of this part is not necessary in order to understand the later parts. On the other hand, the reader may glance through this part in order to gain perspective and motivation for later considerations.

CHAMBER 0

Finite Graphs

I don't know how you all see it, when it comes to the children.
Wu-Tang is for the children. We teach the children.
ODB.

The concept of a graph is one of the most fundamental mathematical concepts ever conceived. Graphs inherently appear in many branches of mathematics and natural sciences. Occurrences of graphs in real world questions range from the spreading of diseases in biology to computer and electrical networks in engineering to lattice gauge theory in elementary particle physics, amongst other manifestations. In mathematics, graphs are unavoidable as they appear (implicitly) whenever there is a relation between objects. In particular, they play a most prominent role in various combinatorial questions. At the same time, graphs often come about via approximation schemes when dealing with a continuous setting.

At its core, the concept of a graph allows us to give a precise meaning to the notion of a neighbor. This naturally extends to a notion of a neighborhood and, more generally, to the idea of a space being connected. These notions clearly have a geometric flavor, which starts on a local scale and extends outwards. As such, many questions investigated for graphs can be seen as dealing with the interplay between local and global geometric features of the graph. This perspective underlies our book.

A very natural question in the context of graphs concerns the propagation in time of various quantities within a graph. This includes, for example, quantities such as information, energy, or heat. Clearly, the geometry of the graph will determine the change in the distribution of the entity in question over time. Hence, understanding the geometry will allow us to understand the propagation. Conversely, investigation of the propagation can be used to understand the geometry.

The basic model for such propagation is given by a heat equation in Section 5. In analytic terms, the solution of a heat equation is provided by a semigroup of operators satisfying certain positivity properties. The operators which arise from graphs automatically satisfy these properties. Conversely, any semigroup of operators on a discrete space with such positivity properties can be seen to arise from a graph. Hence, there is a one-to-one correspondence between graphs and such semigroups. Having set up this framework, the long-term behavior of systems modeled by the heat equation can then conveniently be described via spectral theory, see Section 7.

It turns out that probability theory allows us to give a completely different (though equivalent) approach to the heat equation via the theory of Markov processes on discrete spaces. This is discussed in Section 10. It is rather remarkable that these two different branches of mathematics give solutions to the same problem and this underlines the relevance of our models.

While propagation deals with dynamics arising from graphs and their geometry, there is also a more static point of view given in Section 4. In this view, graphs serve as a basic model in the description of the electric currents in a system of wires in electrostatic equilibrium. Equivalently, one may think of the flow of a liquid in a system of tubes. In this context, graphs are often referred to as networks. Crucial problems in this context are the Poisson equation and the capacitor problem. Graphs then give rise to solutions of the capacitor problem with certain properties and vice versa. So, here again, we encounter a one-to-one correspondence between graphs and solutions of an analytic problem with specific properties.

It is by no means obvious and, in fact, rather surprising that graphs can serve as models for the description of so many different physical problems. These problems include both the heat equation and electrostatics in the discrete setting. Analytically, the connection comes about via a self-adjoint operator known as a graph Laplacian which is introduced in Section 1. This Laplacian generates the semigroup arising in the study of the heat equation and, at the same time, gives rise to resolvents in the study of the capacitor problem. These semigroups and resolvents are introduced in Section 6. It turns out that graph Laplacians share a feature with the negative of the ordinary second derivative of a function on the real line, specifically, that they are positive at a maximum of the function. More importantly, graph Laplacians are even characterized by this property, as discussed in Section 3. This shows that graph Laplacians and the Laplacian on Euclidean space are connected by deep structural ties rather than just by a superficial analogy.

A convenient way to deal with self-adjoint operators is by means of quadratic forms. These are mappings of pairs of functions to numbers. The forms corresponding to graph Laplacians are characterized by compatibility with normal contractions. Forms with such a compatibility are known as Dirichlet forms. They are in a one-to-one correspondence with graphs, as shown in Section 2.

The preceding discussion unfolds a rather remarkable panorama: the compatibility of a form with normal contractions is equivalent to the operator associated to the form sharing features with the second derivative. This, in turn, can be characterized via solutions to both the heat equation and to the capacitor problem. An alternative, but equivalent, point of view is provided by probability via Markov processes. Any of these features characterizes the structure of a graph.

None of these considerations are restricted to finite graphs. In fact, as investigated in later chapters, they also apply to infinite graphs. More generally, the theory of Dirichlet forms encompasses a variety of other geometric situations which includes Laplacians on Euclidean space and manifolds. However, what is so special about graphs, and especially about finite graphs, is that they allow us to give both a precise and panoramic view of the topic without having to bother with numerous technical details.

In the context of the overall structure of the present chapter and of the book, one more remark may be in order. A very convenient feature of the theory of Dirichlet forms is that it not only deals with the geometry of the underlying object, which in this case is a graph, but, at the same time, also includes the concept of something "outside" of the object. In the language of physics, this means that we are dealing with an open system. In our context, this leads to having an additional ingredient, which we call the killing term, in our definition of a graph when compared to what is usually found as the definition of a graph in textbooks on graph theory. Having this killing term at our disposal, we are able to capture numerous phenomena without having to look at case distinctions.

1. Graphs, Laplacians and Dirichlet forms

In this section we introduce the three key objects of our considerations. These are graphs, Laplacians and Dirichlet forms. We will show in the subsequent sections that these three types of objects are in one-to-one correspondences with each other.

To recall and apply some basic facts from linear algebra and analysis the reader may want to solve the Excavation Exercises 0.1, 0.2, 0.3 and 0.4 found at the end of the chapter. These exercises review the basics of the discrete topology, quadratic forms, the Hilbert space of interest and the notion of self-adjointness.

1.1. Graphs. We introduce the basic concepts of graphs and the arising degree function.

Finite graphs are usually defined as combinatorial objects with a finite set of vertices and a set of edges connecting the vertices. More generally, graphs can be considered as having weights on the edges. In the subsequent definition, these notions are captured via a finite set X and a suitable function b on $X \times X$. Moreover, the definition features an additional ingredient given by a function $c\colon X \longrightarrow [0,\infty)$. This ingredient may come as a surprise to the reader familiar with graphs from other contexts. It is intimately linked to the overall perspective of Dirichlet forms taken in this book. As such, the relevance of c unfolds in detail in subsequent sections. Here, we already comment on this relevance in the remarks following the definition.

DEFINITION 0.1 (Graph over finite X). Let X be a finite set. A *graph over* X or a *finite graph* is a pair (b, c) consisting of a function $b\colon X \times X \longrightarrow [0, \infty)$ satisfying

- $b(x, y) = b(y, x)$ for all $x, y \in X$
- $b(x, x) = 0$ for all $x \in X$

and a function $c\colon X \longrightarrow [0, \infty)$. If $c(x) = 0$ for all $x \in X$, then we speak of b as a graph over X (instead of $(b, 0)$). The elements of X are called the *vertices* of the graph. The map b is called the *edge weight*. More specifically, a pair (x, y) with $b(x, y) > 0$ is called an *edge* with *weight* $b(x, y)$ connecting x to y. The vertices x and y are called *neighbors* if they form an edge. We write $x \sim y$ in this case. The map c is called the *killing term*.

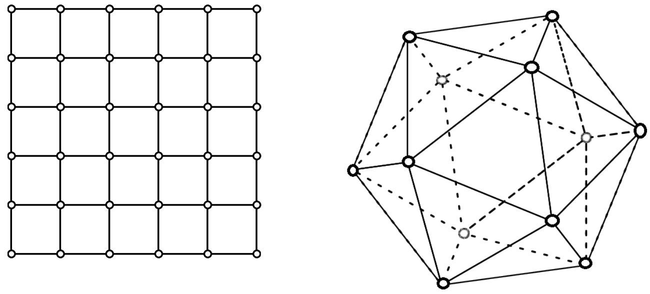

FIGURE 1. A subgraph of the two-dimensional Euclidean lattice and the graph induced by the edges and vertices of the icosahedron.

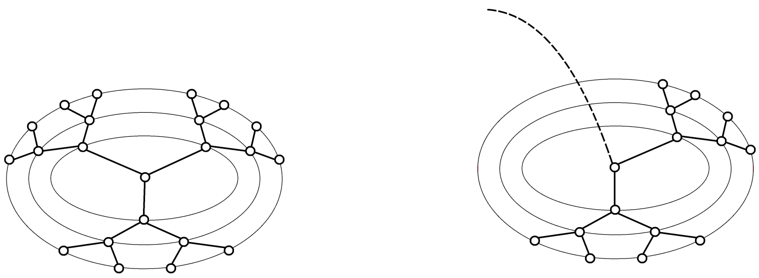

FIGURE 2. The first spheres of a regular tree and a rooted regular tree.

REMARK. In Figures 1 and 2 some basic examples of finite graphs are displayed. While $c = 0$ in the first three examples, in the case of the rooted tree in Figure 2 the dotted line at the root indicates that c does not vanish at this vertex. How these figures relate precisely to our definition of a graph is elaborated in the next remark.

Remark (How our definition compares to definitions found in graph theory). As the function b is symmetric, there is an edge connecting x to y if and only if there is an edge (with the same weight) connecting y to x. As b vanishes on the diagonal, there is no edge from a vertex to itself. Thus, our graphs are weighted undirected graphs without loops in the sense of graph theory. On the other hand, c is not usually included in the definition of a graph. It is a special feature arising from the perspective on graphs we take, that is, the perspective of Dirichlet forms on discrete spaces. As we will show later, with b and c, graphs and symmetric Dirichlet forms over X are in a one-to-one correspondence, see Theorem 0.22. Similarly, graphs and operators satisfying a maximum principle are in a one-to-one correspondence, see Theorem 0.24. Such operators are called graph Laplacians. These two correspondences give analytic characterizations of graphs.

The presence of c as well as b also naturally connects to the stochastic point of view, as seen when we first look at solutions of the heat equation. Here, it is natural to expect that given a positive initial distribution of heat which is bounded above, the amount of heat should remain positive and bounded above (with the same bound). This turns out to be exactly the case when the semigroup is associated to a graph Laplacian arising from a graph involving both b and c, see Theorem 0.49. Analytically, the presence of c captures the possibility of losing heat during the time of the heat flow as can happen geometrically due to boundary conditions. This possibility of losing heat is described in Corollary 0.62 and Theorem 0.65.

We also note that restrictions to subsets of forms associated to graphs are again forms associated to graphs only if we allow for a non-vanishing c (Exercise 0.31).

Finally, the presence of both b and c gives a one-to-one correspondence between graphs and Markov processes (which, in turn, are determined by either the Dirichlet form or the Laplacian). This is discussed towards the end of Subsection 10.1. From this viewpoint, b encodes how the process jumps between points of X and c captures the possibility of the process leaving X. One way of visualizing c is that the value of c indicates the weight of a connection from that vertex to some additional point (often called the graveyard or cemetery) outside of X. This explains the name killing term for c.

In summary, it is only by including c in the definition that we are able to capture natural analytic and probabilistic aspects of discrete Dirichlet spaces.

Example 0.2 (Graphs with standard weights). If b takes values in $\{0,1\}$ and $c(x) = 0$ for all $x \in X$, then we speak of a graph b with *standard weights*. In this case, the set of edges E is given by

$$E = \{(x,y) \in X \times X \mid b(x,y) = 1\}.$$

An important geometric quantity that comes with a graph (b,c) over X is the vertex degree. In our context, this is defined as follows.

Definition 0.3 (Degree). Let (b,c) be a graph over a finite set X. The *degree* is the function $\deg\colon X \longrightarrow [0,\infty)$ given by

$$\deg(x) = \sum_{y\in X} b(x,y) + c(x).$$

Example 0.4 (Combinatorial degree). If b is a graph with standard weights over X, then

$$\deg(x) = \sum_{y\in X, b(x,y)=1} 1 = \#\{y \in X \mid y \sim x\}$$

for $x \in X$. In this case, $\deg(x)$ is the number of neighbors of x and deg is called the *combinatorial degree*.

Notation. Whenever f is a real-valued function on X, we write $f = 0$ if $f(x) = 0$ for all $x \in X$. Likewise, we write $f \geq 0$, $f \leq 0$, $f > 0$ or $f < 0$ whenever these inequalities hold at all vertices of X. We will call functions satisfying $f \geq 0$ *positive* and functions satisfying $f > 0$ *strictly positive*.

Furthermore, we will use the notation $\sum_{x,y\in X}$ for the double sum $\sum_{x\in X}\sum_{y\in X}$ starting with the next subsection and throughout the rest of the book.

1.2. Forms and Laplacians on graphs. Any graph over a finite set comes with a form, a Laplacian and a matrix. We now introduce these objects.

To a finite set X we associate the real vector space $C(X)$ of all functions $f\colon X \longrightarrow \mathbb{R}$. A natural basis for $C(X)$ consists of characteristic functions 1_x which take the value 1 at x and are 0 otherwise. For $f \in C(X)$ we denote by $\operatorname{supp} f$ the *support* of f, which is the set where f is non-zero, that is, $\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$. We say that a function has *full support* if f is non-zero everywhere, that is, $\operatorname{supp} f = X$.

A linear map $L\colon C(X) \longrightarrow C(X)$ is called an *operator* on $C(X)$. Clearly, to any operator L, there exists a unique function $l\colon X \times X \longrightarrow \mathbb{R}$ with

$$Lf(x) = \sum_{y\in X} l(x,y)f(y)$$

for all $f \in C(X)$ and $x \in X$. A direct calculation gives that $l(x,y) = L1_y(x)$. We say that L is the operator on $C(X)$ *induced* by the matrix

l and l is the *matrix associated* to L. Of particular relevance to us is the case when l is *symmetric*, i.e., satisfies

$$l(x, y) = l(y, x)$$

for all $x, y \in X$. We say that L is an operator on $C(X)$ with *symmetric matrix* if l is symmetric or call L a *symmetric operator* in this case.

A *form* over X is a map

$$Q\colon C(X) \times C(X) \longrightarrow \mathbb{R}$$

which is bilinear, i.e., satisfies

$$Q(\alpha f + g, h) = \alpha Q(f, h) + Q(g, h)$$

and

$$Q(f, \alpha g + h) = \alpha Q(f, g) + Q(f, h)$$

for all $f, g, h \in C(X)$ and all $\alpha \in \mathbb{R}$. A form Q is called *symmetric* if Q satisfies $Q(f, g) = Q(g, f)$ for all $f, g \in C(X)$. For the values of Q on the diagonal $\{(f, f) \mid f \in C(X)\}$ of $C(X) \times C(X)$ we will use the notation

$$Q(f) = Q(f, f).$$

In particular, when Q is symmetric, we get

$$Q(f + g) = Q(f) + 2Q(f, g) + Q(g).$$

If Q is a form, then there exists a unique function $l\colon X \times X \longrightarrow \mathbb{R}$ with

$$Q(f, g) = \sum_{x,y \in X} l(x, y) f(x) g(y)$$

for all $f, g \in C(X)$. We call Q the form *induced by* the matrix l and l the *matrix associated* to Q. We note that $Q(1_x, 1_y) = l(x, y)$ for all $x, y \in X$ and $Q(1_x, 1) = \sum_{z \in X} l(x, z)$ where 1 denotes the function which is 1 on all vertices. In particular, Q is symmetric if and only if the associated matrix l is symmetric.

If l is a symmetric matrix over X with associated form Q and associated operator L, then

$$Q(f, g) = \sum_{x,y \in X} l(x, y) f(x) g(y) = \sum_{y \in X} (Lf)(y) g(y) = \sum_{x \in X} f(x)(Lg)(x)$$

for all $f, g \in C(X)$. In this case, we will speak of L as being the operator associated to the form Q and Q being the form associated to the operator L. Hence, defining any one of these three objects, namely, either the matrix l, the form Q or the operator L, uniquely determines the other two associated objects.

After this general introduction to matrices, operators and forms, we now focus on the matrix, operator and form which arise naturally from a graph (b, c) over X. We start with the form $Q_{b,c}$.

DEFINITION 0.5 (Form associated to a graph). Let (b, c) be a graph over a finite set X. The form $Q_{b,c}$ acting on $C(X) \times C(X)$ by

$$Q_{b,c}(f, g) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x)f(x)g(x)$$

is called the *form associated to the graph* (b, c) or the *energy form*.

We note by direct calculation that

$$Q_{b,c}(1_x) = \sum_{y \in X} b(x, y) + c(x) = \deg(x)$$

and

$$Q_{b,c}(1_x, 1_y) = -b(x, y)$$

whenever $x \neq y$. Furthermore,

$$Q_{b,c}(1_x, 1) = c(x)$$

for all $x \in X$.

Clearly, $Q_{b,c}$ is symmetric. Furthermore, by definition, $Q_{b,c}$ has the following feature: if $f, g \in C(X)$ satisfy $|f| \leq |g|$ and $|f(x) - f(y)| \leq |g(x) - g(y)|$ for all $x, y \in X$, then

$$Q_{b,c}(f) \leq Q_{b,c}(g).$$

Symmetric forms with this feature are referred to as Dirichlet forms. We will see that all symmetric Dirichlet forms arise as forms associated to graphs.

We next introduce the operator associated to the form $Q_{b,c}$.

DEFINITION 0.6 (Laplacian). Let (b, c) be a graph over a finite set X. The operator $L_{b,c}$ acting on $C(X)$ by

$$L_{b,c}f(x) = \sum_{y \in X} b(x, y)(f(x) - f(y)) + c(x)f(x)$$

is called the *Laplacian associated to the graph* (b, c).

This Laplacian satisfies a remarkable feature known as the *maximum principle*, namely,

$$L_{b,c}f(x) \geq 0$$

whenever f has a non-negative maximum at $x \in X$. While this is a direct consequence of the definition, a rather surprising amount of information can be extracted from this maximum principle. Furthermore, as we will see later, the validity of this maximum principle actually characterizes Laplacians arising from graphs.

A direct computation shows that $Q_{b,c}$ and $L_{b,c}$ are induced by the same matrix $l_{b,c}$, which we introduce next.

Definition 0.7 (Matrix associated to a graph). Let (b,c) be a graph over a finite set X. The matrix $l_{b,c}$ given by

$$l_{b,c}(x,y) = \begin{cases} -b(x,y) & \text{if} \quad x \neq y \\ \sum_{z\in X} b(x,z) + c(x) & \text{if} \quad x = y \end{cases}$$

is called the *matrix associated to the graph* (b,c). We say that (b,c) *induces* the matrix $l_{b,c}$.

As b is symmetric, so is $l_{b,c}$. As the form and the Laplacian associated to a graph (b,c) are both induced by the matrix $l_{b,c}$, we immediately obtain the following relation between them.

Proposition 0.8 (Green's formula). *Let (b,c) be a graph over a finite set X. Let $Q_{b,c}$ and $L_{b,c}$ be the form and Laplacian associated to (b,c). For all $f,g \in C(X)$,*

$$Q_{b,c}(f,g) = \sum_{x\in X}(L_{b,c}f)(x)g(x) = \sum_{x\in X} f(x)(L_{b,c}g)(x).$$

One of the goals of this chapter is to characterize the matrices, forms and operators induced by graphs within the class of all symmetric matrices, forms and operators. We will start by characterizing symmetric matrices induced by graphs.

Lemma 0.9 (Characterizing matrices arising from graphs). *Let X be a finite set. Let $l\colon X\times X \longrightarrow \mathbb{R}$ be a symmetric matrix. Then, the following statements are equivalent:*

(i) *There exists a graph (b,c) such that $l = l_{b,c}$.* (*"Graph"*)

(ii) *The matrix l satisfies*

$$l(x,y) \leq 0$$

for all $x,y \in X$ with $x \neq y$ and

$$\sum_{z\in X} l(x,z) \geq 0$$

for all $x \in X$. (*"Matrix"*)

Moreover, if (i) *and* (ii) *hold, then the graph (b,c) which induces l satisfies $c = 0$ if and only if $\sum_{z\in X} l(x,z) = 0$ for all $x \in X$.*

Proof. (i) $\Longrightarrow$ (ii): Let $l = l_{b,c}$ be the matrix associated to a graph (b,c). By the definition of $l_{b,c}$, $l(x,y) = -b(x,y) \leq 0$ for all $x \neq y$ as $b(x,y) \geq 0$. Furthermore,

$$\begin{aligned}\sum_{z\in X} l(x,z) &= l(x,x) + \sum_{z\neq x} l(x,z) = \sum_{z\in X} b(x,z) + c(x) - \sum_{z\neq x} b(x,z) \\ &= c(x) \geq 0\end{aligned}$$

for all $x \in X$ as $b(x,x) = 0$. This gives (ii).

(ii) $\Longrightarrow$ (i): Define $b\colon X\times X\longrightarrow\mathbb{R}$ for $x\neq y$ by

$$b(x,y)=-l(x,y)\quad\text{and}\quad b(x,x)=0.$$

Define $c\colon X\longrightarrow\mathbb{R}$ by

$$c(x)=\sum_{z\in X}l(x,z).$$

Then, (b,c) is a graph over X by (ii) and the symmetry of l.

Furthermore, by construction, $l_{b,c}(x,y)=-b(x,y)=l(x,y)$ for $x\neq y$ and

$$\begin{aligned}l_{b,c}(x,x)&=\sum_{z\in X}b(x,z)+c(x)=\sum_{z\neq x}b(x,z)+c(x)\\&=-\sum_{z\neq x}l(x,z)+\sum_{z\in X}l(x,z)=l(x,x).\end{aligned}$$

Therefore, l is the matrix associated to the graph (b,c). This gives (i).

The last statement is clear from the considerations above. □

1.3. Laplacians and forms on graphs with a measure. We will next discuss Laplacians as operators on a finite-dimensional Hilbert space. Although these notions will not be used until Section 5, we introduce them at this point because of the importance of this viewpoint for the overall theory.

We start by introducing measures on a finite set X. If $m\colon X\longrightarrow(0,\infty)$ is a strictly positive function on X, then we can extend m to a measure of full support on X via

$$m(A)=\sum_{x\in A}m(x)$$

for all subsets $A\subseteq X$. Therefore, the pair (X,m) can be seen as a measure space.

Remark. Technically, we could allow for the function m to take values in $[0,\infty)$. However, in this case, all further considerations are carried out on the support of m, which is the same as passing from X to the subset of X where m does not vanish. Hence, for convenience, we exclude this and only consider the case when m is strictly positive.

Notation. When m is assumed to be strictly positive, then we say that (X,m) is a *finite measure space*.

Definition 0.10 (Graph over finite (X,m)). If (X,m) is a finite measure space and (b,c) is a graph over X, then (b,c) is called a *graph over* (X,m).

There are some measures which occur naturally in our setting. Two of these are introduced next. The first one is general, the second one requires a graph structure (b,c) over X.

EXAMPLE 0.11 (Counting measure). Let $m = 1$. Then m is called the *counting measure* on X. In this case, the measure of a set $A \subseteq X$ is the number of vertices in the set, i.e.,

$$m(A) = \sum_{x \in A} 1 = \#A.$$

EXAMPLE 0.12 (Normalizing measure). Given a graph (b, c) over X, we let $m(x) = \deg(x) = \sum_{y \in X} b(x, y) + c(x)$. Whenever we use deg in the spirit of a measure, we denote it by n and call n the *normalizing measure*, i.e., $n = \deg$ is given by

$$n(x) = \sum_{y \in X} b(x, y) + c(x).$$

We note that in the case of graphs with standard weights, i.e., b taking values in $\{0, 1\}$ and $c = 0$, the normalizing measure n satisfies

$$n(A) = \#E_A + \frac{1}{2}\#\partial_E A$$

for $A \subseteq X$ where $E_A = \{(x, y) \in A \times A \mid x \sim y\}$ and

$$\partial_E A = \{(x, y) \in (A \times (X \setminus A)) \cup ((X \setminus A) \times A) \mid x \sim y\}.$$

That is, the normalizing measure counts the number of edges within A plus the number of edges leaving A (Exercise 0.28).

An important geometric quantity which comes with a graph (b, c) over (X, m) is another type of a vertex degree. This is introduced next.

DEFINITION 0.13 (Weighted degree). Let (b, c) be a graph over a finite measure space (X, m). The *weighted degree* is the function $\mathrm{Deg}\colon X \longrightarrow [0, \infty)$ given by

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right).$$

We note that

$$\mathrm{Deg} = \frac{\deg}{m},$$

where deg is the degree function.

EXAMPLE 0.14 (Weighted degree for counting and normalizing measure). Let us discuss the function Deg in the case of the counting and normalizing measures introduced above. In the case of the counting measure $m = 1$, we have

$$\mathrm{Deg} = \frac{\deg}{m} = \deg.$$

In particular, for standard weights and $m = 1$, Deg is the same as the combinatorial degree.

For the normalizing measure $m = n = \deg$, we have

$$\mathrm{Deg} = \frac{\deg}{n} = 1,$$

which justifies the name of n. In this case, the weighted degree does not distinguish between vertices.

Given a measure m, the space $C(X)$ inherits a Hilbert space structure in a natural way. In this context, we can make full use of the theory of self-adjoint operators on Hilbert spaces in order to analyze Laplacians on graphs, especially in the case of infinite graphs considered later. Here, we introduce the corresponding notations and concepts in the finite setting.

The vector space $C(X)$ with inner product

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x)m(x)$$

and induced norm

$$\|f\| = \langle f, f\rangle^{1/2}$$

is complete and, therefore, a Hilbert space. This Hilbert space will be denoted by $\ell^2(X,m)$. Note that we work here with spaces of real-valued functions.

A linear map $L_m\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ is called an *operator* on $\ell^2(X,m)$. Such an operator L_m can be uniquely represented by a matrix $l\colon X\times X \longrightarrow \mathbb{R}$ with

$$\langle L_m f, g\rangle = \sum_{x,y\in X} l(x,y)f(y)g(x)$$

for all $f, g \in \ell^2(X,m)$. Equivalently,

$$L_m f(x) = \frac{1}{m(x)} \sum_{y\in X} l(x,y)f(y)$$

for all $f \in \ell^2(X,m)$ and $x \in X$. In fact, a direct calculation gives

$$l(x,y) = \langle L_m 1_y, 1_x\rangle.$$

We call L_m the *operator induced by the matrix l on $\ell^2(X,m)$* and we call l the *matrix associated to L_m*.

An operator L_m on $\ell^2(X,m)$ is called *self-adjoint* if L_m satisfies

$$\langle L_m f, g\rangle = \langle f, L_m g\rangle$$

for all $f, g \in \ell^2(X,m)$. Clearly this holds if and only if $\langle L_m 1_y, 1_x\rangle = \langle 1_y, L_m 1_x\rangle$, that is, if and only if

$$l(x,y) = l(y,x).$$

Hence, an operator L_m is self-adjoint on $\ell^2(X,m)$ if and only if the matrix associated to L_m is symmetric.

It is clear from the preceding discussion that self-adjoint operators are in one-to-one correspondence with symmetric matrices. Furthermore, if L_m is a self-adjoint operator with an associated symmetric matrix l, then we can associate a symmetric form Q induced by the matrix l as before by

$$Q(f,g) = \sum_{x,y\in X} l(x,y)f(x)g(y).$$

This form will then satisfy

$$Q(f,g) = \langle L_m f, g\rangle = \langle f, L_m g\rangle.$$

In this case, we denote the form by Q_{L_m} and note that the map $L_m \mapsto Q_{L_m}$ provides a one-to-one correspondence between self-adjoint operators and symmetric forms.

Let us emphasize that we do not need the measure m in order to define the form or to define matrices. The measure only enters when we want to speak about operators on a Hilbert space.

We next define the operator on $\ell^2(X,m)$ that is most prominent throughout our work.

Definition 0.15 (Laplacian on $\ell^2(X,m)$). Let (b,c) be a graph over a finite measure space (X,m). The operator $L_{b,c,m}$ acting on $\ell^2(X,m)$ via

$$L_{b,c,m}f(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y)) + \frac{c(x)}{m(x)}f(x)$$

is called the *Laplacian on $\ell^2(X,m)$ associated to the graph* (b,c).

We note the following immediate relationship between the Laplacian $L_{b,c}$ and the Laplacian $L_{b,c,m}$ on $\ell^2(X,m)$:

$$L_{b,c,m}f(x) = \frac{1}{m(x)} L_{b,c}f(x)$$

for all $f \in \ell^2(X,m)$ and all $x \in X$. In particular, we note that

$$\langle L_{b,c,m}1_y, 1_x\rangle = L_{b,c,m}1_y(x)m(x) = L_{b,c}1_y(x).$$

Therefore, the matrix associated to $L_{b,c,m}$ is just the matrix $l_{b,c}$ associated to the graph (b,c) as in Definition 0.7. In particular, $L_{b,c,m}$ is self-adjoint as $l_{b,c}$ is symmetric. Furthermore, $Q_{b,c}$ is the form associated to both $L_{b,c}$ and $L_{b,c,m}$ and Green's formula, Proposition 0.8, transfers to $L_{b,c,m}$ as

$$Q_{b,c}(f,g) = \langle L_{b,c,m}f, g\rangle = \langle f, L_{b,c,m}g\rangle$$

for all $f, g \in \ell^2(X,m)$.

2. Characterizing forms associated to graphs

In this section we give a structural characterization of forms associated to graphs. This will be based on the concept of a normal contraction. In fact, we will show that forms associated to graphs are exactly the forms compatible with normal contractions.

A map $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called a *normal contraction* if

$$C(0) = 0 \quad \text{and} \quad |C(s) - C(t)| \leq |s - t|$$

for all $s, t \in \mathbb{R}$. In particular, we note that $|C(s)| \leq |s|$ for all $s \in \mathbb{R}$ when C is a normal contraction.

In the context of normal contractions it is convenient to define

$$s \wedge t = \min\{s, t\} \quad \text{and} \quad s \vee t = \max\{s, t\}$$

for real numbers or for real-valued functions s and t.

EXAMPLE 0.16. The following maps $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ are normal contractions.

(a) $C(s) = |s|$.
(b) $C(s) = (\pm s) \vee 0$.
(c) $C(s) = s \wedge 1$.
(d) $C(s) = 0 \vee (s \wedge 1)$.

The last normal contraction in the example above maps a real number s to the number closest to s in $[0, 1]$. We will denote this normal contraction as $C_{[0,1]}$, that is,

$$C_{[0,1]}(s) = 0 \vee (s \wedge 1).$$

The normal contraction $C_{[0,1]}$ will play a special role in some of the characterizations below.

Given a form Q on $C(X)$ and a normal contraction C, we will say that Q is *compatible* with C if

$$Q(C \circ f) \leq Q(f)$$

for all $f \in C(X)$. Here $C \circ f$ denotes the composition of C and f. From the defining properties of a normal contraction we directly infer the following compatibility of normal contractions and forms associated to graphs.

PROPOSITION 0.17 (Compatibility of graph forms with normal contractions). *Let (b, c) be a graph over a finite set X and let $Q_{b,c}$ be the form associated to (b, c). If $f \in C(X)$ and C is a normal contraction, then*

$$Q_{b,c}(C \circ f) \leq Q_{b,c}(f).$$

PROOF. As C is a normal contraction, we clearly have $|C(f(x))| \leq |f(x)|$ and $|C(f(x)) - C(f(y))| \leq |f(x) - f(y)|$ for all $x, y \in X$. Taking squares we obtain

$$(C(f(x)))^2 \leq f^2(x) \quad \text{and} \quad (C(f(x)) - C(f(y)))^2 \leq (f(x) - f(y))^2.$$

This gives the desired statement after multiplying by $c(x)$ and $b(x, y)$ and taking sums. $\square$

The characterization of forms associated to graphs that we are after will follow from a converse to this proposition. Our proof of this converse is based on studying forms Q which are compatible with suitable normal contractions. This will be of interest in other situations as well. We will need the following auxiliary proposition.

PROPOSITION 0.18 (Representing forms via differences). *Let X be a finite set. Let Q be a symmetric form over X with associated matrix $l \colon X \times X \longrightarrow \mathbb{R}$. Define $b_Q \colon X \times X \longrightarrow \mathbb{R}$ and $c_Q \colon X \longrightarrow \mathbb{R}$ by*

$$b_Q(x, y) = \begin{cases} -l(x, y) & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

and

$$c_Q(x) = \sum_{y \in X} l(x, y).$$

The form Q satisfies

$$Q(f, g) = \frac{1}{2} \sum_{x,y \in X} b_Q(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c_Q(x) f(x) g(x)$$

for all $f, g \in C(X)$.

PROOF. This follows by a direct computation. By definition,

$$Q(f, g) = \sum_{x,y \in X} l(x, y) f(x) g(y).$$

Furthermore, by using the definitions of c_Q and b_Q, we get

$$l(x, x) = \sum_{y \in X} l(x, y) - \sum_{y \neq x} l(x, y) = c_Q(x) + \sum_{y \in X} b_Q(x, y).$$

Therefore,

$$\begin{aligned}
&Q(f, g) \\
&\quad = \sum_{x,y \in X} l(x, y) f(x) g(y) \\
&\quad = \sum_{x \in X} l(x, x) f(x) g(x) + \sum_{x \in X} \sum_{y \neq x} l(x, y) f(x) g(y) \\
&\quad = \sum_{x \in X} \left(c_Q(x) + \sum_{y \in X} b_Q(x, y) \right) f(x) g(x) - \sum_{x,y \in X} b_Q(x, y) f(x) g(y)
\end{aligned}$$

$$= \sum_{x,y\in X} b_Q(x,y)f(x)(g(x)-g(y)) + \sum_{x\in X} c_Q(x)f(x)g(x)$$
$$= \frac{1}{2}\sum_{x,y\in X} b_Q(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c_Q(x)f(x)g(x),$$

where in the last equality we use the symmetry of b_Q, which follows from the symmetry of l. □

We note that whenever (b_Q, c_Q) is a graph over X, the proposition above says that $Q = Q_{b_Q,c_Q}$. We will now show that compatibility with certain normal contractions implies that (b_Q, c_Q) is indeed a graph. We start by characterizing the symmetric forms which are compatible with the absolute value and those which are compatible with the normal contraction $C_{[0,1]} \circ f = 0 \vee (f \wedge 1)$ introduced above.

LEMMA 0.19 (Characterization of compatibility with normal contractions). *Let X be a finite set. Let Q be a symmetric form over X with associated matrix $l\colon X \times X \longrightarrow \mathbb{R}$.*

(a) *The following statement are equivalent:*
 (i) *The form Q satisfies, for all $f \in C(X)$,*
$$Q(|f|) \leq Q(f).$$
 (ii) *The matrix l satisfies, for all $x \neq y$,*
$$l(x,y) \leq 0.$$
(b) *The following statements are equivalent:*
 (i) *The form Q satisfies, for all $f \in C(X)$,*
$$Q(C_{[0,1]} \circ f) \leq Q(f).$$
 (ii) *The matrix l satisfies, for all $x \in X$ and $y \in X$ with $x \neq y$,*
$$l(x,y) \leq 0 \quad \textit{and} \quad \sum_{z\in X} l(x,z) \geq 0.$$

REMARK. The proof of the implication (i) $\Longrightarrow$ (ii) in (a) given below actually shows that $Q(|f|) \leq Q(f)$ for all $f \in C(X)$ is also equivalent to
$$Q(f_+, f_-) \leq 0$$
for all $f \in C(X)$, where the positive and negative part $f_\pm$ of f are defined by
$$f_+ = f \vee 0 \quad \text{and} \quad f_- = (-f) \vee 0.$$

PROOF. As shown in Proposition 0.18, we have
$$Q(f) = \frac{1}{2}\sum_{x\neq y} b_Q(x,y)(f(x)-f(y))^2 + \sum_{x\in X} c_Q(x)f^2(x)$$
with
$$b_Q(x,y) = -l(x,y) \text{ for } x \neq y$$

and

$$c_Q(x) = \sum_{z\in X} l(x,z).$$

This shows the implication (ii) $\Longrightarrow$ (i) in both (a) and (b), compare the reasoning in the proof of Proposition 0.17.

(i) $\Longrightarrow$ (ii) in (a): Assume that Q satisfies $Q(|f|) \le Q(f)$ for all $f \in C(X)$. Let $x, y \in X$ with $x \neq y$ and consider $f = 1_x - 1_y$, where 1_z denotes the characteristic function of $z \in X$. Then, $|f| = 1_x + 1_y$. Hence, the assumption on Q gives

$$Q(1_x + 1_y) \le Q(1_x - 1_y).$$

Invoking the bilinearity and symmetry of Q, we can easily infer

$$4Q(1_x, 1_y) \le 0.$$

Since $l(x,y) = Q(1_x, 1_y)$, the desired statement follows.

(i) $\Longrightarrow$ (ii) in (b): Assume that Q satisfies $Q(C_{[0,1]} \circ f) \le Q(f)$ for all $f \in C(X)$.

We start by showing that $l(x,y) \le 0$ for all $x \neq y$. By part (a), which has already been proven, it suffices to show that $Q(|f|) \le Q(f)$ holds for all $f \in C(X)$.

Let $f \in C(X)$. After replacing f by αf with a suitable $\alpha > 0$, we can assume without loss of generality that $|f| \le 1$. Now, consider the decomposition of f into positive and negative parts $f = f_+ - f_-$ where $f_+(x) = f(x) \vee 0$ and $f_-(x) = -f(x) \vee 0$. Clearly, $|f| = f_+ + f_-$. For $s > 0$ set

$$f_s = f_+ - sf_-.$$

Then, $C_{[0,1]} \circ f_s = f_+$ for all $s > 0$. Thus, our assumption gives

$$Q(f_+) = Q(C_{[0,1]} \circ f_s) \le Q(f_s) = Q(f_+ - sf_-).$$

Invoking the bilinearity of Q and dividing by $s > 0$, we can then easily infer

$$0 \le -2Q(f_+, f_-) + sQ(f_-)$$

for all $s > 0$. Letting $s \to 0$, we obtain

$$0 \le -Q(f_+, f_-).$$

Given this inequality, it follows that

$$\begin{aligned}
Q(|f|) &= Q(f_+ + f_-)\\
&= Q(f_+) + 2Q(f_+, f_-) + Q(f_-)\\
&\le Q(f_+) - 2Q(f_+, f_-) + Q(f_-)\\
&= Q(f).
\end{aligned}$$

This gives the desired compatibility of Q with $|\cdot|$.

We now turn to proving that $\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Let $x \in X$ and consider $f = 1 + s1_x$ with $s > 0$. Then, $C_{[0,1]} \circ f = 1$ for all $s > 0$ and we obtain by assumption that

$$Q(1) = Q(C_{[0,1]} \circ f) \leq Q(f) = Q(1 + s1_x).$$

By the bilinearity of Q and dividing by s, this implies

$$0 \leq 2Q(1, 1_x) + sQ(1_x).$$

Letting $s \to 0$, we obtain

$$0 \leq Q(1, 1_x) = \sum_{z\in X} l(x,z).$$

This gives the desired inequality for every $x \in X$. $\square$

We are now in position to prove our characterization of symmetric forms associated to graphs in terms of compatibility with normal contractions.

Theorem 0.20 (Characterization of forms associated to graphs). *Let Q be a symmetric form over a finite set X. Then, the following statements are equivalent:*

(i) *There exists a graph (b,c) over X with*

$$Q = Q_{b,c}. \qquad \text{(“Graph”)}$$

(ii) *The matrix l associated to Q satisfies, for $x, y \in X$ with $x \neq y$,*

$$l(x,y) \leq 0 \qquad \text{and} \qquad \sum_{z\in X} l(x,z) \geq 0. \qquad \text{(“Matrix”)}$$

(iii) *For all $f \in C(X)$,*

$$Q(C_{[0,1]} \circ f) \leq Q(f).$$

(*“Form compatible with one normal contraction”*)

(iv) *For all normal contractions C and $f \in C(X)$,*

$$Q(C \circ f) \leq Q(f).$$

(*“Form compatible with normal contractions”*)

(v) *If $f, g \in C(X)$ satisfy, for all $x, y \in X$,*

$$|f| \leq |g| \qquad \text{and} \qquad |f(x) - f(y)| \leq |g(x) - g(y)|,$$

then

$$Q(f) \leq Q(g).$$

Remark. Note that the above shows that compatibility with a particular normal contraction, namely $C_{[0,1]}$, is equivalent to compatibility with all normal contractions. It can also be shown that this is equivalent to compatibility with the contraction $C_{(-\infty,1]}$ given by $C_{(-\infty,1]}(s) = s \wedge 1$, i.e., $Q(f \wedge 1) \leq Q(f)$ for all $f \in C(X)$ (Exercise 0.29).

PROOF. This follows from the preceding considerations. Indeed, Lemma 0.9 gives the equivalence between (i) and (ii). The equivalence between (ii) and (iii) is the content of Lemma 0.19 (b). The implication (i) $\Longrightarrow$ (v) can be directly read off from the definition of $Q_{b,c}$ and was also already noted in Subsection 1.2 (compare Proposition 0.17 for a similar reasoning as well). The implication (v) $\Longrightarrow$ (iv) is clear from the definition of a normal contraction. Finally, (iv) $\Longrightarrow$ (iii) is obvious as $C_{[0,1]}$ is a normal contraction. $\square$

The previous result indicates that we should single out forms satisfying any one of the equivalent conditions appearing in Theorem 0.20.

DEFINITION 0.21 (Dirichlet form over a finite set). Let X be a finite set. A form Q on $C(X)$ is called a *Dirichlet form* if Q satisfies

$$Q(C \circ f) \leq Q(f)$$

for all $f \in C(X)$ and all normal contractions $C\colon \mathbb{R} \longrightarrow \mathbb{R}$.

Given the notion of a Dirichlet form, the preceding considerations directly imply the following result.

THEOREM 0.22 (Correspondence Dirichlet forms and graphs). *Let X be a finite set. The map $(b, c) \mapsto Q_{b,c}$ gives a bijective correspondence between graphs (b, c) over X and symmetric Dirichlet forms over X.*

REMARK. It is also possible to characterize graphs with $c = 0$ via forms which are compatible with certain contractions. This will be discussed in Section 8.

3. Characterizing Laplacians associated to graphs

The Laplacian Δ acting on smooth functions on Euclidean space via $\Delta f = -f''$ has the property that $\Delta f(x) \geq 0$ whenever a smooth function f has a maximum at x. Here, we are going to see that Laplacians on graphs are characterized by a very similar feature. This feature is called the maximum principle. The validity of this principle means that Laplacians on graphs can be seen as the negative of taking the second derivative in a discrete setting. The maximum principle also has strong consequences for solutions u of equations of the form $(L + \alpha)u = f$ for $\alpha \geq 0$.

Excavation Exercise 0.5 recalls the equivalence of injectivity, surjectivity and bijectivity for an operator on a finite-dimensional vector space, which will be used throughout this section.

We start by defining the maximum principle that will characterize all Laplacians on finite graphs.

Definition 0.23 (Maximum principle). Let X be a finite set and let L be an operator on $C(X)$. The operator L is said to satisfy the *maximum principle* if

$$Lf(x) \geq 0$$

whenever $f \in C(X)$ has a non-negative maximum at $x \in X$.

Remark. Clearly, L satisfies the maximum principle if and only if $Lf(x) \leq 0$ whenever $f \in C(X)$ has a non-positive minimum at $x \in X$.

Remark. We have phrased the definition of the maximum principle without any reference to a measure. However, the inequality in question remains unchanged if both sides are multiplied by the inverse of the measure of x. Thus, the operator L associated to a matrix l satisfies the maximum principle if and only if for one (all) $m\colon X \longrightarrow (0,\infty)$ the operator L_m associated to the matrix l on $\ell^2(X,m)$ satisfies

$$L_m f(x) \geq 0$$

whenever $f \in \ell^2(X,m)$ has a non-negative maximum at $x \in X$.

Remark. The definition raises the question if $Lf(x) > 0$ whenever f has a non-negative maximum at x. Our subsequent discussion will show that the vanishing of $Lf(x)$ for all such x is indeed possible if f is constant. Moreover, under suitable connectedness assumptions, we will see that the only case when $Lf(x)$ vanishes is when f is constant.

We now show that the maximum principle characterizes Laplacian operators within the set of symmetric operators on $C(X)$.

Theorem 0.24 (Maximum principle and graphs). *Let X be a finite set and let L be a symmetric operator on $C(X)$. Then, the following statements are equivalent:*

(i) *The operator L satisfies the maximum principle.*
(ii) *There exists a graph (b,c) over X such that $L = L_{b,c}$ is the Laplacian associated to (b,c).*

Proof. (i) $\Longrightarrow$ (ii): Let l be the matrix associated to L. By Lemma 0.9 it suffices to show that $l(x,y) \leq 0$ for all $x \neq y$ and $\sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle to $f = 1$, we directly obtain $L1(x) = \sum_{z\in X} l(x,z) \geq 0$ for all $x \in X$. Applying the maximum principle at $x \in X$ to $f = -1_y$ for an arbitrary $y \in X$ with $y \neq x$ we infer $-L1_y(x) = -l(x,y) \geq 0$ so that $l(x,y) \leq 0$ for all $x \neq y$.

(ii) $\Longrightarrow$ (i): As $L = L_{b,c}$ is the Laplacian associated to a graph (b,c) it follows that if f has a non-negative maximum at x, then

$$Lf(x) = \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x) \geq 0,$$

which completes the proof. □

The maximum principle discussed in the previous theorem is not a strict analogue to the maximum principle satisfied by the Laplacian Δ on Euclidean space alluded to at the beginning of this section. In fact, there is no restriction on the sign of the maximum in the Euclidean case. This is due to the presence of c in our setting. We now give a strict analogue.

DEFINITION 0.25 (Strong maximum principle). Let X be a finite set and let L be an operator on $C(X)$. The operator L is said to satisfy the *strong maximum principle* if $Lf(x) \geq 0$ holds whenever $f \in C(X)$ has a maximum at $x \in X$.

Hence, the strong maximum principle removes the assumption found in the maximum principle that the maximum attained by f at x is non-negative. As such, the maximum principle holds whenever the strong maximum principle holds. In fact, the relationship between these two principles can be described as follows.

LEMMA 0.26 (Maximum principle and strong maximum principle). *Let X be a finite set and let L be an operator on $C(X)$ satisfying the maximum principle. The operator L satisfies the strong maximum principle if and only if*

$$L1 = 0.$$

PROOF. Assume L satisfies the strong maximum principle. Considering $f = 1$ and any $x \in X$, we then obtain $L1(x) \geq 0$. Similarly, considering $f = -1$ and any $x \in X$ we obtain $-L1(x) \geq 0$. This implies $L1 = 0$.

Conversely, assume $L1 = 0$. Let $f \in C(X)$ have a maximum at $x \in X$. Then, for any $s \in \mathbb{R}$, $f + s1$ also has a maximum at x. Choosing s so that this maximum is non-negative then gives

$$Lf(x) = Lf(x) + sL1(x) = L(f + s1)(x) \geq 0,$$

where the last inequality is due to the fact that L satisfies the maximum principle. Therefore, L satisfies the strong maximum principle. □

Note that for a Laplacian $L_{b,c}$ associated to a graph (b, c), $L_{b,c}1 = 0$ if and only if $c = 0$. Therefore, combining the previous lemma with Theorem 0.24, we immediately infer the following characterization of the vanishing of the killing term.

COROLLARY 0.27 (Strong maximum principle and vanishing c). *Let X be a finite set and let L be a symmetric operator on $C(X)$. Then, the following statements are equivalent:*

(i) *L satisfies the strong maximum principle.*
(ii) *There exists a graph (b, c) over X with $c = 0$ such that $L = L_b$ is the Laplacian associated to b.*

The maximum principle has strong consequences for solutions u of equations of the form $(L+\alpha)u = f$ for f with a fixed sign and $\alpha \geq 0$. Here, $L+\alpha$ is shorthand notation for $L+\alpha I$, where I is the identity operator on $C(X)$. We will now discuss these consequences in some detail. We will need the following topological assumption in order to deal with the existence and uniqueness of solutions, i.e., bijectivity of $L+\alpha$. The maximum principle will then yield additional features of the solutions.

Definition 0.28 (Connected component and paths). Let (b, c) be a graph over a finite set X. Given $x, y \in X$ we call a sequence $(x_0, x_1, \ldots, x_n)$ of pairwise distinct vertices a *path* from x to y if $x_0 = x, x_n = y$ and $x_j \sim x_{j+1}$ for $j = 0, 1, \ldots, n-1$. We say that a path $(x_0, x_1, \ldots, x_n)$ *connects* the vertices x_0 and x_n. We call a subset Y of X *connected* if any two vertices in Y can be connected by a path of vertices in Y. Furthermore, Y is a *connected component* of X if Y is connected and Y is not contained in a strictly larger connected subset of X. A graph (b, c) is called *connected* if (b, c) has only one connected component.

Remark. There is an equivalent (and maybe even more elegant) definition of connected components via saturated sets for which one does not have to define what it means for a set to be connected first (Exercise 0.30).

Remark. Whenever a graph without a killing term is connected, taking a proper subset and restricting the associated form to functions on the subset gives rise to a graph with a non-vanishing killing term. In this sense, graphs with non-vanishing c are unavoidable if one wants compatibility of the forms with restrictions to subsets (Exercise 0.31).

It turns out that connectedness of the graph together with non-triviality of c makes $L_{b,c}$ injective (and, therefore, bijective). Specifically, the following holds.

Lemma 0.29 (Non-vanishing c characterizes the bijectivity of $L_{b,c}$). *Let (b, c) be a graph over a finite set X and let $L_{b,c}$ be the associated Laplacian on $C(X)$. The operator $L_{b,c}$ is bijective if and only if c does not vanish identically on any connected component of (b, c).*

Proof. As $L_{b,c}$ is a linear operator on a finite dimensional vector space, bijectivity is equivalent to injectivity. Thus, it suffices to characterize injectivity. By restricting attention to a specific connected component, we can assume without loss of generality that the graph is connected.

If $c = 0$, then clearly $L_{b,c}1 = 0$. Therefore, $L_{b,c}$ is not injective in this case.

Now, suppose that c does not vanish at all $x \in X$. Let $u \in C(X)$ satisfy $L_{b,c}u = 0$. Green's formula, Proposition 0.8, gives

$$\begin{aligned} 0 &= \sum_{x \in X} u(x) L_{b,c} u(x) = Q_{b,c}(u) \\ &= \frac{1}{2} \sum_{x,y \in X} b(x,y)(u(x) - u(y))^2 + \sum_{x \in X} c(x) u^2(x). \end{aligned}$$

As all terms appearing in the sums are non-negative, we infer $u(x) = u(y)$ whenever $b(x,y) > 0$ and $u(x) = 0$ whenever $c(x) \neq 0$. As the graph is connected, the first set of conditions implies u is constant and the second set of conditions implies $u = 0$ as c does not vanish identically. Therefore, $L_{b,c}$ is injective. □

Remark. Note that injectivity of $L_{b,c}$ is equivalent to $Q_{b,c}$ being an inner product.

Theorem 0.30 (Maximum principle and solutions to $(L+\alpha)u = f$). *Let X be a finite set and let L be a symmetric operator on $C(X)$ which satisfies the maximum principle. For any $\alpha > 0$ and $f \in C(X)$ the equation*

$$(L + \alpha)u = f$$

has a unique solution u. Furthermore, $0 \leq u \leq 1/\alpha$ if $0 \leq f \leq 1$.

Proof. As L satisfies the maximum principle, by Theorem 0.24 there exists a graph (b, c) over X such that $L = L_{b,c}$. Therefore, $L + \alpha$ is the Laplacian associated to the graph $(b, c + \alpha)$. As $c + \alpha > 0$ for $\alpha > 0$, the operator $L + \alpha$ is bijective by Lemma 0.29. This gives the existence and uniqueness of the solution u as $u = (L + \alpha)^{-1} f$.

Assume now additionally that $0 \leq f \leq 1$. We first show $u \geq 0$. Let u have a minimum at $x \in X$ and assume $u(x) < 0$. We can then apply the maximum principle to $-u$ at x to obtain

$$-Lu(x) \geq 0.$$

As $u(x) < 0$, this gives the contradiction

$$0 \leq f(x) = Lu(x) + \alpha u(x) \leq \alpha u(x) < 0.$$

By a similar reasoning we can show $u \leq 1/\alpha$ as follows: Let u have a maximum at $x \in X$ and assume $u(x) > 1/\alpha > 0$. We can then directly apply the maximum principle to u at x to obtain the contradiction

$$1 \geq f(x) = Lu(x) + \alpha u(x) \geq \alpha u(x) > 1.$$

This completes the proof. □

Remark. By the characterization of Theorem 0.24 the assumption that L is symmetric and satisfies the maximum principle can be replaced by the assumption that $L = L_{b,c}$ for a graph (b, c) over X. A converse to this theorem also holds, as will be discussed in Section 6.

The preceding theorem deals with the case $\alpha > 0$. Thus, it raises the question of what happens when $\alpha = 0$, that is, when we wish to solve $Lu = f$ for a given function $f \in C(X)$. In order to address this problem, we will look at the injectivity of the operator L, i.e., we look at solutions of $Lu = 0$.

For Laplacians on graphs, this question has already been addressed above. Functions $u \in C(X)$ which satisfy

$$L_{b,c}u = 0$$

for the Laplacian associated to a graph (b, c) over X are called *harmonic*. It is clear that if $L_{b,c}$ is bijective, then $u = 0$ is the only harmonic function. By Lemma 0.29, the operator $L_{b,c}$ is bijective if and only if $c \neq 0$ on every connected component of (b, c). In the case of $c = 0$,

$$L_b 1 = 0,$$

so that all constant functions are harmonic. If, furthermore, the graph b is connected, then these are the only harmonic functions, as we will show below, see also the proof of Lemma 0.29.

This discussion implies that the existence and uniqueness as well as the estimates found in Theorem 0.30 for solutions u of $(L+\alpha)u = f$ for $\alpha > 0$ cannot be valid for $\alpha = 0$ and all symmetric operators satisfying the maximum principle. However, the existence and uniqueness of solutions to $Lu = f$ is clear when L is a bijective operator. Furthermore, when $L = L_{b,c}$ is bijective and the graph is connected we recover a variant of the estimates found in Theorem 0.30 for the solution u.

In order to show this, we will first discuss some versions of a "Liouville property" for Laplacians associated to graphs.

LEMMA 0.31 (Liouville-type properties). *Let (b, c) be a connected graph over a finite set X.*

(a) *If $u \in C(X)$ is harmonic, then u is constant.*
(b) *If $L_{b,c}$ on $C(X)$ is bijective, $u \in C(X)$ has a non-negative maximum and $L_{b,c}u(x) = 0$ for all $x \in X$ at which u takes this maximum, then $u = 0$.*

REMARK. The famous Liouville Theorem asserts that harmonic functions (i.e., functions satisfying $\Delta f = 0$) in the plane are constant if they are bounded. Lemma 0.31 gives some variants of this theorem.

PROOF. (a) The argument for this already appeared in the proof of Lemma 0.29. Namely, if u is harmonic, then Proposition 0.8 gives

$$0 = \sum_{x \in X} u(x) L_{b,c}u(x) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(u(x) - u(y))^2 + \sum_{x \in X} c(x)u^2(x).$$

As the graph is connected, we obtain that u is constant.

(b) If u has a non-negative maximum at x, then

$$0 = L_{b,c}u(x) = \sum_{y \in X} b(x,y)(u(x) - u(y)) + c(x)u(x)$$

implies $u(y) = u(x)$ for all $y \sim x$ and $c(x)u(x) = 0$. Repeating this argument, we infer that u is constant by the connectedness of the graph. Since $L_{b,c}$ is assumed to be bijective, it follows that $c \neq 0$ by Lemma 0.29. Letting x be such that $c(x) \neq 0$, we obtain $u(x) = 0$. As u is constant, $u = 0$, which gives the conclusion. $\square$

COROLLARY 0.32. *Let (b, c) be a connected graph over a finite set X such that the associated Laplacian $L_{b,c}$ on $C(X)$ is bijective. For any $f \in C(X)$, the equation*

$$L_{b,c}u = f$$

has a unique solution. Furthermore, if f satisfies $f \geq 0$ and $f \neq 0$, then $u > 0$.

PROOF. As $L_{b,c}$ is bijective, $u = L_{b,c}^{-1}f$ is the unique solution of $L_{b,c}u = f$.

Assume now that $f \geq 0$ with $f \neq 0$. Then, u satisfies $u \neq 0$ as otherwise $L_{b,c}u = 0$. It remains to show $u > 0$. Assume there exists a $y \in X$ with $u(y) \leq 0$. Consider $v = -u$. Then, v has a non-negative maximum. Moreover, at each x where v attains this maximum we have $L_{b,c}v(x) = -L_{b,c}u(x) = -f(x) \leq 0$. Since $L_{b,c}$ satisfies the maximum principle by Theorem 0.24, it follows that $L_{b,c}v(x) \geq 0$ and thus $L_{b,c}v(x) = 0$ at every maximum. Therefore, by Lemma 0.31, we infer $v = 0$ and thus $u = 0$. This is a contradiction. $\square$

REMARK. The proof of the corollary uses the strong Liouville property. In fact, it is not hard to generalize the proof to even characterize a variant of this Liouville property in the following way: If L is a bijective operator on $C(X)$ which satisfies the maximum principle, then the following statements are equivalent:

(i) The inverse L^{-1} is positivity improving, i.e., $L^{-1}f > 0$ whenever $f \geq 0$ and $f \neq 0$.
(ii) Any function u with a non-negative maximum and $Lu \leq 0$ satisfies $u = 0$.

Indeed, the implication (ii) $\Longrightarrow$ (i) follows exactly as in the proof of the corollary. To show the implication (i) $\Longrightarrow$ (ii) let u with $Lu \leq 0$ have a non-negative maximum. Assume that $u \neq 0$. It follows that $f = -Lu$ does not vanish identically (as L is injective) and satisfies $f \geq 0$. Therefore, (i) implies $L^{-1}f = -u > 0$, which is a contradiction to the assumption that u has a non-negative maximum.

4. Networks and electrostatics

In this section we will discuss a context from physics in which graphs appear naturally. More specifically, we will show how graphs serve as the right objects to study electrostatics in a discrete setting. We will first introduce the necessary background and notations and then turn to the basic equations of electrostatics and their solutions. The main focus will be on harmonic functions and three fundamental problems of electrostatics: the Poisson problem, the Dirichlet problem and the capacitor problem.

We start by describing our situation and fixing terminology. For now, we consider a graph (b, c) with $c = 0$ over a finite set X. Such a setting is sometimes referred to as a *network* and written as (X, b). We will write b for $(b, 0)$, Q_b for $Q_{b,0}$ and L_b for $L_{b,0}$. A pair $(x, y) \in X \times X$ with $b(x, y) > 0$ is called an edge. Since (x, y) is an ordered pair, it is natural to think of edges as being directed, that is, (x, y) is an edge going from x to y. The set of all edges is denoted by $E = E(X, b)$. The function $w \colon E \longrightarrow \mathbb{R}$ given by

$$w((x, y)) = \frac{1}{b(x, y)}$$

is called the *resistance* and b is called the *conductance* in the context of networks.

For an edge $e = (x, y)$, we call $x = s(e)$ the *source* of e, $y = r(e)$ the *range* of e and $\overline{e} = (y, x)$ the *reverse edge* of e. As b is symmetric, it follows that $e \in E$ if and only if $\overline{e} \in E$. An n-tuple $(e_1, \ldots, e_n)$ of edges is called a *cycle* if

$$r(e_j) = s(e_{j+1}), \quad j = 1, \ldots, n,$$

where we set $e_{n+1} = e_1$.

A map $\varphi \colon E \longrightarrow \mathbb{R}$ is called a *flow* if $\varphi(e) = -\varphi(\overline{e})$. The *energy* of a flow φ is defined by

$$\mathcal{E}(\varphi) = \frac{1}{2} \sum_{e \in E} \varphi^2(e) w(e).$$

For our subsequent considerations, it may be helpful to keep the following interpretations of the quantities introduced above in mind: Consider a static situation of currents in a system of wires or water in a system of tubes connected at certain joints. This is modeled by a network with the following correspondences:

- Functions on the vertices correspond to potentials, i.e., (differences in) voltage or pressure on the joints.
- Flows correspond to electrical currents or water flows.
- Resistance corresponds to electrical resistance or thickness of tubes.

- Charge distribution corresponds to the Laplacian applied to the potential (Poisson equation of electrostatics).

In this setting, *Ohm's law* applies and says that the potential difference U and the current I are connected to the resistance R via $R = U/I$, i.e.,

$$\text{resistance} = \frac{\text{potential difference}}{\text{flow}}.$$

The corresponding *energy* is then given by

$$\frac{1}{2}UI = \frac{1}{2}\frac{U^2}{R} = \frac{1}{2}I^2R.$$

We will come back to these interpretations from time to time in what follows.

We will now investigate flows satisfying certain additional properties. The first property states that the total flow times the resistance of edges, i.e., the total potential difference, is equal to 0 along any cycle.

Definition 0.33 (Kirchhoff cycle rule). Let b be a graph over a finite set X and let $\varphi\colon E \longrightarrow \mathbb{R}$ be a flow on (X, b). Then, φ is said to satisfy the *Kirchhoff cycle rule* (KCR) if

$$\sum_{j=1}^{n} \varphi(e_j)w(e_j) = 0$$

for any cycle $(e_1, \ldots, e_n)$.

Example 0.34 (Flows induced by functions). Let $f \in C(X)$. One checks directly that $\Psi_f\colon E \longrightarrow \mathbb{R}$ defined by

$$\Psi_f(e) = (f(r(e)) - f(s(e)))b(s(e), r(e)) = \frac{f(r(e)) - f(s(e))}{w(e)}$$

is a flow satisfying the Kirchhoff cycle rule. It is called the *flow induced by f*. Clearly,

$$\Psi_{f+\lambda g} = \Psi_f + \lambda\Psi_g$$

for all $f, g \in C(X)$ and $\lambda \in \mathbb{R}$.

In fact, the preceding is not just an example but rather the example of a flow satisfying the Kirchhoff cycle rule. This is the content of the next proposition.

Proposition 0.35 (Characterization of flows satisfying (KCR)). *Let b be a graph over a finite set X and let $\varphi\colon E \longrightarrow \mathbb{R}$ be a flow on (X, b). Then, the following statements are equivalent:*

(i) *The flow φ satisfies the Kirchhoff cycle rule.*
(ii) *There exists an $f \in C(X)$ with $\varphi = \Psi_f$.*

In this case, $\Psi_{f_1} = \Psi_{f_2}$ if and only if $f_1 - f_2$ is constant on each connected component of the graph.

Remark. The last statement is known in physics as the arbitrariness in fixing the zero of the potential.

Proof. (ii) $\Longrightarrow$ (i): This is discussed in Example 0.34.

(i) $\Longrightarrow$ (ii): Without loss of generality, let (X, b) be connected as, otherwise, we argue on each connected component of the graph separately. Fix $o \in X$ and let f be a function on X with $f(o) = 0$. For any $x \in X$ let $(x_0, \ldots, x_n)$ be a path in X with $x_0 = o$ and $x_n = x$ and define

$$f(x) = \sum_{j=0}^{n-1} \varphi((x_j, x_{j+1}))w((x_j, x_{j+1})).$$

This is well-defined since φ satisfies the Kirchhoff cycle rule. By construction, we then have for x, y with $x \sim y$

$$f(y) = f(x) + \varphi((x, y))w((x, y)),$$

that is,

$$\frac{f(y) - f(x)}{w((x, y))} = (f(y) - f(x))b(x, y) = \varphi((x, y)).$$

This gives (ii).

We now turn to the last statement: Assume again without loss of generality that the graph is connected and let $\Psi_{f_1} = \Psi_{f_2}$. Thus,

$$0 = \Psi_{f_1 - f_2}.$$

Letting $f = f_1 - f_2$, we infer

$$0 = \frac{f(r(e)) - f(s(e))}{w(e)}$$

for any edge e. As the graph is connected, we conclude that f is constant. $\square$

The proposition says that functions on the vertices are in one-to-one correspondence with flows on edges satisfying the Kirchhoff cycle rule. Accordingly, it is possible to "translate" statements from the world of functions to the world of flows and vice versa. This will be studied next.

Proposition 0.36 (Energy via flows and via functions). *Let b be a graph over a finite set X with associated form Q_b. If $\varphi\colon E \longrightarrow \mathbb{R}$ is a flow on (X, b) with $\varphi = \Psi_f$ for $f \in C(X)$, then*

$$\mathcal{E}(\varphi) = Q_b(f).$$

Remark. This is a version of the equality $\frac{1}{2}\frac{U^2}{R} = \frac{1}{2}I^2R$ discussed in connection with Ohm's Law.

Proof. This follows by a direct computation as

$$\begin{aligned}\mathcal{E}(\varphi) &= \frac{1}{2}\sum_{e\in E}\varphi^2(e)w(e)\\ &= \frac{1}{2}\sum_{(x,y)\in E}\varphi^2((x,y))\frac{1}{b(x,y)}\\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))^2 = Q_b(f).\end{aligned}$$

Here, we used $\varphi = \Psi_f$, i.e., $\varphi((x,y)) = b(x,y)(f(y)-f(x))$ in the next to last line. □

We now turn to a second important property that a flow may satisfy. This property may be interpreted as stating that the flow into a vertex equals the flow out of the vertex.

Definition 0.37 (Kirchhoff vertex rule). Let b be a graph over a finite set X and let $x \in X$. A flow $\varphi\colon E \longrightarrow \mathbb{R}$ on (X,b) satisfies the *Kirchhoff vertex rule* (KVR) *at* x if

$$\sum_{e\in E, r(e)=x}\varphi(e) = 0.$$

If a flow satisfies the Kirchhoff vertex rule at every vertex, then it is said to satisfy the *Kirchhoff vertex rule* (KVR).

Remark. If a flow φ satisfies the Kirchhoff vertex rule at $x \in X$, then

$$\sum_{e\in E, s(e)=x}\varphi(e) = 0$$

(and conversely). This follows since $e \in E$ if and only if $\overline{e} \in E$ and $\varphi(e) = -\varphi(\overline{e})$.

Furthermore, for any decomposition of the edge set $E_1 \dot\cup E_2 = E_x = \{e \mid r(e) = x\}$ we have

$$\sum_{e\in E_1}\varphi(e) = \sum_{e\in E_2}\varphi(\overline{e}).$$

This gives the interpretation that the flow into a vertex equals the flow out of a vertex mentioned above.

Remark. By the laws of electrostatics, the current in a network of wires satisfies both the Kirchhoff cycle rule and the Kirchhoff vertex rule. Similarly, both Kirchhoff rules are “obvious” for the (static) flow of water in a network of pipes.

We now give an interpretation of the Kirchhoff vertex rule for flows coming from functions. We start with the definition of a harmonic function for a graph. This concept was already introduced in Section 3. We now extend the definition to subsets of the vertex set.

Definition 0.38 (Harmonic functions on graphs). Let (b, c) be a graph over a finite set X with associated Laplacian $L_{b,c}$. Let $A \subseteq X$. A function $f \in C(X)$ is called *harmonic on A (with respect to the graph (b, c))* if

$$L_{b,c}f(x) = 0$$

for all $x \in A$. If f is harmonic on $A = X$, then f is called *harmonic*.

Remark. The concept of a harmonic function is defined without reference to a measure on X. However, f is clearly harmonic on $A \subseteq X$ with respect to (b, c) if and only if $L_{b,c,m}f = 0$ on A for the operator $L_{b,c,m}$ associated to (b, c) over the measure space (X, m) for one (all) choices of $m : X \longrightarrow (0, \infty)$.

It is not hard to characterize under which conditions $\varphi = \Psi_f$ satisfies KVR. Note that by Lemma 0.35 this characterizes flows satisfying KVR within the class of flows satisfying KCR.

Lemma 0.39 (Harmonic functions and Kirchhoff vertex rule). *Let b be a graph over a finite set X with associated Laplacian L_b. Let $f \in C(X)$ and let $\varphi = \Psi_f$ be the flow induced by f. Then, the following statements are equivalent:*

(i) *The flow φ satisfies the Kirchhoff vertex rule at $x \in X$.*
(ii) *$L_b f(x) = 0$.*

In particular, $\varphi = \Psi_f$ satisfies the Kirchhoff vertex rule if and only if f is harmonic.

Proof. Due to

$$\varphi((x, y)) = (f(y) - f(x))b(x, y)$$

this is immediate from the definitions. □

We now study a fundamental problem in electrostatics of networks. This problem consists of finding the flow generated by a given charge distribution and subject to fixed voltages at certain points. Giving a mathematical description of the ideas from physics behind this problem leads to equations involving the Laplacian of the network. Three instances of such equations have received special attention. These are presented next. To do so, we will assume that our network is modeled by a connected graph (b, c) over a finite set X (even though we could restrict to the case $c = 0$).

The Dirichlet problem (DP). There are no charges in the interior and we are given the voltage at certain points. The desired flow will then satisfy the Kirchhoff cycle rule. Thus, it is induced by a function. This function must then be harmonic at all points where there is no voltage given since the flow satisfies the Kirchhoff vertex rule at all such points. Thus, we are led to the Dirichlet problem:

Given a subset $B \subseteq X$ ("the boundary") and a function g on B, find a function u on X satisfying:

- $L_{b,c}u = 0$ on $A = X \setminus B$ ("u is harmonic on A")
- $u = g$ on B. ("u takes the value g on the boundary")

The capacitor problem (CP). There are no charges outside of two given sets (e.g., metal plates) on which the voltage is fixed as zero and one, respectively. The desired flow will then satisfy the Kirchhoff cycle rule. Thus, it is induced by a function. This function must then be harmonic at all points where there is no voltage given since the flow satisfies the Kirchhoff vertex rule at all such points. Thus, we are led to the capacitor problem:

Given two subsets $F, G \subseteq X$ ("the metal plates"), find a function u on X satisfying:

- $L_{b,c}u = 0$ on $X \setminus (F \cup G)$ ("u is harmonic on $X \setminus (F \cup G)$")
- $u = 1$ and $L_{b,c}u \geq 0$ on F ("u is 1 and is superharmonic on F")
- $u = 0$ and $L_{b,c}u \leq 0$ on G. ("u is 0 and is subharmonic on G")

The Poisson problem (PP). We are given charges but no further conditions on the voltage. Thus, we are led to the Poisson problem:

Given a function g on X, find a function u on X satisfying:

- $L_{b,c}u = g$.

We will now show how these problems can be solved. In fact, we will show even more and discuss how unique solvability of the capacitor problem characterizes Laplacians on graphs.

We begin with a discussion of the Poisson problem. In fact, we have already discussed problems of this type in Section 3. As the constant function 1 is in the kernel of $L_{b,c}$ for $c = 0$, in general, there is neither uniqueness nor existence of the solution of $L_{b,c}u = g$ for a graph (b, c) over X. However, a slight strengthening of the requirements will give both existence and uniqueness. This strengthening consists in fixing the voltage to be zero at one point. This is known as "fixing the gauge."

THEOREM 0.40 (The Poisson problem with a fixed gauge). *Let (b, c) be a connected graph over a finite set X. Let $p \in X$ and let $g\colon X \setminus \{p\} \longrightarrow \mathbb{R}$. Then, the Poisson problem with a fixed gauge:*

- *$L_{b,c}u = g$ on $X \setminus \{p\}$*
- *$u(p) = 0$*

has a unique solution. Moreover, if $g \geq 0$, then $u \geq 0$ on $X \setminus \{p\}$.

PROOF. Set $\tilde{X} = X \setminus \{p\}$ and consider the graph $(\tilde{b}, \tilde{c})$ over $\tilde{X}$ with $\tilde{b}(x, y) = b(x, y)$ and $\tilde{c}(x) = c(x) + b(x, p)$ for $x, y \in \tilde{X}$. A direct calculation shows that u is the desired solution of the Poisson problem with a fixed gauge if and only the restriction of u to $\tilde{X}$, that is, $v = u_{|\tilde{X}}$ satisfies

$$L_{\tilde{b},\tilde{c}}v = g.$$

Note that if we start with v defined on $\tilde{X}$, we extend it by 0 to define u on X.

Now, $\tilde{c}$ does not vanish identically on any connected component of $(\tilde{b}, \tilde{c})$ since the graph (b, c) is connected and, hence, p must have at least one neighbor in every connected component of $\tilde{X}$. Thus, by Lemma 0.29, $L_{\tilde{b},\tilde{c}}$ is bijective so the equation $L_{\tilde{b},\tilde{c}}v = g$ on $\tilde{X}$ has a unique solution v.

Now, suppose $g \geq 0$. We wish to show that $u \geq 0$. Let $x \in X \setminus \{p\}$ be a minimum for u on $X \setminus \{p\}$ such that $u(x) \leq 0$. We obtain

$$0 \leq g(x) = L_{b,c}u(x) = \sum_{y \in X} b(x, y)(u(x) - u(y)) + c(x)u(x) \leq 0$$

and thus $L_{b,c}u(x) = 0$. Therefore, $u(y) = u(x)$ for all $y \sim x$. Repeating this argument shows that u is constant on the connected component of $X \setminus \{p\}$ which contains x. Now, this connected component has at least one vertex x_p which is connected to p. As $u(p) = 0$, we have $u = 0$. Therefore, $u \geq 0$. $\square$

After this discussion of the Poisson problem we now turn to a discussion of the Dirichlet problem. We note that our analysis of the Dirichlet problem below actually yields unique solvability of the capacitor problem. It also gives the existence of the effective resistance metric found in the literature on networks.

THEOREM 0.41 (The Dirichlet problem). *Let (b, c) be a connected graph over a finite set X. Let $B \subseteq X$ with $B \neq \emptyset$, $A = X \setminus B$ and $g\colon B \longrightarrow \mathbb{R}$. Then, the Dirichlet problem* (DP)*:*
- $L_{b,c}u = 0$ *on* A
- $u = g$ *on* B

has a unique solution. Moreover, for the set

$$\mathcal{A}_g = \{h \in C(X) \mid h = g \text{ on } B\}$$

and $f \in \mathcal{A}_g$ the following statements are equivalent:
(i) $Q_{b,c}(f) = \inf\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}$.
(ii) *The function f solves the Dirichlet problem* (DP).

In particular, there exists a unique minimizer in (i). *Moreover, if $0 \leq g \leq 1$, then $0 \leq f \leq 1$.*

REMARK. The theorem above says that the solution of the Dirichlet problem minimizes energy, as is sensible for a solution to a physical problem.

REMARK. For $B = \emptyset$, the corresponding statement is wrong in general. For example, $L_{b,c}u = 0$ does not have a unique solution if $c = 0$.

PROOF. We will show a series of claims which will prove the theorem (and a bit more).

Claim 1. The solution of (DP) exists and is unique.

Proof of Claim 1. We transform the problem to an equivalent problem for which we will establish existence and uniqueness. Let f be a solution of $L_{b,c}f = 0$ on A with $f = g$ on B, that is, let f solve (DP). For any $x \in A$, we then have

$$\begin{aligned}
0 &= L_{b,c}f(x) \\
&= \sum_{y\in X} b(x,y)(f(x)-f(y)) + c(x)f(x) \\
&= \sum_{y\in A} b(x,y)(f(x)-f(y)) + \sum_{y\in B} b(x,y)(f(x)-f(y)) + c(x)f(x) \\
&= \sum_{y\in A} b(x,y)(f(x)-f(y)) + \left(c(x) + \sum_{y\in B} b(x,y)\right) f(x) - \sum_{y\in B} b(x,y)g(y) \\
&= \sum_{y\in A} b(x,y)(f(x)-f(y)) + d(x)f(x) - h(x)
\end{aligned}$$

with

$$d(x) = c(x) + \sum_{y\in B} b(x,y) \quad \text{and} \quad h(x) = \sum_{y\in B} b(x,y)g(y).$$

Note that both d and h do not depend on f.

We let $L_A^{(D)} = L_{b_A,d}$, which we call the *Dirichlet Laplacian* associated to the graph (b_A, d) over A, given by $b_A(x,y) = b(x,y)$ for $x, y \in A$, d as above and the restriction f_A of f to A, we obtain from the above that

(P) $L_A^{(D)} f_A = h$ on A.

Now, if f is a solution of (DP), then f_A solves (P), as shown by the above calculation. Conversely, any solution $\widetilde{f}$ of (P) becomes a solution f to (DP) after extending $\widetilde{f}$ by g on B. This gives:

$$f \text{ solves (DP)} \quad \Longleftrightarrow \quad f_A \text{ solves (P)}.$$

Therefore, it suffices to show that (P) has a unique solution, that is, $L_A^{(D)}$ is bijective. By construction, $L_A^{(D)}$ is the Laplacian associated to the graph (b_A, d) over A. Thus, by Lemma 0.29, it suffices to show that d does not vanish on any connected component of A, where the connected components are defined with respect to b_A. Let Z be such a connected component. Invoking the definition of d, it suffices to find $x \in Z$ and $y \in B$ with $b(x,y) > 0$. First, we choose an arbitrary $y' \in B$ and $o \in Z$. As the graph is connected there exists a path $(x_0, x_1, \ldots, x_n)$ in (X, b) with $x_0 = o$ and $x_n = y'$. Let j be the smallest index such that x_j does not belong to Z. Then, letting $y = x_j$, y belongs to B as otherwise it would belong to Z since Z is a connected component. Thus, $x = x_{j-1} \in Z$ and $y = x_j \in B$ satisfy $b(x,y) > 0$. This finishes the proof of Claim 1.

Claim 2. Any minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP).

Proof of Claim 2. Suppose that there exists an $f \in \mathcal{A}_g$ with

$$Q_{b,c}(f) = \min\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Let φ be an arbitrary function supported on A. Then, $f + \lambda\varphi$ belongs to $\mathcal{A}_g$ for all $\lambda \in \mathbb{R}$. Thus, the function

$$\lambda \mapsto Q_{b,c}(f + \lambda\varphi) = Q_{b,c}(f) + 2\lambda Q_{b,c}(f, \varphi) + \lambda^2 Q_{b,c}(\varphi)$$

has a minimum at $\lambda = 0$. Taking the derivative at $\lambda = 0$ yields

$$0 = Q_{b,c}(f, \varphi) = \sum_{x \in X} L_{b,c} f(x)\varphi(x)$$

by Green's formula, Proposition 0.8. As φ supported in A was arbitrary, we conclude that $L_{b,c} f = 0$ on A.

Claim 3. There exists a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

Proof of Claim 3. Let (f_n) be a sequence in $\mathcal{A}_g$ with

$$\lim_{n\to\infty} Q_{b,c}(f_n) = \min\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

It follows that $(Q_{b,c}(f_n))$ is a bounded sequence. Let o be an arbitrary point in B. Then, $f_n(o) = g(o)$ for all $n \in \mathbb{N}$ as $f_n \in \mathcal{A}_g$. As we will show below, the boundedness of $(Q_{b,c}(f_n))$ together with the boundedness of $(f_n(o))$ implies that $(f_n(x))$ is bounded for any $x \in X$. By choosing a suitable subsequence we can, without of loss of generality, assume that (f_n) converges pointwise to a function f. Obviously, $f \in \mathcal{A}_g$ and

$$Q_{b,c}(f) = Q_{b,c}\left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} Q_{b,c}(f_n) = \min\{Q_{b,c}(h) \mid h \in \mathcal{A}_g\}.$$

Thus, f is a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$.

It remains to show the desired boundedness of $(f_n(x))$ for $x \in X$. Let $x \in X$ and let $\gamma = (x_0, \ldots, x_n)$ with $x_0 = o$ and $x_n = x$ be a path from o to x. Then, for any function u, we have by the Cauchy–Schwarz inequality

$$\begin{aligned}
&|u(x) - u(o)| \\
&\quad\leq \sum_{j=0}^{n-1} |u(x_j) - u(x_{j+1})| \\
&\quad= \sum_{j=0}^{n-1} |u(x_j) - u(x_{j+1})| b(x_j, x_{j+1})^{1/2} \cdot \frac{1}{b(x_j, x_{j+1})^{1/2}} \\
&\quad\leq \left(\sum_{j=0}^{n-1} (u(x_j) - u(x_{j+1}))^2 b(x_j, x_{j+1})\right)^{1/2} \left(\sum_{j=0}^{n-1} b(x_j, x_{j+1})^{-1}\right)^{1/2} \\
&\quad\leq Q_{b,c}^{1/2}(u) C(\gamma)
\end{aligned}$$

with $C(\gamma) = \left(\sum_{j=1}^{n} b(x_j, x_{j+1})^{-1}\right)^{1/2}$. Applying this to f_n and noting that $f_n(o) = g(o)$ for all n since $o \in B$, we get

$$|f_n(x) - g(o)| \leq C(\gamma) Q_{b,c}^{1/2}(f_n).$$

As $(Q_{b,c}(f_n))_n$ is bounded and $C(\gamma)$ does not depend on n, it follows that $(f_n(x))_n$ is bounded.

Claim 4. If $0 \leq g \leq 1$, then $0 \leq f \leq 1$.

Proof of Claim 4. Recall that $C_{[0,1]} \circ f = 0 \vee f \wedge 1$. If $f \in \mathcal{A}_g$, then $C_{[0,1]} \circ f \in \mathcal{A}_g$ since $C_{[0,1]} \circ g = g$. Therefore, $C_{[0,1]} \circ f$ is also a minimizer of $Q_{b,c}$ as $Q_{b,c}$ is a Dirichlet form and thus $Q_{b,c}(C_{[0,1]} \circ f) \leq Q_{b,c}(f)$. The already proven uniqueness then gives $f = C_{[0,1]} \circ f$, which is equivalent to $0 \leq f \leq 1$.

By combining the preceding statements we now prove the theorem: Claim 1 yields the existence and uniqueness of solutions to (DP). Claim 2 shows the implication (i) $\Longrightarrow$ (ii). Furthermore, in Claim 3, we have shown the existence of a minimizer of $Q_{b,c}$ on $\mathcal{A}_g$. We next turn to (ii) $\Longrightarrow$ (i): The solution of (DP) and the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ both exist and are unique by the considerations above. As the minimizer of $Q_{b,c}$ on $\mathcal{A}_g$ solves (DP) by Claim 2, it coincides with the unique solution of (DP). Thus, this unique solution minimizes $Q_{b,c}$ on $\mathcal{A}_g$. Finally, the last statement of the theorem follows from Claim 4. □

A consequence of Theorem 0.41 is the existence of the so-called effective resistance W_{eff}. We discuss this next. By letting $B = \{x, y\}$ for $x, y \in X$ with $x \neq y$ and $g \colon B \longrightarrow \mathbb{R}$ by $g(x) = 0$ and $g(y) = 1$, we obtain the following result.

COROLLARY 0.42 (Existence of effective resistance). *Let b be a connected graph over a finite set X and let $x, y \in X$ with $x \neq y$. Then, there exists a unique $f = f_{x,y}$ with $f(x) = 0$, $f(y) = 1$ and $L_b f = 0$ on $X \setminus \{x, y\}$. This f is the minimizer of Q_b on*

$$\mathcal{A}_{x,y} = \{h \in C(X) \mid h(x) = 0, h(y) = 1\}.$$

REMARK (Effective resistance and the resistance metric). The name effective resistance arises from an interpretation in electrostatics as follows: Put a normalized voltage between x and y. The effective resistance $W_{\text{eff}}(x, y)$ of the entire network is then determined via $W_{\text{eff}}(x, y) = U/I$ where $U = 1 - 0$ is the difference in voltage between x and y and I is the arising current. As the energy E is given by $E = UI$, we can replace I by E/U and obtain $W_{\text{eff}}(x, y) = U^2/E = 1/E$. Now, the energy E is given by $Q_b(f_{x,y})$ and the formula

$$W_{\text{eff}}(x, y) = \frac{1}{Q_b(f_{x,y})}$$

follows.

The effective resistance can be expressed by the following remarkable formula (Exercise 0.32 (a))

$$W_{\text{eff}}(x,y) = \max\{(f(x)-f(y))^2 \mid Q_b(f) \le 1\}.$$

Indeed, the effective resistance defines a metric (Exercise 0.32 (c*)). However, it is somewhat easier to see that

$$r(x,y) = W_{\text{eff}}^{1/2}(x,y), \qquad x \neq y$$

and $r(x,y) = 0$ for $x = y$ defines a metric as well (Exercise 0.32 (b)).

As another consequence of Theorem 0.41, we also obtain the existence and uniqueness of solutions to the capacitor problem.

Corollary 0.43 (Capacitor problem). *Let (b,c) be a connected graph over a finite set X with associated Laplacian $L_{b,c}$. Let $F, G \subseteq X$ be subsets of X with $F \cap G = \emptyset$ and $F \cup G \neq \emptyset$. Then, the capacitor problem* (CP)*:*

- *$u = 1$ and $L_{b,c}u \ge 0$ on F*
- *$u = 0$ and $L_{b,c}u \le 0$ on G*
- *$L_{b,c}u = 0$ on $X \setminus (F \cup G)$*

has a unique solution. This solution is given by the unique minimizer of $Q_{b,c}$ on

$$\mathcal{A} = \{h \in C(X) \mid h \ge 1 \text{ on } F,\ h \le 0 \text{ on } G\}$$

and satisfies $0 \le u \le 1$.

Proof. By Theorem 0.41, the problem

- $L_{b,c}u = 0$ on $X \setminus (F \cup G)$
- $u = 1$ on F and $u = 0$ on G

has a unique solution which satisfies $0 \le u \le 1$. Furthermore, for $x \in F$,

$$\begin{aligned} L_{b,c}u(x) &= \sum_{y \in X} b(x,y)(u(x)-u(y)) + c(x)u(x) \\ &= \sum_{y \in X} b(x,y)(1-u(y)) + c(x) \ge 0. \end{aligned}$$

For $x \in G$,

$$L_{b,c}u(x) = \sum_{y \in X} b(x,y)(u(x)-u(y)) + c(x)u(y) = -\sum_{y \in X} b(x,y)u(y) \le 0.$$

Therefore, u is a solution of the capacitor problem. It is unique as the solution of the Dirichlet problem is unique by Theorem 0.41.

Moreover, this solution is the unique minimizer of $Q_{b,c}$ on

$$\mathcal{A}_g = \{h \in C(X) \mid h = 1 \text{ on } F,\ h = 0 \text{ on } G\}$$

with $g = 1_F$. Now, obviously $\mathcal{A}_g \subseteq \mathcal{A}$. As $Q_{b,c}$ is a Dirichlet form and $C_{[0,1]}\mathcal{A} = \mathcal{A}_g$, the desired statement on the minimizer follows. $\square$

We are now ready to prove a characterization of Dirichlet forms in electrostatics. Note that, although we have always formulated the capacitor problem for $L_{b,c}$, it can just as well be formulated for a general symmetric operator L. However, if we have unique solutions, L is immediately the Laplacian associated to a connected graph, as the following result shows.

THEOREM 0.44 (Characterization of graphs in electrostatics). *Let X be a finite set and let Q be a symmetric form over X with associated operator L. Then, the following statements are equivalent:*

(i) *There exists a graph (b, c) over X such that b is connected or c does not vanish identically on any connected component with $L = L_{b,c}$ and $Q = Q_{b,c}$.*
(ii) *Every capacitor problem* (CP) *for L on X has a unique solution.*

PROOF. (i) $\Longrightarrow$ (ii): By Corollary 0.43 there is a unique solution on every connected component of the graph whose intersection with the set $F \cup G$ from the capacitor problem is non-empty. In the case of a connected component whose intersection with $F \cup G$ is empty, the capacitor problem reduces to finding a harmonic function on this component. Clearly, the constant function 0 is harmonic and non-vanishing c on this component yields uniqueness of this solution.

(ii) $\Longrightarrow$ (i): Let l be the matrix of L, so $Lf(x) = \sum_{y \in X} l(x, y)f(y)$ for all $f \in C(X)$ and $x \in X$. By Lemma 0.9, in order for L to be equal to $L_{b,c}$ for a graph (b, c) over X we need to show $l(x, y) \leq 0$ for $x \neq y$ and $\sum_{y \in X} l(x, y) \geq 0$ for all $x \in X$.

Let $x \in X$ and let 1_x be the characteristic function of $\{x\}$. If $F = \{x\}$ and $G = X \setminus \{x\}$, then the unique solution u of (CP) must satisfy $u = 1_x$. Since u solves (CP) for L, we get $Lu \leq 0$ on G and, hence, for all $y \neq x$,
$$l(y, x) = L1_x(y) \leq 0.$$
By the symmetry of Q it also follows that $l(x, y) \leq 0$. If $F = X$, then u, the unique solution of (CP), must satisfy $u = 1$ so we obtain from (CP) that $Lu \geq 0$ and, hence,
$$\sum_{y \in X} l(x, y) = L1(x) \geq 0$$
for all $x \in X$. This shows that L and Q are associated to a graph by Lemma 0.9.

We now show that the graph (b, c) must be connected if c vanishes on some connected component. Suppose not and let U be a connected component of (b, c) where c vanishes. Let $F, G \subseteq X \setminus U$ be such that $F \cup G \neq \emptyset$ and $F \cap G = \emptyset$, which is possible since we assumed that (b, c) is not connected. Since $(F \cup G) \cap U = \emptyset$ the capacitor problem reduces to $L_{b,c}u = 0$ on U. Thus, all constant functions on U are solutions to the capacitor problem on U since $c = 0$ on U. This contradicts the uniqueness of the solutions of the capacitor problem and, therefore, implies that (b, c) must be connected. $\square$

From Lemma 0.29 we know that c being non-vanishing on any connected component is equivalent to injectivity of the operator $L_{b,c}$, which is equivalent to $Q_{b,c}$ being an inner product. This gives the following immediate corollary.

COROLLARY 0.45. *Let X be a finite set and let Q be a symmetric form over X which is an inner product with associated operator L. Then, the following statements are equivalent:*

(i) *There exists a graph (b, c) over X with $L = L_{b,c}$ and $Q = Q_{b,c}$.*
(ii) *Every capacitor problem* (CP) *for L on X has a unique solution.*

5. The heat equation and the Markov property

In this section we present another way of looking at graphs and their associated forms and Laplacians. More specifically, we will show that Laplacians on graphs are exactly the operators describing a "heat equation" on a finite set. The mathematical formulation of this connection requires the concepts of a semigroup and of the Markov property.

See Excavation Exercises 0.6, 0.7 and 0.8 for some of the required background. More specifically, these exercises review the concept of the norm of an operator on a Hilbert space, the basics of the semigroup associated to an operator and how the semigroup behaves for commuting operators.

Let (X, m) be a finite measure space. We will deal with operators on $\ell^2(X, m)$. In order to simplify the notation, we will write L instead of L_m for such operators. Let $L\colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ be a self-adjoint operator. A continuously differentiable function $\varphi\colon [0, \infty) \longrightarrow \ell^2(X, m)$ is a *solution of the parabolic equation* associated to L with initial condition $f \in \ell^2(X, m)$ if φ satisfies

- $\partial_t \varphi_t = -L\varphi_t$ for $t \geq 0$
- $\varphi_0 = f$.

In this context we think of $x \in X$ as a space variable and $t \in [0, \infty)$ as time.

We want to investigate conditions on L such that the preceding equation can be thought of as a "heat equation" and the time evolution gives a "heat diffusion" on the graph. A rather detailed discussion of such conditions will be given in Section 10. Here, we just note that it is natural to aim for the following properties:

- If $f \geq 0$, then $\varphi_t \geq 0$ for all $t \geq 0$.
- If $f \leq 1$, then $\varphi_t \leq 1$ for all $t \geq 0$.

Indeed, the first condition states that the amount of heat remains positive if the initial distribution is positive and the second condition implies that the diffusion process does not contribute to the total amount of heat but rather distributes it in time. If L is such that the above are satisfied for any solution of the parabolic equation, then we will call the associated equation the *heat equation.*

Given $f \in \ell^2(X, m)$, there exists a unique solution of the parabolic equation above given by

$$\varphi_t = e^{-tL} f.$$

Here, e^{-tL} is defined via the power series

$$e^{-tL} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} L^n.$$

We call e^{-tL} for $t \geq 0$ the *semigroup associated to the operator* L. Motivated by the above considerations, we say that the semigroup is *positivity preserving* if $f \geq 0$ implies

$$e^{-tL} f \geq 0$$

for all $t \geq 0$. Recall that a function satisfying $f \geq 0$ is called positive. Therefore, the semigroup e^{-tL} is positivity preserving if it maps positive functions to positive functions.

We say that the semigroup is *contracting* if $f \leq 1$ implies

$$e^{-tL} f \leq 1$$

for all $t \geq 0$. If the semigroup e^{-tL} is both positivity preserving and contracting, then e^{-tL} is called a *Markov semigroup* and is said to satisfy the *Markov property*. We note that the Markov property corresponds exactly to the two properties aimed at above. As e^{-tL} satisfies the parabolic equation associated to L, it follows that if e^{-tL} is a Markov semigroup, then $e^{-tL} f$ is a solution of the heat equation with initial condition f.

We now start towards characterizing the Markov property for semigroups e^{-tL}. We will need an auxiliary lemma which does not involve graphs. In what follows, if $A\colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ is an operator on $\ell^2(X, m)$, then $\|A\|$ denotes the operator norm of A which is defined by $\|A\| = \sup\{\|Af\| \mid f \in \ell^2(X, m), \|f\| = 1\}$. In particular, if A and B are operators on $\ell^2(X, m)$, then $\|AB\| \leq \|A\|\|B\|$.

LEMMA 0.46 (Lie–Trotter product formula on finite sets). *Let (X, m) be a finite measure space. If A and B are operators on $\ell^2(X, m)$, then*

$$e^{A+B} = \lim_{n\to\infty} (e^{\frac{1}{n}A} e^{\frac{1}{n}B})^n.$$

PROOF. Set $S_n = e^{\frac{1}{n}(A+B)}$ and $T_n = e^{\frac{1}{n}A} e^{\frac{1}{n}B}$ for $n \in \mathbb{N}$. We want to show that $\|S_n^n - T_n^n\| \to 0$ as $n \to \infty$.

We first note that for any operator L on $\ell^2(X,m)$ we have $\|e^L\| \leq e^{\|L\|}$. Consequently, it follows that

$$\|T_n\| \leq \|e^{\frac{1}{n}A}\|\|e^{\frac{1}{n}B}\| \leq e^{\frac{1}{n}\|A\|}e^{\frac{1}{n}\|B\|} = e^{\frac{1}{n}(\|A\|+\|B\|)}$$

and

$$\|S_n\| \leq e^{\frac{1}{n}\|A+B\|} \leq e^{\frac{1}{n}(\|A\|+\|B\|)}.$$

A telescoping argument gives

$$S_n^n - T_n^n = \sum_{j=0}^{n-1} S_n^j(S_n - T_n)T_n^{n-1-j}.$$

Therefore,

$$\|S_n^n - T_n^n\| \leq C_1 n\|S_n - T_n\|,$$

where $C_1 = e^{(\|A\|+\|B\|)}$. Moreover,

$$\begin{aligned}
\|S_n - T_n\| &= \left\| \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{A+B}{n}\right)^j - \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{A}{n}\right)^k \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{B}{n}\right)^l \right\| \\
&= \left\| \sum_{j=2}^{\infty} \frac{1}{j!}\left(\frac{A+B}{n}\right)^j - \sum_{k+l\geq 2} \frac{1}{k!l!}\left(\frac{A}{n}\right)^k \left(\frac{B}{n}\right)^l \right\| \\
&\leq C\frac{1}{n^2}
\end{aligned}$$

for some constant C. Therefore,

$$\|S_n^n - T_n^n\| \leq \frac{C_1 C}{n},$$

which yields the desired statement. □

We now characterize when a semigroup is positivity preserving in terms of the matrix and the form associated to a self-adjoint operator.

THEOREM 0.47 (First Beurling–Deny criterion). *Let (X,m) be a finite measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix elements of the operator L satisfy, for all $x, y \in X$ with $x \neq y$,*

$$l(x,y) \leq 0. \qquad \text{(“Operator”)}$$

(ii) *The form satisfies, for all $f \in \ell^2(X,m)$,*

$$Q(|f|) \leq Q(f). \qquad \text{(“Form”)}$$

(iii) *The semigroup satisfies, for all $f \geq 0$ and $t \geq 0$,*

$$e^{-tL}f \geq 0. \qquad \text{(“Semigroup”)}$$

REMARK. The proof below gives yet another characterization of when a semigroup e^{-tL}, $t \geq 0$, is positivity preserving, namely, if and only if

(iv) $|e^{-tL}f| \leq e^{-tL}|f|$ for all $f \in C(X)$.

Indeed, the "if" is clear and the "only if" is shown at the beginning of the proof of (iii) $\Longrightarrow$ (ii). This is a useful characterization in various situations.

PROOF. (i) $\Longrightarrow$ (iii): We first decompose L into a diagonal and an off-diagonal part. More specifically, we write

$$L = \widetilde{L} + \widetilde{D},$$

where $\widetilde{L}$ has matrix elements equal to those of L on the off-diagonal and matrix elements equal to zero on the diagonal and $\widetilde{D}$ has matrix elements equal to those of L on the diagonal and matrix elements equal to zero on the off-diagonal. The Lie–Trotter formula, Lemma 0.46, then gives

$$e^{-tL} = \lim_{n\to\infty} \left(e^{-\frac{t}{n}\widetilde{L}} e^{-\frac{t}{n}\widetilde{D}}\right)^n .$$

Now, by assumption, $-\widetilde{L}$ has only non-negative entries. This is then also true of $e^{-\frac{t}{n}\widetilde{L}}$. Also, $e^{-\frac{t}{n}\widetilde{D}}$ has only non-negative entries as it is a diagonal matrix with exponential functions on the diagonal. Putting this together, we infer that e^{-tL} has only non-negative matrix entries. This gives (iii).

(iii) $\Longrightarrow$ (ii): From (iii) we easily obtain

$$|e^{-tL}f| \leq e^{-tL}|f|.$$

Indeed, write $f = f_+ - f_-$ with $f_+ = f \vee 0$ and $f_- = -f \vee 0$. Note that $f_+ \geq 0$, $f_- \geq 0$ and $|f| = f_+ + f_-$. Now, a direct computation gives

$$\begin{aligned}
|e^{-tL}f| &= |e^{-tL}f_+ - e^{-tL}f_-| \\
&\leq |e^{-tL}f_+| + |e^{-tL}f_-| \\
&= e^{-tL}f_+ + e^{-tL}f_- \\
&= e^{-tL}|f|.
\end{aligned}$$

Here, we used assumption (iii) in the next to last step. From this preliminary consideration we infer

$$\langle e^{-tL}f, f\rangle \leq |\langle e^{-tL}f, f\rangle| \leq \langle e^{-tL}|f|, |f|\rangle.$$

Moreover, $\langle |f|, |f|\rangle = \langle f, f\rangle$. This gives

$$\langle (e^{-tL} - I)|f|, |f|\rangle \geq \langle (e^{-tL} - I)f, f\rangle.$$

Dividing by $t > 0$ we infer

$$\langle \frac{1}{t}(e^{-tL} - I)|f|, |f|\rangle \geq \langle \frac{1}{t}(e^{-tL} - I)f, f\rangle.$$

Noting that $\partial_t e^{-tL} = -Le^{-tL}$ so that $\partial_t e^{-tL}|_{t=0} = -L$ and letting $t \to 0^+$ in the inequality above then yields

$$-Q(|f|) = \langle -L|f|, |f|\rangle \geq \langle -Lf, f\rangle = -Q(f).$$

This gives (ii).

(ii) $\implies$ (i): This has already been shown in Lemma 0.19 (a). $\square$

REMARK. There is another characterization of the first Beurling–Deny criterion involving the form and taking the maximum and minimum of two functions (Exercise 0.33).

Having dealt with the positivity preserving part of the Markov property, we are now going to characterize the contracting part.

THEOREM 0.48 (Second Beurling–Deny criterion). *Let (X, m) be a finite measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with associated matrix l and form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *The matrix elements of the operator L satisfy, for all $x, y \in X$ with $x \neq y$,*

$$l(x,y) \leq 0 \quad \text{and} \quad \sum_{z\in X} l(x,z) \geq 0. \qquad (\text{“Operator”})$$

(ii) *The form satisfies, for all $f \in \ell^2(X, m)$,*

$$Q(0 \vee f \wedge 1) \leq Q(f). \qquad (\text{“Form”})$$

(iii) *The semigroup satisfies, for all $t \geq 0$ and $0 \leq f \leq 1$,*

$$0 \leq e^{-tL} f \leq 1. \qquad (\text{“Semigroup”})$$

PROOF. (i) $\iff$ (ii): This was already shown in Theorem 0.20.

(i) $\iff$ (iii): The equivalence of $l(x, y) \leq 0$ for $x \neq y$ and the semigroup being positivity preserving was already shown in Theorem 0.47. For the remaining part, we start with a preliminary consideration. Set $f = L1$ so that the statement of (i) is equivalent to $f \geq 0$. Consider now the function $u_t = e^{-tL}1$. This function satisfies $u_0 = 1$ and

$$\partial_t u_t = -Le^{-tL}1 = -e^{-tL}L1 = -e^{-tL}f$$

for all $t \geq 0$. In particular,

$$\lim_{t\to 0^+} \frac{1}{t}(u_t - u_0) = \partial_t u_t|_{t=0} = -f.$$

We now turn to proving the desired equivalence. If (i) holds, then u satisfies $u_0 = 1$ and $\partial_t u_t = -e^{-tL} f \leq 0$, where the last inequality follows as e^{-tL} is positivity preserving and $f \geq 0$ due to (i). This shows that u_t is non-increasing in t and gives

$$e^{-tL}1 \leq 1.$$

Now, let $0 \leq f \leq 1$. Then the inequality above implies

$$0 \leq e^{-tL} f \leq e^{-tL} 1 \leq 1$$

as e^{-tL} is positivity preserving. This shows (iii).

Conversely, if (iii) holds, then we infer

$$-L1 = \partial_t e^{-tL} 1|_{t=0} = \lim_{t \to 0^+} \frac{1}{t} \left(e^{-tL} - 1 \right) \leq 0$$

from which $\sum_{z \in X} l(x,z) \geq 0$ follows. □

REMARK. From the proofs of Theorem 0.20 and Theorem 0.48 above we actually see that, under the assumption $l(x,y) \leq 0$ for $x \neq y$, the following statements are equivalent:

(i) The matrix elements of the operator L satisfy, for all $x \in X$,

$$\sum_{z \in X} l(x,z) \geq 0.$$

(ii) The form satisfies, for all $f \geq 0$,

$$Q(f \wedge 1) \leq Q(f).$$

(iii) The semigroup satisfies, for all $t \geq 0$,

$$e^{-tL} 1 \leq 1.$$

Furthermore, we note that the condition $Q(f \wedge 1) \leq Q(f)$ is equivalent to the fact that Q is a Dirichlet form (Exercise 0.29).

We now conclude this section with a characterization of the validity of the Markov property via graphs.

THEOREM 0.49 (Characterization of the Markov property). *Let (X,m) be a finite measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with associated form $Q = Q_L$. Then, the following statements are equivalent:*

(i) *There exists a graph (b,c) over (X,m) with*

$$Q = Q_{b,c} \quad \text{and} \quad L = L_{b,c,m}. \qquad \text{(“Graph”)}$$

(ii) *The semigroup e^{-tL}, $t \geq 0$, satisfies the Markov property, i.e.,*

$$0 \leq e^{-tL} f \leq 1 \quad \text{for all} \quad 0 \leq f \leq 1. \qquad \text{(“Semigroup”)}$$

PROOF. The statement directly follows by combining the first and second Beurling–Deny criteria, that is, Theorems 0.47 and 0.48, with Lemma 0.9. □

REMARK. By Theorem 0.22 graphs are in a one-to-one correspondence with Dirichlet forms. Therefore, the preceding theorem implies that Dirichlet forms are in a one-to-one correspondence with semigroups satisfying the Markov property.

6. Resolvents and heat semigroups

It is rather remarkable that the same mathematical structure (i.e., Dirichlet forms) appears prominently in both the theory of the heat equation and electrostatics. This is not only true in the discrete setting considered in this book but also in the continuous setting. In the continuous setting, instead of the Laplacian L, one considers the continuous Laplacian Δ.

In this section, we will discuss some of the general mathematics connecting the heat equation and electrostatics. Although the results hold for general self-adjoint positive operators on an arbitrary Hilbert space, we will stick to the setting of a finite set X with a measure m and the associated Hilbert space $\ell^2(X, m)$. In order to simplify the notation, we will write L instead of L_m for operators on this Hilbert space.

The Excavation Exercises 0.9, 0.10 and 0.11 review some facts of linear algebra, in particular, the spectral theorem for self-adjoint operators which is used below.

Our considerations in Section 5 show that the heat equation leads to the study of semigroups e^{-tL} for $t \geq 0$ with the Markov property. This Markov property means that L is associated to a Dirichlet form or, equivalently, to a graph. On the other hand, as discussed in Section 4, electrostatics deals with the energy of a network which is encoded by a Dirichlet form and leads one to consider basic problems of electrostatics. These problems, which have various manifestations such as the Poisson problem, the Dirichlet problem and the capacitor problem, all involve the Laplacian L associated to a graph. As seen in our discussion in Section 4, this yields equations of the form

$$Lu = g,$$

where it is sometimes necessary to modify the underlying graph.

In this sense, electrostatics naturally leads to the study of the inverse of the operator L. As L itself may not be invertible, see Lemma 0.29 for a characterization of the invertibility of L, this leads to the study of the operators $(L+\alpha)^{-1}$ for $\alpha > 0$. As by Green's formula all of the eigenvalues of L are non-negative, it follows that $L+\alpha$ is always invertible for $\alpha > 0$, in fact, even for $\alpha > -\lambda_0$, where λ_0 denotes the smallest eigenvalue of L. The operators $(L+\alpha)^{-1}$ are known as *resolvents* associated to L.

Mathematically, semigroups and resolvents are intimately related. In fact, each one can be obtained from the other. The corresponding formulae which we prove in this section are the following:

$$(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt$$

for $\alpha > 0$ and

$$e^{-tL} = \lim_{n\to\infty} \left(\frac{n}{t} \left(L + \frac{n}{t} \right)^{-1} \right)^n$$

for all $t > 0$. The first formula above is referred to as the "Laplace transform."

To make sense of these formulae, we think of the arising operators as matrices and treat the formulae as being meant to hold in each component separately. A more structural interpretation is possible but not necessary for the subsequent considerations of this chapter. We will, however, need this more general interpretation in later chapters.

We will require some preparation in order to provide a proof of these formulae. Although we think of L as being the Laplacian arising from a graph, the connection between semigroups and resolvents mentioned above hold for general self-adjoint operators with non-negative eigenvalues.

The Laplacian L is self-adjoint on $\ell^2(X, m)$. As such, all eigenvalues of L are real. The set of eigenvalues of L is called the *spectrum* of L and denoted by $\sigma(L)$. For any $\lambda \in \sigma(L)$, we let E_λ be the orthogonal projection onto the eigenspace of λ. In this situation, the following simple version of the "spectral theorem" is known from basic linear algebra:

- $E_\lambda E_\mu = 0$ for $\lambda \neq \mu$.
- $I = \sum_{\lambda\in\sigma(L)} E_\lambda$.
- $L = \sum_{\lambda\in\sigma(L)} \lambda E_\lambda$.

Moreover, by

$$\langle f, Lf \rangle = Q(f, f) \geq 0$$

we infer that all of the eigenvalues of L are non-negative.

This allows us to express both the semigroup e^{-tL} and the resolvents $(L + \alpha)^{-1}$ easily via the projections E_λ. Indeed, the above formulae directly give

$$L^n = \sum_{\lambda\in\sigma(L)} \lambda^n E_\lambda$$

for any natural number n. This immediately implies

$$\sum_{n=0}^{\infty} c_n L^n = \sum_{\lambda\in\sigma(L)} \varphi(\lambda) E_\lambda$$

whenever $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ is a power series converging for all $z \in \mathbb{C}$. In particular, we infer for the semigroup that

$$e^{-tL} = \sum_{\lambda\in\sigma(L)} e^{-t\lambda} E_\lambda$$

for any $t \geq 0$.

As for the resolvents, we note for $\alpha > 0$ that

$$R_\alpha = \sum_{\lambda \in \sigma(L)} (\lambda + \alpha)^{-1} E_\lambda = \sum_{\lambda \in \sigma(L)} \frac{1}{\lambda + \alpha} E_\lambda$$

clearly satisfies the equations

$$R_\alpha (L + \alpha) = (L + \alpha) R_\alpha = I.$$

Thus, R_α is the inverse of $(L + \alpha)$ and we obtain

$$(L + \alpha)^{-1} = \sum_{\lambda \in \sigma(L)} (\lambda + \alpha)^{-1} E_\lambda.$$

Furthermore, as above,

$$(L + \alpha)^{-n} = \sum_{\lambda \in \sigma(L)} (\lambda + \alpha)^{-n} E_\lambda$$

for all natural numbers n.

The above considerations clearly hold for any self-adjoint operator with non-negative eigenvalues and not only the Laplacian L. For the proof of the following lemma, we recall the elementary identity $e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n$, which implies that

$$e^{-t\lambda} = \frac{1}{e^{t\lambda}} = \lim_{n \to \infty} \left(\frac{1}{1 + \frac{t\lambda}{n}} \right)^n = \lim_{n \to \infty} \left(\frac{n}{t} \left(\lambda + \frac{n}{t} \right)^{-1} \right)^n.$$

Lemma 0.50 (Laplace transform). *Let (X, m) be a finite measure space. Let L be a self-adjoint operator on $\ell^2(X, m)$ with non-negative eigenvalues.*

(a) *For all $\alpha > 0$,*

$$(L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt.$$

(*"Laplace transform"*)

(b) *For all $t > 0$,*

$$e^{-tL} = \lim_{n \to \infty} \left(\frac{n}{t} \left(L + \frac{n}{t} \right)^{-1} \right)^n.$$

Proof. (a) The discussion above gives

$$e^{-t\alpha} e^{-tL} = \sum_{\lambda \in \sigma(L)} e^{-t(\alpha + \lambda)} E_\lambda \quad \text{and} \quad (L + \alpha)^{-1} = \sum_{\lambda \in \sigma(L)} \frac{1}{\lambda + \alpha} E_\lambda.$$

Now, the desired statement follows easily by integration.

(b) As follows from the discussion above, for all natural numbers n we have

$$e^{-tL} = \sum_{\lambda \in \sigma(L)} e^{-t\lambda} E_\lambda \quad \text{and} \quad \left(\frac{n}{t} \left(\frac{n}{t} + L \right)^{-1} \right)^n = \sum_{\lambda \in \sigma(L)} \left(\frac{\frac{n}{t}}{\frac{n}{t} + \lambda} \right)^n E_\lambda.$$

Now, the desired statement follows easily from

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{t\lambda}{n}}\right)^n = e^{-t\lambda}.$$

This completes the proof. □

The previous lemma is valid for any self-adjoint operator with non-negative eigenvalues. If L is the Laplacian associated to a graph, then e^{-tL} also satisfies the Markov property and this gives another characterization of the Laplacian on graphs as follows.

COROLLARY 0.51. *Let (X,m) be a finite measure space. Let L be a self-adjoint operator on $\ell^2(X,m)$ with non-negative eigenvalues. Then, the following statements are equivalent:*

(i) *For all $t \geq 0$ and all $f \in \ell^2(X,m)$ with $0 \leq f \leq 1$,*
$$0 \leq e^{-tL} f \leq 1.$$
(ii) *For all $\alpha > 0$ and all $f \in \ell^2(X,m)$ with $0 \leq f \leq 1$,*
$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1.$$
(iii) *There exists a graph (b,c) over (X,m) with $L = L_{b,c,m}$.*

PROOF. The equivalence between (i) and (ii) follows easily from the formulae given in Lemma 0.50 above. The equivalence between (i) and (iii) was shown in Theorem 0.49. □

7. A Perron–Frobenius theorem and large time behavior

In this section we study positivity improving semigroups, existence of ground states and large time behavior of the heat equation. In a sense, we study how heat spreads both instantaneously (small time behavior) and as time goes to infinity (large time behavior).

Excavation Exercises 0.9, 0.10 and 0.11 giving the spectral theorem used for the previous section will also be helpful for this section. Furthermore, Exercise 0.12 recalls the variational characterization of the bottom of the spectrum while Exercise 0.13 reviews the concepts of direct sums of Hilbert spaces and operators.

We start by identifying the property of operators which will be of interest.

DEFINITION 0.52 (Positivity improving). An operator $A\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ is called *positivity improving* if $Af > 0$ whenever $f \geq 0$ with $f \neq 0$.

Recall that functions satisfying $f \geq 0$ are called positive and functions satisfying $f > 0$ are called strictly positive. Hence an operator is positivity improving if it maps non-trivial positive functions to strictly positive functions.

We next show that, for semigroups and resolvents, positivity improvement can be characterized by connectedness of the graph. Recall from Definition 0.28 that we call a subset of a graph connected if any two vertices in the subset can be connected by a path in the subset. A maximal connected subset is called a connected component of the graph and a graph is called connected if the graph consists of a single connected component.

We note that for any subset $U \subseteq X$, the space $\ell^2(X,m)$ can naturally be decomposed into $\ell^2(U,m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$, where m_U and $m_{X\setminus U}$ denote the restrictions of m to U and $X \setminus U$, respectively. Furthermore, we note that if U is a connected component of X, then $L = L_{b,c,m}$ maps $\ell^2(U,m_U)$ to $\ell^2(U,m_U)$ and $\ell^2(X \setminus U, m_{X\setminus U})$ to $\ell^2(X \setminus U, m_{X\setminus U})$. Hence, L can be written as $L_U \oplus L_{X\setminus U}$ acting on the direct product $\ell^2(U,m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$, where L_U means that L is restricted to $\ell^2(U,m_U)$. This will be used in what follows.

Proposition 0.53 (Characterization of positivity improving semigroups and resolvents). *Let (b,c) be a graph over a finite measure space (X,m) with associated Laplacian $L = L_{b,c,m}$. Then, the following statements are equivalent:*

(i) *The semigroup e^{-tL} is positivity improving for one (all) $t > 0$.*
(ii) *The resolvent $(L+\alpha)^{-1}$ is positivity improving for one (all) $\alpha > 0$.*
(iii) *The graph (b,c) is connected.*

Proof. (i) $\Longrightarrow$ (ii): This follows immediately from the fact that $(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt$, which is shown in Lemma 0.50 (b).

(ii) $\Longrightarrow$ (iii): Suppose that (b,c) is not connected so that there exists a non-empty connected component U of X with $U \neq X$. We may then write $L = L_U \oplus L_{X\setminus U}$, where L_U is the restriction of L to $\ell^2(U,m_U)$ and $L_{X\setminus U}$ of L to $\ell^2(X \setminus U, m_{X\setminus U})$. It follows that

$$(L+\alpha)^{-1} = (L_U + \alpha)^{-1} \oplus (L_{X\setminus U} + \alpha)^{-1}.$$

Let $f \in \ell^2(U,m_U)$ be positive and non-trivial. Then $(f,0) \in \ell^2(U,m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$, which can be unitarily identified with $\ell^2(X,m)$, is positive and non-trivial but

$$(L+\alpha)^{-1}(f,0) = ((L_U+\alpha)^{-1} f, (L_{X\setminus U}+\alpha)^{-1} 0) = ((L_U+\alpha)^{-1} f, 0)$$

is not strictly positive. Hence $(L+\alpha)^{-1}$ is not positivity improving.

(iii) $\Longrightarrow$ (i): Let $f \geq 0$ with $f \neq 0$. Let $\varphi \colon [0,\infty) \times X \longrightarrow [0,\infty)$ via

$$\varphi_t(x) = e^{-tL} f(x).$$

By Corollary 0.51 we have $\varphi_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$. We wish to show that $\varphi_t(x) > 0$ for all $t > 0$ and $x \in X$.

Assume that $\varphi_{t_0}(x_0) = 0$ for some $t_0 > 0$ and some $x_0 \in X$. Then, $t \mapsto \varphi_t(x_0)$ has a minimum at t_0. Thus,

$$\partial_t \varphi_{t_0}(x_0) = 0.$$

As φ_t solves $\partial_t \varphi_t = -L\varphi_t$, this implies

$$\begin{aligned} 0 &= L\varphi_{t_0}(x_0) \\ &= \frac{1}{m(x_0)} \sum_{y \in X} b(x_0, y)(\varphi_{t_0}(x_0) - \varphi_{t_0}(y)) + \frac{c(x_0)}{m(x_0)} \varphi_{t_0}(x_0) \\ &= -\frac{1}{m(x_0)} \sum_{y \in X} b(x_0, y)\varphi_{t_0}(y). \end{aligned}$$

By $\varphi \geq 0$ we conclude $\varphi_{t_0}(y) = 0$ for all $y \sim x_0$. By connectedness of the graph, we obtain inductively that $\varphi_{t_0} = 0$. This gives the contradiction $f = e^{t_0 L}\varphi_{t_0} = 0$. $\square$

REMARK. If the heat semigroup is positivity improving, then heat spreads "instantaneously" over the entire space. This is often referred to as the *infinite propagation speed* for the heat equation. Positivity improving semigroups are also sometimes called *ergodic*.

REMARK. A positivity preserving semigroup $P_t = e^{-tL}$ is positivity improving if and only if only the trivial subspaces of $\ell^2(X, m)$ are invariant under the semigroup and multiplication by functions on X (Exercise 0.34).

We will now focus on the behavior of the semigroup as time goes to infinity. This will be investigated in two steps. We first show convergence of the semigroup to the eigenspace of the smallest eigenvalue and then study this eigenspace.

LEMMA 0.54 (Speed of convergence). *Let L be a self-adjoint operator on $\ell^2(X, m)$. Let $\lambda_0 < \lambda_1$ be the smallest and second smallest eigenvalues of L, respectively, and let $\alpha = \lambda_1 - \lambda_0$. If E_0 is the orthogonal projection onto the eigenspace of λ_0, then*

$$\|e^{\lambda_0 t} e^{-tL} - E_0\| \leq e^{-\alpha t}.$$

In particular,

$$\|e^{-tL} - E_0\| \leq e^{-\lambda_1 t}$$

if $\lambda_0 = 0$.

PROOF. We write $L = \sum_{j=0}^{n} \lambda_j E_j$ with pairwise different eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ of L and E_j the associated pairwise orthogonal spectral projections onto the eigenspaces. These are the projections denoted by E_λ in Section 6. As discussed there,

$$e^{-tL} = \sum_{j=0}^{n} e^{-t\lambda_j} E_j.$$

This yields

$$e^{\lambda_0 t} e^{-tL} = E_0 + \sum_{j=1}^{n} e^{-t(\lambda_j - \lambda_0)} E_j.$$

From this we derive

$$\|e^{\lambda_0 t}e^{-tL} - E_0\| \leq e^{-(\lambda_1-\lambda_0)t}$$

as follows: Let $f \in \ell^2(X,m)$. We use the fact that the E_j are pairwise orthogonal twice to get

$$\begin{aligned}
\|(e^{\lambda_0 t}e^{-tL} - E_0)f\|^2 &= \sum_{j,k=1}^{n} e^{-t(\lambda_j-\lambda_0)}e^{-t(\lambda_k-\lambda_0)}\langle E_j f, E_k f\rangle \\
(E_j \text{ pairwise orthogonal}) \qquad &= \sum_{j=1}^{n} e^{-2t(\lambda_j-\lambda_0)}\|E_j f\|^2 \\
&\leq e^{-2\alpha t}\sum_{j=0}^{n}\|E_j f\|^2 \\
(E_j \text{ pairwise orthogonal}) \qquad &= e^{-2\alpha t}\|\sum_{j=0}^{n} E_j f\|^2 \\
&= e^{-2\alpha t}\|f\|^2.
\end{aligned}$$

Since this holds for all $f \in \ell^2(X,m)$, taking square roots yields the conclusion. □

The result above shows that $e^{\lambda_0 t}e^{-tL}$ converges exponentially to E_0, the orthogonal projection onto the eigenspace of λ_0. In particular, if $\lambda_0 = 0$, we get that the semigroup e^{-tL} converges exponentially to E_0.

We will now investigate the properties of E_0 in the case when the graph is connected. The following result is known as the Perron–Frobenius theorem and states that the eigenspace of λ_0 is one-dimensional. We recall that by the variational characterization of the bottom of the spectrum we have $\lambda_0 = \inf Q(f)$, where the infimum is taken over all $f \in \ell^2(X,m)$ with $\|f\| = 1$.

THEOREM 0.55 (Perron–Frobenius). *Let (b,c) be a connected graph over a finite measure space (X,m). Let $L = L_{b,c,m}$ be the associated Laplacian with form $Q = Q_{b,c}$ and let λ_0 be the smallest eigenvalue of L with E_0 the associated orthogonal projection. Then, the eigenspace of λ_0 is one-dimensional and there exists a unique normalized strictly positive eigenfunction u corresponding to λ_0 with*

$$E_0 f = \langle u, f\rangle u$$

for all $f \in \ell^2(X,m)$.

PROOF. We first note the following general fact.

Claim. A normalized function u is an eigenfunction corresponding to λ_0 if and only if $Q(u) = \lambda_0$.

Proof of the claim. If $Lu = \lambda_0 u$ with $\|u\| = 1$, then $Q(u) = \langle Lu, u\rangle = \lambda_0\|u\|^2 = \lambda_0$.

Conversely, let u be normalized with $Q(u) = \lambda_0$. Let $\lambda_0 < \ldots < \lambda_n$ denote the eigenvalues of L. Writing $L = \sum_{j=0}^n \lambda_j E_j$, we note that

$$\lambda_0 = Q(u) = \langle u, Lu\rangle = \langle u, \sum_{j=0}^n \lambda_j E_j u\rangle = \sum_{j=0}^n \lambda_j \|E_j u\|^2$$

with $\sum_{j=0}^n \|E_j u\|^2 = \|u\|^2 = 1$. This shows $E_j u = 0$ for $j \geq 1$ and $E_0 u = u$, so that $Lu = \lambda_0 u$.

We now show that any eigenfunction corresponding to λ_0 is either strictly positive or strictly negative:

Let u be a normalized eigenfunction corresponding to λ_0. Then,

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0.$$

Here, we used the variational characterization of λ_0 in the first inequality and that Q is a Dirichlet form in the second inequality. Therefore,

$$\lambda_0 = Q(|u|).$$

As $|u|$ is normalized as well, we infer that $|u|$ is also an eigenfunction corresponding to λ_0 by the claim.

We now write $u = u_+ - u_-$, where $u_+ = u \vee 0$ and $u_- = -u \vee 0$, so that $|u| = u_+ + u_-$. Then

$$u_+ = \frac{1}{2}(|u| + u) \quad \text{and} \quad u_- = \frac{1}{2}(|u| - u)$$

are also eigenfunctions corresponding to λ_0 (or vanish identically). Assume, without loss of generality, that $u_+ \neq 0$. As e^{-tL} is positivity improving for all $t > 0$ by Proposition 0.53, we infer

$$0 < e^{-L} u_+ = e^{-\lambda_0} u_+.$$

This implies

$$u_+ > 0 \quad \text{and} \quad u_- = 0.$$

These considerations show that any eigenfunction corresponding to λ_0 has a strict sign. We conclude that the eigenspace of λ_0 is one-dimensional as eigenfunctions with a strict sign cannot be orthogonal to one another.

Now, as the eigenspace of λ_0 is one-dimensional, we then obtain

$$E_0 f = \langle u, f\rangle u$$

for any normalized eigenfunction u and $f \in \ell^2(X, m)$. Hence, any normalized strictly positive u has the desired properties and is uniquely determined by these properties. $\square$

We note that $\lambda_0 = 0$ is equivalent to L being not invertible. We now use Theorem 0.55 above to give another proof of Lemma 0.29 which characterizes this property in the case when the graph is connected.

Corollary 0.56 (Characterization of $\lambda_0 = 0$). *Let (b,c) be a connected graph over a finite measure space (X,m), $L = L_{b,c,m}$, $Q = Q_{b,c}$ and λ_0 be the smallest eigenvalue of L. Then, $\lambda_0 = 0$ if and only if $c = 0$.*

Proof. From Theorem 0.55 and its proof, we know that

$$\lambda_0 = Q(u) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(u(x)-u(y))^2 + \sum_{x\in X} c(x)u^2(x),$$

where u is the unique strictly positive normalized eigenfunction corresponding to λ_0 which minimizes $Q(f)$ over $\|f\| = 1$. If $\lambda_0 = 0$, then $c = 0$ since $u > 0$. Conversely, if $c = 0$, then taking u to be a constant function such that $\|u\| = 1$ will minimize $Q(f)$ with value $\lambda_0 = 0$. $\square$

Remark (The case $c = 0$). In fact, if $c = 0$, the dimension of the eigenspace of $\lambda_0 = 0$ is equal to the number of connected components of the graph (Exercise 0.35).

We now introduce some terminology related to the quantities presented above.

Definition 0.57 (Ground state and ground state energy). Let (b,c) be a connected graph over a finite measure space (X,m) with associated Laplacian $L = L_{b,c,m}$. The smallest eigenvalue λ_0 of L is called the *ground state energy* and the normalized positive eigenfunction u corresponding to λ_0 is called the *ground state*.

We also introduce the heat kernel, which arises from the heat semigroup e^{-tL}.

Definition 0.58 (Heat kernel). Let (b,c) be a graph over a finite measure space (X,m) with associated Laplacian $L = L_{b,c,m}$. The map

$$p\colon [0,\infty) \times X \times X \longrightarrow [0,\infty)$$

defined by

$$e^{-tL}f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $t \geq 0$, $f \in \ell^2(X,m)$ and $x \in X$ is called the *heat kernel*.

Remark. From the symmetry of the semigroup, which follows from the self-adjointness of the operator, we note that

$$\begin{aligned}p_t(x,y) = e^{-tL}1_y(x)/m(y) &= \frac{1}{m(x)m(y)}\langle 1_x, e^{-tL}1_y\rangle \\ &= \frac{1}{m(x)m(y)}\langle e^{-tL}1_x, 1_y\rangle = e^{-tL}1_x(y)/m(x).\end{aligned}$$

The next result connects the heat kernel and the ground state and ground state energy.

THEOREM 0.59 (Convergence to the ground state and ground state energy). *Let (b, c) be a connected graph over the finite measure space (X, m). Let $L = L_{b,c,m}$ be the associated Laplacian with ground state energy λ_0, ground state u and heat kernel p. Let $\lambda_1 > \lambda_0$ be the second smallest eigenvalue of L and let $\alpha = \lambda_1 - \lambda_0$.*

(a) *For all $x, y \in X$,*

$$|e^{\lambda_0 t} p_t(x, y) - u(x)u(y)| \leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}.$$

(*"Theorem of Chavel–Karp for finite graphs"*)

(b) *For all $x, y \in X$,*

$$\lim_{t\to\infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0.$$

(*"Theorem of Li for finite graphs"*)

PROOF. To prove (a), first observe that for any $f \in \ell^2(X, m)$ we have $|f(x)| \leq \|f\|/\sqrt{m(x)}$. Now, the formula for E_0 in Theorem 0.55 gives $E_0 1_y(x)/m(y) = u(x)u(y)$ while $p_t(x, y) = e^{-tL}1_y(x)/m(y)$ by definition. From Lemma 0.54 we then obtain

$$\begin{aligned} |e^{\lambda_0 t} p_t(x, y) - u(x)u(y)| &= \frac{|e^{\lambda_0 t} e^{-tL} 1_y(x) - E_0 1_y(x)|}{m(y)} \\ &\leq \frac{\|e^{\lambda_0 t} e^{-tL} - E_0\| \|1_y\|}{m(y)\sqrt{m(x)}} \\ &\leq \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}. \end{aligned}$$

This gives (a).

To prove (b), note from the above that

$$u(x)u(y) - \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}} \leq e^{\lambda_0 t} p_t(x, y) \leq u(x)u(y) + \frac{e^{-\alpha t}}{\sqrt{m(x)m(y)}}.$$

As u is strictly positive by Theorem 0.55, (b) follows after taking logarithms for large t, dividing by t and letting $t \to \infty$. □

We now give an immediate corollary which states that the only eigenvalue which has a strictly positive eigenfunction is the ground state energy.

COROLLARY 0.60 (Positive eigenfunctions are multiples of ground states). *Let (b, c) be a connected graph over the finite measure space (X, m). Let $L = L_{b,c,m}$ be the associated Laplacian with ground state energy λ_0 and ground state u. If there exists $\lambda \in \mathbb{R}$ and $v \geq 0$ which is non-trivial and satisfies $Lv = \lambda v$, then*

$$\lambda = \lambda_0 \qquad \text{and} \qquad v = \alpha u$$

for $\alpha > 0$.

Proof. As λ is an eigenvalue, it follows that $\lambda_0 \leq \lambda$ since λ_0 is the smallest eigenvalue of L by definition. Now, from $Lv = \lambda v$ we get

$$e^{-tL}v = e^{-\lambda t}v.$$

Since $v \geq 0$, it follows that

$$e^{-\lambda t}v(x) = e^{-tL}v(x) = \sum_{y \in X} p_t(x,y)v(y)m(y) \geq p_t(x,y)v(y)m(y)$$

for all $x \in X$. Now, choose $x_0 \in X$ such that $v(x_0) \neq 0$. Then, by Theorem 0.59 (b) and the estimate above, we get

$$-\lambda = \lim_{t\to\infty} \frac{1}{t}\log\left(e^{-\lambda t}v(x_0)\right) \geq \lim_{t\to\infty} \frac{1}{t}\log\left(p_t(x_0,x_0)v(x_0)m(x_0)\right) = -\lambda_0.$$

Therefore, $\lambda \leq \lambda_0$. Combining the two inequalities gives $\lambda = \lambda_0$. That $v = \alpha u$ for $\alpha > 0$ then follows as the eigenspace of λ_0 is one-dimensional by Theorem 0.55. $\square$

We finish this section by looking at consequences for the case when the killing term c vanishes.

Corollary 0.61 (The case $c = 0$). *Let b be a connected graph over a finite measure space (X,m). If $L = L_{b,0,m}$ is the associated Laplacian, λ_1 is the second smallest eigenvalue of L and e^{-tL} is the heat semigroup with heat kernel p, then*

$$\left| p_t(x,y) - \frac{1}{m(X)} \right| \leq \frac{e^{-t\lambda_1}}{\sqrt{m(x)m(y)}}.$$

Proof. Since $c = 0$, the ground state energy is 0 by Corollary 0.56 and the normalized strictly positive eigenfunction is given by the constant function with value $1/\sqrt{m(X)}$ by connectedness. Now, the statement follows from Theorem 0.59 (a). $\square$

Combining the above results yields the following characterization of the case of a vanishing killing term.

Corollary 0.62 (Characterization of $c = 0$). *Let (b,c) be a connected graph over a finite measure space (X,m) and let $L = L_{b,c,m}$ be the associated Laplacian with heat semigroup e^{-tL} and heat kernel p. Then,*

$$\lim_{t\to\infty} p_t(x,y) = 0$$

for all $x,y \in X$ if and only if

$$c \neq 0.$$

Proof. If $c = 0$, then $p_t(x,y) \to 1/m(X) \neq 0$ as $t \to \infty$ by Corollary 0.61. On the other hand, if $c \neq 0$, then $\lambda_0 > 0$ by Corollary 0.56. Therefore, $p_t(x,y) \to 0$ as $t \to \infty$ by Theorem 0.59. $\square$

Remark. In Theorem 0.59 and Corollary 0.61 one obtains exponential convergence towards the ground state. The rate depends on the distance between the first two eigenvalues, i.e., the so-called *spectral gap*. This motivates the study of the spectral gap, which is an important topic of research.

8. When there is no killing

In the previous sections we have seen various characterizations for matrices, forms and operators associated to graphs where both an edge weight and a killing term are present. In this section we consider characterizations for the case when the killing term vanishes. As such, this section will provide both a summary of the preceding material and introduce several new ideas.

We let l be a symmetric matrix on X with associated symmetric form Q and operator L. That is,

$$\sum_{x,y\in X} l(x,y)f(x)g(y) = Q(f,g) = Q(g,f)$$
$$= \sum_{x\in X} Lf(x)g(x) = \sum_{x\in X} f(x)Lg(x)$$

for all $f, g \in C(X)$. We note that if any one of l, Q or L is associated to a graph, then all three are associated to the same graph. That is, any one of the equalities $l = l_{b,c}$, $Q = Q_{b,c}$ or $L = L_{b,c}$ for a graph (b, c) over X, implies that all three equalities are true. The same is clearly true for a graph b over X.

We will first recall the characterizations for matrices, forms and operators associated to graphs (b, c). We will then discuss the case of no killing, i.e., when $c = 0$ for each of the objects. In some cases, this has already been done in the previous sections, in other cases, we will introduce new ideas.

We start with matrices. Lemma 0.9 shows that l is associated to a graph (b, c) if and only if

$$l(x,y) \le 0 \quad \text{for all } x \neq y \text{ and}$$
$$\sum_{y\in X} l(x,y) \ge 0 \quad \text{for all } x \in X.$$

Furthermore, Lemma 0.9 also shows that l is a matrix associated to a graph b if and only if

$$l(x,y) \le 0 \quad \text{for all } x \neq y \text{ and}$$
$$\sum_{y\in X} l(x,y) = 0 \quad \text{for all } x \in X.$$

Hence, we see that the difference between graphs with $c \neq 0$ and $c = 0$ is precisely encoded in the sum $\sum_{y \in X} l(x, y)$. Indeed, it is the case that

$$c(x) = \sum_{y \in X} l(x, y)$$

when connecting graphs and matrices. This gives the matrix perspective on both graphs (b, c) as well as graphs b over X.

We next discuss the Dirichlet form characterization for graphs. We recall that Q is a Dirichlet form if and only if Q is compatible with all normal contractions C, that is,

$$Q(C \circ f) \leq Q(f)$$

for all $f \in C(X)$ and all normal contractions C. Theorem 0.20 shows that $Q = Q_{b,c}$ if and only if Q is a Dirichlet form. In fact, Theorem 0.20 gives even more information as it states that Q is a Dirichlet form if and only if Q is compatible with $C_{[0,1]}$, where $C_{[0,1]} \circ f = 0 \vee f \wedge 1$, if and only if $|f(x) - f(y)| \leq |g(x) - g(y)|$ and $|f| \leq |g|$ imply that $Q(f) \leq Q(g)$ for all $f, g \in C(X)$. This gives the form perspective on graphs (b, c).

For the form perspective on graphs b over X, i.e., graphs without killing, we start by defining the notion of a contraction. We call a map $C \colon \mathbb{R} \longrightarrow \mathbb{R}$ a *contraction* if

$$|C(s) - C(t)| \leq |s - t|$$

for all $s, t \in \mathbb{R}$. Hence, the difference between a contraction and a normal contraction is that we do not require that $C(0) = 0$ for a contraction. Note, in particular, that

$$C(s) = s \vee 1 = \max\{s, 1\}$$

is a contraction which is not normal. We now present a counterpart to Theorem 0.20 characterizing symmetric forms which are associated to graphs b over X.

Theorem 0.63 (Characterization of forms associated to graphs with no killing). *Let Q be a symmetric form over a finite set X. Then, the following statements are equivalent:*

(i) *There exists a graph b over X such that $Q = Q_b$.*
(ii) *For all $f \in C(X)$,*

$$Q(f) \geq 0 \quad \text{and} \quad Q(f \vee 1) \leq Q(f).$$

(iii) *For all contractions C and $f \in C(X)$,*

$$Q(C \circ f) \leq Q(f).$$

(iv) *If $f, g \in C(X)$ satisfy $|f(x) - f(y)| \leq |g(x) - g(y)|$ for all $x, y \in X$, then*

$$Q(f) \leq Q(g).$$

PROOF. (i) $\Longrightarrow$ (iv): If $|f(x)-f(y)| \leq |g(x)-g(y)|$ for all $x, y \in X$, then clearly

$$b(x,y)(f(x)-f(y))^2 \leq b(x,y)(g(x)-g(y))^2$$

so that $Q_b(f) \leq Q_b(g)$.

(iv) $\Longrightarrow$ (iii): Since $|C(f(x)) - C(f(y))| \leq |f(x) - f(y)|$ for all contractions C, it follows that $Q(C \circ f) \leq Q(f)$.

(iii) $\Longrightarrow$ (ii): Since $C \circ f = f \vee 1$ is a contraction, it follows that $Q(f \vee 1) \leq Q(f)$. Furthermore, taking the contraction $C \circ f = 0$ for all $f \in C(X)$ gives

$$0 = Q(0) = Q(C \circ f) \leq Q(f)$$

for all $f \in C(X)$.

(ii) $\Longrightarrow$ (i): We first note that

$$0 \leq Q(1) = Q(0 \vee 1) \leq Q(0) = 0$$

so that $Q(1) = 0$. By applying the Cauchy–Schwarz inequality to the matrix elements of Q, it then follows that $Q(f,1) = 0$ for all $f \in C(X)$.

Next, we observe that $f_+ = f \vee 0 = \lim_{s \to 0}(f \vee s)$. For $s \neq 0$, $f \vee s = s((f/s) \vee 1)$ so that

$$Q(s((f/s) \vee 1)) = s^2 Q((f/s) \vee 1) \leq s^2 Q(f/s) = Q(f)$$

by our assumption on Q. Therefore,

$$Q(f_+) = Q(f \vee 0) \leq Q(f)$$

for all $f \in C(X)$ by letting $s \to 0$. A similar reasoning for $f_- = -f \vee 0$, gives that

$$Q(f_-) \leq Q(f).$$

Now, by checking cases, we obtain

$$f \wedge 1 = -(f-1)_- + 1.$$

Therefore,

$$\begin{aligned}
Q(f \wedge 1) &= Q(-(f-1)_- + 1) \\
&= Q((f-1)_-) - 2Q((f-1)_-, 1) + Q(1) \\
&= Q((f-1)_-) \\
&\leq Q(f-1) \\
&= Q(f) - 2Q(f,1) + Q(1) \\
&= Q(f).
\end{aligned}$$

Hence, as we have shown that both $Q(f \vee 0) \leq Q(f)$ and $Q(f \wedge 1) \leq Q(f)$ for all $f \in C(X)$, it follows that

$$Q(C_{[0,1]} \circ f) \leq Q(f).$$

Now, Theorem 0.20 implies that there exists a graph (b, c) over X such that $Q = Q_{b,c}$. As $Q(1) = \sum_{x \in X} c(x) = 0$, it follows that $c = 0$. This completes the proof. $\square$

We now recall some of the operator characterizations for graphs (b, c). First, by Theorem 0.24, $L = L_{b,c}$ if and only if L satisfies the maximum principle, that is,

$$Lf(x) \geq 0$$

for any f which achieves a non-negative maximum at $x \in X$. Furthermore, by Corollary 0.51, from the heat semigroup and resolvent viewpoint, we get that $L = L_{b,c}$ if and only if

$$0 \leq e^{-tL} f \leq 1$$

if and only if

$$0 \leq (L + \alpha)^{-1} f \leq \frac{1}{\alpha}$$

for all $f \in \ell^2(X, m)$ with $0 \leq f \leq 1$ and all $t \geq 0$ and $\alpha > 0$.

We now turn to the operator perspective on $c = 0$. In Corollary 0.27 we have proven the so-called strong maximum principle, which says that $Lf(x) \geq 0$ for any $x \in X$ which is a maximum for f. We just recall it here.

Theorem 0.64 (Characterization of operators associated to graphs with no killing). *Let L be a symmetric operator over a finite set X. Then, the following statements are equivalent:*

(i) *There exists a graph b over X such that $L = L_b$.*
(ii) *The operator L satisfies the strong maximum principle.*

Furthermore, in the case of $L = L_{b,c}$ where (b, c) is a connected graph and λ_0 is the smallest eigenvalue of L, Lemma 0.29 and Corollary 0.56 give that $c = 0$ is equivalent to L being not bijective, which is equivalent to $\lambda_0 = 0$. In particular, this is also equivalent to the existence of non-zero harmonic functions for L (which are the constant functions).

We will now look at the semigroup and resolvent viewpoint on the lack of a killing term. We have already seen one manifestation of this in Corollary 0.62 which stated that, in the long term, the heat kernel will tend to 0 if and only if there is a killing term. Thus, as X is a finite set, it follows that

$$e^{-tL} 1(x) = \sum_{y \in X} p_t(x, y) m(y) \to 0$$

as $t \to \infty$ if and only if $c \neq 0$. This gives the long-term perspective on heat loss in the presence of a killing term.

We will now consider the case of short-term behavior. In particular, we will look at the validity of the equation

$$e^{-tL}1 = 1$$

for all $t > 0$. We note that $e^{-tL}1$ can be interpreted as the total amount of heat found within the graph at time t. Hence, the equality above indicates that no heat is lost at any time during the heat evolution. Heat semigroups which satisfy this equation are called *stochastically complete* or *conservative*. Otherwise, the heat semigroup is called *stochastically incomplete* or *non-conservative*.

A natural way for this property to fail is to have a killing boundary condition at some vertices, that is, that heat is removed as soon as it reaches a vertex where killing occurs. As, by Proposition 0.53, heat spreads instantaneously over any space, it follows that heat is lost instantaneously in this case. As the result below shows, this killing is exactly encoded in c and is one of the reasons for the name killing term.

THEOREM 0.65 (Characterization of semigroups associated to graphs with no killing). *Let L be a self-adjoint operator on $\ell^2(X, m)$. Then, the following statements are equivalent:*

(i) *There exists a graph b over (X, m) such that $L = L_b$.*
(ii) *The semigroup e^{-tL} is positivity preserving and satisfies*

$$e^{-tL}1 = 1$$

for all $t > 0$.
(iii) *The resolvent $(L + \alpha)^{-1}$ is positivity preserving and satisfies*

$$\alpha(L + \alpha)^{-1}1 = 1$$

for all $\alpha > 0$.

PROOF. (i) $\Longrightarrow$ (ii): That e^{-tL} for $L = L_b$ is positivity preserving for all $t \geq 0$ follows from Corollary 0.51. Furthermore, as $c = 0$ we get $L1 = 0$ and thus $L^n1 = 0$ for all $n \in \mathbb{N}$. As $e^{-tL} = \sum_{n=0}^{\infty}(-tL)^n/n!$ it follows that

$$e^{-tL}1 = 1.$$

(ii) $\Longrightarrow$ (i): We first note that $e^{-tL}1 = 1$ and the positivity preserving property of e^{-tL} implies that e^{-tL} is contracting, i.e., $e^{-tL}f \leq 1$ for all $f \in \ell^2(X, m)$ with $f \leq 1$. This follows as when $f \leq 1$, we obtain $0 \leq e^{-tL}(1 - f) = e^{-tL}1 - e^{-tL}f = 1 - e^{-tL}f$ so that

$$e^{-tL}f \leq 1$$

for all $f \in \ell^2(X, m)$ with $f \leq 1$ and all $t \geq 0$.

Therefore, as $0 \leq e^{-tL}f \leq 1$ for all $f \in \ell^2(X, m)$ with $0 \leq f \leq 1$ it follows that $L = L_{b,c}$ for a graph (b, c) over (X, m) by Corollary 0.51. Our aim is now to show that $c = 0$.

Suppose not, i.e., suppose that $c \neq 0$. Without loss of generality, we may assume that (b,c) is connected as otherwise we work on a connected component where $c \neq 0$. It follows from Corollary 0.56 that $\lambda_0 > 0$ where λ_0 is the smallest eigenvalue of L. Therefore, the semigroup e^{-tL} converges exponentially to 0 by Theorem 0.59. As such, there exists some $t > 0$ such that $e^{-tL}1 < 1$. The contradiction shows that $c = 0$ so that $L = L_b$.

(ii) $\Longleftrightarrow$ (iii): This is immediate from the identities

$$e^{-tL} = \lim_{n\to\infty} \left(\frac{n}{t}\left(L + \frac{n}{t}\right)^{-1}\right)^n \qquad \text{and} \qquad (L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha}e^{-tL}dt$$

found in Lemma 0.50. □

Remark. It is also possible to base a proof of the fact that $c = 0$ for (ii) $\Longrightarrow$ (i) in Theorem 0.65 above on the Lie–Trotter product formula (Exercise 0.36). Furthermore, stochastic incompleteness is an instantaneous phenomenon, i.e., if it happens for one $t > 0$, then it happens for all $t > 0$ (Exercise 0.37).

9. Turning graphs into other graphs*

In this section we study the effect that changing the graph has on the resolvent. We first show that sending the potential or killing term to infinity at a point introduces a boundary with Dirichlet boundary conditions. While no convergence for the operators can be expected, we show convergence of the resolvents. Secondly, we show how gradually disconnecting a graph decouples the resolvent.

Let (X,m) be a finite measure space. For two bijective operators A and B on $\ell^2(X,m)$ the equalities

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1} = B^{-1}(B-A)A^{-1}$$

hold. These equalities can be checked directly and are essential for the considerations of this section. Whenever A^{-1} and B^{-1} are resolvents, these equalities are called the *resolvent identities.*

Proposition 0.66 (Monotonicity of resolvents). *Let (b,c) be a graph over a finite measure space (X,m) and let $L = L_{b,c,m}$ be the associated Laplacian. If $c' \leq c$ and $L' = L'_{b,c',m}$, then*

$$\langle (L+\alpha)^{-1}f, g\rangle \leq \langle (L'+\alpha)^{-1}f, g\rangle$$

for all $\alpha > 0$ and $f, g \in \ell^2(X,m)$ with $f, g \geq 0$.

Proof. It suffices to consider the case of c and c' being equal except at a single vertex o where $c'(o) \leq c(o)$. Let $\varphi_x = 1_x/m(x)$, $\lambda = c(o) - c'(o)$ and let $\Lambda\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ be given by

$$\Lambda(\cdot) = \langle \cdot, \varphi_o\rangle\varphi_o.$$

A direct calculation gives $L - L' = \lambda\Lambda$. By the resolvent identity, for all $f, g \in \ell^2(X, m)$ with $f, g \geq 0$, we have

$$\begin{aligned}\langle (L' + \alpha)^{-1} f, g\rangle - \langle (L + \alpha)^{-1} f, g\rangle \\ &= \lambda\langle (L' + \alpha)^{-1}\Lambda(L + \alpha)^{-1} f, g\rangle \\ &= \lambda\langle (L + \alpha)^{-1} f, \varphi_o\rangle\langle (L' + \alpha)^{-1}\varphi_o, g\rangle.\end{aligned}$$

The statement follows since $\lambda \geq 0$ and resolvents are positivity preserving, that is, they map positive functions to positive functions by Corollary 0.51. □

Given a graph (b, c) over (X, m) we recall the definition of the Dirichlet Laplacian for a subset $K \subseteq X$. We let $b_K = b|_{K\times K}$, $m_K = m|_K$ and

$$d_K(x) = c(x) + \sum_{y\in X\setminus K} b(x, y)$$

for $x \in K$. The *Dirichlet Laplacian* with respect to K is the Laplacian

$$L_K^{(D)} = L_{b_K, d_K, m_K}$$

on $\ell^2(K, m_K)$. Equivalently, if $\pi_K \colon \ell^2(X, m) \longrightarrow \ell^2(K, m_K)$ is the canonical projection and $i_K \colon \ell^2(K, m_K) \longrightarrow \ell^2(X, m)$ is continuation by zero on K, then $L_K^{(D)} = \pi_K L_{b,c,m} i_K$.

Theorem 0.67 (Turning the potential up to infinity at a vertex yields a Dirichlet Laplacian). *Let (b, c) be a connected graph over a finite measure space (X, m) and let $o \in X$. Let $X' = X \setminus \{o\}$ and for $\lambda \geq 0$ let (b, c_λ) be the graph on (X, m) with $c_\lambda = c + \lambda 1_o$. If $f, g \in \ell^2(X, m)$ and $\alpha > 0$, then*

$$\lim_{\lambda\to\infty} \langle (L_{b,c_\lambda,m} + \alpha)^{-1} f, g\rangle = \langle (L_{X'}^{(D)} + \alpha)^{-1}\pi_{X'} f, \pi_{X'} g\rangle.$$

Proof. For $x \in X$, let $\varphi_x \in \ell^2(X, m)$ be given by $\varphi_x = 1_x/m(x)$ and denote $i = i_{X'}$ and $\pi = \pi_{X'}$. Let $L_\lambda = L_{b,c_\lambda,m}$ and $G_\lambda = (L_\lambda + \alpha)^{-1}$ for $\lambda \geq 0$. The function

$$\lambda \mapsto G_\lambda(x, y) = (L_\lambda + \alpha)^{-1}\varphi_x(y) = \langle (L_\lambda + \alpha)^{-1}\varphi_x, \varphi_y\rangle$$

is monotone decreasing by Proposition 0.66 and bounded below by 0 by Corollary 0.51. Thus, the following limit exists for all $x, y \in X$

$$G_\infty(x, y) = \lim_{\lambda\to\infty} G_\lambda(x, y).$$

Let $\Lambda(\cdot) = \langle \cdot, \varphi_o\rangle\varphi_o$ and note that $L_\mu - L_\lambda = (\mu - \lambda)\Lambda$. From the resolvent formula we infer

$$G_\lambda - G_\mu = G_\lambda(L_\mu - L_\lambda)G_\mu = (\mu - \lambda)G_\lambda\Lambda G_\mu.$$

Taking the matrix elements at $x, y \in X$, setting $\mu = 0$ and using the symmetry of the operators involved gives

$$\begin{aligned} G_\lambda(x,y) - G_0(x,y) &= -\lambda\langle G_\lambda \Lambda G_0 \varphi_x, \varphi_y\rangle \\ &= -\lambda\langle \varphi_x, G_\lambda \Lambda G_0 \varphi_y\rangle \\ &= -\lambda\langle \varphi_x, G_\lambda \varphi_o\rangle\langle G_0\varphi_y, \varphi_o\rangle \\ &= -\lambda G_\lambda(x,o) G_0(o,y). \end{aligned}$$

Since the graph is connected, by Proposition 0.53 the resolvents are positivity improving so that $G_0(o, y) > 0$. Therefore, we obtain

$$\lim_{\lambda\to\infty} \lambda G_\lambda(x,o) = -\lim_{\lambda\to\infty} \frac{G_\lambda(x,y) - G_0(x,y)}{G_0(o,y)} = -\frac{G_\infty(x,y) - G_0(x,y)}{G_0(o,y)}$$

for all $x, y \in X$. In particular, we get $\lim_{\lambda\to\infty} G_\lambda(x, o) = 0$ for all $x \in X$. Therefore, if $f = 1_o$ or $g = 1_o$, then, since $\pi 1_o = 0$,

$$\lim_{\lambda\to\infty} \langle (L_\lambda + \alpha)^{-1} f, g\rangle = 0 = \langle (L^{(D)} + \alpha)^{-1} \pi f, \pi g\rangle.$$

Furthermore, setting $y = o$ in the limit above, we get

$$\lim_{\lambda\to\infty} \lambda G_\lambda(x,o) = \frac{G_0(x,o)}{G_0(o,o)}.$$

Since $G_\infty(x, o) = 0$, we arrive at

$$\begin{aligned} G_\infty(x,y) &= G_0(x,y) - \lim_{\lambda\to\infty} \lambda G_\lambda(x,o) G_0(o,y) \\ &= G_0(x,y) - \frac{G_0(x,o)}{G_0(o,o)} G_0(o,y). \end{aligned}$$

We now show that

$$\pi G_\infty i (L^{(D)}_{X'} + \alpha) = (L^{(D)}_{X'} + \alpha) \pi G_\infty i = I$$

on $\ell^2(X', m')$, where $m' = m|_{X'}$, which will complete the proof. Note that $L^{(D)}_{X'} f(x) = L_0 f(x) + \frac{b(x,o)}{m(x)} f(o)$. Let $g_\infty(\cdot) = \pi G_\infty(x, \cdot) = G_\infty(x, \cdot)$, $g_0(\cdot) = G_0(x, \cdot)$ and $h_0(\cdot) = G_0(o, \cdot)$. By what we have shown above

$$g_\infty = g_0 - \frac{g_0(o)}{h_0(o)} h_0.$$

We calculate, for $y \neq o$,

$$\begin{aligned} (L^{(D)}_{X'} + \alpha) g_\infty(y) &= (L^{(D)}_{X'} + \alpha) g_0(y) - \frac{g_0(o)}{h_0(o)} (L^{(D)}_{X'} + \alpha) h_0(y) \\ &= (L_0 + \alpha) g_0(y) - \frac{g_0(o)}{h_0(o)} (L_0 + \alpha) h_0(y) + \frac{b(y,o)}{m(y)} \left(g_0(o) - \frac{g_0(o)}{h_0(o)} h_0(o) \right) \\ &= \varphi_x(y) - \frac{g_0(o)}{h_0(o)} \varphi_o(y) = \varphi_x(y). \end{aligned}$$

Since $\{\varphi_x\}_{x\in X, x\neq o}$ is a basis for $\ell^2(X', m')$ and $g_\infty(y) = \pi G_\infty i\varphi_x(y)$, this yields

$$(L_{X'}^{(D)} + \alpha)\pi G_\infty i = I.$$

On the other hand, since $i\pi = I - m(o)\Lambda$, $L_{X'}^{(D)} = \pi L_0 i$ on $\ell^2(X', m')$, $G_\infty \Lambda = 0$ and $\Lambda i\varphi = 0$ for $\varphi \in \ell^2(X', m')$, we get

$$\begin{aligned} G_\infty i(L_{X'}^{(D)} + \alpha)\varphi &= G_\infty i\pi(L_0 + \alpha)i\varphi \\ &= G_\infty(I - m(o)\Lambda)(L_0 + \alpha)i\varphi \\ &= \lim_{\lambda\to\infty} G_\lambda(L_0 + \alpha)i\varphi. \end{aligned}$$

Using the resolvent formula $G_\lambda = G_0 - \lambda G_\lambda \Lambda G_0$ we proceed

$$\begin{aligned} \ldots &= G_0(L_0 + \alpha)i\varphi - \lim_{\lambda\to\infty} \lambda G_\lambda \Lambda G_0(L_0 + \alpha)i\varphi \\ &= i\varphi - \lim_{\lambda\to\infty} \lambda G_\lambda \Lambda i\varphi \\ &= i\varphi. \end{aligned}$$

Hence,

$$\pi G_\infty i(L_{X'}^{(D)} + \alpha) = I.$$

Thus, we have shown

$$\pi G_\infty i = (L_{X'}^{(D)} + \alpha)^{-1},$$

which finishes the proof □

Theorem 0.68 (Turning off an edge disconnects the graph in the resolvents). *Let (b_0, c) be a graph over a finite measure space (X, m) with two connected components X_1 and X_2 and let $x_1 \in X_1$ and $x_2 \in X_2$. Let $b_\lambda = b_0 + \lambda 1_{\{(x_1,x_2),(x_2,x_1)\}}$. Then, for all $\alpha > 0$,*

$$\lim_{\lambda\to 0^+} \langle (L_{b_\lambda,c,m} + \alpha)^{-1} f, g\rangle = 0$$

whenever $\operatorname{supp} f \subseteq X_1$ *and* $\operatorname{supp} g \subseteq X_2$.

Proof. Let $L_\lambda = L_{b_\lambda,c,m}$ for $\lambda \geq 0$. We start with a claim.

Claim. There exists a $C \geq 0$ such that for all $\lambda \geq 0$ and all $f, g \in \ell^2(X, m)$

$$|\langle (L_\lambda + \alpha)^{-1} f, g\rangle| \leq C\|f\|\|g\|.$$

Proof of the claim. Let Q_λ be the form associated to L_λ for $\lambda \geq 0$. Then, since $b_\lambda(x_1, x_2) \geq 0 = b_0(x_1, x_2)$ and $b_\lambda = b_0$ otherwise, one has

$$Q_\lambda(\varphi) \geq Q_0(\varphi)$$

for $\varphi \in \ell^2(X, m)$. In particular, for the smallest eigenvalue $\mu_1^{(\lambda)}$ of L_λ with normalized eigenfunction $\psi_1^{(\lambda)}$,

$$\mu_1^{(\lambda)} = \langle L_\lambda \psi_1^{(\lambda)}, \psi_1^{(\lambda)}\rangle = Q_\lambda(\psi_1^{(\lambda)}) \geq Q_0(\psi_1^{(\lambda)}) \geq \mu_1^{(0)}.$$

Let the eigenvalues of L_λ be given by $0 \le \mu_1^{(\lambda)} \le \ldots \le \mu_N^{(\lambda)}$ with orthonormal eigenfunctions $\psi_1^{(\lambda)}, \ldots, \psi_N^{(\lambda)}$. It follows that, for all $f, g \in \ell^2(X, m)$,

$$
\begin{aligned}
|\langle f, (L_\lambda + \alpha)^{-1} g\rangle| &= \left| \sum_{k=1}^N \frac{\langle f, \psi_k^{(\lambda)}\rangle \langle g, \psi_k^{(\lambda)}\rangle}{\mu_k^{(\lambda)} + \alpha} \right| \\
&\le \frac{1}{\mu_1^{(\lambda)} + \alpha} \sum_{k=1}^N \left| \langle f, \psi_k^{(\lambda)}\rangle \langle g, \psi_k^{(\lambda)}\rangle \right| \\
&\le \frac{1}{\mu_1^{(0)} + \alpha} \left(\sum_{k=1}^N \langle f, \psi_k^{(\lambda)}\rangle^2 \right)^{1/2} \left(\sum_{k=1}^N \langle g, \psi_k^{(\lambda)}\rangle^2 \right)^{1/2} \\
&= \frac{1}{\mu_1^{(0)} + \alpha} \|f\| \|g\|,
\end{aligned}
$$

using $\|h\|^2 = \| \sum_\mu E_\mu^{(\lambda)} h\|^2 = \sum_k |\langle h, \psi_k^{(\lambda)}\rangle|^2$ for $h \in \ell^2(X, m)$ since the eigenfunctions are orthonormal and $(L_\lambda + \alpha)^{-1} \psi_k^{(\lambda)} = \frac{1}{\mu_k^{(\lambda)} + \alpha} \psi_k^{(\lambda)}$. This proves the claim.

Now, let $\varphi_x = 1_x / m(x)$ and let x_1, x_2 be as assumed. Set

$$\Lambda(\cdot) = \varphi_{x_1}(\langle \cdot, \varphi_{x_1}\rangle - \langle \cdot, \varphi_{x_2}\rangle) + \varphi_{x_2}(\langle \cdot, \varphi_{x_2}\rangle - \langle \cdot, \varphi_{x_1}\rangle)$$

and, for $\lambda \ge 0$,

$$R_\lambda = (L_\lambda + \alpha)^{-1}.$$

It follows that $L_\lambda - L_0 = \lambda \Lambda$ and, by the resolvent identity, $R_\lambda = R_0 - \lambda R_\lambda \Lambda R_0$. Therefore, we get for f, g such that $\operatorname{supp} f \subseteq X_1$ and $\operatorname{supp} g \subseteq X_2$,

$$
\begin{aligned}
\langle f, R_\lambda g\rangle &= \langle f, R_0 g\rangle - \lambda \langle f, R_\lambda \Lambda R_0 g\rangle \\
&= \langle f, R_0 g\rangle - \lambda \langle f, R_\lambda \varphi_{x_1}\rangle \left(\langle R_0 g, \varphi_{x_1}\rangle - \langle R_0 g, \varphi_{x_2}\rangle \right) \\
&\qquad - \lambda \langle f, R_\lambda \varphi_{x_2}\rangle \left(\langle R_0 g, \varphi_{x_2}\rangle - \langle R_0 g, \varphi_{x_1}\rangle \right) \\
&= \lambda \langle f, R_\lambda \varphi_{x_1}\rangle \langle R_0 g, \varphi_{x_2}\rangle,
\end{aligned}
$$

where the other terms vanish since L_0 leaves the subspaces $\ell^2(X_1, m_1)$ and $\ell^2(X_2, m_2)$ invariant (where $m_1 = m|_{X_1}$ and $m_2 = m|_{X_2}$) and so does $R_0 = (L_0 + \alpha)^{-1}$. Thus, by the claim above, there exists a $C \ge 0$ such that

$$|\langle f, (L_\lambda + \alpha)^{-1} g\rangle| \le \lambda C^2 \|f\| \|g\| \to 0$$

as $\lambda \to 0$ and the statement follows. $\square$

10. Markov processes and the Feynman–Kac formula*

In this section we discuss the connection between the semigroups e^{-tL} arising from the Laplacian $L = L_{b,c,m}$ and a Markov process. Therefore, we connect the analytic perspective of Laplacians on graphs presented thus far with a probabilistic view. Although this viewpoint is both interesting and of conceptual importance, the results presented here are not used in most of the book.

The reader may want to solve Excavation Exercise 0.14 for the purposes of this section.

We start by giving an idea of how to think about the process. To this end, we will take the point of view that we already know that e^{-tL} is a semigroup giving the transition probabilities of a Markov process. We then sketch how the key quantities of the Markov process can be identified in terms of the graph.

These consideration will be made precise later when we introduce the corresponding process in detail. We will first recall some basic notions from probability. We then construct an explicit process and calculate some basic properties such as the expected jumping times and jumping probabilities.

The link between the semigroup and the constructed process is then established via the Feynman–Kac formula, which we prove at the end of the section.

10.1. A basic intuition. In this subsection we give an idea of how to think about the process associated to the semigroup. We will not go into too much technical detail since this will be taken care of in later parts of the section.

A continuous time Markov process on X consists of a memoryless particle moving in time between the vertices of X. This process is essentially characterized by two functions

$$p\colon X \times X \longrightarrow [0, \infty) \quad \text{and} \quad q\colon X \longrightarrow [0, \infty)$$

with the following interpretations:

- $p(x, y)$ is the probability that the particle jumps from x to y.
- $e^{-tq(x)}$ is the probability that a particle starting at x at time 0 is still at x at time t.

Given these quantities, we can define $P\colon [0, \infty) \times X \times X \longrightarrow [0, 1]$, $(t, x, y) \mapsto P_t(x, y)$, as

$$\begin{aligned} P_t(x, y) = {} & \text{the probability that the particle is at } y \text{ at time } t \\ & \text{if the particle starts at } x \text{ at time } 0. \end{aligned}$$

We can then compute the quantities p and q from the short time behavior of P_t as follows.

First, $P_t(x, x)$ is the probability that the particle that started at x is found at x at time t. This means that the particle has either stayed at x up to time t or has returned to x after leaving x. The probability of staying at x is given by $e^{-tq(x)}$. On the other hand, the probability of leaving is $1 - e^{-tq(x)}$ and the probability of returning to x is given by a probability $r(t)$ which tends to zero as $t \to 0$. Hence, we infer

$$P_t(x, x) = e^{-tq(x)} + \varphi_x(t)$$

where φ_x expresses the probability of returning to x and is given by $(1-e^{-tq(x)})r(t)$. Therefore, φ_x has derivative zero at $t = 0$. We conclude that

$$\partial_t P_t(x, x)|_{t=0} = -q(x) + \varphi_x'(0) = -q(x).$$

In a similar way, one argues that the probability $P_t(x, y)$ for small t and $x \neq y$ is governed by the probability $\pi_t(x, y)$ of the event that the particle starts at x and reaches y in one jump and then stays there up to time t. That is,

$$P_t(x, y) = \pi_t(x, y) + \psi_{x,y}(t),$$

where $\psi_{x,y}(t)$ has derivative 0 at $t = 0$. The probability $\pi_t(x, y)$ for this event can be bounded by

$$(1 - e^{-tq(x)})p(x, y)e^{-tq(y)} \leq \pi_t(x, y) \leq (1 - e^{-tq(x)})p(x, y),$$

where the term $e^{-tq(y)}$ in the lower bound accounts for the probability of not leaving y after reaching y. This leads to

$$\partial_t P_t(x, y)|_{t=0} = q(x)p(x, y) + \psi_{x,y}'(0) = q(x)p(x, y).$$

Now, we connect these findings with the structure of the underlying graph b over (X, m) via the semigroup. For the sake of simplicity, we assume that $c = 0$ first and discuss the case of arbitrary $c \geq 0$ later. We assume that the process given above is linked to a semigroup via the identity

$$e^{-tL}1_y(x) = P_t(x, y)$$

for $x, y \in X,\ t \geq 0$ where e^{-tL} is the semigroup of the operator $L = L_{b,0,m}$ on $\ell^2(X, m)$ for the form Q. In particular, for $f, g \in \ell^2(X, m)$,

$$\langle e^{-tL}f, g\rangle = \sum_{x\in X} e^{-tL}f(x)g(x)m(x) = \sum_{x\in X}\left(\sum_{y\in X} P_t(x, y)f(y)\right)g(x)m(x).$$

Thus, we may compute q and p using this identity as follows

$$\begin{aligned}\sum_{z\in X} b(x, z) &= Q(1_x, 1_x)\\ &= -\partial_t\langle e^{-tL}1_x, 1_x\rangle|_{t=0}\\ &= -\partial_t e^{-tL}1_x(x)m(x)|_{t=0}\\ &= -\partial_t P_t(x, x)m(x)|_{t=0}\\ &= q(x)m(x).\end{aligned}$$

Similarly, for $x \neq y$,

$$\begin{aligned} b(x,y) &= -Q(1_y, 1_x) \\ &= \partial_t \langle e^{-tL} 1_y, 1_x \rangle|_{t=0} \\ &= \partial_t e^{-tL} 1_y(x) m(x)|_{t=0} \\ &= \partial_t P_t(x,y) m(x)|_{t=0} \\ &= q(x) p(x,y) m(x). \end{aligned}$$

This gives

$$p(x,y) = \frac{b(x,y)}{\sum_{z \in X} b(x,z)} \quad \text{and} \quad q(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)$$

for $x, y \in X$. Note that the symmetry of b does not imply the symmetry of p but rather that

$$q(x)p(x,y)m(x) = q(y)p(y,x)m(y).$$

If c is non-vanishing, then the considerations above yield

$$\begin{aligned} p(x,y) &= \frac{b(x,y)}{\sum_{z \in X} b(x,z) + c(x)} \quad \text{and} \\ q(x) &= \frac{1}{m(x)} \left(\sum_{z \in X} b(x,z) + c(x) \right) \end{aligned}$$

for $x, y \in X$. This has two consequences. First, the process jumps faster, as can be seen by the increase in $q(x)$. Second, there is a probability of

$$k(x) = 1 - \frac{\sum_{z \in X} b(x,z)}{\sum_{z \in X} b(x,z) + c(x)}$$

that if the particle jumps away from x it does not jump to any vertex of X but rather leaves the system whenever $c(x) > 0$. The point to which the particle leaves in this case is often referred to as the graveyard or cemetery. In probability, one often says that the particle is *killed* at x due to the presence of $c(x) > 0$. For this reason c is often referred to as the killing term.

Note that the preceding discussion has shown that any Markov process on a discrete set naturally comes with a graph. The aim of the subsequent subsections is to show that, conversely, any graph gives a Markov process. Thus, putting these together, we will show a one-to-one correspondence between Markov processes on discrete spaces and graphs.

In Section 7 we introduced the heat kernel of the semigroup e^{-tL} for $t \geq 0$ as the function $p\colon [0,\infty) \times X \times X \longrightarrow [0,\infty)$ such that

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x,y) f(y) m(y)$$

for $x \in X, t \geq 0$. By the calculations above, one sees that we have the identity

$$p_t(x, y) = \frac{1}{m(y)} P_t(x, y)$$

for $x, y \in X, t \geq 0$.

10.2. Some probabilistic background. In this subsection we want to make the considerations above rigorous. To this end, we need some basic notions from probability, in particular, from the theory of stochastic processes. For the convenience of the reader we briefly recall these notions. For more details and background we refer to [**Nor98**].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a finite or countably infinite set. This also covers the case of stochastic processes on infinite sets because the definitions of this section will be used in later parts of the book on infinite graphs. We denote by $\mathbb{P}(A \mid B)$ the probability of the event A conditioned on event B. Moreover, for a random variable $Z\colon \Omega \longrightarrow X$, we denote by $\mathbb{E}(Z)$ the expected value of Z and by $\mathbb{E}(Z \mid A)$ the expected value of Z conditioned on A.

A family Y of random variables Y_n, $n \in \mathbb{N}_0$, taking values in X is called a *discrete time Markov chain* if for all $x_1, \ldots, x_k, y \in X$ we have

$$\mathbb{P}(Y_{k+1} = y \mid Y_0 = x_1, \ldots, Y_k = x_k) = \mathbb{P}(Y_{k+1} = y \mid Y_k = x_k).$$

The distribution of Y_0 is called the *initial distribution* of Y. If the initial distribution is supported on $x \in X$, then we say that Y *starts* at x.

To define a continuous time Markov chain or Markov process, we consider a right-continuous process $\mathbb{X} = (\mathbb{X}_t)_{t>0}$ on X with an initial distribution. We define the sequence of *jump times* $J\colon \mathbb{N}_0 \longrightarrow [0, \infty)$ by $J_0 = 0$ and

$$J_{n+1} = \inf\{t \geq J_n \mid \mathbb{X}_t \neq \mathbb{X}_{J_n}\}$$

for $n \in \mathbb{N}_0$, where $\inf \emptyset = \infty$. We define the sequence of *holding times* $S\colon \mathbb{N} \longrightarrow [0, \infty]$ by

$$S_n = \begin{cases} J_n - J_{n-1} & \text{if } J_n < \infty \\ \infty & \text{otherwise.} \end{cases}$$

The *lifetime* or *explosion time* ζ is defined by

$$\zeta = \sup_{n \in \mathbb{N}_0} J_n.$$

After explosion, the process can be thought to have left X. It is convenient to introduce an additional point x_∞ to X which is often called the *cemetery.* We then set $\mathbb{X}_t = x_\infty$ for $t \geq \zeta$. Such a process is called *minimal.* The terminology minimal refers to the fact that after leaving X the process does not return to X again. We call $Y = (Y_n)_{n \in \mathbb{N}_0}$ given by $Y_n = X_{J_n}$ for $n \in \mathbb{N}_0$ the *jump chain* associated to $\mathbb{X}$.

A *continuous time Markov chain* or *Markov process* is a minimal right-continuous process $\mathbb{X} = (\mathbb{X}_t)_{t>0}$ on X such that the jump chain Y of $\mathbb{X}$ is a discrete time Markov chain and, for each $n \in \mathbb{N}$, the holding times $S_1, \ldots, S_n$ conditioned on $Y_0, \ldots, Y_{n-1}$ are independent exponential random variables where the parameter of S_j is given by $q(Y_j)$ for $j = 1, \ldots n$ and for some function $q \colon X \longrightarrow [0, \infty)$.

Such a process can be constructed via a discrete time Markov chain Y over X and a sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ of independent exponentially distributed random variables of parameter 1 that are also independent of Y. Setting

$$S_n = \frac{1}{q(Y_{n-1})}\xi_n, \qquad J_n = S_1 + \ldots + S_n, \qquad \zeta = \sup_{n\in\mathbb{N}_0} J_n$$

we can define a Markov process $\mathbb{X} \colon [0, \infty) \times \Omega \longrightarrow X$ via

$$\mathbb{X}|_{[J_n,J_{n+1})\times\Omega} = Y_n \qquad \text{and} \qquad \mathbb{X}|_{[\zeta,\infty)\times\Omega} = x_\infty$$

for $n \in \mathbb{N}_0$.

A random variable $T \colon \Omega \longrightarrow [0, \infty]$ is called a *stopping time* for $\mathbb{X}$ if the event $\{T = t_0\}$ depends only on $(\mathbb{X}_t)_{t\leq t_0}$. It turns out, and here we only give a reference to the book of Norris [**Nor98**, Theorem 6.5.4] as the proof is highly probabilistic, that every Markov process on X is a *strong Markov process*, that is, for any stopping time T, the process $\mathbb{X}_T = (\mathbb{X}_t)_{t>0}$ conditioned on $T < \infty$ and $X_T = x$ is a Markov process.

Given a Markov process $\mathbb{X}$ over X and $x \in X$, we use the notation

$$\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid \mathbb{X}_0 = x) \quad \text{and} \quad \mathbb{E}_x(\cdot) = \mathbb{E}(\cdot \mid \mathbb{X}_0 = x).$$

10.3. Construction of the process associated to the semigroup. We next construct a process which we later show to be associated to the semigroup. We start with the case $c = 0$ and discuss the case of non-vanishing c at the end of this subsection. Exercise 0.14 concerning the number of jumps of a process will be used in this subsection.

Let b be a graph over a finite measure space (X, m). If $c = 0$, the two degree functions deg and Deg are given, for $x \in X$, by

$$\deg(x) = \sum_{y\in X} b(x, y) \quad \text{and} \quad \mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y\in X} b(x, y).$$

We next define the Markov process $\mathbb{X} = \mathbb{X}^b$ associated to Q_b via the semigroup e^{-tL} of the operator $L = L_{b,0,m}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be a discrete time Markov chain over X such that

$$\mathbb{P}\left(Y_n = y \,|Y_{n-1} = x\right) = \frac{b(x, y)}{\deg(x)}$$

for $n \in \mathbb{N}_0$. This corresponds to the quantity $p(x, y)$ which we calculated in the first subsection.

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables of parameter 1 which are also independent of Y. We define the sequence of holding times S_n, $n \in \mathbb{N}$, and jumping times J_n, $n \in \mathbb{N}_0$, via

$$S_n = \frac{1}{\operatorname{Deg}(Y_{n-1})}\xi_n \qquad \text{and} \qquad J_n = S_1 + \ldots + S_n$$

with the convention that $J_0 = 0$. That is, in the notation of the subsection above, the function q is given by Deg. The probability for $S_n = S_{n+1}$ vanishes for some $n \in \mathbb{N}_0$ so, for convenience, we restrict the process to when this does not happen.

Since X is assumed to be finite, it is easy to check that $\zeta = \sup_{n\in\mathbb{N}_0} J_n = \infty$ $\mathbb{P}$-almost surely. So, for convenience, we will consider the process only for $\zeta = \infty$.

We define $\mathbb{X} = \mathbb{X}^b \colon [0,\infty) \times \Omega \longrightarrow X$ via

$$\mathbb{X}_t = Y_n \quad \text{ if } t \in [J_n, J_{n+1}).$$

Let us make some basic observations which will help us to interpret the behavior of the process. First, we calculate that the expected holding time from a vertex x is given by $\operatorname{Deg}(x)$. Hence, the larger the sum $\sum_{y\in X} b(x,y)$ and the smaller $m(x)$, the faster the particle jumps when at x.

Lemma 0.69 (Expected holding time). *Let b be a graph over a finite measure space (X,m) and let $\mathbb{X} = \mathbb{X}^b$ be the associated process. Then, for all $n \in \mathbb{N}_0$,*

$$\mathbb{E}(S_{n+1} \mid \mathbb{X}_{J_n} = x) = \frac{1}{\operatorname{Deg}(x)}.$$

Proof. The random variables ξ_n in $S_n = \xi_n/\operatorname{Deg}(Y_{n-1})$ are exponentially distributed with parameter 1. So, we compute

$$\mathbb{E}(S_{n+1} \mid \mathbb{X}_{J_n} = x) = \mathbb{E}(S_{n+1} \mid Y_n = x) = \frac{1}{\operatorname{Deg}(x)} \int_0^\infty s e^{-s} ds = \frac{1}{\operatorname{Deg}(x)}.$$

This gives the statement. $\square$

Next, we compute the probability of making zero, one or more jumps from a vertex x at time t. To this end, we denote the random variable counting the number of jumps up to time t by $N(t)$, i.e.,

$$N(t) = \sup\{n \in \mathbb{N}_0 \mid J_n \le t\}.$$

With the help of the next lemma, we can make the considerations of the first subsection rigorous. In particular, as the first statement shows, the function q of the first subsection coincides with the function q chosen here.

LEMMA 0.70 (Probability of jumping). *Let b be a graph over a finite measure space (X, m) and let $\mathbb{X} = \mathbb{X}^b$ be the associated process. Then,*

$$\mathbb{P}_x\big(N(t) = 0\big) = e^{-\mathrm{Deg}(x)t}$$

and

$$\begin{aligned}\mathbb{P}_x\big(N(t) = 1\big) &= \sum_{y \in X} \mathbb{P}_x(N(t) = 1 \wedge \mathbb{X}_{J_1} = y) \\ &= \sum_{\substack{y \in X, \\ \mathrm{Deg}(x) \neq \mathrm{Deg}(y)}} \frac{b(x,y)}{m(x)} \frac{(e^{-\mathrm{Deg}(y)t} - e^{-\mathrm{Deg}(x)t})}{(\mathrm{Deg}(x) - \mathrm{Deg}(y))} + \sum_{\substack{y \in X, \\ \mathrm{Deg}(x) = \mathrm{Deg}(y)}} \frac{b(x,y)}{m(x)} t e^{-\mathrm{Deg}(y)t}.\end{aligned}$$

Additionally, equality holds for each term under the sum over $y \in X$. In particular,

$$\lim_{t \to 0} \frac{\mathbb{P}_x(N(t) \geq 2)}{t} = 0.$$

PROOF. We calculate in a straightforward manner

$$\mathbb{P}_x(N(t) = 0) = \mathbb{P}_x(S_1 \geq t) = \mathbb{P}(\xi_1 \geq t\mathrm{Deg}(x)) = \int_{t\mathrm{Deg}(x)}^{\infty} e^{-s} ds = e^{-\mathrm{Deg}(x)t}.$$

This gives the first statement. For the second statement, we calculate

$$\begin{aligned}\mathbb{P}_x(N(t) = 1) &= \sum_{y \in X} \mathbb{P}_x(N(t) = 1 \wedge \mathbb{X}_{J_1} = y) \\ &= \sum_{y \in X} \mathbb{P}_x(N(t) = 1 \mid \mathbb{X}_{J_1} = y) \mathbb{P}_x(\mathbb{X}_{J_1} = y).\end{aligned}$$

Now, by the definition of $\mathbb{X}$, we get

$$\mathbb{P}_x(\mathbb{X}_{J_1} = y) = \mathbb{P}(Y_1 = y \mid Y_0 = x) = \frac{b(x,y)}{\deg(x)}.$$

Using the independence of Y_n and ξ_n we proceed to compute

$$\begin{aligned}&\mathbb{P}_x(N(t) = 1 \mid \mathbb{X}_{J_1} = y) \\ &= \mathbb{P}(S_1 \leq t < S_1 + S_2 \mid Y_1 = y, Y_0 = x) \\ &= \mathbb{P}\Big(\frac{1}{\mathrm{Deg}(Y_0)}\xi_1 \leq t < \frac{1}{\mathrm{Deg}(Y_0)}\xi_1 + \frac{1}{\mathrm{Deg}(Y_1)}\xi_2 \mid Y_1 = y, Y_0 = x\Big) \\ &= \mathbb{P}\Big(\frac{1}{\mathrm{Deg}(x)}\xi_1 \leq t < \frac{1}{\mathrm{Deg}(x)}\xi_1 + \frac{1}{\mathrm{Deg}(y)}\xi_2\Big) \\ &= \int_0^t \mathrm{Deg}(x) e^{-\mathrm{Deg}(x)s} \int_{t-s}^{\infty} \mathrm{Deg}(y) e^{-\mathrm{Deg}(y)r} dr ds \\ &= \mathrm{Deg}(x) e^{-\mathrm{Deg}(y)t} \int_0^t e^{-(\mathrm{Deg}(x) - \mathrm{Deg}(y))s} ds,\end{aligned}$$

where the last two equalities stem from the fact that for an exponentially distributed random variable ξ with parameter 1, the density of ξ/a for $a > 0$ is exponentially distributed with parameter a.

Plugging these findings into the calculation above and distinguishing between the cases $\mathrm{Deg}(x) \neq \mathrm{Deg}(y)$ and $\mathrm{Deg}(x) = \mathrm{Deg}(y)$ we arrive at

$$\begin{aligned}
&\mathbb{P}_x(N(t) = 1) \\
&= \sum_{y \in X} \mathbb{P}_x(N(t) = 1 \mid \mathbb{X}_{J_1} = y)\mathbb{P}_x(\mathbb{X}_{J_1} = y) \\
&= \sum_{y \in X, \mathrm{Deg}(x) \neq \mathrm{Deg}(y)} \frac{\mathrm{Deg}(x)}{(\mathrm{Deg}(x) - \mathrm{Deg}(y))}(e^{-\mathrm{Deg}(y)t} - e^{-\mathrm{Deg}(x)t})\frac{b(x,y)}{\deg(x)} \\
&\quad + \sum_{y \in X, \mathrm{Deg}(x) = \mathrm{Deg}(y)} \mathrm{Deg}(x) t e^{-\mathrm{Deg}(y)t} \frac{b(x,y)}{\deg(x)}.
\end{aligned}$$

Given the definitions of Deg and deg, we obtain the statement for $\mathbb{P}_x(N(t) = 1)$. The statement about the limit of $\mathbb{P}_x(N(t) \geq 2)/t$ as $t \to 0$ follows easily via the formula $\mathbb{P}_x(N(t) \geq 2) = 1 - \mathbb{P}_x(N(t) \leq 1)$ and the first two statements. □

We next define the Markov process $\mathbb{X}^{b,c}$ associated to a graph (b, c) over (X, m) with a possibly non-vanishing c. In this case, the definitions of the two degree functions deg and Deg read as

$$\deg(x) = \sum_{y \in X} b(x,y) + c(x) \quad \text{and} \quad \mathrm{Deg}(x) = \frac{1}{m(x)}\left(\sum_{y \in X} b(x,y) + c(x)\right)$$

for $x \in X$. Let $Y = (Y_n)_{n \in \mathbb{N}_0}$ be a discrete time Markov chain over X such that for $n \in \mathbb{N}_0$

$$\mathbb{P}\left(Y_n = y \,|Y_{n-1} = x\right) = \begin{cases} b(x,y)/\deg(x) & \text{if } x, y \in X \\ c(x)/\deg(x) & \text{if } x \in X, y = x_\infty \\ 1 & \text{if } x = y = x_\infty \\ 0 & \text{else,} \end{cases}$$

where x_∞ is the cemetery. Let $(\xi_n)_{n \in \mathbb{N}}$ again be a sequence of independent exponentially distributed random variables of parameter 1 which are also independent of Y. The sequence of holding times S_n, $n \in \mathbb{N}$, and jumping times J_n, $n \in \mathbb{N}_0$, are given via

$$S_n = \frac{1}{\mathrm{Deg}(Y_{n-1})}\xi_n \qquad \text{and} \qquad J_n = S_1 + \ldots + S_n,$$

with the convention that $J_0 = 0$. In the case that $c \neq 0$ it can be checked rather easily that $\zeta < \infty$ $\mathbb{P}$-almost surely since the process will reach x_∞ in a finite time almost surely.

As before, we define $\mathbb{X} = \mathbb{X}^b \colon [0, \infty) \times \Omega \longrightarrow X \cup \{x_\infty\}$ via

$$\mathbb{X}_t = Y_n \quad \text{if } t \in [J_n, J_{n+1}).$$

10.4. The Feynman–Kac semigroup. In this subsection we introduce the Feynman–Kac semigroup. We will later show that this semigroup is equal to the semigroup of the Laplacian. To do so, we compute the generator of the Feynman–Kac semigroup and discover that the generator is the Laplacian.

We will introduce the Feynman–Kac semigroups for subgraphs and show that their generator is the corresponding Laplacian with Dirichlet boundary condition. This is slightly more general than what we need for the Feynman–Kac formula on finite graphs. However, this consideration for subgraphs is not very complicated and will be used later to extend the result to infinite graphs.

Let (b,c) be a graph over (X,m) and let $K\subseteq X$ be a subset. We denote the restriction of m to K by m_K. The graph (b_K,c_K) over (K,m_K) is given by $b_K=b|_{K\times K}$ and $c_K\colon K\longrightarrow[0,\infty)$ by

$$c_K(x)=c(x)+\sum_{y\in X\setminus K}b(x,y).$$

The corresponding Dirichlet Laplacian of (b,c) over (K,m_K) with respect to K is defined as an operator $L_K^{(D)}\colon \ell^2(K,m_K)\longrightarrow\ell^2(X,m)$

$$L_K^{(D)}=L_{b_K,c_K,m_K}.$$

Let $\mathbb{X}=\mathbb{X}^b$ be the process associated to a graph b over (X,m) and let $K\subseteq X$ be a subset. Let $\tau_K\colon\Omega\longrightarrow[0,\infty)$ be the first *exit time*, i.e., τ_K is the time where the process first leaves the set K or

$$\tau_K=\inf\{t\geq 0\mid \mathbb{X}_t\in X\setminus K\}.$$

Clearly, τ_K is a stopping time and $\tau_X=J_\infty=\infty$ almost surely.

Next, we define the operators which turn out to be a semigroup. We say that a family of operators $S_t\colon\ell^2(X,m)\longrightarrow\ell^2(X,m)$ for $t\geq 0$ defines a semigroup if for all $t,t'\geq 0$ we have $S_{t+t'}=S_tS_{t'}$. This implies, in particular, that $S_0=I$.

Lemma 0.71 (Feynman–Kac semigroup). *Let (b,c) be a graph over a finite measure space (X,m) and let $\mathbb{X}=\mathbb{X}^b$ be the associated process. For $K\subseteq X$, let $L_K^{(D)}$ be the Dirichlet Laplacian of (b,c) over (K,m_K) and let τ_K be the first exit time. If $T_t\colon\ell^2(K,m_K)\longrightarrow\ell^2(K,m_K)$ for $t\geq 0$ is defined by*

$$T_tf(x)=\mathbb{E}_x\Big(1_{\{t<\tau_K\}}e^{-\int_0^t(c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\Big),$$

then T_t is a semigroup such that

$$\lim_{t\to 0^+}\frac{f(x)-T_tf(x)}{t}=L_K^{(D)}f(x)$$

for $f\in\ell^2(K,m_K)$ and $x\in K$.

REMARK. Note that for $v\colon X \longrightarrow [0,\infty)$ the integral

$$\int_0^t v(\mathbb{X}_s)ds = \sum_{n\le N(t)} (J_n - J_{n-1})v(\mathbb{X}_{J_n}) + (t - J_{N(t)})v(\mathbb{X}_{J_{N(t)}})$$

defines a random variable which is almost surely finite since $\zeta < \infty$ $\mathbb{P}$-almost surely.

PROOF. We first use the Markov property of $\mathbb{X}$ to show that T_t is a semigroup with $T_0 = I$. Denote by $\mathbb{X}'$ a copy of the process $\mathbb{X}$. Furthermore, we define the stopping time τ_K' to be the first time that $\mathbb{X}'$ leaves K. We then compute, using the strong Markov property of $\mathbb{X}$, that

$$\begin{aligned}
T_tT_{t'}f(x) &= \mathbb{E}_x\Big(1_{\{t<\tau_K\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}T_{t'}f(\mathbb{X}_t)\Big)\\
&= \mathbb{E}_x\Big(1_{\{t<\tau_K\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\mathbb{E}_{\mathbb{X}_t}\Big(1_{\{t'<\tau_K'\}}e^{-\int_0^{t'} (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}'_{t'})\Big)\Big)\\
&= \mathbb{E}_x\Big(1_{\{t+t'<\tau_K\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}e^{-\int_t^{t+t'} (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_{t+t'})\Big)\\
&= \mathbb{E}_x\Big(1_{\{t+t'<\tau_K\}}e^{-\int_0^{t+t'} (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_{t+t'})\Big)\\
&= T_{t+t'}f(x).
\end{aligned}$$

Therefore, $T_tT_{t'} = T_{t+t'}$. Furthermore, the equality $T_0 = I$ is clear.

We next compute the derivative of $T_tf(x)$ at time $t = 0$. Since X is finite, we can assume without loss of generality that $f \ge 0$. To take the derivative of $T_tf(x)$ at $t = 0$, we divide the expected value into three parts according to the number of jumps up to time t. We compute

$$\begin{aligned}
\frac{T_tf(x) - f(x)}{t} =& \frac{1}{t}\mathbb{E}_x\left(1_{\{t<\tau_K, N(t)=0\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)\\
&+ \frac{1}{t}\mathbb{E}_x\left(1_{\{t<\tau_K, N(t)=1\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right) + \psi_t(x)
\end{aligned}$$

with

$$\begin{aligned}
\psi_t(x) &= \frac{1}{t}\mathbb{E}_x\left(1_{\{t<\tau_K, N(t)\ge 2\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)\\
&\le C\frac{1}{t}\mathbb{P}(t<\tau_K, N(t)\ge 2)\\
&\to 0
\end{aligned}$$

as $t \to 0^+$, where $C = \sup_{x\in K} f(x)$ and we used Lemma 0.70 in taking the limit.

Next, we turn to the first term on the right-hand side of the equality above and use Lemma 0.70 again in the third step to get

$$\begin{aligned}
&\frac{1}{t}\left(\mathbb{E}_x\left(1_{\{t<\tau_K, N(t)=0\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)-f(x)\right)\\
&=\frac{1}{t}\Big(\mathbb{E}_x\left(1_{\{N(t)=0\}}e^{-t(c/m)(x)}f(x)\right)-f(x)\Big)\\
&=\frac{1}{t}\big(e^{-t(c/m)(x)}f(x)\mathbb{P}_x(N(t)=0)-f(x)\big)\\
&=\frac{1}{t}(e^{-t(\sum_{y\in X}b(x,y)+c(x))/m(x)}f(x)-f(x))\\
&\to -\frac{1}{m(x)}\Big(\sum_{y\in X}b(x,y)+c(x)\Big)f(x)
\end{aligned}$$

as $t\to 0^+$.

Finally, we turn to the second term on the right-hand side of the equality above and calculate

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}_x\left(1_{\{t<\tau_K, N(t)=1\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)\\
&\qquad=\frac{1}{t}\sum_{y\in K}\mathbb{E}_x\left(1_{\{N(t)=1,\mathbb{X}_{J_1}=y\}}e^{-J_1(c/m)(x)-(t-J_1)(c/m)(y)}\right)f(y).
\end{aligned}$$

To estimate the exponential terms on the right hand side we introduce $q_0=\min_{y\in K}(c/m)(y)$ and $q_1=\max_{y\in K}(c/m)(y)$. Then, for every $y\in K$, we obtain the two-sided estimate on the summands of the right hand side above

$$\begin{aligned}
\mathbb{P}_x\left(N(t)=1,\mathbb{X}_{J_1}=y\right)e^{-tq_1}&\le \mathbb{E}_x\left(1_{\{N(t)=1,\mathbb{X}_{J_1}=y\}}e^{-J_1(c/m)(x)-(t-J_1)(c/m)(y)}\right)\\
&\le \mathbb{P}_x\left(N(t)=1,\mathbb{X}_{J_1}=y\right)e^{-tq_0}.
\end{aligned}$$

Summing over $y\in K$ and dividing by t, we can use Lemma 0.70 to estimate

$$\begin{aligned}
&\frac{1}{t}\sum_{y\in K}\mathbb{P}_x\left(N(t)=1,\mathbb{X}_{J_1}=y\right)f(y)\cdot e^{-tq_j}\\
&\quad=\frac{1}{t}\cdot\frac{1}{m(x)}\Bigg(\sum_{\substack{y\in K,\\ \mathrm{Deg}(x)\neq\mathrm{Deg}(y)}}b(x,y)\frac{(e^{-\mathrm{Deg}(y)t}-e^{-\mathrm{Deg}(x)t})}{(\mathrm{Deg}(x)-\mathrm{Deg}(y))}f(y)\\
&\qquad\qquad+\sum_{\substack{y\in K,\\ \mathrm{Deg}(x)=\mathrm{Deg}(y)}}b(x,y)te^{-\mathrm{Deg}(y)t}f(y)\Bigg)\cdot e^{-tq_j}\\
&\quad\to\frac{1}{m(x)}\sum_{y\in K}b(x,y)f(y)
\end{aligned}$$

as $t\to 0^+$ for $j=0,1$. As these sums are lower and upper bounds for the term we are interested in, we get by recalling the equation on the expected value above

$$\frac{1}{t}\mathbb{E}_x\Big(1_{\{t<\tau_K, N(t)=1\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\Big) \to \frac{1}{m(x)}\sum_{y\in K} b(x,y)f(y),$$

as $t \to 0^+$.

Putting all of these calculations together yields

$$\begin{aligned}\lim_{t\to 0^+}\frac{f(x)-T_tf(x)}{t} &= \frac{1}{m(x)}\left(\left(\sum_{y\in X} b(x,y)+c(x)\right)f(x)-\sum_{y\in K} b(x,y)f(y)\right)\\ &= L_K^{(D)}f(x).\end{aligned}$$

This finishes the proof. □

10.5. The Feynman–Kac formula. We now prove the main result of this section. It links the semigroup of the Laplacian of a graph with the corresponding process. We start with a general version from which we deduce two corollaries that both offer different perspectives.

Excavation Exercise 0.7 will be relevant for this subsection.

The following general version, which is formulated for subgraphs, will also serve to prove a corresponding result for infinite graphs.

THEOREM 0.72 (Feynman–Kac formula for subgraphs). *Let (b, c) be a graph over a finite measure space (X, m) and let $\mathbb{X} = \mathbb{X}^b$ be the process associated to b. For a subset $K \subseteq X$, let $L_K^{(D)} = L_{b_K, c_K, m_K}$ be the Dirichlet Laplacian and τ_K be the first exit time. Then,*

$$e^{-tL_K^{(D)}}f(x) = \mathbb{E}_x\left(1_{\{t<\tau_K\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)$$

for all $f \in \ell^2(K, m_K)$, $x \in K$ and $t \geq 0$.

PROOF. By Lemma 0.71, the semigroup

$$T_tf(x) = \mathbb{E}_x\left(1_{\{t<\tau_K\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}f(\mathbb{X}_t)\right)$$

satisfies

$$\lim_{t\to 0}\frac{T_tf(x)-f(x)}{t} = -L_K^{(D)}f(x).$$

Thus, $L_K^{(D)}$ generates both semigroups T_t and $e^{-tL_K^{(D)}}$. Since for any given linear operator A on the finite-dimensional Hilbert space $\ell^2(X, m)$ the ordinary differential equation

$$\partial_t\varphi(t) = A\varphi(t), \quad \varphi(0) = u$$

has a unique solution $\varphi\colon [0,\infty) \longrightarrow \ell^2(K, m)$ for any $u \in \ell^2(K, m)$, the claim follows. □

Next, we come to two corollaries. The first corollary shows how, on the level of processes, the killing term can be decoupled from the process. That is, we link the semigroup for a graph (b, c) with the process associated to b.

COROLLARY 0.73 (Feynman–Kac formula for finite graphs for $\mathbb{X}^b$). *Let (b, c) be a graph over a finite measure space (X, m) with associated Laplacian $L = L_{b,c,m}$. Let $\mathbb{X} = \mathbb{X}^b$ be the process associated to b. Then,*

$$e^{-tL}f(x) = \mathbb{E}_x\left(e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\right)$$

for all $f \in \ell^2(X, m)$, $x \in X$ and $t \geq 0$.

PROOF. The statement follows immediately with the choice $K = X$ since $\tau_X = \zeta = \infty$ almost surely. $\square$

REMARK. A direct consequence of the Feynman–Kac formula is the inequality

$$e^{-tL_{b,c,m}}f \leq e^{-tL_{b,0,m}}f$$

for all positive $f \in \ell^2(X, m)$. The inequality is strict if $c \neq 0$ and $t > 0$. In particular, if $c \neq 0$, then

$$e^{-tL_{b,c,m}}1 < 1$$

for $t > 0$, where 1 denotes the function which is constantly one on X. This gives a probabilistic proof that $e^{-tL_{b,c,m}}1 < 1$ for all $t > 0$ if and only if $c \neq 0$, which was already shown via analysis as Theorem 0.65 in Section 8. We shall further explore the question of how such a strict inequality can occur in the case of infinite graphs even when $c = 0$ in Chapter 7.

The final result of this section connects the semigroup of the Laplacian directly with its associated process.

COROLLARY 0.74 (Feynman–Kac formula for finite graphs for $\mathbb{X}^{b,c}$). *Let (b, c) be a graph over a finite measure space (X, m) with associated Laplacian $L = L_{b,c,m}$. Let $\mathbb{X} = \mathbb{X}^{b,c}$ be the process associated to (b, c). Then,*

$$e^{-tL}f(x) = \mathbb{E}_x\Big(1_{\{\zeta > t\}} f(\mathbb{X}_t)\Big)$$

for all $f \in \ell^2(X, m)$, $x \in X$ and $t \geq 0$.

PROOF. We embed the graph (b, c) over (X, m) into a supergraph $(b', 0)$ over (X', m') via

$$X' = X \dot\cup \{x_\infty\}$$

$$b'|_{X \times X} = b, \quad b'(x_\infty, x) = b(x, x_\infty) = c(x)$$

for $x \in X$ with $m'|_X = m$ and $m(x_\infty)$ arbitrary, where x_∞ is the cemetery introduced in Subsection 10.2. We apply Theorem 0.72 with the choice $K = X \subseteq X'$ and observe that the restriction of the Laplacian $L' = L_{b',0,m'}$ to X with Dirichlet boundary conditions is exactly $L = L_{b,c,m}$. Furthermore, for the first exit time τ'_X of the process $\mathbb{X}'$ associated to b' and the explosion time ζ of the process $\mathbb{X}$ associated to (b,c), we have

$$\{\zeta > t\} = \{\tau'_X > t\}.$$

Thus, we have by Theorem 0.72 with the choice $K = X \subseteq X'$,

$$e^{-tL}f(x) = \mathbb{E}_x\Big(1_{\{\tau_X > t\}} f(\mathbb{X}'_t)\Big) = \mathbb{E}_x\Big(1_{\{\zeta > t\}} f(\mathbb{X}_t)\Big),$$

where the last equality follows as the processes $\mathbb{X}_t$ and $\mathbb{X}'_t$ agree before they leave X. This finishes the proof. $\square$

REMARK. The Feynman–Kac formula can also be presented via supergraphs (Exercise 0.38). Furthermore, the formula can be used to characterize the lack of killing (Exercise 0.39).

We finish the section with yet another characterization of graphs which is an immediate consequence of what we have proven above.

THEOREM 0.75 (Characterization of Markov semigroups and Markov processes). *Let (X,m) be a finite measure space and let L be a self-adjoint operator on $\ell^2(X,m)$. Then, the following statements are equivalent:*

(i) *e^{-tL} is a Markov semigroup for $t \geq 0$.*

(ii) *There exists a Markov process $\mathbb{X} = \mathbb{X}^{b,c}$ associated to a graph (b,c) over (X,m) such that*

$$e^{-tL}f(x) = \mathbb{E}_x\Big(1_{\{\zeta > t\}} f(\mathbb{X}_t)\Big)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$.

PROOF. (i) $\Longrightarrow$ (ii): If e^{-tL} satisfies the Markov property, then $L = L_{b,c,m}$ for a graph (b,c) over (X,m) by Theorem 0.49. Hence, the statement follows directly from the corollary above.

(ii) $\Longrightarrow$ (i): For $0 \leq f \leq 1$, we obviously have

$$0 \leq \mathbb{E}_x\Big(1_{\{\zeta > t\}} f(\mathbb{X}_t)\Big) \leq 1.$$

Therefore, e^{-tL} is a Markov semigroup for $t \geq 0$. $\square$

Exercises

Excavation exercises.

EXERCISE 0.1 (Discrete topology and continuity). Consider a finite set X with the discrete topology which comes from the discrete metric $d_{\mathrm{disc}}(x,y) = 1$ if $x \neq y$ and $d_{\mathrm{disc}}(x,y) = 0$ if $x = y$.

(a) Show that every function $f\colon X \longrightarrow \mathbb{R}$ is continuous.
(b) Show that the space $C(X)$ of real-valued functions on X is a real vector space with respect to pointwise addition and scalar multiplication.
(c) Give an example of a basis for $C(X)$.

EXERCISE 0.2 (Quadratic form). Let (b,c) be a graph over a finite set X. Show that $Q_{b,c}\colon C(X) \longrightarrow [0,\infty)$, $f \mapsto Q_{b,c}(f) = Q_{b,c}(f,f)$ is a quadratic form, i.e., for $s \in \mathbb{R}$ and $f, g \in C(X)$, $Q_{b,c}$ satisfies

$$Q_{b,c}(sf) = s^2 Q_{b,c}(f)$$

and

$$Q_{b,c}(f+g) + Q_{b,c}(f-g) = 2(Q_{b,c}(f) + Q_{b,c}(g)).$$

EXERCISE 0.3 (Hilbert space). Let (X,m) be a finite measure space.

(a) Show that
$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x)m(x)$$
defines a scalar product on $C(X)$ and
$$\|f\| = \langle f, f\rangle^{1/2}$$
defines a norm on $C(X)$.
(b) Let $\ell^2(X,m)$ be $C(X)$ equipped with $\langle\cdot,\cdot\rangle$. Show that $\ell^2(X,m)$ is a Hilbert space, that is, $\ell^2(X,m)$ is complete with respect to the norm $\|\cdot\|$.
(c) Show that $\{e_x \mid x \in X\}$ with $e_x = 1_x/m^{1/2}(x)$ for $x \in X$, where 1_x is the characteristic function of $\{x\}$, is an orthonormal basis of $\ell^2(X,m)$.
(d) Show that $\ell^2(X,m)$ is unitarily equivalent to $\mathbb{R}^{|X|}$ where $|X|$ denotes the cardinality of X.

EXERCISE 0.4 (Laplacian is self-adjoint). Let (b,c) be a graph over a finite measure space (X,m). Show that the Laplacian $L_{b,c,m}$ is a self-adjoint operator on $\ell^2(X,m)$.

EXERCISE 0.5 (Characterization of bijectivity). Let X be a finite set and let $A\colon C(X) \longrightarrow C(X)$ be an operator. Show that the following statements are equivalent:

(i) A is bijective.

(ii) A is injective.
(iii) A is surjective.

EXERCISE 0.6 (Operator norm bound for the Laplacian). Let (X, m) be a finite measure space.

(a) Let A be an operator on $\ell^2(X, m)$. Show that
$$\|A\| = \sup\{\|Af\| \mid f \in \ell^2(X, m), \|f\| = 1\}$$
defines a norm on the vector space of operators on $\ell^2(X, m)$. We call $\|\cdot\|$ the operator norm.

(b) Show that if A and B are operators on $\ell^2(X, m)$, then
$$\|AB\| \leq \|A\|\|B\|.$$

(c) Let (b, c) be a graph over (X, m) and let $L_{b,c,m}$ be the associated Laplacian. Prove that
$$\|L_{b,c,m}\| \leq 2 \sup_{x \in X} \operatorname{Deg}(x)$$
where $\operatorname{Deg}(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right)$ is the weighted vertex degree.

EXERCISE 0.7 (The semigroup solves the heat equation). Let (X, m) be a finite measure space and let A be a self-adjoint operator on $\ell^2(X, m)$.

(a) Show that the sum
$$e^{-tA} = \sum_{n=0}^{\infty} \frac{1}{n!}(-tA)^n$$
converges absolutely for all $t \geq 0$ with respect to the operator norm $\|\cdot\|$ defined in Exercise 0.6.

(b) Show that the function $\varphi_t = e^{-tA} f$ for $f \in \ell^2(X, m)$ is the unique solution of the differential equation
$$\partial_t \varphi_t = -A\varphi_t$$
for $t \geq 0$ with $\varphi_0 = f$.

EXERCISE 0.8 (Commuting operators and the exponential function). Let (X, m) be a finite measure space. Let A and B be operators on $\ell^2(X, m)$ such that $AB = BA$. Show that
$$e^{A+B} = e^A e^B.$$

Exercise 0.9 (Existence of eigenvalues for self-adjoint operators). Let (X, m) be a finite measure space. Let A be a self-adjoint operator on $\ell^2(X, m)$. Let $\|A\|$ be the operator norm of A as defined in Exercise 0.6 above. Show that either $\|A\|$ or $-\|A\|$ is an eigenvalue for A.

(Hint: Suppose that $\|A\| \neq 0$. Let $f \in \ell^2(X, m)$ be such that $\|f\| = 1$ and $\|Af\| = \|A\|$ (why does this exist?). Let $g = Af/\|Af\|$ so that $Af = \|A\|g$ and use this to show that $Ag = \|A\|f$. Then either $f - g \neq 0$ or $f + g \neq 0$, which can be used to give an eigenvector for $\pm\|A\|$.)

Exercise 0.10 (Reducing subspaces). Let (X, m) be a finite measure space. Let A be a self-adjoint operator on $\ell^2(X, m)$. Suppose that M is a subspace of $\ell^2(X, m)$ such that $AM \subseteq M$. Let $M^\perp$ denote the orthogonal complement of M, that is, $M^\perp = \{f \in \ell^2(X, m) \mid \langle f, g\rangle = 0 \text{ for all } g \in M\}$. Show that $M^\perp$ is a subspace of $\ell^2(X, m)$ and $AM^\perp \subseteq M^\perp$.

Such a subspace is called a *reducing subspace* for A.

Exercise 0.11 (Spectral theorem). Let (X, m) be a finite measure space. Let A be a self-adjoint operator on $\ell^2(X, m)$ and let $\sigma(A)$ denote the set of eigenvalues of A. For every $\lambda \in \sigma(A)$, let E_λ denote the orthogonal projection onto the eigenspace of λ. That is, if $\{f_1^\lambda, f_2^\lambda, \ldots, f_n^\lambda\}$ is an orthonormal basis for the eigenspace of λ, then $E_\lambda(f) = \sum_{i=1}^n \langle f, f_i^\lambda\rangle f_i^\lambda$. Show that:

(a) $E_\lambda E_\mu = 0$ if $\lambda \neq \mu$.
(b) $I = \sum_{\lambda\in\sigma(A)} E_\lambda$.
(c) $A = \sum_{\lambda\in\sigma(A)} \lambda E_\lambda$.

(Hint: Use Exercises 0.9 and 0.10 above, note that eigenspaces are reducing subspaces for A and use induction.)

Exercise 0.12 (Variational characterization of bottom of the spectrum). Let (X, m) be a finite measure space. Let Q be a symmetric quadratic form with associated self-adjoint operator L on $\ell^2(X, m)$. Let λ_0 be the smallest eigenvalue of L. Show that

$$\lambda_0 = \min_{f\in\ell^2(X,m), \|f\|=1} Q(f).$$

Exercise 0.13 (Direct sums of Hilbert spaces and operators). Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ denote Hilbert spaces, that is, complete inner product spaces.

(a) Show that $H_1 \oplus H_2$, which is defined as $H_1 \times H_2$ with inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle(x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle_1 + \langle y_1, y_2\rangle_2,$$

is a Hilbert space.

(b) Let A_1 be an operator on H_1 and A_2 be an operator on H_2. Show that $A_1 \oplus A_2$, which is defined by

$$(A_1 \oplus A_2)(x, y) = (A_1 x, A_2 y),$$

is an operator on $H_1 \oplus H_2$.

(c) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the operator norm on the space of operators on H_1 and H_2, respectively, as defined in Exercise 0.6 above. If $\|\cdot\|$ denotes the operator norm for operators on $H_1 \oplus H_2$, show that

$$\|A_1 \oplus A_2\| = \max\{\|A_1\|_1, \|A_1\|_2\}.$$

EXERCISE 0.14 (Finitely many jumps in finite time almost surely). Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real positive random variables that take values in a finite set and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables of parameter 1 which are independent of θ_n. Show that the random variable $\zeta = \sup_{n\in\mathbb{N}} (\theta_1\xi_1 + \ldots + \theta_n\xi_n)$ satisfies $\zeta = \infty$ almost surely.

Example exercises.

EXERCISE 0.15 (Normal contractions). Show that the following functions $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ are normal contractions.

(a) $C(s) = |s|$.
(b) $C(s) = (\pm s) \vee 0$.
(c) $C(s) = s \wedge 1$.
(d) $C(s) = 0 \vee (s \wedge 1)$.

EXERCISE 0.16 (Positivity preserving but non-contracting). Give an example of a self-adjoint operator on $\ell^2(X, m)$ whose semigroup is positivity preserving but not contracting.

EXERCISE 0.17 (Non-positivity preserving and non-contracting). Give an example of a self-adjoint operator on $\ell^2(X, m)$ whose semigroup is neither positivity preserving nor contracting.

EXERCISE 0.18 (Smallest eigenvalue 0 but no graph). Give an example of a self-adjoint operator on $\ell^2(X, m)$ whose smallest eigenvalue is $\lambda_0 = 0$ with constant eigenfunction $\varphi_0 = 1$ but which is not associated to a graph.

Next, we present various examples which illustrate our theory. As usual, it takes work to compute anything concrete. The examples below are presented in order of increasing difficulty.

EXERCISE 0.19 (Complete graphs). Let $X = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. The *complete graph* with N vertices is given by $b_K(x, y) = 1$ for all $x, y \in X$ with $x \neq y$ and $c_K = 0$. Take your favorite number N with $N \geq 5$.

(a) Draw the graph.
(b) Write down the matrix l_{b_K,c_K} of the Laplacian L_{b_K,c_K}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_K,c_K,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = 1$ and $g = 1_N - 1_{N-1}$.
(f) Solve the Dirichlet problem for $B = \{1, \ldots, \lceil N/2 \rceil\}$ and $g = 1_B$.
(g) Solve the capacitor problem for
$$F = \{1, \ldots, \lfloor N/4 \rfloor\} \text{ and } G = \{\lfloor N/4 \rfloor, \ldots, \lceil N/2 \rceil\}.$$
(h) Solve the heat equation for the initial distributions $f = 1_{\{N\}}$.

EXERCISE 0.20 (Star graphs). Let $X = \{0, 1, \ldots, N\}$ for some $N \in \mathbb{N}$. The *star graph* with $N+1$ vertices is given by $b_S(0, x) = b_S(x, 0) = 1$ for all $x \in X$, $x \neq 0$ and $b_S(x, y) = 0$ for all $x, y \neq 0$ and $c_S = 0$. Take your favorite number N with $N \geq 5$.

(a) Draw the graph.
(b) Write down the matrix l_{b_S,c_S} of the Laplacian L_{b_S,c_S}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_S,c_S,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = N$ and $g = 1_{\{1,\ldots,N-1\}}$.
(f) Solve the Dirichlet problem for $B = \{0, 1, \ldots, \lceil N/2 \rceil\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{1, \ldots, \lfloor N/2 \rfloor\}$ and $G = \{N\}$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{N\}}$.

EXERCISE 0.21 (Line graphs). Let $X = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. The *line graph* with N vertices is given by $b_L(x, y) = 1$ for all $x, y \in X$ with $|x - y| = 1$ and $b_L(x, y) = 0$ otherwise and $c_L = 0$. Take your favorite number N with $N \geq 5$.

(a) Draw the graph.
(b) Write down the matrix l_{b_L,c_L} of the Laplacian L_{b_L,c_L}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_L,c_L,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = N$ and $g = 1_{\{N\}}$.
(f) Solve the Dirichlet problem for $B = \{1, N\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{1\}$ and $G = \{N\}$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{N\}}$.

EXERCISE 0.22 (Cycle graphs). Let $X = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$. The *cycle graph* with N vertices is given by $b_C(x, y) = 1$ for all $x, y \in X$ with $|x-y| = 1$ or $\{x, y\} = \{1, N\}$ and $b_C(x, y) = 0$ otherwise and $c_C = 0$. Take your favorite number N with $N \geq 5$.

(a) Draw the graph.
(b) Write down the matrix l_{b_C,c_C} of the Laplacian L_{b_C,c_C}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_C,c_C,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = N$ and $g = 1_{\{N\}}$.
(f) Solve the Dirichlet problem for $B = \{1, \lfloor N/2 \rfloor\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{1\}$, $G = \{\lfloor N/2 \rfloor\}$ and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{N\}}$.

EXERCISE 0.23 (Wheel graphs). Let $X = \{0, 1, \ldots, N\}$ for some $N \in \mathbb{N}$. The *wheel graph* with $N+1$ vertices is given by $b_W(x, y) = 1$ for all $x, y \in X$ with $|x-y| = 1$ or $\{x, y\} = \{1, N\}$, $b_W(0, x) = b_W(x, 0) = 1$ for all $x \neq 0$ and $b_W(x, y) = 0$ otherwise and $c_W = 0$. Take your favorite number N with $N \geq 4$.

(a) Draw the graph.
(b) Write down the matrix l_{b_W,c_W} of the Laplacian L_{b_W,c_W}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_W,c_W,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = N$ and $g = 1_{\{1,\ldots,N-1\}}$.
(f) Solve the Dirichlet problem for $B = \{0, 1, \ldots, \lfloor N/2 \rfloor\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{1, \ldots, \lfloor N/2 \rfloor\}$, $G = \{N\}$ and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{N\}}$.

EXERCISE 0.24 (Hypercube graphs). Let $X = \{0, 1\}^N$ for some $N \in \mathbb{N}$. The *N-dimensional hypercube* graph is given by $b_H(x, y) = 1$ for all $x, y \in X$ with $|x-y| = 1$ and $b_H(x, y) = 0$ otherwise and $c_H = 0$. Take your favorite number N with $N \geq 3$.

(a) Draw the graph.
(b) Write down the matrix l_{b_H,c_H} of the Laplacian L_{b_H,c_H}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_H,c_H,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem with fixed gauge for $p = (1, \ldots, 1)$ and $g = 1_{\{(0,\ldots,0)\}}$.
(f) Solve the Dirichlet problem for $B = \{x \in X \mid |x| \leq N/2\}$ and $g = 1_B$.

(g) Solve the capacitor problem for $F = \{(0, \dots, 0)\}$, $G = \{(1, \dots, 1)\}$ and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{(0,\dots,0)\}}$.

EXERCISE 0.25 (Tree graphs). Take your favorite numbers k, N with $N, k \geq 2$. Let b_T over X be the graph given by the *first N spheres of the rooted k-regular tree with root o* with edge weights equal to 1 and let $c_T = 0$. (A tree is a graph without cycles. A tree is k-regular rooted if every vertex except for the root o has $k+1$ neighbors while o has k neighbors. The n-th sphere is the subset of vertices whose combinatorial graph distance is less than n, where the combinatorial graph distance between two vertices is the smallest number n such that the vertices can be connected by a path of $n+1$ vertices.)

(a) Draw the graph and write down b_T.
(b) Write down the matrix l_{b_T,c_T} of the Laplacian L_{b_T,c_T}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_T,c_T,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem for $g = 1_{\{B\}}$ where B is the N-th sphere and with $p = o$ where o is the root of the tree.
(f) Solve the Dirichlet problem where B is the N-th sphere of the tree and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{o\}$ with the root o, G being the N-th sphere and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{o\}}$ for the root o.

EXERCISE 0.26 (Complete bipartite graphs). Let $X = \{1, \dots, N, N+1, \dots, N+M\}$ for some $N, M \in \mathbb{N}$. The *complete bipartite graph* with $N+M$ vertices is given by $b_B(x, y) = b_B(y, x) = 1$ for all $x, y \in X$ with $x \leq N$, $y > N$ and $b_B(x, y) = 0$ otherwise and $c_B = 0$. Take your favorite numbers N, M with $N \geq 2$, $M \geq 3$.

(a) Draw the graph.
(b) Write down the matrix l_{b_B,c_B} of the Laplacian L_{b_B,c_B}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_B,c_B,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem for $g = 1_{\{1,\dots,N\}}$.
(f) Solve the Dirichlet problem for $B = \{1, \dots, \lfloor N/2 \rfloor\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{1, \dots, \lfloor N/2 \rfloor\}$, $G = \{M\}$ and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{M\}}$.

EXERCISE 0.27 (Petersen graph*). Let

$$X = \{(j,k) \mid j = 0,1, k = 1,\ldots,5\}.$$

The *Petersen graph* is given by b_P which defines a cycle graph on $\{(0,1),\ldots,(0,5)\}$, $b_P((1,k),(1,l)) = 1$ if $k-l \bmod 5 = 2$, $b_P((0,k),(1,k)) = b_P((1,k),(0,k)) = 1$ for $k = 1,\ldots,5$ and $c_T = 0$.

(a) Draw the graph.
(b) Write down the matrix l_{b_P,c_P} of the Laplacian L_{b_P,c_P}.
(c) Let $m = 1$. Determine the eigenvalues and a set of orthonormal eigenfunctions of the Laplacian $L_{b_P,c_P,m}$.
(d) Compute the resolvents and the semigroup.
(e) Solve the Poisson problem for $g = 1_{\{(0,1),\ldots,(0,5)\}}$ and $p = (0,0)$.
(f) Solve the Dirichlet problem for $B = \{(1,1),\ldots,(1,5)\}$ and $g = 1_B$.
(g) Solve the capacitor problem for $F = \{(0,1)\}$, $G = \{(1,5)\}$ and $g = 1_G$.
(h) Solve the heat equation for the initial distributions $f = 1_{\{(0,1)\}}$.

Challenge!

Extension exercises.

EXERCISE 0.28 (The normalizing measure counts edges). Let X be a finite set. Let b be a graph with standard weights over X, i.e., b takes values in $\{0,1\}$ and $c = 0$. Let $A \subseteq X$. Show that the normalizing measure $n(x) = \sum_{y \in X} b(x,y) = \#\{y \mid y \sim x\}$ satisfies

$$n(A) = \#E_A + \frac{1}{2}\#\partial_E A,$$

where $E_A = \{(x,y) \in A \times A \mid x \sim y\}$ and $\partial_E A = \{(x,y) \in (A \times (X \setminus A)) \cup ((X \setminus A) \times A) \mid x \sim y\}$.

EXERCISE 0.29 (Characterizing Dirichlet forms). Let Q be a symmetric form over X. Show that the following statements are equivalent:

(i) $Q(f \wedge 1) \leq Q(f)$ for all $f \in C(X)$.
(ii) $Q(C_{[0,1]} \circ f) \leq Q(f)$ for all $f \in C(X)$ where $C_{[0,1]}f = 0 \vee f \wedge 1$.

(Hint: Show that $\frac{1}{-\varepsilon}((-\varepsilon f) \wedge 1) \to f_+ = f \vee 0$ as $\varepsilon \to \infty$.)

EXERCISE 0.30 (Saturated sets and connected components). Let (b,c) be a graph over a finite set X. A subset Y of X is called *saturated* in (b,c) if $x \in X$ with $x \sim y$ for $y \in Y$ implies that $x \in Y$. Show that a subset Y saturated in (b,c) is a connected component of X if and only if Y cannot be decomposed into two disjoint non-empty saturated sets.

Exercise 0.31 (Restricting forms to subsets). Let b be a connected graph over a finite set X and let $Q = Q_{b,0}$ be the form associated to b. Let U be a proper subset of X. For a function $f\colon U \longrightarrow \mathbb{R}$, we define $\widetilde{f}$, the extension of f to X, via $\widetilde{f}(x) = f(x)$ for $x \in U$ and $\widetilde{f}(x) = 0$ otherwise. Define the form Q_U on $C(U)$ by $Q_U(f) = Q(\widetilde{f})$. Show that Q_U is associated to a graph (b_U, c_U) with non-vanishing c_U.

Exercise 0.32 (Effective Resistance). Let b be a connected graph over a finite set X and let $L = L_b$ be the associated Laplacian with associated form $Q = Q_b$.

(a) Show that the effective resistance defined by
$$W_{\mathrm{eff}} = \frac{1}{Q(f_{x,y})},$$
where $f = f_{x,y}$ is the unique function satisfying $f(x) = 0$, $f(y) = 1$ and $Lf = 0$ on $X \setminus \{x, y\}$ for $x \neq y$, satisfies
$$W_{\mathrm{eff}}(x,y) = \max\{(f(x) - f(y))^2 \mid Q(f) \leq 1\}.$$
(Hint: Let $x, y \in X$ with $x \neq y$. Using $Q(f) = Q(f + \lambda 1)$ for any $\lambda \in \mathbb{R}$ and $Q(f) = Q(-f)$ it is possible to show (how?) that
$$\min_{f(x)=0, f(y)=1} Q(f) = \min_{f(x)\neq f(y)} \frac{Q(f)}{(f(x) - f(y))^2}.$$
This allows us to conclude the statement.)

(b) Show that
$$r(x,y) = W_{\mathrm{eff}}^{1/2}(x,y), \qquad x \neq y$$
and $r(x,y) = 0$ for $x = y$ defines a metric on the graph.

(c*) It can actually be shown that
$$r^2(x,y) = W_{\mathrm{eff}}(x,y), \qquad x \neq y$$
and $r(x,y) = 0$ for $x = y$ is a metric. Challenge!

Exercise 0.33 (Characterizing the first Beurling–Deny criterion). Let Q be a positive quadratic form. Show that Q satisfies the first Beurling–Deny criterion, i.e., $Q(|f|) \leq Q(f)$ for all $f \in C(X)$ if and only if for all $f, g \in C(X)$
$$Q(f \vee g) + Q(f \wedge g) \leq Q(f) + Q(g).$$

Exercise 0.34 (Positivity improving semigroups and invariant subspaces). Show that a positivity preserving semigroup $P_t = e^{-tL}$ is positivity improving if and only if only the trivial subspaces of $\ell^2(X, m)$ are invariant under the semigroup and multiplication by functions on X.

EXERCISE 0.35 (Characterization of $\lambda_0 = 0$). Let (b, c) be a graph over X with $c = 0$ and let λ_0 be the smallest eigenvalue of $L_{b,c,m} = L_{b,0,m}$. Prove the following statements:

(a) $\lambda_0 = 0$.
(b) The space of eigenfunctions V_0 corresponding to $\lambda_0 = 0$ consists of all functions that are constant on each connected component.
(c) The dimension of V_0 is equal to the number of connected components of the graph.
(d) Show that $L_{b,0,m}$ is not surjective and determine its range.

EXERCISE 0.36 (Stochastic incompleteness and the Lie–Trotter product formula). Let (b, c) be a connected graph over (X, m) and let $L = L_{b,c,m}$ denote the associated Laplacian. Use the Lie–Trotter formula to show that $e^{-tL}1 < 1$ for all $t > 0$ if and only if $c \neq 0$.

(Hint: A symmetric matrix with non-negative entries whose rows (or columns) sum up to 1 is called stochastic and whose rows sum up to less than 1 is called substochastic. Show that these properties are preserved under taking products of matrices so that $e^{-tc}e^{-tL_b}$ is substochastic if and only if $c \neq 0$.)

EXERCISE 0.37 (Stochastic incompleteness is instantaneous). Let (b, c) be a connected graph over (X, m) and let $L = L_{b,c,m}$ be the associated Laplacian. Show that if $e^{-tL}1 < 1$ for some $t > 0$, then $e^{-tL}1 < 1$ for all $t > 0$.

(Hint: Use the semigroup property, i.e., that $e^{-(s+t)L} = e^{-sL}e^{-tL}$ for all $s, t \geq 0$.)

EXERCISE 0.38 (Feynman–Kac formula via supergraphs). Let (b', c') be a graph over a finite measure space (X', m') with associated Laplacian $L' = L_{b',c',m'}$. Show that there is a graph b over a finite (X, m) such that $X' \subseteq X$, $m|_{X'} = m'$, $b|_{X'\times X'} = b'$ and that for the Markov process $\mathbb{X}$ associated to b we have

$$e^{-tL'}f(x) = \mathbb{E}_x\left(1_{t<\tau_{X'}} f(\mathbb{X}_t)\right)$$

for all $f \in \ell^2(X', m')$, $x \in X'$, where $\tau_{X'}$ is the first exit time of X'.

EXERCISE 0.39 (Characterizing $c = 0$). Let (b, c) be a graph over a finite measure space (X, m). Show that the process $(\mathbb{X}_t)$ does not leave X for all $t > 0$ if and only if $c = 0$.

Notes

With the exception of Section 9, the material found in this chapter is certainly well known, though scattered throughout the existing literature (see the discussion at the end of these notes) and not necessarily presented via our perspective of bringing together Dirichlet forms, geometry of graphs and spectral theory.

The crucial references for us are the papers of Beurling/Deny [**BD58, BD59**]. These works announce and outline a general theory connecting electrostatics and heat diffusion through what is there called a Dirichlet form. The setting for these papers is that of locally compact topological spaces allowing for a Radon measure of full support. They do not provide proofs in this general setting. These proofs were provided later in various papers, see the monograph of Fukushima [**Fuk80**] for a detailed treatment and further references. However, in [**BD58**], Beurling/Deny give a complete treatment for the special case in which the locally compact space in question is a finite set and the measure is 1 at every point. Roughly speaking, we follow the treatment of [**BD58**] by considering a finite set but with an arbitrary measure.

The notions introduced in Section 1 are completely standard.

Section 2 rephrases the basic setting of Beurling/Deny [**BD58**] using the language of graphs. In particular, Part (a) of Lemma 0.19 is Remarque 2 in [**BD58**] and Part (b) which characterizes the compatibility with normal contractions can already be found in the proof of Théorème 1 in [**BD58**]. The correspondence between Dirichlet forms and graphs found in Theorem 0.22 (and the underlying equivalence of (i) and (iii) in Theorem 0.20) are Théorème 1 in [**BD58**].

Although maximum principles for Laplacians on graphs appear throughout the literature, we could not find an earlier treatment of the material presented in Section 3 which fully characterizes Laplacians on graphs in terms of maximum principles.

The use of graphs in electrostatics as found in Section 4 is standard at this point and goes back to at least the work of Kirchhoff [**Kir45**]. In particular, the characterization of graphs given via the capacitor problem in Theorem 0.44 is Théorème 2 in [**BD58**]. The resistance metric discussed in the remark following Corollary 0.42 appears in Remarque 3 in [**BD58**].

Section 5 is essentially contained in Beurling/Deny [**BD58**]. In particular, Theorem 0.49 characterizing the Markov property is Théorème 6 in [**BD58**]. The splitting of the Beurling–Deny criteria into two, as found in Theorems 0.47 and 0.48, cannot be found in the quoted works of Beurling/Deny. We have not been able to ascertain the first source of this splitting. It can be found under the name of "Beurling–Deny criteria" in [**RS78**] or [**Dav89**].

Resolvents and semigroups are connected by general principles, the special features of the Markov property found in Corollary 0.51 in Section 6 can be found on page 219 of [**BD58**].

The Perron–Frobenius Theorem found in Section 7 goes back at least to the works of Perron [**Per07**] and Frobenius [**Fro08, Fro09, Fro12**]. This material can be found in textbooks. We essentially follow the presentation in [**RS78**] for the proof of Theorem 0.55. It is also standard to use the Perron–Frobenius Theorem to treat convergence to the ground state of Markov chains as treated in many places, e.g. [**Nor98**]. This is also found for finite Markov chains in [**SC97**]. Theorem 0.59 and Corollary 0.61 are certainly well known; however, we have not been able to find them in this form in any textbook.

Section 8 partially serves as a summary of previous considerations. As such, we refer to the notes above concerning previously discussed results. Many graph theory textbooks discuss graphs without killing as their basic object, therefore, disappearance of the killing term is not an issue. As such, we have not found the characterization of graphs in terms of special Dirichlet forms that we present in Theorem 0.63 in any standard reference. On the other hand, conservativeness or stochastic completeness of semigroups, which is characterized in Theorem 0.65, is a standard topic in the theory of Markov chains on a discrete space state with continuous time. See, for example, [**Nor98**].

Section 9 is not standard. It provides a study of certain geometric questions which arise naturally in our perspective.

Section 10 is standard and discussed in any textbook on Markov processes in continuous time, e.g., [**Nor98**].

Of course there is a great body of excellent textbooks that intersect with some of the topics treated in this and the forthcoming chapters. However, the notes at the end of the chapters have a primarily historical character, rather than attempting to provide an extensive bibliography. Nevertheless, we take this opportunity to give the reader at least a partial glimpse of the broad variety of the subject by listing some standard references that we are aware of. For the books that we missed, we apologize in advance for our ignorance.

For finite graphs, various aspects of the geometry and spectral theory of Laplacian and Markov processes have been studied and presented in books by Chung [**Chu83**], Biggs [**Big93**] and Colin de Verdière [**CdV98**] and the recent and delightful book by Grigor′yan [**Gri18**], which also deals with infinite graphs. The book chapters by Saloff-Coste [**SC97**] study the connections between analytic inequalities and geometry in the context of mixing times of continuous time Markov chains.

The main focus of the remaining texts we mention is on infinite graphs. The textbook of Woess [**Woe00**] provides a standard reference for discrete time Markov chains with a strong focus on discrete groups, see also [**Woe09**]. Moreover, there is an excellent survey article on the spectral theory of graphs by Mohar/Woess [**MW89**]. The potential theory and the electrical network point of view have been developed in the books of Doyle/Snell [**DS84**], Soardi [**Soa94**] and, more recently, in the text by Levin/Peres/Wilmer [**LPW09**] and by Jorgensen/Pearse [**JP**]. Percolation, electric networks, random walks and other stochastic aspects are covered, with a particular focus on trees, in the book by Lyons/Peres [**LP16**]. There is also a text by Barlow [**Bar17**] that is particularly worth mentioning because it complements this book in the sense that it treats heat kernel estimates, which are completely omitted here. A further topic which is not covered in this book concerns discrete notions of curvature, for which we refer the reader to [**NR17**]. Moreover, we also mention the recent book of Kostenko/Nicolussi [**KN21**], which presents some connections between discrete and metric graphs.

Part 1

Foundations and Fundamental Topics

Synopsis

This is the first part of our general study of infinite graphs. We introduce basic quantities associated to graphs such as Dirichlet forms, Laplacians and semigroups in Chapter 1. Chapter 2 expands upon the material developed in Chapter 1 and collects several useful tools that are needed at later points. The main focus of the subsequent chapters in Part 1 is the investigation of certain features of graphs and their Laplacians via solutions (or their absence) to generalized eigenvalue equations. We start with a discussion of essential self-adjointness in Chapter 3, then turn to studying characterizations of the ground state in Chapter 4 and the convergence of the semigroup to the ground state in Chapter 5. The final two chapters of this part deal with fundamental stochastic properties in terms of generalized solutions. These are recurrence in Chapter 6 and stochastic completeness in Chapter 7. Our general point of view in this part is that of functional analysis based on Dirichlet forms. Various topics of this part will be taken up in Part 3. The main focus there will be the investigation of the behavior of solutions to generalized eigenvalue equations in terms of the underlying geometry of the graph.

CHAMBER 1

Infinite Graphs – Key Concepts

First things first, ...
ODB.

In this chapter we discuss key concepts in the spectral geometry of infinite graphs. We first introduce in Section 1 the setting and the main objects of study found throughout the remainder of the book. These include graphs, the associated Laplacians and Dirichlet forms, and the induced semigroups and resolvents. Our definition of a graph includes weights on the edges as well as a killing term. We also introduce a few key tools, such as minimum principles, which will be used throughout.

We then turn to the connection between graphs and Dirichlet forms in Section 2 where we show that graphs are in a one-to-one correspondence with regular Dirichlet forms. In Section 3 we use tools such as approximation by finite graphs, domain monotonicity and maximum principles to prove the Markov property of the semigroup and resolvent associated to a regular Dirichlet form. An additional property of the semigroup and resolvent, namely, that they are positivity improving, is shown to be equivalent to the connectedness of the graph in Section 4.

We discuss certain special cases of the general theory in the subsequent two sections: In Section 5 we give criteria for when the associated Laplacians are bounded operators and in Section 6 we discuss what we call graphs with standard weights. These are graphs where the edge weights are either one or zero and the killing term is absent.

1. The setting in a nutshell

In this section we introduce our basic setting. We will use the material and notation of this section tacitly throughout the remainder of the book. Thus, we assume that the reader is familiar with this section throughout. On the other hand, given familiarity with this section, the reader should be able to read essentially any other part of the book.

Excavation Exercises 1.1 and 1.2 recall basic facts about the Hilbert space which will be introduced in this section.

Throughout, we let X be a discrete and countable set. More precisely, we equip X with the discrete topology and by countable we mean that there is an injective map from X to $\mathbb{N}$. We denote the set of all real-valued functions on X by $C(X)$. For $f \in C(X)$, we write $\operatorname{supp} f$ for the *support of* f, i.e.,

$$\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}.$$

We denote the set of all functions on X with finite support by $C_c(X)$. For $x \in X$, we denote the characteristic function of the set which consists of the element x by 1_x. We call a function $f \in C(X)$ which satisfies $f \geq 0$ *positive* and a function which satisfies $f > 0$ *strictly positive*.

If, additionally, there is a measure m on X, we call (X, m) a *discrete measure space*. To avoid pathologies we will always assume that the measure m has full support, i.e., that every point of X has positive measure. In this situation the set of square summable functions

$$\ell^2(X, m) = \{f \in C(X) \mid \sum_{x \in X} f^2(x)m(x) < \infty\}$$

has a natural Hilbert space structure with inner product given by

$$\langle f, g \rangle = \sum_{x \in X} f(x)g(x)m(x)$$

for $f, g \in \ell^2(X, m)$ and norm $\|f\| = \sqrt{\langle f, f \rangle}$.

Definition 1.1 (Graph over X). A *graph over* X is a pair (b, c) consisting of a function $b \colon X \times X \longrightarrow [0, \infty)$ satisfying

- $b(x, y) = b(y, x)$ for all $x, y \in X$
- $b(x, x) = 0$ for all $x \in X$
- $\sum_{y \in X} b(x, y) < \infty$ for all $x \in X$

and a function $c \colon X \longrightarrow [0, \infty)$. Whenever $c = 0$, when referring to $(b, 0)$ we speak instead of b as a graph over X. We call the elements of X the *vertices* of the graph. We call a pair (x, y) with $b(x, y) > 0$ an *edge with weight* $b(x, y)$. We will also say that x and y are *connected by an edge with weight* $b(x, y)$. We call the vertices x and y *neighbors* if there exists an edge connecting them and write $x \sim y$ in this case. We call the map c the *killing term*.

We note that we speak of neighbors as being connected by an edge. More generally, we say that two vertices x and y are *connected* if there exists a sequence $(x_k)_{k=0}^n$ in X with x_k pairwise distinct, $b(x_k, x_{k+1}) > 0$ for $k = 0, \ldots, n-1$, $x_0 = x$ and $x_n = y$. We call such a sequence a *path* connecting x and y. We call a subset of X *connected* if all pairs of vertices in the subset are connected by a path consisting of vertices in the subset. A *connected component* of the graph is a maximal connected subset of X. If X has only one connected component, i.e., if any two vertices $x, y \in X$ are connected, then we say that the graph (b, c) is *connected*.

We say that a graph (b, c) is *locally finite* if for every $x \in X$ the number of neighbors of x is finite, i.e.,

$$\#\{y \in X \mid y \sim x\} < \infty$$

for all $x \in X$. In general, we will not assume that graphs are locally finite.

The *degree* of a vertex $x \in X$ is the function $\deg : X \longrightarrow [0, \infty)$ defined by

$$\deg(x) = \sum_{y \in X} b(x, y) + c(x).$$

If (b, c) is a graph over X and m is a measure on X with full support we refer to (b, c) as a *graph over* (X, m). In this case, we will also refer to the *weighted degree* $\mathrm{Deg}\colon X \longrightarrow [0, \infty)$ as

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right).$$

EXAMPLE 1.2 (Counting and normalizing measure). Let (b, c) be a graph over X. We now introduce two natural choices for a measure on X. The first is the *counting measure* given by the constant function $m = 1$. In this case, $\mathrm{Deg} = \deg$. The second natural measure, called the *normalizing measure* n, is defined as $n = \deg$. Whenever we use deg in the spirit of a measure we denote it by n. In this case, $\mathrm{Deg} = 1$.

To a graph (b, c) over X, we associate the subspace $\mathcal{D} = \mathcal{D}_{b,c}$ of $C(X)$ given by

$$\mathcal{D} = \{f \in C(X) \mid \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f^2(x) < \infty\}$$

and the bilinear map

$$\mathcal{Q} = \mathcal{Q}_{b,c}\colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{Q}(f, g) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))(g(x) - g(y)) + \sum_{x \in X} c(x) f(x) g(x).$$

We note that the sums defining $\mathcal{Q}(f, g)$ are absolutely convergent on $\mathcal{D}$ due to

$$|b(x, y)(f(x) - f(y))(g(x) - g(y))| \leq \frac{1}{2} b(x, y)(f(x) - f(y))^2 + \frac{1}{2} b(x, y)(g(x) - g(y))^2$$

and

$$|c(x) f(x) g(x)| \leq \frac{1}{2} c(x) \left(f^2(x) + g^2(x) \right).$$

We call $\mathcal{Q}$ the *energy form* and refer to elements of $\mathcal{D}$ as *functions of finite energy*. Clearly, $\mathcal{Q}$ is *symmetric*, i.e., satisfies

$$\mathcal{Q}(f,g) = \mathcal{Q}(g,f)$$

for all $f, g \in \mathcal{D}$. The form $\mathcal{Q}$ is also *positive*, i.e., satisfies

$$\mathcal{Q}(f,f) \geq 0$$

for all $f \in \mathcal{D}$.

We will often be interested in the values of $\mathcal{Q}$ on the diagonal only. In this case, we will use the notation

$$\mathcal{Q}(f) = \mathcal{Q}(f,f)$$

for $f \in \mathcal{D}$. We can then extend $\mathcal{Q}$ to a map on $C(X)$, again denoted by $\mathcal{Q}$, defined by $\mathcal{Q}\colon C(X) \longrightarrow [0,\infty]$ via

$$\mathcal{Q}(f) = \begin{cases} \mathcal{Q}(f) & \text{if } f \in \mathcal{D} \\ \infty & \text{else.} \end{cases}$$

This map has the following semi-continuity property.

Proposition 1.3 (Lower semi-continuity of $\mathcal{Q}$). *Let (b,c) be a graph over X. If a sequence (f_n) in $C(X)$ converges pointwise to $f \in C(X)$, i.e., $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in X$, then*

$$\mathcal{Q}(f) \leq \liminf_{n\to\infty} \mathcal{Q}(f_n).$$

Proof. This is a direct consequence of Fatou's lemma. Indeed, consider the measure space $X \times X$ with the measure B and X with the measure C given by

$$B(M) = \frac{1}{2}\sum_{(x,y)\in M} b(x,y) \quad \text{and} \quad C(N) = \sum_{x\in N} c(x)$$

for $M \subseteq X \times X$, $N \subseteq X$, and the functions $F_n, F\colon X \times X \longrightarrow [0,\infty)$ defined by

$$F_n(x,y) = (f_n(x) - f_n(y))^2 \quad \text{and} \quad F(x,y) = (f(x) - f(y))^2.$$

Then, clearly $F_n(x,y) \to F(x,y)$ for all $x, y \in X$, $f_n^2(x) \to f^2(x)$ for all $x \in X$ as $n \to \infty$ and

$$\int_{X\times X} F dB + \int_X f^2 dC = \mathcal{Q}(f), \qquad \int_{X\times X} F_n dB + \int_X f_n^2 dC = \mathcal{Q}(f_n).$$

Now, Fatou's lemma gives the desired statement. □

Besides the energy form $\mathcal{Q}$ associated to (b,c) we will also consider the *formal Laplacian* $\mathcal{L}_{b,c}$ acting on

$$\mathcal{F} = \mathcal{F}_b = \{f \in C(X) \mid \sum_{y\in X} b(x,y)|f(y)| < \infty \ \text{ for all } x \in X\}$$

by

$$\mathcal{L}_{b,c}f(x) = \sum_{y\in X} b(x,y)(f(x) - f(y)) + c(x)f(x).$$

We note that the formal Laplacian $\mathcal{L}_{b,c}$ depends on both b and c while the domain $\mathcal{F}$ depends only on b.

The operator $\mathcal{L}_{b,c}$ has a certain symmetry property and the form $\mathcal{Q}$ and operator $\mathcal{L}_{b,c}$ are related by an integration by parts formula which we refer to as Green's formula. This is the content of the next proposition.

Proposition 1.4 (Green's formula). *Let (b,c) be a graph over X.*

(a) *Every $\varphi \in C_c(X)$ belongs to $\mathcal{F}$ and for all $f \in \mathcal{F}$*

$$\sum_{x\in X} \varphi(x)\mathcal{L}_{b,c}f(x) = \sum_{x\in X} \mathcal{L}_{b,c}\varphi(x)f(x)$$
$$= \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi(x) - \varphi(y))(f(x) - f(y)) + \sum_{x\in X} c(x)\varphi(x)f(x),$$

where all of the sums are absolutely convergent.

(b) *We have*

$$\mathcal{D} \subseteq \mathcal{F}$$

and thus for all $f \in \mathcal{D}$ and $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}_{b,c}f(x) = \sum_{x\in X} \mathcal{L}_{b,c}\varphi(x)f(x).$$

Proof. (a) By the assumptions on f, φ and b we have

$$\sum_{x,y\in X} |b(x,y)f(y)\varphi(x)| = \sum_{x\in X} |\varphi(x)| \sum_{y\in X} b(x,y)|f(y)| < \infty$$

and

$$\sum_{x,y\in X} |b(x,y)f(x)\varphi(x)| = \sum_{x\in X} |f(x)\varphi(x)| \sum_{y\in X} b(x,y) < \infty.$$

Given this finiteness, the desired equalities follow easily by direct computations.

(b) Given (a), it suffices to show that every $f \in \mathcal{D}$ belongs to $\mathcal{F}$. To see this, we calculate

$$\sum_{y\in X} b(x,y)|f(y)| \leq \sum_{y\in X} b(x,y)|f(x) - f(y)| + \sum_{y\in X} b(x,y)|f(x)|.$$

Now, the first term can be seen to be finite via the Cauchy–Schwarz inequality as

$$\left(\sum_{y\in X} b(x,y)\right)^{1/2} \left(\sum_{y\in X} b(x,y)(f(x) - f(y))^2\right)^{1/2} \leq \deg^{1/2}(x)\mathcal{Q}^{1/2}(f)$$

and the second term is bounded by $\deg(x)|f(x)| < \infty$. This gives the desired statement. $\square$

In most of our subsequent considerations we not only have a graph (b, c) over X but also a measure m of full support on X. In this situation, suitable restrictions of the form $\mathcal{Q}$ will yield self-adjoint operators on the Hilbert space $\ell^2(X, m)$. These operators will be our prime concern. To describe how these operators arise we will need the norm $\|\cdot\|_{\mathcal{Q}} \colon \mathcal{D} \cap \ell^2(X, m) \longrightarrow [0, \infty)$ given by

$$\|f\|_{\mathcal{Q}} = \left(\mathcal{Q}(f) + \|f\|^2\right)^{1/2},$$

where $\|f\|$ is the $\ell^2(X, m)$ norm of f.

We define the form $Q^{(N)} = Q^{(N)}_{b,c,m}$ as the restriction of $\mathcal{Q}$ to

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m).$$

Then, clearly, $Q^{(N)}$ is symmetric and positive as $\mathcal{Q}$ has these properties. As above, we set

$$Q^{(N)}(f) = Q^{(N)}(f, f)$$

and extend $Q^{(N)}$ to all of $\ell^2(X, m)$ by setting it to be ∞ outside of $\mathcal{D} \cap \ell^2(X, m)$. We think of $Q^{(N)}$ as arising from some sort of Neumann boundary conditions and this is the reason for the superscript (N). We will refer to $Q^{(N)}$ as the *Neumann form*.

If a sequence (f_n) from $\ell^2(X, m)$ converges to f in $\ell^2(X, m)$, then it clearly converges pointwise and from Proposition 1.3 we obtain

$$Q^{(N)}(f) \leq \liminf_{n \to \infty} Q^{(N)}(f_n).$$

Thus, $Q^{(N)}$ is a lower semi-continuous map on a subspace of $\ell^2(X, m)$. By standard theory, see Theorem B.9 in Appendix B, $Q^{(N)}$ is closed, i.e., $D(Q^{(N)})$ is complete with respect to $\|\cdot\|_{\mathcal{Q}}$.

In some sense, $Q^{(N)}$ is the "maximal" form associated to a graph. We will be even more concerned with the "minimal" form. This form comes about by considering all symmetric closed forms which are restrictions of $Q^{(N)}$ (or $\mathcal{Q}$) and whose domain contains $C_c(X)$. The intersection over the domains of all such forms will be a closed subspace of $D(Q^{(N)})$. Hence, the restriction of $\mathcal{Q}$ to this domain will yield a positive closed form. We denote this form by $Q^{(D)} = Q^{(D)}_{b,c,m}$ and its domain by $D(Q^{(D)}) = D(Q^{(D)}_{b,c,m})$.

By construction $Q^{(D)}$ is the smallest closed form extending the restriction of $\mathcal{Q}$ to $C_c(X) \times C_c(X)$. Thus, we can also obtain $D(Q^{(D)}_{b,c,m})$ by taking the closure with respect to $\|\cdot\|_{\mathcal{Q}}$ of $C_c(X)$, that is,

$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}.$$

We think of $Q^{(D)}$ as arising from some sort of Dirichlet boundary conditions and this is the reason for the superscript (D).

By the standard theory of closed forms, see Lemma B.7 and Corollary B.12, there exists a unique self-adjoint operator $L^{(D)} = L^{(D)}_{b,c,m}$ on $\ell^2(X,m)$ whose domain $D(L^{(D)})$ is contained in $D(Q^{(D)})$ and which satisfies

$$\langle g, L^{(D)} f\rangle = Q^{(D)}(g,f)$$

for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. We call $L^{(D)}$ the *Dirichlet Laplacian* or just *the Laplacian* associated to a graph. We denote the *spectrum* of $L^{(D)}$ by $\sigma(L^{(D)})$ and the bottom of the spectrum of $L^{(D)}$ by $\lambda_0(L^{(D)})$. We note that $L^{(D)}$ is positive and thus $\sigma(L^{(D)}) \subseteq [0,\infty)$ and $\lambda_0(L^{(D)}) \geq 0$.

In general, it is rather hard to describe explicitly the domain of $L^{(D)}$. Still, the action of this operator is easy to describe. To do so, we introduce the formal operator $\mathcal{L} = \mathcal{L}_{b,c,m}$ associated to a graph (b,c) over the measure space (X,m). This operator has domain $\mathcal{F}$ and acts via

$$\mathcal{L}f(x) = \frac{1}{m(x)}\mathcal{L}_{b,c}f(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(f(x)-f(y)) + c(x)f(x)\right).$$

From Proposition 1.4, we immediately infer the following variant of Green's formula in the case when we have a measure.

Proposition 1.5 (Green's formula for $\mathcal{L}$). *Let (b,c) be a graph over (X,m). For all $f \in \mathcal{F}$ and $\varphi \in C_c(X)$ we have*

$$\begin{aligned}\sum_{x\in X}\varphi(x)\mathcal{L}f(x)m(x) &= \sum_{x\in X}\mathcal{L}\varphi(x)f(x)m(x)\\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi(x)-\varphi(y))(f(x)-f(y)) + \sum_{x\in X} c(x)\varphi(x)f(x).\end{aligned}$$

If $f \in \mathcal{D}$, then the last term reads as $\mathcal{Q}(\varphi,f)$. In particular, if $f\in\mathcal{D}$ satisfies $\mathcal{L}f \in \ell^2(X,m)$, then for all $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi,f) = \langle \varphi, \mathcal{L}f\rangle.$$

Finally, if $f \in \mathcal{D}\cap\ell^2(X,m)$ and $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then for all $\varphi \in C_c(X)$

$$\mathcal{Q}(\varphi,f) = \langle \mathcal{L}\varphi, f\rangle.$$

Comparing this with the defining property of $L^{(D)}$ and using the fact that $Q^{(D)}$ is a restriction of $\mathcal{Q}$, we immediately infer the following theorem.

Theorem 1.6 (Action of the Dirichlet Laplacian). *Let (b,c) be a graph over (X,m) and let $L^{(D)}$ be the Dirichlet Laplacian. Then,*

$$L^{(D)}f(x) = \mathcal{L}f(x)$$

for all $f \in D(L^{(D)})$ and $x \in X$.

Proof. By definition, $L^{(D)}$ is the unique self-adjoint operator with $D(L^{(D)}) \subseteq D(Q^{(D)})$ which satisfies $\langle g, L^{(D)}f\rangle = Q^{(D)}(g,f)$ for all $f \in D(L^{(D)})$ and $g \in D(Q^{(D)})$. Furthermore, as $Q^{(D)}$ is a restriction of $\mathcal{Q}$ and $C_c(X) \subseteq D(Q^{(D)}) \subseteq \mathcal{D} \subseteq \mathcal{F}$, from Green's formula, Proposition 1.5, we have

$$\langle \varphi, L^{(D)}f\rangle = Q^{(D)}(\varphi, f) = \mathcal{Q}(\varphi, f) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x)$$

for all $\varphi \in C_c(X)$ and $f \in D(Q^{(D)})$. The conclusion follows by choosing $\varphi = 1_x/m$ for arbitrary x. □

Next, we discuss an innocent-looking feature of the form $\mathcal{Q}$ whose surprising consequences will unfold later. We let C denote a normal contraction, i.e., $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $C(0) = 0$ and $|C(s) - C(t)| \leq |s - t|$ for all $s, t \in \mathbb{R}$. Then, the form $\mathcal{Q}$ is *compatible* with C in the sense that if $f \in \mathcal{D}$, then

$$\mathcal{Q}(C \circ f) \leq \mathcal{Q}(f).$$

In particular, it follows that $C \circ f \in \mathcal{D}$ for all $f \in \mathcal{D}$. It is obvious that this formula also holds if $\mathcal{Q}$ is replaced by $Q^{(N)}$. It is less clear but still true that it also holds for $Q^{(D)}$. A proof can be found in Section 2. A closed form which is compatible all with normal contractions is called a *Dirichlet form*. Thus, we will see that $Q^{(D)}$ and $Q^{(N)}$ are Dirichlet forms. We further discuss Dirichlet forms in Section 2.

We mention here that this compatibility has strong consequences for both the *semigroup* $e^{-tL^{(D)}}$ and *resolvent* $(L^{(D)} + \alpha)^{-1}$ associated to $L^{(D)}$ where $t \geq 0$ and $\alpha > 0$. Namely, this semigroup and resolvent satisfy

$$0 \leq e^{-tL^{(D)}} f \leq 1 \qquad \text{and} \qquad 0 \leq \alpha(L^{(D)} + \alpha)^{-1} f \leq 1$$

for all $f \in \ell^2(X, m)$ with $0 \leq f \leq 1$. This is known as the *Markov property* of the semigroup and resolvent. With this property, we can extend the semigroup and resolvent to all $\ell^p(X, m)$ for $p \in [1, \infty]$. Details will be discussed in Section 1.

For $\alpha \in \mathbb{R}$ we say that a function u is *α-subharmonic* if $u \in \mathcal{F}$ and

$$(\mathcal{L} + \alpha)u \leq 0.$$

We say that u is *α-superharmonic* if $-u$ is α-subharmonic, i.e., $u \in \mathcal{F}$ satisfies $(\mathcal{L} + \alpha)u \geq 0$. We say that u is *α-harmonic* if u is both α-sub and α-superharmonic, i.e., $u \in \mathcal{F}$ satisfies

$$(\mathcal{L} + \alpha)u = 0.$$

When $\alpha = 0$, we say that u is *(sub/super)harmonic*. We will see that various features of such functions are intimately related to the geometric, spectral and stochastic properties of graphs.

We next present three basic results concerning solutions of the equation

$$(\mathcal{L} + \alpha)u = f$$

which will be used in various later considerations. We refer to this equation as the *Poisson equation*. As a special case, we note that u is α-harmonic when $f = 0$.

We will use the notation $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$ for the minimum and maximum of two functions, respectively. We start with a minimum principle for certain supersolutions of the Poisson equation.

THEOREM 1.7 (Minimum principle). *Let (b, c) be a graph over (X, m). Let $U \subseteq X$. Assume that a function $u \in \mathcal{F}$ satisfies*

- *$(\mathcal{L} + \alpha)u \geq 0$ on U for some $\alpha \geq 0$*
- *$u \wedge 0$ attains a minimum on U*
- *$u \geq 0$ on $X \setminus U$.*

If $\alpha > 0$ or if every connected component of U is connected to $X \setminus U$, then $u \geq 0$. In fact, on each connected component of U either $u = 0$ or $u > 0$.

PROOF. Without loss of generality we can assume that U is connected. If $u > 0$ there is nothing to show. Therefore, assume there exists a vertex $x \in U$ with $u(x) \leq 0$. As $u \wedge 0$ attains a minimum on U, there then exists a vertex $x_0 \in U$ with $u(x_0) \leq 0$ and $u(x_0) \leq u(y)$ for all $y \in U$. As $u(y) \geq 0$ for $y \in X \setminus U$, we obtain $u(x_0) - u(y) \leq 0$ for all $y \in X$. By the supersolution assumption we then find

$$\begin{aligned} 0 &\leq (\mathcal{L} + \alpha)u(x_0) \\ &= \frac{1}{m(x_0)} \left(\sum_{y \in X} b(x_0, y)(u(x_0) - u(y)) + c(x_0)u(x_0) \right) + \alpha u(x_0) \leq 0. \end{aligned}$$

Therefore, if $\alpha > 0$, then $0 = u(x_0)$ and $u(y) = u(x_0) = 0$ for all $y \sim x_0$. As U is connected, iteration of this argument shows that $u = 0$ on U.

On the other hand, for $\alpha = 0$, we obtain by the same argument that u is constant on U. As U is connected to $X \setminus U$, namely there exist $x \in U$ and $y \in X \setminus U$ such that $x \sim y$, we conclude that $u = 0$ on U. $\square$

For the following lemma, given a sequence of functions (u_n) and a function u we write

$$u_n(x) \nearrow u(x)$$

as $n \to \infty$ if $u_n(x) \leq u_{n+1}(x)$ for all $n \in \mathbb{N}_0$ and if $u_n(x) \to u(x)$ as $n \to \infty$ for $x \in X$. In other words, the sequence converges at x in a monotonically increasing manner. We will write $u_n \nearrow u$ pointwise if this happens at all $x \in X$.

LEMMA 1.8 (Monotone convergence of solutions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$ and let $u, f \in C(X)$. Let (u_n) be a sequence of functions in $\mathcal{F}$ with $u_n \geq 0$. Assume that $u_n(x) \nearrow u(x)$ and $(\mathcal{L} + \alpha)u_n(x) \to f(x)$ for all $x \in X$ as $n \to \infty$. Then, $u \in \mathcal{F}$ and*

$$(\mathcal{L} + \alpha)u = f.$$

PROOF. Without loss of generality, we assume that $m = 1$. By assumption

$$(\mathcal{L} + \alpha)u_n(x) = \sum_{y \in X} b(x, y)(u_n(x) - u_n(y)) + (c(x) + \alpha)u_n(x)$$

converges to $f(x)$ for any $x \in X$. As $\sum_{y \in X} b(x, y)u_n(x)$ converges increasingly to $u(x) \sum_{y \in X} b(x, y) < \infty$, the assumptions on (u_n) show that $\sum_{y \in X} b(x, y)u_n(y)$ must converge as well and, in fact, must converge to $\sum_{y \in X} b(x, y)u(y)$ by the monotone convergence theorem. From this, we easily obtain the conclusion. $\square$

We let

$$u_+ = u \vee 0 \qquad \text{and} \qquad u_- = -u \vee 0$$

denote the positive and negative parts of u so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. The next lemma then shows that the positive and negative parts of a α-harmonic function are α-subharmonic.

LEMMA 1.9 (α-subharmonic and α-superharmonic functions). *Let (b, c) be a graph over (X, m). Let $\alpha \in \mathbb{R}$. If $u, v \in \mathcal{F}$ are α-subharmonic (α-superharmonic, respectively), then $u \vee v$ is α-subharmonic ($u \wedge v$ is α-superharmonic, respectively). In particular, if u is α-harmonic, then u_+, u_- and $|u|$ are all α-subharmonic.*

PROOF. Let u, v be α-subharmonic for some $\alpha \in \mathbb{R}$ and let $w = u \vee v$. Let $x \in X$ and assume without loss of generality that $w(x) = u(x) \geq v(x)$. Then,

$$\begin{aligned} w(x) - w(y) &= \begin{cases} u(x) - u(y) & \text{if } u(y) \geq v(y) \\ u(x) - v(y) & \text{else} \end{cases} \\ &\leq u(x) - u(y). \end{aligned}$$

Thus, $(\mathcal{L} + \alpha)w \leq (\mathcal{L} + \alpha)u \leq 0$ so that w is α-subharmonic.

Now, let u, v be α-superharmonic. We first observe that $u \wedge v = -((-u) \vee (-v))$. Hence, by what we have shown above, $(-u) \vee (-v)$ is α-subharmonic as $-u$ and $-v$ are α-subharmonic. Therefore, $u \wedge v$ is α-superharmonic. The "in particular" statement follows as $u_\pm = (\pm u) \vee 0$ and $|u| = u_+ + u_-$. $\square$

We now introduce the heat equation. More specifically, a function $u \colon [0, \infty) \times X \longrightarrow \mathbb{R}$ is called a *solution of the heat equation* if, for every $x \in X$, the mapping $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t > 0$. The equation $(\mathcal{L} + \partial_t)u = 0$ is called the *heat equation*. If u has all of the properties above but instead of equality in the heat equation satisfies $(\mathcal{L}+\partial_t)u \geq 0$, then we call u a *supersolution* of the heat equation. If u is a solution of the heat equation and $u_0 = f$ for $f \in C(X)$, then f is called the *initial condition* for u. We will say that u satisfies the heat equation with initial condition f in this case. We think of x as a space variable and t as time.

We note that if $f \in \ell^2(X,m)$, then the function

$$u_t(x) = e^{-tL^{(D)}} f(x)$$

is a solution of the heat equation with initial condition f, as follows from the spectral theorem. For details and a proof, see Theorem A.33 in Appendix A.

We now prove a minimum principle for the heat equation. In particular, for supersolutions of the heat equation on certain subsets, positivity on the boundary propagates to positivity on the subset. This will be used later to establish the minimality of certain solutions.

THEOREM 1.10 (Minimum principle for the heat equation). *Let (b,c) be a graph over (X,m). Let $U \subseteq X$ be a connected subset and suppose that U contains a vertex which is connected to a vertex outside of U. Let $T \geq 0$ and let $u\colon [0,T] \times X \longrightarrow \mathbb{R}$ be such that $t \mapsto u_t(x)$ is continuously differentiable on $(0,T)$ for every $x \in U$ and $u_t \in \mathcal{F}$ for all $t \in (0,T]$. Assume u satisfies*
- *$(\mathcal{L} + \partial_t)u \geq 0$ on $(0,T) \times U$*
- *$u \wedge 0$ attains a minimum on $U \times [0,T]$*
- *$u \geq 0$ on $((0,T] \times (X \setminus U)) \cup (\{0\} \times U)$.*

Then, $u \geq 0$ on $[0,T] \times U$.

PROOF. Let (t,x) be a point where $u \wedge 0$ attains a minimum on $U \times [0,T]$. If $u_t(x) \geq 0$, the conclusion follows so we assume $u_t(x) < 0$. Since u is positive on $\{0\} \times U$ we have $t > 0$. Furthermore, since u attains a minimum at (t,x) with respect to t we obtain $\partial_t u_t(x) = 0$ if $t < T$ and $\partial_t u_t(x) \leq 0$ if $t = T$.

Since u also attains a negative minimum at (t,x) with respect to x, we have

$$\mathcal{L}u_t(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(u_t(x) - u_t(y)) + \frac{c(x)}{m(x)} u_t(x) \leq 0.$$

Therefore, $(\mathcal{L} + \partial_t)\, u_t(x) \leq 0$. As u also satisfies $(\mathcal{L} + \partial_t)u \geq 0$ we obtain

$$(\mathcal{L} + \partial_t)\, u_t(x) = 0$$

and hence $\mathcal{L}u_t(x) = 0$. Therefore, $u_t(y) = u_t(x) < 0$ for all $y \sim x$. Iterating this argument and using the assumption that U is connected implies that u_t is a negative constant on U. At the vertex $x \in U$ which has a neighbor not in U, the equation $\mathcal{L}u_t(x) = 0$ then contradicts the assumption $u \geq 0$ on $(0, T] \times (X \setminus U)$. □

In most parts of this book, we focus on the form $Q^{(D)}$ and the operator $L^{(D)}$. However, as has already been seen, other forms inducing operators naturally appear, e.g., the Neumann form $Q^{(N)}$. A convenient way to deal with this situation is to introduce the following more general notion.

Definition 1.11 (Associated forms and operators). We say that a form Q with domain $D(Q)$ is *associated to a graph* if Q is closed,

$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)}) \qquad \text{and} \qquad Q = Q^{(N)}$$

on $D(Q)$. We then say that the arising operator L is *associated to a graph* or an *associated operator*.

Remark. An equivalent formulation is that Q is a restriction of $\mathcal{Q}_{b,c}$, the domain $D(Q)$ of Q contains $C_c(X)$ and $D(Q)$ is complete with respect to $\|\cdot\|_{\mathcal{Q}}$ (Exercise 1.10). As Q is a symmetric positive closed form, L is a self-adjoint operator with spectrum contained in $[0, \infty)$, see Appendix B for details.

The statement and proof of Theorem 1.6 directly carry over to operators associated to graphs by replacing $Q^{(D)}$ by Q and $L^{(D)}$ by L.

Theorem 1.12 (Action of associated operators). *Let (b, c) be a graph over (X, m). Let L be an associated operator. Then,*

$$Lf(x) = \mathcal{L}f(x)$$

for all $f \in D(L)$.

We note, in particular, that the result above applies to the operator $L^{(N)} = L^{(N)}_{b,c,m}$ with domain $D(L^{(N)})$ arising from the Neumann form $Q^{(N)} = Q^{(N)}_{b,c,m}$. We will refer to $L^{(N)} = L^{(N)}_{b,c,m}$ as the *Neumann Laplacian*.

Notation. As already seen in the preceding discussion, we will often suppress the subscripts b, c or b, c, m in various quantities if the graph is clear from the context. The most prominent role in the book will be played by Dirichlet boundary conditions, i.e., the form $Q^{(D)} = Q^{(D)}_{b,c,m}$ and the operator $L^{(D)} = L^{(D)}_{b,c,m}$. For this reason, we will often suppress the superscript (D) if no confusion should arise. Thus, if not stated otherwise (as is the case, for example, in the next section) we will often write Q instead of $Q^{(D)}$ and L instead of $L^{(D)}$.

Remark. We note that our definition of a graph allows for the vertex set X to be finite. However, most of the statements from this point on are trivially true if X is finite. As this is not the focus of the remaining parts of the book, we will not discuss the finite case explicitly. However, there are a few instances when we need X to be infinite for the statement to be true. Whenever this is the case, we include this explicitly in our assumptions and discuss the finite case via remarks. On the other hand, we note that any set which allows for a connected graph structure b must be countable (Exercise 1.11).

2. Graphs and (regular) Dirichlet forms

In this section we show that graphs and regular Dirichlet forms are in a one-to-one correspondence. To this end, we use some of the fundamental theory of closed forms on Hilbert spaces, which is discussed in Appendix B.

Let X be a countable set and let m be a measure on X with full support. A *symmetric positive form over* (X, m) is given by a dense subspace $D(Q)$ of $\ell^2(X, m)$ called the *domain of the form* and a bilinear map

$$Q : D(Q) \times D(Q) \longrightarrow \mathbb{R}$$

satisfying

- $Q(f, g) = Q(g, f)$ (*"Symmetry"*)
- $Q(f, f) \geq 0$ (*"Positivity"*)

for all $f, g \in D(Q)$. From now on, all forms are assumed to be symmetric and positive so we do not mention this explicitly.

We note that such a map is already determined by its values on the diagonal as

$$Q(f, g) = \frac{1}{4}\big(Q(f + g, f + g) - Q(f - g, f - g)\big).$$

For $f \in \ell^2(X, m)$, we then define $Q(f)$ by

$$Q(f) = \begin{cases} Q(f, f) & \text{if } f \in D(Q) \\ \infty & \text{otherwise.} \end{cases}$$

If the map $\ell^2(X, m) \longrightarrow [0, \infty]$, $f \mapsto Q(f)$, is lower semi-continuous, then Q is called *closed*. If Q has a closed extension, then Q is called *closable* and the smallest closed extension is called the *closure* of Q.

The form Q is closed if and only if $D(Q)$ with the *form norm* $\| \cdot \|_Q : D(Q) \longrightarrow [0, \infty)$ given by

$$\|f\|_Q = \big(Q(f) + \|f\|^2\big)^{1/2}$$

is complete. If Q' is closable with closure Q, then for any $f \in D(Q)$, there exists a sequence (f_n) in $D(Q')$ with

$$\lim_{n\to\infty} \|f - f_n\|_Q = 0.$$

For details and further background on these concepts for general Hilbert spaces we refer the reader to Appendix B. Here, we only note the following direct consequence of lower semi-continuity in our case.

PROPOSITION 1.13 (Consequence of lower semi-continuity). *Let Q be a closed form on $\ell^2(X,m)$. If (f_n) is a sequence in $D(Q)$ satisfying*
- $f_n \to f$ *in* $\ell^2(X,m)$
- $(Q(f_n))$ *is bounded,*

then $f \in D(Q)$ and
$$Q(f) \le \liminf_{n\to\infty} Q(f_n).$$

PROOF. By $f_n \to f$ in $\ell^2(X,m)$ it follows that $f_n \to f$ pointwise and we can invoke lower semi-continuity of Q to infer that
$$Q(f) \le \liminf_{n\to\infty} Q(f_n) < \infty.$$
This is the desired inequality, which also implies $f \in D(Q)$. □

Let $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a normal contraction, i.e., a map with $C(0) = 0$ and $|C(s) - C(t)| \le |s-t|$. If Q is both closed and satisfies
$$Q(C \circ f) \le Q(f)$$
for all $f \in D(Q)$ and all normal contractions C, then Q is called a *Dirichlet form* on (X,m).

For a graph (b,c) over (X,m), we show next that $Q^{(N)} = Q^{(N)}_{b,c,m}$ is a Dirichlet form. This form was introduced in the last section as the restriction of $\mathcal{Q} = \mathcal{Q}_{b,c}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$.

PROPOSITION 1.14 ($Q^{(N)}$ is a Dirichlet form). *Let (b,c) be graph over (X,m). Then, $Q^{(N)}_{b,c,m}$ is a Dirichlet form.*

PROOF. As $Q^{(N)}$ is a restriction of $\mathcal{Q}$, it is lower semi-continuous by Proposition 1.3. By Theorem B.9 in Appendix B this implies that $Q^{(N)}$ is closed. Clearly, for all normal contractions C and $f \in \ell^2(X,m)$, it follows that $C \circ f \in \ell^2(X,m)$. Furthermore, for $f \in D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$,
$$Q^{(N)}(C \circ f) = \mathcal{Q}(C \circ f) \le \mathcal{Q}(f) = Q^{(N)}(f).$$
Thus, $Q^{(N)}$ is closed and compatible with normal contractions. Therefore, $Q^{(N)}$ is a Dirichlet form. □

Let $\|\cdot\|_\infty$ denote the supremum norm on $C_c(X)$. A Dirichlet form Q on (X,m) is called *regular* if $D(Q) \cap C_c(X)$ is dense in both $C_c(X)$ with respect to $\|\cdot\|_\infty$ and in $D(Q)$ with respect to the form norm $\|\cdot\|_Q$.

It turns out that a Dirichlet form Q on (X,m) is regular if and only if Q is the closure of the restriction of Q to the subspace $C_c(X)$. The "if" direction is immediate from the definition of a regular Dirichlet form. The "only if" direction is shown next.

LEMMA 1.15. *Let Q be a regular Dirichlet form over (X,m). Then, $C_c(X)$ is contained in $D(Q)$. In particular, Q is the closure of the restriction of Q to $C_c(X) \times C_c(X)$.*

Proof. Let $x \in X$ be arbitrary and let $\varphi = 2 \cdot 1_x$ so that $\varphi \in C_c(X)$. We will show that $\varphi \in D(Q)$. As x is chosen arbitrarily, this will imply the first statement.

As Q is regular, $C_c(X) \cap D(Q)$ is dense in $C_c(X)$ with respect to the supremum norm, so there exists a $\psi \in D(Q)$ with $1 < \psi(x) < 3$ and $|\psi(y)| < 1$ for all $y \neq x$, i.e.,

$$\|\varphi - \psi\|_\infty < 1.$$

As Q is a Dirichlet form, $D(Q)$ is invariant under taking the modulus and we can assume $\psi \geq 0$. Furthermore, as taking the minimum with 1 is also a normal contraction, $\psi \wedge 1 \in D(Q)$. As $D(Q)$ is a vector space it contains $\psi - \psi \wedge 1$ and this is a nonzero multiple of φ by construction. Thus $\varphi \in D(Q)$ and as $x \in X$ is arbitrary, the first statement follows.

As Q was assumed to be regular, the space $C_c(X) = C_c(X) \cap D(Q)$ is dense in $D(Q)$ with respect to the form norm and the "in particular" statement follows. $\square$

We have already encountered a regular Dirichlet form. More specifically, whenever (b, c) is a graph over (X, m) and $Q = Q^{(D)} = Q^{(D)}_{b,c,m}$ is the form defined in the previous section with domain

$$D(Q^{(D)}_{b,c,m}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$$

and acting as a restriction of $\mathcal{Q}_{b,c}$, i.e., $Q^{(D)}_{b,c,m}$ is the closure of $\mathcal{Q}_{b,c}$ restricted to $C_c(X) \times C_c(X)$, then Q is a regular Dirichlet form as we now show. In particular, we will show that the domain of $Q^{(D)}_{b,c,m}$ is preserved by normal contractions.

Lemma 1.16 ($Q^{(D)}$ is a regular Dirichlet form). *Let (b, c) be a graph over (X, m). Then, $Q^{(D)}_{b,c,m}$ is a regular Dirichlet form.*

Proof. We first show that $Q = Q^{(D)}_{b,c,m}$ is a Dirichlet form. We denote the restriction of Q to $C_c(X) \times C_c(X)$ by $\mathcal{Q}^{(\mathrm{comp})}_{b,c}$. Whenever C is a normal contraction and $\varphi \in C_c(X)$, we find by a direct computation

$$\begin{aligned}
\mathcal{Q}^{(\mathrm{comp})}_{b,c}(C \circ \varphi) = \frac{1}{2} \sum_{x,y \in X} & b(x,y)(C \circ \varphi(x) - C \circ \varphi(y))^2 \\
& + \sum_{x \in X} c(x) \left(C \circ \varphi(x)\right)^2 \\
\leq \frac{1}{2} & \sum_{x,y \in X} b(x,y)(\varphi(x) - \varphi(y))^2 + \sum_{x \in X} c(x)\varphi(x)^2 \\
= \mathcal{Q}&^{(\mathrm{comp})}_{b,c}(\varphi).
\end{aligned}$$

Here, we used the defining properties of a normal contraction in the middle step.

We will extend this inequality to $D(Q)$. In particular, we will show that $C \circ f \in D(Q)$ for $f \in D(Q)$. As Q is the closure of its restriction $\mathcal{Q}_{b,c}^{(\mathrm{comp})}$ to $C_c(X) \times C_c(X)$, there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ with respect to $\| \cdot \|_Q$. In particular, $\varphi_n \to f$ in $\ell^2(X,m)$. Then, clearly, the sequence $(C \circ \varphi_n)$ belongs to $C_c(X)$ and converges to $C \circ f$ in $\ell^2(X,m)$. Moreover, the sequence $(Q(C \circ \varphi_n))$ is bounded as

$$Q(C \circ \varphi_n) = \mathcal{Q}_{b,c}^{(\mathrm{comp})}(C \circ \varphi_n) \le \mathcal{Q}_{b,c}^{(\mathrm{comp})}(\varphi_n) = Q(\varphi_n) \to Q(f)$$

as $n \to \infty$. From Proposition 1.13, we then infer $C \circ f \in D(Q)$ and

$$Q(C \circ f) \le Q(f).$$

Therefore, Q is a Dirichlet form.

By construction, Q is the closure of $\mathcal{Q}_{b,c}^{(\mathrm{comp})}$. Hence, Q is regular. This finishes the proof. □

It turns out that the converse to the previous lemma holds as well.

Lemma 1.17 (Regular Dirichlet forms arise from graphs). *Let Q be a regular Dirichlet form over (X,m). Then, there exists a graph (b,c) over (X,m) with $Q = Q_{b,c,m}^{(D)}$.*

Proof. By Lemma 1.15, $C_c(X)$ is contained in $D(Q)$. Define $b \colon X \times X \longrightarrow \mathbb{R}$ by

$$b(x,y) = -Q(1_x, 1_y)$$

for $x \neq y$ and $b(x,x) = 0$ and define $c \colon X \longrightarrow \mathbb{R}$ by

$$c(x) = Q(1_x) - \sum_{y \in X} b(x,y).$$

We will show that (b,c) is a graph with $Q_{b,c,m}^{(D)} = Q$. This will also show that the sum appearing in the definition of c is absolutely convergent.

Claim. (a) For any $x, y \in X$ with $x \neq y$, we have $Q(1_x, 1_y) \le 0$. In particular, $b(x,y) \ge 0$.

(b) For any finite $K \subseteq X$ and $x \in K$, we have $Q(1_K, 1_x) \ge 0$.

Proof of the claim. (a): Consider for $x \neq y$ the function $f = 1_x - 1_y$. As the modulus is a normal contraction and Q is a Dirichlet form we obtain

$$Q(1_x + 1_y) = Q(|f|) \le Q(f) = Q(1_x - 1_y).$$

As Q is bilinear, this gives

$$Q(1_x) + 2Q(1_x, 1_y) + Q(1_y) \le Q(1_x) - 2Q(1_x, 1_y) + Q(1_y).$$

Therefore,

$$4Q(1_x, 1_y) \le 0,$$

which gives the conclusion.

(b): Consider now for $x \in K$ the function $g_s = 1_K + s1_x$ for $s \geq 0$. As taking the minimum with 1 is a normal contraction we infer

$$Q(1_K) = Q(1 \wedge g_s) \leq Q(g_s) = Q(1_K + s1_x).$$

As Q is bilinear,

$$0 \leq 2Q(1_K, 1_x) + sQ(1_x).$$

Letting $s \to 0^+$ then yields

$$0 \leq Q(1_K, 1_x).$$

From (a) of the claim, b is positive. Moreover, for any $K \subseteq X$ finite and any $x \in K$ we compute

$$\begin{aligned} Q(1_x) &= Q(1_K, 1_x) - \sum_{y \in K, y \neq x} Q(1_y, 1_x) \\ &= Q(1_K, 1_x) + \sum_{y \in K, y \neq x} b(x, y) \\ &= Q(1_K, 1_x) + \sum_{y \in K} b(x, y). \end{aligned}$$

As, by the claim, both $Q(1_K, 1_x)$ and b are positive, we can now conclude

$$\sum_{y \in K} b(x, y) \leq Q(1_x)$$

for any $K \subseteq X$ finite and this gives

$$\sum_{y \in X} b(x, y) \leq Q(1_x) < \infty.$$

From this we infer

$$Q(1_x) - \sum_{y \in X} b(x, y) \geq 0$$

for all $x \in X$. Thus, c defined at the beginning of the proof exists and is positive. Hence, (b, c) is indeed a graph.

Moreover, from the very definitions of b and c we conclude for $x, y \in X$ with $x \neq y$

$$Q(1_x, 1_y) = -b(x, y) = Q_{b,c,m}^{(D)}(1_x, 1_y)$$

and for $x \in X$

$$Q(1_x) = c(x) + \sum_{y \in X} b(x, y) = Q_{b,c,m}^{(D)}(1_x).$$

By bilinearity, Q and $Q_{b,c,m}^{(D)}$ agree on $C_c(X)$. As both are regular Dirichlet forms, they must then be equal. $\square$

Remark. In the preceding proof we have shown the following: Let $Q^{(\mathrm{comp})}$ be a form on $C_c(X)\times C_c(X)$ with $Q^{(\mathrm{comp})}(C\circ\varphi)\le Q^{(\mathrm{comp})}(\varphi)$ for all $\varphi\in C_c(X)$ and all normal contractions C. Then, there exists a graph (b,c) over X such that $Q^{(\mathrm{comp})}$ is the restriction of $\mathcal{Q}_{b,c}$ to $C_c(X)\times C_c(X)$ (Exercise 1.12).

Theorem 1.18 (Regular Dirichlet forms and graphs). *The map $(b,c)\mapsto Q^{(D)}_{b,c,m}$ is a bijective correspondence between graphs (b,c) over (X,m) and regular Dirichlet forms over (X,m).*

Proof. This is a direct consequence of Lemmas 1.16 and 1.17. In particular, injectivity of the map follows directly from the first lines of the proof of Lemma 1.17. □

We finish this section by providing a structural characterization of the domain of the unique regular Dirichlet form associated to a graph. We recall that $\mathcal{D}$ denotes the space of functions of finite energy. We let $\mathcal{D}_0$ denote the subspace of $f\in\mathcal{D}$ for which there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n\to f$ pointwise and $\mathcal{Q}(f-\varphi_n)\to 0$ as $n\to\infty$.

Theorem 1.19 (Domain of $D(Q^{(D)})$). *Let (b,c) be a graph over (X,m) with associated energy form $\mathcal{Q}_{b,c}$. Then, $Q^{(D)}=Q^{(D)}_{b,c,m}$ is the restriction of $\mathcal{Q}_{b,c}$ to*

$$D(Q^{(D)})=\mathcal{D}_0\cap\ell^2(X,m).$$

Remark. To put this result into perspective, we compare it with the corresponding statement for the Neumann form $Q^{(N)}=Q^{(N)}_{b,c,m}$. By definition, $Q^{(N)}$ arises as a restriction of $\mathcal{Q}_{b,c}$ to

$$D(Q^{(N)})=\mathcal{D}\cap\ell^2(X,m).$$

So, we see that the difference between the Dirichlet and Neumann boundary conditions comes from a corresponding difference between $\mathcal{D}$ and $\mathcal{D}_0$.

Proof. We let $Q=Q^{(D)}$ in the proof. To show

$$D(Q)=\mathcal{D}_0\cap\ell^2(X,m)$$

we will prove two inclusions.

$D(Q)\subseteq\mathcal{D}_0\cap\ell^2(X,m)$: By definition, $D(Q)$ is the closure of $C_c(X)$ with respect to $\|\cdot\|_Q$ given by $\|f\|_Q=(Q(f)+\|f\|)^{1/2}$. This immediately gives the statement as $\ell^2(X,m)$ convergence implies pointwise convergence.

$\mathcal{D}_0\cap\ell^2(X,m)\subseteq D(Q)$: Let $f\in\mathcal{D}_0\cap\ell^2(X,m)$. As Q is a closed form, the restriction of Q to the diagonal is lower semi-continuous. Thus, by Proposition 1.13, it suffices to find a sequence (χ_n) in $C_c(X)$ with $\chi_n\to f$ in $\ell^2(X,m)$ as $n\to\infty$ and $(Q(\chi_n))$ bounded.

Since $f \in \mathcal{D}_0$ we can find a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$. This implies, in particular, that the sequence $(Q(\varphi_n)) = (\mathcal{Q}(\varphi_n))$ is bounded. We will modify the sequence (φ_n) in order to obtain a sequence (χ_n) converging to f in $\ell^2(X, m)$. Consider

$$\psi_n = \varphi_n \wedge |f|.$$

Claim. We have:

- $\psi_n \in C_c(X)$ for all n.
- $\psi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q(\psi_n))_n$ is bounded.

Proof of the claim. The first two statements are straightforward. The last statement follows from

$$\begin{aligned} |\psi_n(x) - \psi_n(y)| &\leq |\varphi_n(x) - \varphi_n(y)| + |\ |f(x)| - |f(y)|\ | \\ &\leq |\varphi_n(x) - \varphi_n(y)| + |f(x) - f(y)|. \end{aligned}$$

Consider now $\chi_n = \psi_n \vee -|f|$. Then, we clearly have

$$\chi_n = -(-\psi_n \wedge |f|).$$

Thus, we can apply the reasoning of the previous claim to obtain:

- $\chi_n \in C_c(X)$ for all n.
- $\chi_n \to f$ pointwise as $n \to \infty$.
- The sequence $(Q(\chi_n))_n$ is bounded.

Moreover, by construction the sequence (χ_n) satisfies

$$-|f| \leq \chi_n \leq |f|.$$

Thus, by Lebesgue's dominated convergence theorem, the sequence (χ_n) converges to f in $\ell^2(X, m)$. Hence, the sequence (χ_n) has all of the desired properties. This finishes the proof. □

REMARK. It is possible to elaborate on the approximation of f and $C \circ f$ by a sequence in $C_c(X)$ (Exercise 1.13).

3. Approximation, domain monotonicity and the Markov property

A basic idea in the study of regular Dirichlet forms is to first investigate their restrictions to compact sets. In our context, this amounts to looking at restrictions to finite subsets of X. We will discuss various instances of this idea in this section. Together, they will provide very basic features of regular Dirichlet forms and the associated operators.

Excavation Exercise 1.3 discussing convergence of operators will be used in the proof of Lemma 1.21 below.

We recall that if (b, c) is a graph over (X, m), then the Laplacian $L = L^{(D)}$ associated to the regular Dirichlet form $Q = Q^{(D)}$ satisfies $\sigma(L) \subseteq [0, \infty)$, where $\sigma(L)$ denotes the spectrum of L. This follows as Q is a symmetric positive closed form, see Appendix B for details. In particular, we can then use the spectral theorem to define the semigroup and resolvent associated to L, that is, e^{-tL} for $t \geq 0$ and $(L+\alpha)^{-1}$ for $\alpha > 0$.

Using the spectral theorem, we can also show that both the semigroup and resolvent are bounded operators on $\ell^2(X,m)$ with $\|e^{-tL}\| \leq 1$ and $\|\alpha(L+\alpha)^{-1}\| \leq 1$ for all $t \geq 0$ and $\alpha > 0$, see Propositions A.32 and A.34 in Appendix A. We refer to the fact that the norm of both the semigroup and resolvent is uniformly bounded by 1 by saying that the semigroup is a *contraction semigroup* and the resolvent is a *contraction resolvent*.

The facts mentioned above follow from the general theory of forms and operators. In this section we will use the graph structure and the method of exhaustion via finite sets to establish a further property of both the semigroup and the resolvent associated to L. More specifically, if (X,m) is a discrete measure space and $A\colon \ell^2(X,m) \longrightarrow \ell^2(X,m)$ is a bounded operator, then A is said to have the *Markov property* or to be *Markov* if

$$0 \leq Af \leq 1$$

for any $f \in \ell^2(X,m)$ with $0 \leq f \leq 1$. We will show that both e^{-tL} and $\alpha(L+\alpha)^{-1}$ are Markov for every $t \geq 0$ and $\alpha > 0$.

We note that the Markov property consists of two separate inequalities. We recall that a function $f \in C(X)$ is called positive if $f \geq 0$. We call an operator mapping positive functions to positive functions *positivity preserving.* We call an operator mapping functions bounded above by 1 to functions bounded above by 1 *contracting.* Hence, we see that a bounded operator has the Markov property if and only if the operator is positivity preserving and contracting. For an abstract treatment of the Markov property and its relation to Dirichlet forms see Appendix C.

After this discussion of the property of interest, we now introduce the basic ideas for the exhaustion process. For (b,c) over (X,m) let $\mathcal{Q} = \mathcal{Q}_{b,c}$ be the associated energy form. For any finite set $K \subseteq X$, we denote the restriction of m to K by m_K and let $Q_K^{(D)}$ be the form defined on $\ell^2(K,m_K)$ by

$$Q_K^{(D)}(f) = \mathcal{Q}(i_K f)$$

for $f \in \ell^2(K,m_K)$. Here, $i_K\colon C(K) \longrightarrow C(X)$ is the canonical embedding, i.e., $i_K f$ is the extension of $f \in C(K)$ to X by setting $i_K f$ to be identically zero outside of K. Clearly, $Q_K^{(D)}$ is a closed form on $\ell^2(K,m_K)$ since the domain of $Q_K^{(D)}$ is the entire Hilbert space $\ell^2(K,m_K)$.

A short calculation then gives

$$Q_K^{(D)}(f) = \mathcal{Q}(i_K f) = \mathcal{Q}_{b_K,c_K}(f) + \sum_{x\in K} d_K(x) f^2(x),$$

where b_K is the restriction of b to $K \times K$, c_K is the restriction of c to K and

$$d_K(x) = \sum_{y \in X\setminus K} b(x,y)$$

describes the edge deficiency of a vertex in K compared to the same vertex in X. Thus, $Q_K^{(D)}$ is the Dirichlet form associated to the graph $(b_K, c_K + d_K)$ over (K, m_K), i.e.,

$$Q_K^{(D)} = \mathcal{Q}_{b_K, c_K + d_K}.$$

Clearly, $Q_K^{(D)}$ is regular as K is finite.

We denote the self-adjoint operator associated to $Q_K^{(D)}$ by $L_K^{(D)}$ and call it the *Dirichlet Laplacian with respect to* K. As K is finite, this operator is bounded and defined on the entire Hilbert space $\ell^2(K, m_K)$. We infer from Theorem 1.6 that

$$L_K^{(D)} f(x) = \frac{1}{m(x)} \left(\sum_{y\in K} b(x,y)(f(x) - f(y)) + \big(d_K(x) + c(x)\big) f(x) \right)$$

for all $f \in \ell^2(K, m_K)$ and $x \in K$. In particular,

$$\mathcal{L}(i_K f)(x) = L_K^{(D)} f(x)$$

for all $x \in K$ and thus $\mathcal{L} f = L_K^{(D)} f$ if f is supported on K.

We can use this explicit formula to obtain some information on the semigroup and resolvent of $L_K^{(D)}$, which is gathered in the next proposition. Part of this proposition can be inferred from the material presented in Chapter 0. To make the presentation self-contained, we provide a complete proof here.

Proposition 1.20 (Features of restrictions to finite sets). *Let (b,c) be a graph over (X,m).*

(a) *If $K \subseteq X$ is finite, the eigenvalues of $L_K^{(D)}$ are non-negative. If the graph is additionally connected and infinite, then the eigenvalues of $L_K^{(D)}$ are strictly positive and $L_K^{(D)}$ is invertible.* (*"Positivity"*)

(b) *If $K \subseteq X$ is finite and $f \in \ell^2(K, m_K)$ with $0 \le f \le 1$, then*

$$0 \le \alpha (L_K^{(D)} + \alpha)^{-1} f \le 1 \qquad \text{(“Markov”)}$$

for $\alpha > 0$. Furthermore, $(L_K^{(D)})^{-1} f \ge 0$ if the graph is connected and infinite.

(c) *If $K \subseteq H$ are finite subsets of X and $\alpha > 0$, then*

$$(L_K^{(D)} + \alpha)^{-1} f \le (L_H^{(D)} + \alpha)^{-1} f$$

on K for all $f \in \ell^2(K, m_K)$ with $f \geq 0$, where f is extended by zero on $H \setminus K$. If the graph is connected and infinite, then the statement holds also for $\alpha = 0$. (*"Domain monotonicity"*)

PROOF. (a) The first statement is clear from the inequality

$$\langle f, L_K^{(D)} f\rangle = Q_K^{(D)}(f) = \mathcal{Q}(i_K f) \geq 0$$

applied to the eigenfunctions of $L_K^{(D)}$. Now, assume that there exists an f_0 such that $L_K^{(D)} f_0 = 0$. Then,

$$\begin{aligned} 0 &= \langle f_0, L_K^{(D)} f_0\rangle \\ &= Q_K^{(D)}(f_0) \\ &= \sum_{x,y\in K} b(x,y)(f_0(x) - f_0(y))^2 + \sum_{x\in K}(c(x) + d_K(x))f_0^2(x) \end{aligned}$$

yields that f_0 is constant on every connected component of K. By the fact that the graph is connected and infinite, there exists a vertex $x \in K$ with $y \in X \setminus K$ such that $b(x,y) > 0$. Therefore, $d_K(x) > 0$. Indeed, this is true for every connected component of K. Hence, we have

$$0 = \langle f_0, L_K^{(D)} f_0\rangle \geq d_K(x) f_0^2(x) \geq 0.$$

Thus, $f_0 = 0$ and, therefore, 0 is not an eigenvalue. This shows the strict positivity of the eigenvalues of $L_K^{(D)}$ and invertibility follows.

(b) By (a) the resolvent $(L_K^{(D)} + \alpha)^{-1}$ exists for every $\alpha > 0$ and also for $\alpha = 0$ if the graph is connected and infinite. Consider now $f \in \ell^2(K, m_K)$ with $0 \leq f \leq 1$ and set $u = (L_K^{(D)} + \alpha)^{-1} f$. Then,

$$\begin{aligned} f(x)m(x) &= (L_K^{(D)} + \alpha)u(x)m(x) \\ &= \sum_{y\in K} b(x,y)(u(x) - u(y)) + \big(d_K(x) + c(x) + \alpha m(x)\big)u(x). \end{aligned}$$

We will investigate this equality for $x_M \in K$ such that $u(x_M)$ is the maximum of u and $x_0 \in K$ such that $u(x_0)$ is the minimum of u on K.

For x_0, we have

$$\sum_{y\in K} b(x_0, y)(u(x_0) - u(y)) \leq 0$$

and we infer from $f(x_0) \geq 0$ that $u(x_0) \geq 0$. For x_M we have

$$\sum_{y\in K} b(x_M, y)(u(x_M) - u(y)) \geq 0$$

and we infer from $f(x_M) \leq 1$ and $u(x_M) \geq u(x_0) \geq 0$ that $\alpha u(x_M) \leq 1$. This gives

$$0 \leq u(x_0) \leq u(x) \leq u(x_M) \leq \frac{1}{\alpha}$$

for all $x \in K$ and we have shown (b).

(c) Set $u_K = (L_K^{(D)} + \alpha)^{-1} f$, $u_H = (L_H^{(D)} + \alpha)^{-1} f$ and $v = u_H - u_K$. Then, on K we have

$$\begin{aligned}(L_H^{(D)} + \alpha)v &= (L_H^{(D)} + \alpha)(u_H - u_K)\\ &= f - (L_H^{(D)} + \alpha)u_K\\ &= f - \big((L_H^{(D)} + \alpha) - (L_K^{(D)} + \alpha) + (L_K^{(D)} + \alpha)\big)u_K\\ &= f - (L_H^{(D)} - L_K^{(D)})u_K - f\\ &= (L_K^{(D)} - L_H^{(D)})u_K\\ &= 0.\end{aligned}$$

We use this to show $v \geq 0$.

Clearly, $v \geq 0$ on $H \setminus K$ as u_K vanishes outside of K and $u_H \geq 0$ by (b). Consider now $x_0 \in K$ such that $v(x_0)$ is the minimum of v on K. Assume that $v(x_0) < 0$. Then, we obtain from the previous equality and the explicit formula for $L_H^{(D)}$ the contradiction

$$\begin{aligned}0 &= (L_H^{(D)} + \alpha)v(x_0)m(x_0)\\ &= \sum_{y \in H} b(x_0, y)(v(x_0) - v(y)) + (d_H(x_0) + c(x_0) + \alpha m(x_0))\, v(x_0)\\ &< 0.\end{aligned}$$

This contradiction shows $v(x_0) \geq 0$ and thus $v \geq 0$. $\square$

Our next result will show convergence of the restrictions to finite subsets for both the resolvent and the semigroup. In order to be able to state the result conveniently we will use the following notation.

Notation. Let (b, c) be a graph over (X, m), let $Q = Q_{b,c,m}^{(D)}$ be the associated regular Dirichlet form and $Q_K^{(D)}$ be the restriction of Q to the finite set $K \subseteq X$ with associated Dirichlet Laplacian $L_K^{(D)}$ acting on $\ell^2(K, m_K)$ as defined above. We extend $L_K^{(D)}$ by zero on the orthogonal complement of $\ell^2(K, m_K)$ in $\ell^2(X, m)$. We will extend functions Φ of $L_K^{(D)}$ accordingly, that is, for $f \in \ell^2(X, m)$, we write $\Phi(L_K^{(D)})$ for $i_K \Phi(L_K^{(D)})(f|_K)$. This is, in particular, used for the function $\Phi(\lambda) = (\lambda + \alpha)^{-1}$, i.e.,

$$(L_K^{(D)} + \alpha)^{-1} f \quad \text{for} \quad i_K (L_K^{(D)} + \alpha)^{-1}(f|_K),$$

but also applies to $\Phi(\lambda) = (\lambda + \alpha)$ or $\Phi(\lambda) = e^{-t\lambda}$. The extended operators will be denoted by the same symbols as the original ones.

Lemma 1.21 (Convergence of finite approximations). *Let (b, c) be a graph over (X, m) and let Q be the associated regular Dirichlet form with Laplacian L. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_n K_n$.*

(a) *If $f \in \ell^2(X,m)$ and $\alpha > 0$, then*

$$\lim_{n\to\infty} (L^{(D)}_{K_n} + \alpha)^{-1} f = (L+\alpha)^{-1} f.$$

(b) *If $f \in \ell^2(X,m)$ and $t \geq 0$, then*

$$\lim_{n\to\infty} e^{-tL^{(D)}_{K_n}} f = e^{-tL} f.$$

Furthermore, if additionally $f \geq 0$, then the sequences in both statements converge not only in $\ell^2(X,m)$ but also pointwise monotonically increasingly, i.e.,

$$(L^{(D)}_{K_n} + \alpha)^{-1} f \nearrow (L+\alpha)^{-1} f \quad \text{ and } \quad e^{-tL^{(D)}_{K_n}} f \nearrow e^{-tL} f$$

pointwise as $n \to \infty$.

Remark. The proof of (b) will actually show

$$\lim_{n\to\infty} \Phi(L^{(D)}_{K_n}) f = \Phi(L) f$$

for any $f \in \ell^2(X,m)$ and any function $\Phi\colon [0,\infty) \longrightarrow \mathbb{R}$ which is continuous and satisfies $\lim_{x\to\infty} \Phi(x) = 0$. We say that such functions vanish at infinity.

Proof. (a) In the proof we will use the following characterization of resolvents: Whenever Q is a positive closed form with associated self-adjoint operator L, the function f is an arbitrary element of the underlying Hilbert space and $\alpha > 0$, then $u = (L+\alpha)^{-1} f$ is the unique minimizer of

$$Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2$$

over $v \in D(Q)$. See Theorem E.1 for a proof of this characterization.

After decomposing f into positive and negative parts, we can restrict attention to $f \geq 0$. Define

$$u_n = (L^{(D)}_{K_n} + \alpha)^{-1} f.$$

Then, $u_n \geq 0$ by the Markov property in Proposition 1.20 (b).

By domain monotonicity, Proposition 1.20 (c), the sequence $(u_n(x))$ is monotone increasing for any $x \in X$. Moreover, we have $\|u_n\| \leq \alpha^{-1}\|f\|$ since the operators $(L^{(D)}_{K_n} + \alpha)^{-1}$ are bounded uniformly in norm by $1/\alpha$, as follows from the spectral theorem, see Proposition A.34. This implies that $(u_n(x))$ is also bounded for any $x \in X$. Thus, the sequence (u_n) converges pointwise and in $\ell^2(X,m)$ to a function $u \in \ell^2(X,m)$ by Lebesgue's dominated convergence theorem.

Let $\varphi \in C_c(X)$. Assume without loss of generality that the support of φ is contained in K_1. Then, $Q(\varphi) = Q^{(D)}_{K_n}(\varphi)$ for all n sufficiently large. Since Q is closed and thus lower semi-continuous, convergence of (u_n) to u and the minimizing property of u_n then give

$$\begin{aligned} Q(u) + \alpha \left\| u - \tfrac{1}{\alpha} f \right\|^2 &\leq \liminf_{n\to\infty} \left(Q(u_n) + \alpha \left\| u - \tfrac{1}{\alpha} f \right\|^2 \right) \\ &= \liminf_{n\to\infty} \left(Q(u_n) + \alpha \left\| u_n - \tfrac{1}{\alpha} f \right\|^2 \right) \\ &= \liminf_{n\to\infty} \left(Q_{K_n}^{(D)}(u_n) + \alpha \left\| u_n - \tfrac{1}{\alpha} f \right\|^2 \right) \\ &\leq \liminf_{n\to\infty} \left(Q_{K_n}^{(D)}(\varphi) + \alpha \left\| \varphi - \tfrac{1}{\alpha} f \right\|^2 \right) \\ &= Q(\varphi) + \alpha \left\| \varphi - \tfrac{1}{\alpha} f \right\|^2 . \end{aligned}$$

As $\varphi \in C_c(X)$ is arbitrary and Q is regular, this implies

$$Q(u) + \alpha \left\| u - \tfrac{1}{\alpha} f \right\|^2 \leq Q(v) + \alpha \left\| v - \tfrac{1}{\alpha} f \right\|^2$$

for any $v \in D(Q)$. Thus, u is a minimizer of

$$Q(v) + \alpha \left\| v - \tfrac{1}{\alpha} f \right\|^2 ,$$

so that u must then be equal to $(L+\alpha)^{-1} f$ by the characterization of the resolvent stated at the start of the proof.

(b) Let $C_0([0,\infty))$ be the vector space of all continuous functions $\Phi\colon [0,\infty) \longrightarrow \mathbb{R}$ with $\lim_{x\to\infty} \Phi(x) = 0$. Define for $\alpha > 0$ the function $\Phi_\alpha\colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\Phi_\alpha(x) = (x+\alpha)^{-1}.$$

Then, clearly $\Phi_\alpha \in C_0([0,\infty))$ for any $\alpha > 0$ and $\Phi_\alpha(L) = (L+\alpha)^{-1}$ by the functional calculus, see Definition A.21 in Appendix A.

Let $\mathcal{A}$ be the closure in the supremum norm of the linear span of Φ_α for $\alpha > 0$. Then, by (a) we have

$$\lim_{n\to\infty} \Phi(L_{K_n}^{(D)}) f = \Phi(L) f$$

for all $\Phi \in \mathcal{A}$ and $f \in \ell^2(X,m)$. We will show that for every $t \geq 0$, the function $[0,\infty) \longrightarrow \mathbb{R}$ given by $x \mapsto e^{-tx}$ belongs to $\mathcal{A}$, which will complete the proof. The statement for $t = 0$ is clear, so we assume that $t > 0$.

We note that it suffices to show that

$$\mathcal{A} = C_0([0,\infty)).$$

We will do so by proving the following claim and then applying the Stone–Weierstrass theorem.

Claim. The set $\mathcal{A}$ has the following properties:

- $\mathcal{A}$ separates the points of $[0,\infty)$ (i.e., for any $x, y \in [0,\infty)$ with $x \neq y$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \neq \Phi(y)$).
- $\mathcal{A}$ does not vanish identically at any point (i.e., for any $x \in [0,\infty)$ there exists a $\Phi \in \mathcal{A}$ with $\Phi(x) \neq 0$).
- $\mathcal{A}$ is an algebra.

Proof of the claim. The first two points follow directly by considering $\Phi = \Phi_1$. As for the last point, by definition, $\mathcal{A}$ is a vector space. Thus, it suffices to show that $\mathcal{A}$ is closed under multiplication. To show this it suffices to show $\Phi_\alpha \Phi_\beta \in \mathcal{A}$ for any $\alpha, \beta > 0$. For $\alpha \neq \beta$ this is clear as

$$\Phi_\alpha \Phi_\beta = \frac{1}{\alpha - \beta}(\Phi_\beta - \Phi_\alpha).$$

For $\alpha = \beta$ we can consider a sequence (β_n) of positive numbers with $\beta_n \to \beta = \alpha$ and $\beta_n \neq \beta$ for all n. Then, by what we have just shown $\Phi_\alpha \Phi_{\beta_n} \in \mathcal{A}$ as $\beta_n \neq \alpha$. Thus, $\Phi_\alpha \Phi_\beta \in \mathcal{A}$ as $\lim_{n\to\infty} \Phi_\alpha \Phi_{\beta_n} = \Phi_\alpha \Phi_\beta$ in the supremum norm. This finishes the proof of the claim.

Given the claim, the desired statement that $\mathcal{A} = C_0([0,\infty))$ follows directly from the Stone–Weierstrass theorem. This concludes the proof of (b).

In the case of $f \geq 0$, the fact that the sequence (u_n) given by $u_n = (L_{K_n}^{(D)} + \alpha)^{-1} f$ is monotonically increasing pointwise follows from Lemma 1.20 (c). The corresponding statement for $e^{-tL_{K_n}^{(D)}}$ follows from the connection between resolvents and semigroups. That is, from the formula

$$\left(\frac{k}{t}\left(x + \frac{k}{t}\right)^{-1}\right)^{k} = \left(1 + \frac{tx}{k}\right)^{-k} \to e^{-tx}$$

as $k \to \infty$ for any $t > 0$, it follows that

$$e^{-tL_{K_n}^{(D)}} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L_{K_n}^{(D)} + \frac{k}{t}\right)^{-1}\right)^{k} f$$

for any $f \in \ell^2(X,m)$ and $t > 0$, see Theorem A.35 for more details. $\square$

REMARK. The convergence given in the previous lemma is a characterization of regularity (Exercise 1.14).

Combining the Markov property of the resolvents of restrictions to finite sets proven in Lemma 1.20 (b) along with the convergence statements in Lemma 1.21 gives the Markov properties for the semigroups and resolvents associated to the regular form on the entire graph.

COROLLARY 1.22 (Markov property of resolvents and semigroups). *Let (b,c) be a graph over (X,m) with associated regular Dirichlet form Q and Laplacian L. Then, for any $f \in \ell^2(X,m)$ with $0 \leq f \leq 1$,*

$$0 \leq \alpha(L+\alpha)^{-1} f \leq 1 \qquad \text{and} \qquad 0 \leq e^{-tL} f \leq 1$$

for all $\alpha > 0$ and $t \geq 0$.

REMARK. It is not necessary for the function to be bounded in order for the positivity preserving property above to hold (Exercise 1.15).

PROOF. After suitable approximation procedures, it suffices to consider $\varphi \in C_c(X)$ with $0 \leq \varphi \leq 1$. Consider now an increasing sequence K_n of finite subsets of X with $X = \bigcup_n K_n$. In particular, we may assume that the support of φ is contained in K_n for all $n \in \mathbb{N}$. By Lemma 1.21 we have

$$(L+\alpha)^{-1}\varphi = \lim_{n\to\infty} (L_{K_n}^{(D)} + \alpha)^{-1}\varphi.$$

By the Markov property for finite sets, Lemma 1.20 (b), we have $0 \leq \alpha(L_{K_n}^{(D)} + \alpha)^{-1}\varphi \leq 1$. Combining these two observation we obtain the desired statement for the resolvents.

We now turn to proving the statement for the semigroups. The case $t = 0$ is clear so we restrict attention to the case $t > 0$. As above, the equality

$$e^{-tL}f = \lim_{k\to\infty} \left(\frac{k}{t}\left(L + \frac{k}{t}\right)^{-1}\right)^k f$$

for any $f \in \ell^2(X, m)$ given in Theorem A.35 gives the statement from the already shown statement for the resolvents. $\square$

REMARK (Second Beurling–Deny criterion). The preceding corollary also follows immediately from the general theory of Dirichlet forms, where it is referred to as one direction of the second Beurling–Deny criterion. Indeed, this is one of the characterizing features of Dirichlet forms. For finite sets this was discussed in Chapter 0. The case of arbitrary Dirichlet forms is treated in Appendix C. Here, we gave a direct proof as this method of proof is rather instructive and has further consequences which we establish below.

We now show that the resolvent generates the minimal solution to the Poisson problem for a positive function. This follows directly from the considerations above, the minimum principle and the convergence to solutions.

LEMMA 1.23 (Resolvents as minimal solutions to $(\mathcal{L} + \alpha)u = f$). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Let $\alpha > 0$ and $f \in \ell^2(X, m)$. If $u = (L + \alpha)^{-1}f$, then u belongs to $\mathcal{F}$ and satisfies*

$$(\mathcal{L} + \alpha)u = f.$$

Furthermore, if additionally $f \geq 0$, then u is the smallest $v \in \mathcal{F}$ with $v \geq 0$ and $(\mathcal{L} + \alpha)v \geq f$.

PROOF. We first show that u is a solution as stated. For $\alpha > 0$, we note that the resolvent $(L + \alpha)^{-1}$ maps $\ell^2(X, m)$ into $D(L) \subseteq D(Q) \subseteq \mathcal{D} \subseteq \mathcal{F}$, where the last inclusion follows by Proposition 1.4 (b) and the other inclusions follow from the definitions. By Theorem 1.6, the operator L is a restriction of $\mathcal{L}$ so that $u = (L + \alpha)^{-1}f \in \mathcal{F}$ satisfies

$$(\mathcal{L} + \alpha)u = f,$$

as claimed.

We now establish the minimality of u when additionally $f \geq 0$. We first note that $u \geq 0$ whenever $f \geq 0$ as the resolvent is positivity preserving by Corollary 1.22. Now, let $v \geq 0$ be another function with $v \in \mathcal{F}$ and $(\mathcal{L} + \alpha)v \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_n K_n$ and let $L_{K_n}^{(D)}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$. We recall that $L_{K_n}^{(D)}$ agrees with $\mathcal{L}$ on the set of functions supported in K_n. Let $f_n = f 1_{K_n}$,

$$u_n = (L_{K_n}^{(D)} + \alpha)^{-1} f_n$$

and extend u_n by 0 to $X \setminus K_n$. Then, letting $w_n = v - u_n$, w_n satisfies

- $(\mathcal{L} + \alpha)w_n = (\mathcal{L} + \alpha)v - (L_{K_n}^{(D)} + \alpha)u_n \geq f - f_n = 0$ on K_n
- $w_n \wedge 0$ attains a minimum on K_n since K_n is finite
- $w_n = v \geq 0$ on $X \setminus K_n$.

Hence, we can apply the minimum principle, Theorem 1.7, and find $w_n = v - u_n \geq 0$ on X. Therefore, $v \geq u_n$ on X.

Finally, we show that u_n converges to u and thus $v \geq u$, which will complete the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $(L_{K_n}^{(D)} + \alpha)^{-1} f_k$ for $n \geq k$. Then, Lemma 1.21 (a) gives

$$\lim_{n \to \infty} (L_{K_n}^{(D)} + \alpha)^{-1} f_k = (L + \alpha)^{-1} f_k.$$

Furthermore, Proposition A.34 gives

$$\|\alpha(L + \alpha)^{-1}\| \leq 1 \quad \text{ and } \quad \|\alpha(L_{K_n}^{(D)} + \alpha)^{-1}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $\alpha > 0$. Therefore, as $f_k \to f$ in $\ell^2(X, m)$ we have

$$\lim_{k \to \infty} (L + \alpha)^{-1} f_k = (L + \alpha)^{-1} f$$

and

$$\|(L_{K_n}^{(D)} + \alpha)^{-1}(f_n - f_k)\| \leq \frac{1}{\alpha}\|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\begin{aligned}\|u_n - u\| \leq \|(L_{K_n}^{(D)} + \alpha)^{-1}(f_n - f_k)\| + \|(L_{K_n}^{(D)} + \alpha)^{-1} f_k - (L + \alpha)^{-1} f_k\| \\ + \|(L + \alpha)^{-1}(f_k - f)\|,\end{aligned}$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. $\square$

As the resolvent associated to the operator coming from the regular Dirichlet form generates the minimal positive solution of the Poisson equation, so does the semigroup generate the minimal solution of the heat equation. This is discussed next.

We recall that a function

$$u \colon [0, \infty) \times X \longrightarrow \mathbb{R}$$

is called a solution of the heat equation with initial condition f if for all $x \in X$, the mapping $t \mapsto u_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t > 0$ with $u_0 = f$. We call u a supersolution of the heat equation with initial condition f if u satisfies all the assumptions above and instead of equality in the heat equation we have

$$(\mathcal{L} + \partial_t)u_t(x) \geq 0.$$

We now show that if the initial condition is positive, then the semigroup of the associated Laplacian generates the minimal positive supersolution of the heat equation.

Lemma 1.24 (Semigroup as the minimal solution of the heat equation). *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Let $f \in \ell^2(X, m)$. If*

$$u_t(x) = e^{-tL} f(x)$$

for $t \geq 0$ and $x \in X$, then u is a solution of the heat equation with initial condition f.

Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

Proof. As L is a restriction of $\mathcal{L}$ by Theorem 1.6, the fact that $u_t(x) = e^{-tL} f(x)$ is a solution of the heat equation with initial condition f for $f \in \ell^2(X, m)$ is a consequence of the spectral theorem and can be found as Theorem A.33 in Appendix A.

We now show minimality. Let f additionally satisfy $f \geq 0$. Then, by Corollary 1.22 we have $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in X$ as the semigroup is positivity preserving. Thus, u is a positive solution of the heat equation with initial condition f. Now, suppose that w is a positive supersolution of the heat equation with $w_0 \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_n K_n$ and let $L^{(D)}_{K_n}$ be the Dirichlet Laplacian on $\ell^2(K_n, m_{K_n})$. We recall that $L^{(D)}_{K_n}$ agrees with $\mathcal{L}$ on functions supported in K_n. We let $f_n = f 1_{K_n}$ and

$$u^{(n)}_t(x) = e^{-tL^{(D)}_{K_n}} f_n(x)$$

for $x \in K_n$ and $t \geq 0$. We extend $u^{(n)}$ by 0 to $[0, \infty) \times X \setminus K_n$. If $w^{(n)} = w - u^{(n)}$, then $w^{(n)}$ satisfies

- $(\mathcal{L} + \partial_t)w^{(n)} \geq 0$ on $(0, T) \times K_n$
- $w^{(n)} \wedge 0$ attains a minimum on the compact set $[0, T] \times K_n$ since $w^{(n)}$ is continuous
- $w^{(n)} \geq 0$ on $((0, T] \times (X \setminus K_n)) \cup (\{0\} \times K_n)$.

Hence, we can apply the minimum principle for the heat equation, Theorem 1.10, to obtain $w^{(n)} = w - u^{(n)} \geq 0$ on $[0,T] \times K_n$ for all $n \in \mathbb{N}$. Therefore, $w \geq u^{(n)}$ on $[0,T] \times X$ as $u^{(n)}$ vanishes outside of K_n and w is positive.

We now show that $u^{(n)}$ converges to u from which it follows that $w \geq u$, thereby completing the proof. Indeed, this can be seen by first fixing $k \in \mathbb{N}$ and considering $e^{-tL_{K_n}^{(D)}} f_k$ for $n \geq k$. Then, Lemma 1.21 (b) gives

$$\lim_{n\to\infty} e^{-tL_{K_n}^{(D)}} f_k = e^{-tL} f_k.$$

Furthermore, by Proposition A.32 in Appendix A we have

$$\|e^{-tL}\| \leq 1 \quad \text{and} \quad \|e^{-tL_{K_n}^{(D)}}\| \leq 1$$

for all $n \in \mathbb{N}$ and all $t \geq 0$. As $f_k \to f$ in $\ell^2(X,m)$ we have

$$\lim_{k\to\infty} e^{-tL} f_k = e^{-tL} f$$

and

$$\|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| \leq \|f_n - f_k\| \to 0$$

as $k, n \to \infty$. Thus, the triangle inequality implies

$$\|u^{(n)} - u\| \leq \|e^{-tL_{K_n}^{(D)}}(f_n - f_k)\| + \|e^{-tL_{K_n}^{(D)}} f_k - e^{-tL} f_k\| + \|e^{-tL}(f_k - f)\|,$$

where we have shown that all three terms go to 0 as $k, n \to \infty$. □

We have shown that the resolvent of an operator coming from a regular Dirichlet form is positivity preserving for $\alpha > 0$. We will now finish this section by showing that, in some cases, we can even deal with α which are not positive. This will be used later in Chapter 4.

We will write $Q \geq C$ for a Dirichlet form Q and $C \in \mathbb{R}$ if

$$Q(f) \geq C\|f\|^2$$

for all $f \in D(Q)$.

COROLLARY 1.25. *Let (b, c) be a graph over (X, m) with associated regular Dirichlet form Q and Laplacian L. Let $Q \geq C$ for $C \in \mathbb{R}$. If $\alpha > -C$, then $(L + \alpha)^{-1}$ is positivity preserving, i.e.,*

$$(L + \alpha)^{-1} f \geq 0$$

for $f \in \ell^2(X, m)$ with $f \geq 0$.

PROOF. By Corollary 1.22, we have $e^{-tL} f \geq 0$ for all $f \in \ell^2(X, m)$ with $f \geq 0$. From $Q \geq C$ we note by the variational characterization of the bottom of the spectrum that $\lambda_0(L) \geq C$ and thus $\sigma(L) \subseteq [C, \infty)$, see Theorem E.8. In particular, $(L + \alpha)^{-1}$ exists for all $\alpha > -C$.

This allows us to extend the Laplace transform formula from Theorem A.35 as follows: From

$$(x + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tx} dt,$$

which holds for all $x \geq C$ and $\alpha > -C$, we obtain

$$(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tL} dt$$

for all $\alpha > -C$. As $e^{-tL} f \geq 0$, this completes the proof. □

Remark. We note, in particular, that this result extends the corresponding statement from Corollary 1.22 to allow α to be negative when $C > 0$. In particular, for any connected finite set $K \subseteq X$ which contains a vertex which is connected to a vertex outside of K, it follows that the Dirichlet form $Q_K^{(D)}$ satisfies the assumptions of the corollary above by Proposition 1.20 (a). Therefore, the above result applies to the resolvent of any Dirichlet Laplacian $L_K^{(D)}$ for $K \subseteq X$ finite which is connected to a vertex outside of K.

4. Connectedness, irreducibility and positivity improving

In this section we discuss some of the consequences of connectedness of the graph. This geometric property translates directly into properties of the form as well as the associated semigroup and resolvent. Specifically, connectedness of the graph is equivalent to irreducibility of the form, a property which can be understood as stating that the form cannot be decomposed into two orthogonal parts. Furthermore, connectedness is equivalent to the fact that the semigroup or resolvent maps non-vanishing positive functions to strictly positive functions. This property is called positivity improving.

In the previous section we showed that the semigroup and resolvent associated to L always map positive functions to positive functions. This property is called positivity preserving. Here, we will show that connectedness is equivalent to a strict strengthening of this property.

We recall that a subset of X is called connected if any two points in the subset can be connected by a path consisting of vertices in the subset. A maximal connected subset is called a connected component and (b, c) is called connected if it consists of one connected component.

We now introduce the necessary concepts of irreducible forms and positivity improving operators. A quadratic form Q on $\ell^2(X, m)$ with domain $D(Q)$ is called *irreducible* if the only subsets $U \subseteq X$ such that $1_U D(Q) \subseteq D(Q)$ and

$$Q(f) = Q(1_U f) + Q(1_{X \setminus U} f)$$

for all $f \in D(Q)$ are either $U = \emptyset$ or $U = X$.

An operator A on $\ell^2(X, m)$ is called *positivity improving* if $Af > 0$ for all non-trivial $f \in D(A)$ with $f \geq 0$.

Theorem 1.26 (Characterization of connectedness and positivity improving). *Let (b, c) be a graph over (X, m) with associated regular*

Dirichlet form Q and Laplacian L. Then, the following statements are equivalent:

(i) (b, c) *is connected.*
(ii) Q *is irreducible.*
(iii) $(L + \alpha)^{-1}$ *is positivity improving for all* $\alpha > 0$.
(iv) e^{-tL} *is positivity improving for all* $t > 0$.

PROOF. (i) $\Longrightarrow$ (iv): Let $\varphi \in C_c(X)$ be positive and non-trivial. Let (K_n) be an increasing sequence of connected finite sets such that $X = \bigcup_n K_n$. Denote by $L_{K_n}^{(D)}$ the operators corresponding to the restrictions of Q to $C_c(K_n)$. By Proposition 1.20 (b), $(L_{K_n}^{(D)} + \alpha)^{-1}\varphi \geq 0$ for all $\alpha > 0$. Therefore,

$$e^{-tL_{K_n}^{(D)}}\varphi = \lim_{n\to\infty} \left(\frac{n}{t}\left(L_{K_n}^{(D)} + \frac{n}{t}\right)^{-1}\right)^n \varphi \geq 0,$$

so that the semigroup $e^{-tL_{K_n}^{(D)}}$ is positivity preserving.

Now, let $u(x, t) = e^{-tL_{K_n}^{(D)}}\varphi(x) \geq 0$ and assume that n is large enough so that the support of φ is included in K_n. We want to show that $u(x, t) > 0$ for all $x \in K_n$ and $t > 0$. If there exists $x_0 \in K_n$ and $t_0 > 0$ such that $u(x_0, t_0) = 0$, then (x_0, t_0) is a minimum for u in both variables. Having a minimum at t_0 gives

$$0 = \partial_t u(x_0, t_0) = -L_{K_n}^{(D)} u(x_0, t_0).$$

Now, having a minimum at x_0 yields $u(y, t_0) = 0$ for all $y \sim x_0$. As K_n is connected, this implies $u(x, t_0) = 0$ for all $x \in K_n$. However,

$$e^{t_0 L_{K_n}^{(D)}} u(x, t_0) = \varphi(x),$$

so that $\varphi = 0$ on K_n which gives a contradiction to the assumption on φ. Therefore, $e^{-tL_{K_n}^{(D)}}\varphi > 0$ for all $t > 0$, so that $e^{-tL_{K_n}^{(D)}}$ is positivity improving.

As we assume that $\varphi \geq 0$, by Lemma 1.21, we get $e^{-tL_{K_n}^{(D)}}\varphi \to e^{-tL}\varphi$ as $n \to \infty$ where the convergence is pointwise monotonically increasing. Therefore, we infer that $e^{-tL}\varphi > 0$.

For a non-trivial positive function $f \in \ell^2(X, m)$, let $f_n = 1_{K_n} f \in C_c(X)$. Then, f_n converges monotonically increasingly to f in $\ell^2(X, m)$ as $n \to \infty$. By Corollary 1.22 applied to the functions $f_{n+1} - f_n$, we have that $0 < e^{-tL} f_n \to e^{-tL} f$ where the convergence is pointwise monotonically increasing. Therefore, $e^{-tL} f > 0$.

(iv) $\Longrightarrow$ (iii): This follows directly from the Laplace transform formula in Theorem A.35, that is,

$$(L + \alpha)^{-1} = \int_0^\infty e^{-\alpha t} e^{-tL} dt.$$

(iii) $\Longrightarrow$ (ii): If the form Q is not irreducible, then there exists a proper non-trivial subset $U \subseteq X$ such that L decomposes into a direct sum of operators $L_U \oplus L_{X\setminus U}$ on $\ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$. Hence, the resolvent also decomposes into a direct sum $(L_U+\alpha)^{-1} \oplus (L_{X\setminus U}+\alpha)^{-1}$. In this case, taking a non-trivial function $f \in \ell^2(U, m_U)$ with $f \geq 0$ yields a non-trivial function $(f, 0) \in \ell^2(U, m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$ which is non-negative. However,

$$(L+\alpha)^{-1}(f,0) = ((L_U+\alpha)^{-1}f, (L_{X\setminus U}+\alpha)^{-1}0) = ((L_U+\alpha)^{-1}f, 0),$$

which is not strictly positive.

(ii) $\Longrightarrow$ (i): For any connected component U, we clearly have that $1_U\varphi \in D(Q)$ and

$$Q(\varphi) = Q(1_U\varphi) + Q(1_{X\setminus U}\varphi)$$

for any $\varphi \in C_c(X)$. We want to show that the same holds for $f \in D(Q)$ so that we may apply irreducibility to conclude connectedness.

Let $f \in D(Q)$ and let $\varphi_n \in C_c(X)$ be such that $\|\varphi_n - f\|_Q \to 0$ as $n \to \infty$. Then, $(1_U\varphi_n)$ is a Cauchy sequence in $\|\cdot\|_Q$ since

$$\begin{aligned}Q(1_U\varphi_n - 1_U\varphi_m) &\leq Q(1_U(\varphi_n - \varphi_m)) + Q(1_{X\setminus U}(\varphi_n - \varphi_m)) \\ &= Q(\varphi_n - \varphi_m)\end{aligned}$$

and $\|1_U\varphi_n - 1_U\varphi_m\| \leq \|\varphi_n - \varphi_m\|$ for all $n, m \in \mathbb{N}_0$. Hence, $(1_U\varphi_n)$ converges in $D(Q)$ so that $1_U f \in D(Q)$. Furthermore, as $Q(1_U\varphi_n) \to Q(1_U f)$ and $Q(\varphi_n) \to Q(f)$ as $n \to \infty$, it follows that

$$Q(f) = Q(1_U f) + Q(1_{X\setminus U} f).$$

By irreducibility, we infer that either $U = \emptyset$ or $U = X$. This shows that (b, c) is connected. $\square$

5. Boundedness and compactly supported functions

In this section we study basic facts about the domain of the operators. More specifically, we first characterize boundedness of the operators and the form. We then characterize when the formal Laplacian maps the finitely supported functions into ℓ^2.

We start with a characterization of boundedness. For a graph (b, c) over (X, m) we let $\mathcal{Q} = \mathcal{Q}_{b,c}$ and $\mathcal{L} = \mathcal{L}_{b,c,m}$ and recall that

$$\mathrm{Deg}(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y) + c(x)\right)$$

for $x \in X$ denotes the weighted degree. Furthermore, we recall that a closed form Q with domain $D(Q)$ is associated to the graph if $D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)})$ and Q is a restriction of $\mathcal{Q}$. We say that the corresponding operator L, which is a restriction of $\mathcal{L}$, is associated to the graph as well. The boundedness statement below shows that there is a unique such form whenever the weighted degree is bounded.

THEOREM 1.27 (Characterization of boundedness). *Let (b, c) be a graph over (X, m) and let L be a Laplacian associated to the graph with form Q. Then, the following statements are equivalent:*

(i) *The weighted degree* Deg *is a bounded function on X.*
(ii) *The form $\mathcal{Q}$ and, thus, Q is bounded on $\ell^2(X, m)$.*
(iii) *The operator $\mathcal{L}$ and, thus, L is bounded on $\ell^2(X, m)$.*

The equivalent statements, in particular, imply that $Q^{(D)} = Q = Q^{(N)}$. Moreover, if Deg *is bounded by $D < \infty$, then $Q \leq 2D$ and L is bounded by $2D$ on $\ell^2(X, m)$.*

PROOF. (i) $\Longrightarrow$ (ii): Suppose $\mathrm{Deg} \leq D$, i.e., $\sum_{y \in X} b(x, y) + c(x) \leq Dm(x)$ for all $x \in X$. Then for $f \in \ell^2(X, m)$ we have

$$\sum_{x \in X} \left(\sum_{y \in X} b(x, y) + c(x) \right) f^2(x) \leq D \sum_{x \in X} f^2(x) m(x) = D \|f\|^2.$$

Therefore, if $f \in \ell^2(X, m)$, then

$$\begin{aligned} \mathcal{Q}(f) &= \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))^2 + \sum_{x \in X} c(x) f^2(x) \\ &\leq \sum_{x,y \in X} b(x, y)(f^2(x) + f^2(y)) + \sum_{x \in X} c(x) f^2(x) \\ &\leq 2 \sum_{x,y \in X} b(x, y) f^2(x) + \sum_{x \in X} c(x) f^2(x) \\ &\leq 2D \|f\|^2, \end{aligned}$$

where we used the symmetry of b and Fubini's theorem in the second inequality above. This shows that $\ell^2(X, m) \subseteq \mathcal{D}$ and that $\mathcal{Q}$ is bounded by $2D$ on $\ell^2(X, m)$.

(ii) $\Longrightarrow$ (iii): The statements for L follow directly from the fact that

$$\|L\| = \sup_{f \in D(L), \|f\| \leq 1} \langle Lf, f \rangle = \sup_{f \in D(L), \|f\| \leq 1} Q(f)$$

as L is self-adjoint. In particular, if $\mathrm{Deg} \leq D$, then L is bounded by $2D$ from the argument above. As L is thus bounded, it follows that $D(L) = \ell^2(X, m)$ and since L and $\mathcal{L}$ agree on $D(L)$ by Theorem 1.6, the statement for $\mathcal{L}$ follows.

(iii) $\Longrightarrow$ (i): Let $e_x = 1_x / \sqrt{m(x)}$ for $x \in X$ and observe that $\|e_x\| = 1$ and

$$\langle Le_x, e_x \rangle = Q(e_x) = \mathrm{Deg}(x).$$

As we assume that L is bounded, this shows the statement. $\square$

We next investigate the issue of whether the space of functions of compact support $C_c(X)$ is in the domain of an associated operator L. We start by giving an example where $\mathcal{L}$ does not map $C_c(X)$ into $\ell^2(X,m)$, in which case $C_c(X)$ cannot be included in $D(L)$.

EXAMPLE 1.28 ($\mathcal{L}C_c(X)$ not in $\ell^2(X,m)$). Let $X = \mathbb{N}_0$ and $b(0,k) = b(k,0) = k^{-2}$ for $k \geq 1$ and $b(k,l) = 0$ for $k,l \geq 1$ with $c = 0$. Furthermore, let m be given by $m(k) = k^{-3}$ for $k \geq 1$ with $m(0) = 1$. Then,

$$\sum_{k=0}^{\infty} (\mathcal{L}1_0(k))^2 m(k) \geq \sum_{k=1}^{\infty} \frac{b^2(0,k)}{m(k)} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Thus, $\mathcal{L}1_0$ is not in $\ell^2(X,m)$.

The next theorem characterizes $C_c(X) \subseteq D(L)$ and gives some sufficient conditions for this to hold. In particular, $C_c(X) \subseteq D(L)$ if the graph is locally finite.

THEOREM 1.29 (Characterization of $C_c(X) \subseteq D(L)$). *Let (b,c) be a graph over (X,m) and let L be a Laplacian associated to the graph. Then, the following statements are equivalent:*

(i) $C_c(X) \subseteq D(L)$.
(ii) $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.
(iii) *The functions $X \longrightarrow [0,\infty)$, $y \mapsto b(x,y)/m(y)$ are in $\ell^2(X,m)$ for all $x \in X$.*
(iv) $\ell^2(X,m) \subseteq \mathcal{F}$.

Furthermore, the equivalent conditions above are satisfied if

$$\inf_{y \sim x} m(y) > 0$$

for all $x \in X$ which holds, in particular, if the graph is locally finite.

PROOF. (i) $\Longleftrightarrow$ (ii): Let Q be the form associated to L. From general properties, see Theorem B.11 in Appendix B,

$$D(L) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \text{there exists a unique } g \in \ell^2(X,m) \text{ with} \\ Q(h,f) = \langle h, g\rangle \text{ for all } h \in D(Q) \end{array} \right\}.$$

By Green's formula, Proposition 1.5, we have for all $h \in D(Q) \subseteq \mathcal{F}$, $\varphi \in C_c(X)$ and $g = \mathcal{L}\varphi$

$$Q(h,\varphi) = \sum_{x\in X} h(x)g(x)m(x).$$

The right-hand side is equal to $\langle h, g\rangle$ if and only if $g = \mathcal{L}\varphi \in \ell^2(X,m)$. Along with the fact that L is a restriction of $\mathcal{L}$ by Theorem 1.12, this shows the equivalence between (i) and (ii).

(ii) $\Longleftrightarrow$ (iii): Let φ_x be given by $\varphi_x(y) = b(x,y)/m(y)$. We observe that

$$\mathcal{L}1_x = \mathrm{Deg}(x)1_x - \varphi_x,$$

which yields $\mathcal{L}1_x \in \ell^2(X,m)$ if and only if $\varphi_x \in \ell^2(X,m)$.

(iii) $\Longleftrightarrow$ (iv): Assume that φ_x given by $\varphi_x(y) = b(x,y)/m(y)$ are in $\ell^2(X,m)$ for all $x \in X$. Then, for $f \in \ell^2(X,m)$, we get by the Cauchy–Schwarz inequality

$$\sum_{y \in X} b(x,y)|f(y)| = \sum_{y \in X} \varphi_x(y)|f(y)|m(y) \leq \|\varphi_x\|\|f\|.$$

Hence, $f \in \mathcal{F}$.

On the other hand, assume that $\ell^2(X,m) \subseteq \mathcal{F}$ and let $x \in X$. Define $N_x = \{y \in X \mid y \sim x\}$. Then, $\ell^1(N_x, b(x,\cdot)) = \{f \in \mathcal{F} \mid \operatorname{supp} f \subseteq N_x\}$ and we have

$$\ell^2(N_x, m1_{N_x}) \subseteq \ell^1(N_x, b(x,\cdot)).$$

By the closed graph theorem we infer the existence of a constant $C \geq 0$ such that for all $f \in \ell^2(X,m)$ and all $x \in X$

$$\sum_{y \in X} b(x,y)|f(x)| = \|f1_{N_x}\|_{\ell^1(N_x, b(x,\cdot))} \leq C\|f1_{N_x}\| \leq C\|f\|.$$

Therefore,

$$\sum_{y \in X} \varphi_x(y)|f(y)|m(y) = \sum_{y \in X} b(x,y)|f(y)| \leq C\|f\|.$$

Hence, $\varphi_x \in \ell^2(X,m)$ by the Riesz representation theorem.

Finally, the condition $\inf_{y \sim x} m(y) = C_x > 0$ implies that $\varphi_x \in \ell^2(X,m)$ for $x \in X$, since

$$\|\varphi_x\|^2 = \sum_{y \in X} \frac{b^2(x,y)}{m(y)} \leq \frac{1}{C_x} \sum_{y \in X} b^2(x,y) < \infty.$$

This shows the "in particular" statement. □

6. Graphs with standard weights

In this section we discuss a class of examples which have been of special interest in the literature, namely, graphs with standard weights. They appear as a special case in our framework. In particular, we apply the results of the previous section to characterize when the associated forms and operators on such graphs are bounded.

Definition 1.30 (Graphs with standard weights). Let (b,c) be a graph over X. If b takes values in $\{0,1\}$ and $c = 0$, we say that b is a graph with *standard weights*.

We denote the edges of the graph by

$$E = \{(x,y) \in X \times X \mid x \sim y\}.$$

For graphs with standard weights, the degree function deg given by $\deg(x) = \sum_{y\in X} b(x,y)$ is the *combinatorial degree*, i.e., if $x \in X$, then

$$\deg(x) = \#\{y \in X \mid x \sim y\} = \#\left(E \cap (\{x\} \times X)\right).$$

The assumption $\sum_{y\in X} b(x,y) < \infty$ clearly implies that graphs with standard weights are locally finite.

6.1. The energy form and the formal Laplacian for graphs with standard weights. We now explicitly write out the energy form and the Laplacian in the case of standard weights.

For a graph b with standard weights, the energy form $\mathcal{Q}$ is given by

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y\in X, x\sim y} (f(x) - f(y))^2$$

for $f \in C(X)$. Furthermore, by local finiteness, the domain $\mathcal{F}$ of the formal Laplacian consists of all functions, i.e.,

$$\mathcal{F} = C(X).$$

We denote the formal Laplacian $\mathcal{L}_{b,0}$ for graphs b with standard weights by $\widetilde{\Delta}$. This operator acts as

$$\widetilde{\Delta} f(x) = \sum_{y\in X, y\sim x} (f(x) - f(y)).$$

6.2. The counting measure. We now introduce the counting measure and give a boundedness criterion for the resulting combinatorial Laplacian.

The counting measure $m = 1$ counts the number of vertices in a subset of X. In this case, the degree and the weighted degree satisfy

$$\deg = \mathrm{Deg}$$

and are equal to the combinatorial degree.

In this case, we denote the Laplacian $L^{(D)}_{b,0,1}$ associated to $Q^{(D)}_{b,0,1}$ by Δ. By Theorem 1.6, Δ is a restriction of $\widetilde{\Delta}$.

We deduce the following corollaries from the results of the previous sections.

COROLLARY 1.31 (Characterization of boundedness). *Let b be a graph with standard weights and let $m = 1$ be the counting measure. Then, the following statements are equivalent:*

(i) *The combinatorial degree* deg *is a bounded function on X.*
(ii) *The form Q is bounded.*
(iii) *The operator Δ is bounded.*

PROOF. This follows directly from Theorem 1.27 and the equality of the combinatorial and weighted degrees, deg = Deg, in this case. $\square$

COROLLARY 1.32 ($C_c(X) \subseteq D(\Delta)$). *Let b be a graph with standard weights and let $m = 1$ be the counting measure. Then, $C_c(X) \subseteq D(\Delta)$.*

PROOF. As $m = 1$ is uniformly bounded from below by a positive constant, the statement follows directly from Theorem 1.29. $\square$

6.3. The normalizing measure. We now introduce the normalizing measure and discuss how the resulting Laplacian is always bounded.

The normalizing measure n is given by deg which is the combinatorial degree in the case of standard weights. This measure counts the number of edges for a subset of vertices, more specifically,

$$n(A) = \#E_A + \frac{1}{2}\#\partial_E A$$

for $A \subseteq X$, where $E_A = E \cap (A \times A)$ and

$$\partial_E A = E \cap (((X \setminus A) \times A) \cup (A \times (X \setminus A)))$$

(cf. Exercise 0.28). Letting $m = n$, the weighted degree Deg satisfies

$$\mathrm{Deg} = 1.$$

For the normalizing measure $n = \deg$, we denote the Laplacian $L_{b,0,n}^{(D)}$ associated to $Q_{b,0,n}^{(D)}$ by Δ_n and refer to Δ_n as the *normalized Laplacian*. By Theorem 1.6, Δ_n is a restriction of $\frac{1}{n}\widetilde{\Delta}$, that is

$$\Delta_n f(x) = \frac{1}{\deg(x)} \sum_{y \in X, y \sim x} (f(x) - f(y))$$

for $f \in D(\Delta_n)$ and $x \in X$.

COROLLARY 1.33 (Δ_n is bounded). *Let b be a graph with standard weights and let n be the normalizing measure. Then, the normalized Laplacian Δ_n is a bounded operator on $\ell^2(X, n)$. In particular, $C_c(X) \subseteq D(\Delta_n) = \ell^2(X, n)$.*

PROOF. This follows directly from Theorem 1.27 and the equality $\mathrm{Deg} = 1$ in this case. $\square$

Exercises

Excavation exercises.

Exercise 1.1 (The Hilbert space $\ell^2(X,m)$). Let (X,m) be a discrete measure space. Show that

$$\ell^2(X,m) = \{f \colon X \longrightarrow \mathbb{R} \mid \sum_{x\in X} f^2(x)m(x) < \infty\}$$

is a real Hilbert space with inner product given by

$$\langle f, g\rangle = \sum_{x\in X} f(x)g(x)m(x).$$

Exercise 1.2 (Denseness of $C_c(X)$ in $\ell^2(X,m)$). Let (X,m) be a discrete measure space. Show that $C_c(X)$ is dense in $\ell^2(X,m)$.

Exercise 1.3 (Closure convergence). Let (L_n) be a sequence of self-adjoint operators on a Hilbert space and let L be a self-adjoint operator. Assume that for a family $\{\Phi_\alpha\}_{\alpha\in I}$ of measurable bounded functions from $\mathbb{R}$ to $\mathbb{R}$ and some index set I we have

$$\lim_{n\to\infty} \Phi_\alpha(L_n)f = \Phi_\alpha(L)f$$

for all f in the Hilbert space and for all $\alpha \in I$. Let $\mathcal{A}$ be the closure of $\{\Phi_\alpha\}_{\alpha\in I}$ with respect to the supremum norm. Show that

$$\lim_{n\to\infty} \Phi(L_n)f = \Phi(L)f$$

for all $\Phi \in \mathcal{A}$ and f in the Hilbert space.

Example exercises.

Exercise 1.4 (Mutual independence of positivity preserving and contracting). Let (X,m) be a discrete measure space.

(a) Give an example of a self-adjoint operator with a positivity preserving but not contracting semigroup.
(b) Give an example of a self-adjoint operator which is neither positivity preserving nor contracting.
(c) Give an example of a self-adjoint operator whose semigroup is contracting but not positivity preserving.

Exercise 1.5 (Graph with strictly positive b). Give an example of a graph b over an infinite set X with $b(x,y) > 0$ for all $x, y \in X$ with $x \neq y$.

Exercise 1.6 (Infinite star graph). Consider the infinite star-graph with $X = \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m\colon X \longrightarrow (0,\infty)$ and $b\colon X \times X \longrightarrow [0,\infty)$ satisfying $b(x,y) = 0$ whenever $\{x,y\} \subseteq \mathbb{N}$ and $b(x,y) > 0$ whenever $x \neq y$ with $0 \in \{x,y\}$. Let Q be an associated form and L be the corresponding Laplacian.

(a) Characterize the boundedness of the associated form Q.
(b) Characterize the validity of $C_c(X) \subseteq D(L)$.

Exercise 1.7 (Dirichlet form not satisfying $C_c(X) \subseteq D(Q)$). Give an example of a discrete measure space (X,m) and a Dirichlet form Q on (X,m) which does not satisfy $C_c(X) \subseteq D(Q)$.

(Hint: Any expression of the form $Q(f) = \frac{1}{2}\sum_{x,y} b(x,y)(f(x) - f(y))^2$ with symmetric $b \geq 0$ will be compatible with contractions. If $\sum_y b(x,y) < \infty$ does not hold for all $x \in X$, the domain of Q cannot contain all of $C_c(X)$.)

Exercise 1.8 (Dirichlet form with $\mathcal{D} \subseteq \ell^\infty$). Give an example of a Dirichlet form over an infinite set X such that all functions of finite energy are bounded.

(Hint: An example may be given based on $X = \mathbb{N}$ and $b\colon X \times X \longrightarrow [0,\infty)$ with $b(x,y) = 0$ whenever $|x - y| > 1$. To achieve the desired boundedness it suffices (why?) to show finiteness of $\sum_{n\in\mathbb{N}} |f(n) - f(n+1)|$ for all functions of finite energy. Note that such functions satisfy (why?) $\sum_{n\in\mathbb{N}} b(n,n+1)|f(n) - f(n+1)|^2 < \infty$.)

Exercise 1.9 (Dirichlet form with $\mathcal{D} \subseteq \ell^2(X,m)$). Give an example of Dirichlet form over an infinite set such that all functions of finite energy belong to $\ell^2(X,m)$.

(Hint: Have a look at the preceding exercise.)

Extension exercises.

Exercise 1.10 (Forms associated to a graph). Show that a form Q with domain $D(Q) \subseteq \ell^2(X,m)$ is associated to a graph if and only if $C_c(X) \subseteq D(Q)$, the form Q is a restriction of $\mathcal{Q}$ and $D(Q)$ is complete with respect to $\|\cdot\|_{\mathcal{Q}}$.

Exercise 1.11 (Graphs over arbitrary sets). Let X be an arbitrary set and assume that $b\colon X \times X \longrightarrow [0,\infty)$ satisfies $b(x,y) = b(y,x)$, $b(x,x) = 0$ and

$$\sum_{z\in X} b(x,z) = \sup_{U\subseteq X,\text{finite}} \sum_{y\in U} b(x,y) < \infty$$

for all $x,y \in X$. Call a subset Y of X connected if for arbitrary $x,y \in Y$ there exists $n \in \mathbb{N}$ and $x_0,\ldots,x_n \in Y$ with $x_0 = x$, $x_n = y$

and $b(x_k, x_{k+1}) > 0$ for all $k = 0, \ldots, n-1$. Show that any connected subset of X is countable.

EXERCISE 1.12 (Graphs and forms on $C_c(X)$). Let $Q^{(\mathrm{comp})}$ be a form on $C_c(X) \times C_c(X)$ with $Q^{(\mathrm{comp})}(C \circ \varphi) \leq Q^{(\mathrm{comp})}(\varphi)$ for all $\varphi \in C_c(X)$ and all normal contractions $C\colon \mathbb{R} \longrightarrow \mathbb{R}$. Show that there exists a graph (b, c) over X such that $Q^{(\mathrm{comp})}$ is the restriction of $\mathcal{Q}_{b,c}$ to $C_c(X) \times C_c(X)$.

EXERCISE 1.13 (Approximating f and Ff). Let (X, m) be a discrete measure space. Let $Q^{(\mathrm{comp})}$ be a closable form on $C_c(X) \times C_c(X)$ with closure Q. Let $F\colon \ell^2(X, m) \longrightarrow \ell^2(X, m)$ be a continuous map with $F(C_c(X)) \subseteq C_c(X)$ and assume that $Q^{(\mathrm{comp})}(F\varphi) \leq Q^{(\mathrm{comp})}(\varphi)$ holds for all $\varphi \in C_c(X)$.

(a) Show that $Ff \in D(Q)$ for all $f \in D(Q)$ and $Q(Ff) \leq Q(f)$.
(b) Show that for any $f \in D(Q)$ there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ with respect to $\|\cdot\|_Q$ and $F\varphi_n \to Ff$ with respect to $\|\cdot\|_Q$.

EXERCISE 1.14 (Regularity and resolvent convergence). Let (X, m) be a discrete measure space. Let Q be a Dirichlet form on (X, m) such that $C_c(X) \subseteq D(Q)$ and let L be the self-adjoint operator associated to Q. For an increasing sequence of finite sets $K_n \subseteq X$ such that $X = \bigcup_n K_n$, let L_{K_n} be the operators corresponding to the restriction of Q to $C_c(K_n)$. Assume

$$\lim_{n\to\infty} (L_{K_n} + \alpha)^{-1}\varphi = (L + \alpha)^{-1}\varphi$$

for all $\alpha > 0$ and $\varphi \in C_c(X)$. Show that Q is regular.

EXERCISE 1.15 (Positivity preservation). Let A be a bounded operator on $\ell^p(X, m)$ for any $p \in [1, \infty]$ which has the Markov property, i.e., for all $f \in \ell^p(X, m)$ with $0 \leq f \leq 1$ we have $0 \leq Af \leq 1$. Show that A is positivity preserving, i.e., $Af \geq 0$ for all $f \geq 0$.

Notes

Most of the material found in this chapter is known to experts. Very roughly speaking, substantial parts of the chapter can be seen as an elaboration on the well-known Beurling–Deny formulae in the specific (and most simple) situation where the underlying locally compact space is just a discrete topological space. For a treatment of the general case, we refer to the monograph of Fukushima [**Fuk80**] and the subsequent textbook of Fukushima/Ōshima/Takeda [**FŌT11**]. Our presentation here is determined by our perspective of bringing together the geometry of graphs, spectral theory and Dirichlet forms. In this, we follow to a large extent the treatment of Keller/Lenz [**KL12**] and the subsequent discussion in Haeseler/Keller/Lenz/Wojciechowski [**HKLW12**]. Of course, this chapter generalizes the corresponding material in Chapter 0 which deals with finite sets, compare the notes there.

Section 1 can be seen as summarizing the setting and the basic perspective on Dirichlet forms on discrete sets developed in [**KL12, HKLW12**]. The Green's formula as it appears in Propositions 1.4 and 1.5 was first presented in [**HK11**].

The main result of Section 2, Theorem 1.18, shows the correspondence between graphs and regular Dirichlet forms on discrete sets. This can be derived from the Beurling–Deny formula as given in [**FŌT11**]. Here, we follow [**KL12**].

Approximation of regular Dirichlet forms by exhaustions of the space and related topics such a domain monotonicity, as presented in Section 3, appear in many places and are, indeed, a main tool in the study of regular Dirichlet forms. For manifolds, the fundamental paper on the construction of the semigroup by approximation is by Dodziuk [**Dod83**]. For graphs with standard weights and counting measure, the corresponding treatment goes back to the thesis of Wojciechowski [**Woj08**], see the articles [**Woj09, Web10**] for related material as well. The discussion here follows [**KL12**], which in turn is inspired by [**Woj08**]. Here, we also use approximation to prove the Markov property of the semigroup given in Corollary 1.22. This Markov property is, of course, well known and can be derived from abstract theory of Dirichlet forms, see e.g. [**FŌT11**].

The fact that the heat semigroup on a connected Riemannian manifold is positivity improving is shown in [**Dod83**] with corresponding results for graphs with standard weights and counting measure found in [**Woj08**]. The statement for graphs as considered in our presentation can be found in [**KL12**]. The characterization of connectivity in terms of positivity improving semigroups found in Section 4 is from [**HKLW12**]. That the concepts of irreducibility and positivity improving agree for semigroups is standard and can be found, for example, in [**FŌT11, RS75**].

The main result of Section 5 is Theorem 1.27. It is certainly a part of mathematical folklore. In the form stated here, it can be found in [**HKLW12**]. The equivalence between (i), (ii) and (iii) given in Theorem 1.29 goes back to [**KL12**]. The equivalence of these conditions to (iv) seems not to have appeared in print earlier.

Section 6 presents the main classes of examples found in the literature. It seems fair to say that a large part of the existing literature treats the normalized Laplacian as an analogue to the Laplace–Beltrami operator on a manifold. From this point of view, the general weighted graphs presented in this book and their Laplacians then correspond to weighted manifolds and the associated Laplacian.

For complementary textbooks on infinite graphs we refer the reader to the corresponding comments at the end of the notes to Chapter 0. As mentioned above, the textbook of Fukushima/Ōshima/Takeda [**FŌT11**] offers an excellent exposition on the theory of Dirichlet forms, but we also mention [**BH91, Dav89, MR92**]. Furthermore, the theory of Markov diffusion semigroups developed in [**BGL14**] is, in some sense, complementary to the discrete setting we treat here. Textbooks treating analysis on Riemannian manifolds include [**Cha84, Cha06, Gri09, Jos17**].

CHAMBER 2

Infinite Graphs – Toolbox

For cryin' out loud my style is wild so book me.
Not long is how long that this rhyme took me.
Ghostface Killah.

In this chapter we extend the theory of the key concepts introduced in the previous chapter. In particular, we collect various tools that are needed at later parts of the book and provide further conceptual insights. However, in contrast to the previous chapter, the material here is not required for all of the subsequent considerations. So, it is possible to skip over this chapter, dive into the material that follows and only come back here when coming across a topic where the material is needed.

A remarkable feature of semigroups and resolvents associated to graphs is that they can be extended to all ℓ^p spaces. This is ultimately a consequence of the Markov property. We discuss this extension in Section 1. This material is used in parts of Chapters 3 and 8 and is crucial for the considerations in Chapter 7.

In Section 2 we discuss restrictions of forms to subsets. While we already touched upon this topic in Section 3 for restrictions to finite sets, the general case is discussed here. This material puts the results of Section 3 in a wider perspective and is relevant for the material in Chapter 4.

A special feature of graphs, which is going to play a major role in subsequent developments, is the non-locality of the Dirichlet form. A direct consequence of the non-locality is the lack of a pointwise Leibniz rule and, even worse, the lack of a chain rule. This poses various challenges in applying standard techniques from analysis. We discuss this non-locality and give an extensive presentation of ways to deal with it in Section 3. This material will be used in Part 3, specifically in Chapters 12 and 13.

Most of the theory developed in this book needs a discrete set with a graph structure and a measure. However, certain parts can be developed without a measure. While this is not necessary for any of our main applications later, it is quite instructive to see this. We present the corresponding considerations in Section 4. This section can be omitted and is marked as optional.

We present a stochastic interpretation of the semigroup and a Feynman–Kac formula in the also optional Section 5. The general theory as developed in these sections is not really specific to discrete sets. In essence, it holds for general Dirichlet forms.

1. Generators, semigroups and resolvents on ℓ^p

In this section we discuss how a Dirichlet form gives rise to a semigroup and a resolvent on $\ell^p(X, m)$ for every $p \in [1, \infty]$. The basic idea is that a Laplacian L associated to a Dirichlet form induces a contraction Markov semigroup e^{-tL} for $t \geq 0$ and a contraction Markov resolvent $(L + \alpha)^{-1}$ for $\alpha > 0$ on $\ell^2(X, m)$. The Markov property then allows us to extend these to every ℓ^p space while preserving the contraction and Markov properties. The arising semigroups and resolvents agree on their common domains and have a symmetry property in that the semigroup and resolvent on $\ell^p(X, m)$ are adjoint to the semigroup and resolvent on $\ell^q(X, m)$ for $1/p + 1/q = 1$. Moreover, for $p \in [1, \infty)$, the semigroups and resolvents are strongly continuous.

We then discuss the generators of the semigroups and resolvents and show that their action agrees with that of the formal Laplacian. Furthermore, we show that the resolvent generates the minimal solution of the Poisson equation and the semigroup generates the minimal solution of the heat equation. Finally, we give criteria for the boundedness of the generators on $\ell^p(X, m)$.

We start this section with a short discussion of ℓ^p spaces. We then consider semigroups and resolvents associated to graphs and their generators in the subsequent subsections. Throughout this section we will make extensive use of general operator theory. So, for general background on spectral theory see Appendix A, for closed forms see Appendix B, for Dirichlet forms see Appendix C and for general semigroups and resolvents on Banach spaces see Appendix D. Excavation Exercises 2.1, 2.2 and 2.3 recall basic facts about ℓ^p spaces which will be used throughout this section.

Let (X, m) be a discrete measure space. For every $p \in [1, \infty)$, we define

$$\ell^p(X, m) = \{f \in C(X) \mid \sum_{x \in X} |f(x)|^p m(x) < \infty\}.$$

Then, $\ell^p(X, m)$ is a vector space with norm $\|\cdot\|_p$ given by

$$\|f\|_p = \left(\sum_{x \in X} |f(x)|^p m(x) \right)^{1/p}.$$

For $p = \infty$, we define

$$\ell^\infty(X, m) = \{f \in C(X) \mid \sup_{x \in X} |f(x)| < \infty\}.$$

This is a vector space with norm given by

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

In fact, neither $\ell^\infty(X,m)$ nor the norm $\|\cdot\|_\infty$ depend on the underlying measure. Therefore, whenever $\ell^\infty(X,m)$ appears independently of the other $\ell^p(X,m)$ spaces, we will write $\ell^\infty(X)$ for $\ell^\infty(X,m)$. The spaces $\ell^p(X,m)$ are complete with respect to $\|\cdot\|_p$ for $p\in[1,\infty]$.

1.1. Semigroups, resolvents and their generators. In this subsection we give a brief overview of the general theory of semigroups and resolvents on $\ell^p(X,m)$. This will be applied to graphs in the next subsection. For full details and proofs, we refer the reader to Appendix D.

We start with the definition of a strongly continuous contraction Markov semigroup. For $p\in[1,\infty]$, we denote the bounded linear operators on $\ell^p(X,m)$ by

$$B(\ell^p(X,m)) = \{L\colon \ell^p(X,m)\longrightarrow \ell^p(X,m) \mid L \text{ is linear and bounded}\}.$$

Definition 2.1 (Semigroup). Let (X,m) be a discrete measure space and let $p\in[1,\infty]$. A map $S\colon[0,\infty)\longrightarrow B(\ell^p(X,m))$ is called a *semigroup* on $\ell^p(X,m)$ if

$$S(s+t)=S(s)S(t)$$

for all $s,t\geq 0$. A semigroup S is called *strongly continuous* if

$$\lim_{t\to 0^+} S(t)f = f$$

for all $f\in\ell^p(X,m)$. A semigroup S is called a *contraction semigroup* if

$$\|S(t)\|_p \leq 1$$

for all $t\geq 0$. Finally, a semigroup is called a *Markov semigroup* if

$$0\leq S(t)f\leq 1$$

for all $t\geq 0$ and $f\in\ell^p(X,m)$ with $0\leq f\leq 1$.

Remark. Strong continuity of the semigroup on $\ell^p(X,m)$ implies that the map $[0,\infty)\longrightarrow[0,\infty)$ given by

$$t\mapsto \|S(t)f\|_p$$

is continuous for all $f\in\ell^p(X,m)$, see Proposition D.3 in Appendix D.

Example 2.2. If $A\in B(\ell^p(X,m))$, then e^{-tA} for $t\geq 0$ gives a strongly continuous semigroup which generates a solution of the parabolic equation involving A (Exercise 2.7). Furthermore, given an initial condition, this solution is unique (Exercise 2.8).

Any strongly continuous semigroup S defines an operator called the generator of the semigroup.

Definition 2.3 (Generator of a semigroup). Let (X, m) be a discrete measure space and let $p \in [1, \infty]$. If S is a strongly continuous semigroup, then the operator A with

$$D(A) = \{f \in \ell^p(X, m) \mid g = \lim_{t \to 0^+} \frac{f - S(t)f}{t} \text{ exists in } \ell^p(X, m)\}$$

and

$$Af = g$$

for $f \in D(A)$ is called the *generator* of S.

It follows that $D(A)$ is dense in $\ell^p(X, m)$ and that A is a closed operator. If S is additionally a contraction semigroup, then it can be shown that $A + \alpha$ is a bijection for $\alpha > 0$ with inverse given by

$$(A + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t) dt.$$

See Theorem D.9 for further details. We note that, in order to define the integral above, we need the notion of integration of Banach space-valued functions. This is defined as a Riemann integral via approximation by Riemann sums of step functions. Furthermore, we note that the properties of the generator of a strongly continuous semigroup characterize such generators (Exercise 2.9).

We now introduce resolvents and point out their connections to semigroups.

Definition 2.4 (Resolvents). Let (X, m) be a discrete measure space and let $p \in [1, \infty]$. A map $G \colon (0, \infty) \longrightarrow B(\ell^p(X, m))$ for $p \in [1, \infty]$ is called a *resolvent* on $\ell^p(X, m)$ if G satisfies the *resolvent identity*

$$G(\alpha) - G(\beta) = -(\alpha - \beta) G(\alpha) G(\beta)$$

for all $\alpha, \beta > 0$. A resolvent G is called *strongly continuous* if

$$\lim_{\alpha \to \infty} \alpha G(\alpha) f = f$$

for all $f \in \ell^p(X, m)$. A resolvent G is called a *contraction resolvent* if

$$\|\alpha G(\alpha)\|_p \leq 1$$

for all $\alpha > 0$. Finally, a resolvent G is called a *Markov resolvent* if

$$0 \leq \alpha G(\alpha) f \leq 1$$

for all $\alpha > 0$ and $f \in \ell^p(X, m)$ with $0 \leq f \leq 1$.

Remark. Strong continuity of the resolvent on $\ell^p(X, m)$ implies that the map $(0, \infty) \longrightarrow [0, \infty)$ given by

$$\alpha \mapsto \|G(\alpha) f\|_p$$

is continuous for all $f \in \ell^p(X, m)$ (Exercise 2.10).

Resolvents, as semigroups, have generators. In order for the definition to be meaningful, we note that the range of $G(\alpha)$ is independent of α for any resolvent. Furthermore, $G(\alpha)$ is a bijection onto its range and the expression $G(\alpha)^{-1}f - \alpha f$ is independent of α for any strongly continuous resolvent, see Proposition D.15. This allows us to define the generator of a resolvent.

DEFINITION 2.5 (Generator of a resolvent). Let (X, m) be a discrete measure space and let $p \in [1, \infty]$. If G is a strongly continuous resolvent on $\ell^p(X, m)$, then the operator A with $D(A) = \mathrm{Range}(G(\alpha))$ and

$$Af = G(\alpha)^{-1}f - \alpha f$$

for $f \in D(A)$ is called the *generator of the resolvent* G.

It follows that the generator A is closed and that $A+\alpha$ is a bijection with inverse $G(\alpha)$, see Corollary D.17.

We now highlight the connection between resolvents and semigroups. Namely, if S is a strongly continuous contraction semigroup, then

$$G(\alpha) = \int_0^\infty e^{-t\alpha} S(t)dt$$

defines a strongly continuous contraction resolvent. If A denotes the generator of S, then

$$G(\alpha) = (A + \alpha)^{-1}$$

so that A is also the generator of G. Therefore,

$$(A + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t)dt,$$

which is referred to as the Laplace transform formula. For further details and a proof, see Theorem D.18.

1.2. Graphs and Markov semigroups and resolvents on discrete spaces. We have seen that graphs gives rise to a strongly continuous Markov contraction semigroup and resolvent on $\ell^2(X, m)$. In this subsection we discuss that these semigroups and resolvents naturally extend to all $\ell^p(X, m)$ for $p \in [1, \infty]$.

Let (b, c) be a graph over a discrete measure space (X, m) and let L be an operator associated to the graph, see Definition 1.11. Using the functional calculus, see Proposition A.32, we define the operator $S\colon [0, \infty) \longrightarrow B(\ell^2(X, m))$ via

$$S(t) = e^{-tL}.$$

We refer to S as the *semigroup associated to* L. If $L = L^{(D)}$, the Laplacian arising from the regular Dirichlet form, we denote S by $S_{b,c,m}$ and call it the *semigroup associated to the graph* (b, c) *over* (X, m). Indeed, S is a strongly continuous contraction semigroup and whenever the form Q associated to L is a Dirichlet form, S even becomes a Markov semigroup. For the Dirichlet Laplacian $L^{(D)}$, the Markov property was already shown in Corollary 1.22 in the last chapter. For a general Dirichlet form Q this is a consequence of the second Beurling–Deny criterion, Theorem C.4 in Appendix C. These considerations are summarized in the next proposition.

PROPOSITION 2.6 (Markov property on $\ell^2(X,m)$ – semigroup). *Let (b,c) be a graph over (X,m) and let L be an associated operator. Then, S is a strongly continuous contraction semigroup with values in the self-adjoint operators on $\ell^2(X,m)$. If the form associated to L is a Dirichlet form, then S is even a Markov semigroup on $\ell^2(X,m)$. This holds, in particular, for $L^{(D)}$ and $L^{(N)}$.*

PROOF. That S is a strongly continuous contraction semigroup with values in the self-adjoint operators follows from functional calculus, see Proposition A.32 in Appendix A for details. That S is a Markov semigroup follows by Corollary 1.22 for $L = L^{(D)}$ and by Theorem C.4 in Appendix C for a general L. $\square$

The Markov property will allow us to extend S to all $\ell^p(X,m)$ spaces for $p \in [1,\infty]$. A crucial ingredient in this extension process is the fact that a Markov matrix defines a bounded operator on $\ell^p(X,m)$ for $p \in [1,\infty]$.

DEFINITION 2.7 (Markov matrix). Let (X,m) be a discrete measure space. A function $a\colon X \times X \longrightarrow \mathbb{R}$ is called a *Markov matrix* if a satisfies the following properties:

- $a(x,y) = a(y,x)$
- $a(x,y) \geq 0$
- $\sum_{z\in X} a(x,z)m(z) \leq 1$

for all $x,y \in X$.

With this notion we now show that a Markov matrix can be used to define a bounded operator on $\ell^p(X,m)$ for all $p \in [1,\infty]$.

LEMMA 2.8 (General bound for a Markov matrix). *Let (X,m) be a discrete measure space. Let a be a Markov matrix. Then, for all $f \in C(X)$,*

$$\sup_{x\in X} \sum_{y\in X} |a(x,y)f(y)|m(y) \leq \sup_{x\in X} |f(x)|,$$

and for $p \in [1,\infty)$,

$$\sum_{x\in X} \left(\sum_{y\in X} |a(x,y)f(y)|m(y) \right)^p m(x) \leq \sum_{x\in X} |f(x)|^p m(x).$$

Here, the value ∞ is allowed to occur.

In particular, for $p \in [1,\infty]$ the matrix a induces a bounded operator $A^{(p)}$ with norm not exceeding 1 on each $\ell^p(X,m)$ by

$$A^{(p)}f(x) = \sum_{y\in X} a(x,y)f(y)m(y).$$

Proof. The "in particular" statement is a direct consequence of the inequalities. Thus, it suffices to show these inequalities. The case $p=\infty$ is clear. Consider now $p \in (1,\infty)$. Let $q \in (1,\infty)$ satisfy $1/p+1/q=1$. Then, we can estimate

$$\begin{aligned}
&\sum_{x\in X}\left(\sum_{y\in X} a(x,y)|f(y)|m(y)\right)^p m(x)\\
&\quad= \sum_{x\in X}\left(\sum_{y\in X} |a(x,y)m(y)|^{1/q}|a(x,y)m(y)|^{1/p}|f(y)|\right)^p m(x)\\
&\quad\le \sum_{x\in X}\left(\sum_{y\in X} a(x,y)m(y)\right)^{p/q}\left(\sum_{y\in X} a(x,y)|f(y)|^p m(y)\right) m(x)\\
&\quad\le \sum_{x\in X}\sum_{y\in X} a(x,y)|f(y)|^p m(y)m(x)\\
&\quad= \sum_{y\in X} |f(y)|^p m(y)\sum_{x\in X} a(x,y)m(x)\\
&\quad\le \sum_{y\in X} |f(y)|^p m(y),
\end{aligned}$$

where we used the Hölder inequality in the third line, the fact that $\sum_{y\in X} a(x,y)m(y)$
≤ 1 in the fourth line, Fubini's Theorem in the fifth line, and $a(x,y)=a(y,x)$ and $\sum_{x\in X} a(y,x)m(x) \le 1$ in the last line. The case of $p=1$ follows in a similar manner by using Fubini's Theorem. This finishes the proof. □

Remark. Another approach to the result above is to observe that the bound holds easily for $p=1$ and $p=\infty$. The result then follows for general $p \in [1,\infty]$ by interpolation, see Theorem E.21.

We need a further piece of notation. Whenever $p,q \in [1,\infty]$ satisfy $1/p+1/q=1$ (where the cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$ are allowed) we can appeal to the Hölder inequality to infer that

$$(f,g) = \sum_{x\in X} f(x)g(x)m(x)$$

exists as an absolutely convergent sum for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. Then, $(\cdot,\cdot)$ is called the *dual pairing* between $\ell^p(X,m)$ and

$\ell^q(X,m)$. Of course, for $p = q = 2$, we just have

$$(f,g) = \sum_{x\in X} f(x)g(x)m(x) = \langle f,g\rangle.$$

THEOREM 2.9 (Extension theorem – semigroups). *Let (b,c) be a graph over (X,m). Let S be the semigroup of an operator associated to the graph arising from a Dirichlet form. Then, there exists a unique family of contraction Markov semigroups $S^{(p)}$ on $\ell^p(X,m)$ for $p \in [1,\infty]$ satisfying the following properties:*

- *$S^{(2)} = S$.* (*"Extension"*)
- *For all $t \geq 0$ and all $p,q \in [1,\infty]$ with $1/p + 1/q = 1$*
$$(S^{(p)}(t)f,g) = (f,S^{(q)}(t)g)$$
for $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$. (*"Symmetry"*)
- *For all $t \geq 0$ and all $p,q \in [1,\infty]$*
$$S^{(p)}(t)f = S^{(q)}(t)f$$
for all $f \in \ell^p(X,m) \cap \ell^q(X,m)$. (*"Consistency"*)

For $p \in [1,\infty)$, the semigroup $S^{(p)}$ is strongly continuous. For $p = \infty$, the semigroup $S^{(\infty)}$ is weak continuous, i.e., the map*

$$t \mapsto (S^{(\infty)}(t)f,g)$$

is continuous for all $f \in \ell^\infty(X,m)$ and $g \in \ell^1(X,m)$.

REMARK. The weak* continuity of $S^{(\infty)}$ can be seen to be equivalent to pointwise continuity in the discrete setting (Exercise 2.11).

PROOF. We first deal with the uniqueness statement. By consistency and the extension property, the semigroups are defined on $C_c(X)$. As $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ and all $S^{(p)}$ are bounded, this shows that the semigroups are uniquely determined on $\ell^p(X,m)$ for $p \in [1,\infty)$. For $p = \infty$, we note that the semigroup on $\ell^\infty(X,m)$ is uniquely determined by the semigroup on $\ell^1(X,m)$ by the symmetry condition.

We now turn to proving existence. By Proposition 2.6, $S(t) = e^{-tL}$ for $t \geq 0$ is a Markov semigroup of self-adjoint operators on $\ell^2(X,m)$. Now, for every $t \geq 0$, there exists a $p_t \colon X \times X \longrightarrow \mathbb{R}$ with

$$S(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$. This p is called the *heat kernel* of the semigroup of S. We will now show that p_t is a Markov matrix for every $t \geq 0$.

First, note that by direct calculation $S(t)1_y(x) = p_t(x,y)m(y)$. As $S(t)$ is self-adjoint, we get

$$p_t(x,y)m(y)m(x) = \langle S(t)1_y, 1_x\rangle = \langle 1_y, S(t)1_x\rangle = p_t(y,x)m(x)m(y)$$

so that $p_t(x,y) = p_t(y,x)$ for all $x,y \in X$ and $t \geq 0$.

Since $S(t)$ is Markov, it follows that

$$0 \leq S(t)1_y(x) = p_t(x,y)m(y).$$

Therefore, $p_t(x,y) \geq 0$ for all $x, y \in X$ and $t \geq 0$.

Finally, letting $K_n \subseteq X$ be finite such that $K_n \subseteq K_{n+1}$ and $X = \bigcup_n K_n$, it follows by the Markov property that $0 \leq S(t)1_{K_n} \leq 1$ and thus

$$0 \leq \sum_{y \in X} p_t(x,y)1_{K_n}(y)m(y) = \sum_{y \in K_n} p_t(x,y)m(y) \leq 1.$$

By the monotone convergence theorem

$$\sum_{y \in X} p_t(x,y)m(y) \leq 1$$

for every $x \in X$ and $t \geq 0$.

Hence, for every $t \geq 0$, p_t is a Markov matrix and Lemma 2.8 gives for any $p \in [1,\infty]$ that the operator $S^{(p)}(t)\colon \ell^p(X,m) \longrightarrow \ell^p(X,m)$ given by

$$S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y)$$

is bounded with norm not exceeding 1. We now show that these operators have the desired properties.

Markov property. As each p_t is a Markov matrix for every $t \geq 0$, each operator $S^{(p)}$ satisfies

$$0 \leq S^{(p)}(t)f \leq 1$$

whenever $0 \leq f \leq 1$ for $f \in \ell^p(X,m)$.

Consistency. By definition, we have

$$S^{(p)}(t)f(x) = \sum_{y \in X} p_t(x,y)f(y)m(y) = S^{(q)}(t)f(x)$$

for any $t \geq 0$ whenever $f \in \ell^p(X,m) \cap \ell^q(X,m)$.

$S^{(2)} = S$. This is clear from the definition of $S^{(p)}$.

Semigroup property for $S^{(p)}$. By the consistency of the family the space

$$\mathcal{C} = \bigcap_{p \in [1,\infty]} \ell^p(X,m)$$

is invariant under any $S^{(p)}(t)$ for $t \geq 1$ and $p \in [1,\infty]$. Moreover, the action of $S^{(p)}$ on $\mathcal{C}$ agrees with the action of S. As S satisfies $S(s)S(t) = S(s+t)$ for all $s,t \geq 0$, the same will hold for $S^{(p)}$ on $\mathcal{C}$. As $\mathcal{C}$ contains $C_c(X)$, the space $\mathcal{C}$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$. Then, the semigroup property follows on $\ell^p(X,m)$ for $p \in [1,\infty)$ as each $S^{(p)}(t)$ is a bounded operator. To deal with the case $p = \infty$, it suffices to consider $f \geq 0$. Any such function can be written as a monotone limit of functions in $C_c(X)$. By the Markov property, the operators $S^{(\infty)}(t)$ on $\ell^\infty(X,m)$ are compatible with monotone limits and the desired statement follows.

Symmetry. The symmetry property is clear for $f,g \in C_c(X)$. It then follows in the generality stated by approximating $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$, where $1/p + 1/q = 1$, by sequences (f_n) and (g_n) in $C_c(X)$.

Strong continuity for $p \in [1,\infty)$. From the strong continuity of S on $\ell^2(X,m)$ we infer pointwise continuity of p_t for $t \to 0^+$ in the sense that we have

$$p_t(x,y) = \frac{1}{m(y)} e^{-tL} 1_y(x) \to \frac{1}{m(y)} 1_y(x)$$

as $t \to 0^+$ for every $x, y \in X$.

We now treat the general case and let $f \in \ell^p(X,m)$ for $p \in [1,\infty)$. In order to simplify the notation we set

$$u_t(x) = S^{(p)}(t)f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y).$$

Let $\varepsilon > 0$. Since $C_c(X)$ is dense in $\ell^p(X,m)$ for $p \in [1,\infty)$ we can choose as finite subset $K \subseteq X$ with

$$\|(1-1_K)f\|_p^p < \varepsilon$$

so that

$$\|1_K f\|_p^p > \|f\|_p^p - \varepsilon.$$

For t sufficiently close to 0, we then infer from the pointwise continuity and the finiteness of K that

$$\|1_K(u_t - f)\|_p^p < \varepsilon.$$

Therefore, combining with the above, we obtain

$$\|1_K u_t\|_p^p > \|f\|_p^p - 2\varepsilon.$$

Moreover, as each $S^{(p)}(t)$ has norm not exceeding 1 we also have

$$\|u_t\|_p^p \leq \|f\|_p^p.$$

Therefore, for small enough $t > 0$, we infer from the last two inequalities

$$\|f\|_p^p \geq \|u_t\|_p^p = \|1_K u_t\|_p^p + \|(1-1_K)u_t\|_p^p > \|f\|_p^p - 2\varepsilon + \|(1-1_K)u_t\|_p^p.$$

Hence,

$$\|(1-1_K)u_t\|_p^p < 2\varepsilon.$$

This gives the desired continuity at 0 as

$$\|u_t - f\|_p \leq \|(1-1_K)u_t\|_p + \|1_K(u_t - f)\|_p + \|(1-1_K)f\|_p.$$

Weak continuity for $p = \infty$.* This follows from the strong continuity for $p = 1$ and the symmetry of the family. $\square$

As, by the previous theorem, $S^{(p)}$ is a strongly continuous contraction semigroup for all $p \in [1, \infty)$, it follows that $S^{(p)}$ has a generator, see Definition 2.3. We denote the generator of this semigroup by $L^{(p)}$. For $p = \infty$, we do not have a strongly continuous semigroup. However, we can define $L^{(\infty)}$ to be the Banach space adjoint of $L^{(1)}$, i.e.,

$$L^{(\infty)} = (L^{(1)})^*.$$

After this discussion of semigroups we now turn to resolvents. We start by discussing the resolvent on $\ell^2(X, m)$ and the connection between the resolvent and the semigroup in this case. We recall that as a Laplacian L associated to a graph comes from a positive form, it follows that $\sigma(L) \subseteq [0, \infty)$, where $\sigma(L)$ denotes the spectrum of L, see the discussion in Appendix B. In particular, for every $\alpha > 0$, the resolvent $(L + \alpha)^{-1}$ exists and is a bounded operator on $\ell^2(X, m)$. We define $G\colon (0, \infty) \longrightarrow B(\ell^2(X, m))$ by

$$G(\alpha) = (L + \alpha)^{-1}$$

and refer to it as the *resolvent associated* L. For $L = L^{(D)}$ we denote G by $G_{b,c,m}$ and call it the *resolvent associated to the graph* (b, c).

PROPOSITION 2.10 (Markov property on $\ell^2(X, m)$ – resolvents). *Let (b, c) be a graph over (X, m) and let L be an operator associated to the graph. Then, G is a strongly continuous contraction resolvent with generator L which takes values in the bounded self-adjoint operators. Moreover, for $\alpha > 0$,*

$$G(\alpha) = (L + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t)dt, \qquad (\text{“Laplace transform”})$$

where S is the semigroup associated to L. If the form associated to L is a Dirichlet form, then G is even a Markov resolvent. This holds, in particular, for $L^{(D)}$ and $L^{(N)}$.

PROOF. The spectral theorem easily gives that G is a strongly continuous contraction resolvent, see Proposition A.34 in Appendix A. The Laplace transform formula can also be shown by the spectral theorem, see Theorem A.35 in Appendix A. Finally, the Markov property follows from the Markov property of the semigroup, Proposition 2.6. □

Theorem 2.9 shows that the semigroup S on $\ell^2(X, m)$ can be extended to all $\ell^p(X, m)$ for $p \in [1, \infty]$. An analogous extension theorem for the resolvents is discussed next.

THEOREM 2.11 (Extension theorem – resolvents). *Let (b, c) be a graph over (X, m). Let G be the resolvent of L which is associated to the graph and arises from a Dirichlet form. Then, there exists a unique family of strongly continuous contraction Markov resolvents $G^{(p)}$ on $\ell^p(X, m)$ for $p \in [1, \infty]$ satisfying the following properties:*

- $G^{(2)} = G$. (*“Extension”*)

- *For all $\alpha > 0$ and all $p, q \in [1, \infty]$ with $1/p + 1/q = 1$*
$$(G^{(p)}(\alpha)f, g) = (f, G^{(q)}(\alpha)g)$$
for $f \in \ell^p(X, m)$ and $g \in \ell^q(X, m)$. *("Symmetry")*
- *For all $\alpha > 0$ and all $p, q \in [1, \infty]$*
$$G^{(p)}(\alpha)f = G^{(q)}(\alpha)f$$
for $f \in \ell^p(X, m) \cap \ell^q(X, m)$. *("Consistency")*

If $S^{(p)}$ is the contraction Markov semigroup with generator $L^{(p)}$, then $G^{(p)}$ satisfies
$$G^{(p)}(\alpha) = (L^{(p)} + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S^{(p)}(t) dt$$
for all $\alpha > 0$ and $p \in [1, \infty]$. *("Laplace transform")*

In particular, $L^{(p)}$ is also the generator of $G^{(p)}$.

Proof. *Uniqueness.* This is easy by the statement on the generator.

Existence. We define the resolvent, for $\alpha > 0$ and $p \in [1, \infty)$, by
$$G^{(p)}(\alpha) = \int_0^\infty e^{-t\alpha} S^{(p)}(t) dt$$
and we define the resolvent for $p = \infty$ to be the dual of $G^{(1)}$
$$G^{(\infty)}(\alpha) = (G^{(1)})^*(\alpha)$$
for $\alpha > 0$. As $S^{(p)}$ is a strongly continuous contraction semigroups for $p \in [1, \infty)$ by Theorem 2.9, from Theorem D.18 and Proposition D.21 $G^{(p)}$ are strongly continuous contraction resolvents. They are Markov since $S^{(p)}(t)$ is Markov for every $t \geq 0$ and all $p \in [1, \infty]$. Then, integrating the corresponding statements of Theorem 2.9, we find that this family has the claimed properties.

The Laplace transform. We first consider the case $p \in [1, \infty)$. Then, in the existence part of the proof we have defined
$$G^{(p)}(\alpha) = \int_0^\infty e^{-t\alpha} S^{(p)}(t) dt$$
for $\alpha > 0$. Now, as $L^{(p)}$ is the generator of $S^{(p)}$, Theorem D.18 directly gives the formula
$$\int_0^\infty e^{-t\alpha} S^{(p)}(t) dt = (L^{(p)} + \alpha)^{-1}.$$
We now turn to the case $p = \infty$. Here, by Proposition D.21, we have
$$G^{(\infty)}(\alpha) = (G^{(1)}(\alpha))^* = ((L^{(1)}+\alpha)^{-1})^* = ((L^{(1)})^*+\alpha)^{-1} = (L^{(\infty)}+\alpha)^{-1},$$
where the last equality follows by the definition of $L^{(\infty)}$. This finishes the proof. $\square$

REMARK. It is also possible to base a direct proof of the previous result on Lemma 2.8 and the properties of G given in Proposition 2.10 (Exercise 2.12).

1.3. Minimal solutions and the action of the generators. Given the generators on $\ell^p(X,m)$, we now extend several results previously proven for $\ell^2(X,m)$ to $\ell^p(X,m)$ for general p. In particular, we show the existence of minimal solutions to the Poisson and heat equations and describe the action of the generators on $\ell^p(X,m)$.

We start with the Poisson equation. Here, we show that the resolvent on $\ell^p(X,m)$ generates the minimal solution. For $p=2$, this was already shown in Lemma 1.23 in Section 3 by using exhaustion techniques. We now extend this result to cover all $\ell^p(X,m)$ spaces and all operators associated to graphs. To this end, we denote the extension of the resolvent $G_{b,c,m}$ associated to $L^{(D)}$ to $\ell^p(X,m)$ by $G^{(p)}_{b,c,m}$.

THEOREM 2.12 (Resolvents as minimal solutions to $(\mathcal{L}+\alpha)u=f$). *Let (b,c) be a graph over (X,m) and let L be an associated operator arising from a Dirichlet form. Let $p\in[1,\infty]$ and let $G^{(p)}$ be the resolvent on $\ell^p(X,m)$ associated to L. If $f\in\ell^p(X,m)$, $\alpha>0$ and*

$$u=G^{(p)}(\alpha)f,$$

then u belongs to $\mathcal{F}$ and satisfies the Poisson equation

$$(\mathcal{L}+\alpha)u=f. \qquad (\text{“Poisson equation”})$$

Furthermore, if additionally $f\geq 0$, then $u\geq 0$ and for the resolvent $G_{b,c,m}$ associated to $L^{(D)}$ we have that

$$u=G^{(p)}_{b,c,m}(\alpha)f$$

is the smallest $v\in\mathcal{F}$ with $v\geq 0$ and $(\mathcal{L}+\alpha)v\geq f$.

PROOF. Let $f\in\ell^p(X,m)$ for $p\in[1,\infty]$. Without loss of generality, we assume $f\geq 0$. For a general $f\in\ell^p(X,m)$, we can decompose $f=f_+-f_-$ into its negative and positive parts.

Let (K_n) be an increasing sequence of finite subsets of X with $X=\bigcup_n K_n$ and let $f_n=f1_{K_n}$ so that $f_n\in C_c(X)$ for all $n\in\mathbb{N}$. Let

$$u_n=G^{(p)}(\alpha)f_n.$$

As $G^{(p)}$ is Markov by Theorem 2.11, $u_n\geq 0$ for all $n\in\mathbb{N}$ and the sequence (u_n) is monotonically increasing and converges to $u=G^{(p)}(\alpha)f\in\ell^p(X,m)$ by the monotone convergence theorem.

As the resolvents agree on their common domain due to the consistency statement in Theorem 2.11 and $f_n\in C_c(X)\subseteq\ell^p(X,m)$ for all $p\in[1,\infty]$, we have

$$u_n=G^{(p)}(\alpha)f_n=G^{(2)}(\alpha)f_n$$

for all $n \in \mathbb{N}$. By Theorem 1.12, we infer that $L = \mathcal{L}$ on $D(L) = G^{(2)}(\alpha)\ell^2(X,m) \supseteq G^{(2)}C_c(X)$ and thus

$$(\mathcal{L} + \alpha)u_n = (L + \alpha)G^{(2)}(\alpha)f_n = f_n$$

for all $n \in \mathbb{N}$. We conclude that $u \in \mathcal{F}$ solves the Poisson equation by taking monotone limits, Lemma 1.8. Furthermore, $u = G^{(p)}(\alpha)f \geq 0$ for $f \geq 0$ since $G^{(p)}$ is Markov and, therefore, positivity preserving by Theorem 2.11.

Now, let $v \in \mathcal{F}$ with $v \geq 0$ satisfy $(\mathcal{L} + \alpha)v \geq f$. Therefore, $(\mathcal{L} + \alpha)v \geq f_n$ for all $n \in \mathbb{N}$ and as $u_n = G^{(2)}_{b,c,m}(\alpha)f_n$ is the minimum positive solution of this inequality by Lemma 1.23 we obtain $u_n \leq v$ for all $n \in \mathbb{N}$. Taking the limit gives $u \leq v$. $\square$

We now turn to determining the action of the generators on $\ell^p(X,m)$ for $p \in [1,\infty]$. We will show that the generators are restrictions of the formal Laplacian. This generalizes Theorem 1.6, dealing with $L^{(D)}$ on $\ell^2(X,m)$, and Theorem 1.12, which deals with all associated operators on $\ell^2(X,m)$. We recall by Theorem 2.11 that the generators of the extended semigroup and the extended resolvent on $\ell^p(X,m)$ agree.

Theorem 2.13 (Action of the generators on $\ell^p(X,m)$). *Let (b,c) be a graph over (X,m) and let L be an associated operator arising from a Dirichlet form. For $p \in [1,\infty]$, the generator $L^{(p)}$ of the semigroup (and the resolvent) extended to $\ell^p(X,m)$ is a restriction of the formal operator $\mathcal{L}$. In particular, $D(L^{(p)}) \subseteq \mathcal{F}$.*

Proof. If $f \in D(L^{(p)})$, then $g = (L^{(p)}+\alpha)f \in \ell^p(X,m)$. Moreover, by Theorem 2.12, $f = G^{(p)}(\alpha)g = (L^{(p)} + \alpha)^{-1}g$ solves

$$(\mathcal{L} + \alpha)f = g = (L^{(p)} + \alpha)f.$$

This gives $\mathcal{L}f = L^{(p)}f$ for all $f \in D(L^{(p)})$ and $p \in [1,\infty]$. $\square$

Remark. It is a non-trivial problem to determine explicitly the domains of the generators $L^{(p)}$. We will have more to say about this topic in the next subsection for the case of bounded operators as well as in Sections 2, 2 and 3.

We now show that the semigroups generate minimal solutions of the heat equation. This extends the result of Lemma 1.24 from $\ell^2(X,m)$ to $D(L^{(p)}) \subseteq \ell^p(X,m)$. In order to show existence, we restrict our attention to the case of $p \in [1,\infty)$ and use the general theory found in Appendix D for strongly continuous semigroups. The case of $p = \infty$, when the semigroup is not strongly continuous, will be handled in Chapter 7, where we explore bounded solutions of the heat equation.

We recall that a function $u\colon [0,\infty) \times X \longrightarrow \mathbb{R}$ is called a solution of the heat equation with initial condition f if $u_t = u(t,\cdot) \in \mathcal{F}$ for all

$t > 0$, $t \mapsto u_t(x)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ for all $x \in X$ and u satisfies

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for $x \in X$ and $t > 0$ with $u_0 = f$. Furthermore, u is called a supersolution with initial condition f if u satisfies the inequality $(\mathcal{L} + \partial_t)u \geq 0$ instead of equality in the above. For the proof of the next theorem, we invoke general semigroup theory, which is developed in Theorem D.6.

THEOREM 2.14 (Semigroups as minimal solutions to $(\mathcal{L}+\partial_t)u = 0$). *Let (b,c) be a graph over (X,m). Let $p \in [1,\infty)$ and let $S^{(p)}$ be the semigroup on $\ell^p(X,m)$ to an associated operator arising from a Dirichlet form. If $f \in D(L^{(p)})$, $t \geq 0$ and*

$$u_t = S^{(p)}(t)f,$$

then u is a solution of the heat equation

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

with initial condition $u_0 = f$. (*"Heat equation"*)

Furthermore, if additionally $f \geq 0$, then $u \geq 0$ and for the semigroup $S_{b,c,m}$ associated to $L^{(D)}$ we have that

$$u = S^{(p)}_{b,c,m}(\alpha)f$$

is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

PROOF. By Theorem 2.9, $S^{(p)}$ is a strongly continuous semigroup for $p \in [1,\infty)$ with generator $L^{(p)}$. Hence, by Theorem D.6 in Appendix D, $u_t = S^{(p)}(t)f$ is a solution of the equation $(L^{(p)} + \partial_t)u_t = 0$ with $u_0 = f$ and has the required differentiability and continuity properties. Now, by Theorem 2.13 directly above, $L^{(p)}$ is a restriction of $\mathcal{L}$. Thus, $u_t \in \mathcal{F}$ is a solution of the heat equation with initial condition f. This shows the first statement.

If $f \geq 0$, then u is positive as $S^{(p)}$ is Markov by Theorem 2.9. We now show the minimality of $u_t = S^{(p)}_{b,c,m}(t)f$ for the semigroup associated to the Dirichlet Laplacian. Let w be a positive supersolution of the heat equation with $w_0 \geq f$. Let (K_n) be an increasing sequence of finite subsets of X with $X = \bigcup_n K_n$ and let $f_n = f1_{K_n}$ so that $f_n \in C_c(X)$ for all $n \in \mathbb{N}$. Let

$$u^{(n)}_t = S^{(p)}_{b,c,m}(t)f_n.$$

As $S^{(p)}_{b,c,m}(t)$ is Markov for every $t \geq 0$ by Theorem 2.9, the sequence $(u^{(n)}_t)$ consists of positive functions, is monotonically increasing and converges to $u \in \ell^p(X,m)$ by the monotone convergence theorem.

From what we have shown above, $u^{(n)}$ satisfies

$$(\mathcal{L} + \partial_t)u^{(n)}_t = 0$$

for $t > 0$ and $u_0^{(n)} = f_n$ for all $n \in \mathbb{N}$. Furthermore, $u_t^{(n)} = S_{b,c,m}^{(2)}(t) f_n$ by Theorem 2.9 as $f_n \in C_c(X) \subseteq \ell^2(X,m)$ for $n \in \mathbb{N}$. As w is a positive supersolution of the heat equation with $w_0 \geq f_n$ we obtain $u^{(n)} \leq w$ by Lemma 1.24. Letting $n \to \infty$ gives $u \leq w$, which completes the proof. $\square$

1.4. Boundedness of the ℓ^p generators. Having determined the action of the generators on ℓ^p in the previous subsection, we now characterize when the generators are bounded. We also give applications to graphs with standard weights. This extends the considerations of Section 5 from ℓ^2 to ℓ^p.

Given a graph (b,c) over (X,m), we recall the definition of the weighted degree as

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x,y) + c(x) \right).$$

In Section 5 we have shown that any associated Laplacian on $\ell^2(X,m)$ is a bounded operator if and only if Deg is a bounded function on X. We now extend this result to all $\ell^p(X,m)$ spaces. Furthermore, we show that if the formal Laplacian is a bounded operator on one $\ell^p(X,m)$ space, then it is a bounded operator on all $\ell^p(X,m)$ spaces. In order to achieve this, we need the Riesz–Thorin interpolation theorem, see Theorem E.21 in Appendix E.

THEOREM 2.15 (Boundedness of $L^{(p)}$). *Let (b,c) be a graph over (X,m). Let $p \in [1,\infty]$ and $L^{(p)}$ be the generator of the extended semigroup of an associated operator arising from a Dirichlet form. Then, the following statements are equivalent:*

(i) *The weighted degree* Deg *is a bounded function on X.*
(ii) *The operator $\mathcal{L}$ and, thus, $L^{(p)}$ is bounded on $\ell^p(X,m)$ for all $p \in [1,\infty]$.*
(iii) *The operator $\mathcal{L}$ and, thus, $L^{(p)}$ is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$.*

Specifically, if Deg *is bounded by $D < \infty$, then $\mathcal{L}$ and $L^{(p)}$ are bounded by $2D$ on $\ell^p(X,m)$ for $p \in [1,\infty]$.*

In order to prove the theorem we first show a duality statement.

LEMMA 2.16 (Duality and boundedness). *Let (b,c) be a graph over (X,m) and let $p,q \in [1,\infty]$ be such that $1/p+1/q = 1$. If $\ell^p(X,m) \subseteq \mathcal{F}$ and the restriction of $\mathcal{L}$ to $\ell^p(X,m)$ is bounded, then $\ell^q(X,m) \subseteq \mathcal{F}$ and the restriction of $\mathcal{L}$ to $\ell^q(X,m)$ is bounded with the same bound.*

Proof. We denote the dual pairing of p and q with $1/p+1/q=1$ by $(\cdot,\cdot)$, i.e., for $f\in\ell^p(X,m)$ and $g\in\ell^q(X,m)$,

$$(f,g)=\sum_{x\in X}f(x)g(x)m(x).$$

To prove the statement of the lemma we treat the cases $p=1$ and $p>1$ separately.

For $p=1$, we first notice that $\ell^\infty(X)\subseteq\mathcal{F}$. Then, from Green's formula, Proposition 1.5, and Hölder's inequality, we infer that for all $f\in\ell^\infty(X)$ and $\varphi\in C_c(X)$

$$\left|\sum_{x\in X}(\mathcal{L}f)(x)\varphi(x)m(x)\right|=\left|\sum_{x\in X}f(x)(\mathcal{L}\varphi)(x)m(x)\right|\le C\|\varphi\|_1\|f\|_\infty,$$

where C is a bound for $\mathcal{L}$ on $\ell^1(X,m)$. Letting $\varphi=1_x/m(x)$ gives that

$$|\mathcal{L}f(x)|\le C\|f\|_\infty$$

for all $x\in X$ so that C is a bound for $\mathcal{L}$ on $\ell^\infty(X,m)$.

For $p>1$, let $\varphi\in C_c(X)\subseteq\ell^q(X,m)$ and let $f\in\ell^p(X,m)$. Then, again by Green's formula, Proposition 1.5, we get

$$\left|\sum_{x\in X}(\mathcal{L}\varphi)(x)f(x)m(x)\right|=\left|\sum_{x\in X}\varphi(x)(\mathcal{L}f)(x)m(x)\right|\le C\|\varphi\|_q\|f\|_p,$$

where C is a bound for $\mathcal{L}$ on $\ell^p(X,m)$. As $\|g\|_q=\sup_{h\in\ell^p(X,m)}(h,g)$ for $\|h\|_p=1$, the inequality above shows that $\mathcal{L}C_c(X)\subseteq\ell^q(X,m)$ and $\|\mathcal{L}\varphi\|_q\le C\|\varphi\|_q$ for all $\varphi\in C_c(X)$. This yields that $\mathcal{L}$ is a bounded operator on a dense subspace of $\ell^q(X,m)$. Thus, $\mathcal{L}$ can be extended to a bounded operator on $\ell^q(X,m)$ with C as a bound for $\mathcal{L}$ on $\ell^q(X,m)$.

Since functions in $\ell^q(X,m)$ can be approximated monotonically from below by functions in $C_c(X)$, this bounded operator can be seen to agree with the restriction of $\mathcal{L}$ to $\ell^q(X,m)$ by monotone convergence. This proves the lemma. $\square$

We now turn to the proof of our boundedness result.

Proof of Theorem 2.15. By Theorem 2.13 the generator $L^{(p)}$ of an extended semigroup on $\ell^p(X,m)$ is a restriction of $\mathcal{L}$. Thus, boundedness of $\mathcal{L}$ on $\ell^p(X,m)$ implies boundedness of $L^{(p)}$ for $p\in[1,\infty]$.

(i) $\Longrightarrow$ (ii): Assume that Deg is bounded. Then, for $f\in\ell^\infty(X)$ and $x\in X$,

$$\begin{aligned}\mathcal{L}f(x)&\le\frac{1}{m(x)}\sum_{y\in X}b(x,y)\left(|f(x)|+|f(y)|\right)+\frac{c(x)}{m(x)}|f(x)|\\&\le 2\|f\|_\infty\mathrm{Deg}(x).\end{aligned}$$

Thus, if Deg is bounded, then the restriction of $\mathcal{L}$ to $\ell^\infty(X)$ is bounded. By Lemma 2.16, we obtain that the restriction of $\mathcal{L}$ to $\ell^1(X,m)$ is bounded. By the Riesz–Thorin interpolation theorem, Theorem E.21, we obtain that the restriction of $\mathcal{L}$ to $\ell^p(X,m)$ is a bounded operator for all $p \in [1,\infty]$. Furthermore, we note that if Deg is bounded above by D, then the bound on the operator is $2D$. This proves (ii) and the "specifically" statement at the end of the theorem.

(ii) $\Longrightarrow$ (iii): This is obvious.

(iii) $\Longrightarrow$ (i): Assume $\mathcal{L}$ is bounded on $\ell^p(X,m)$ for some $p \in [1,\infty]$. By Lemma 2.16, $\mathcal{L}$ is then also bounded on $\ell^q(X,m)$ for q such that $1/p + 1/q = 1$. By the Riesz–Thorin interpolation theorem, Theorem E.21, $\mathcal{L}$ is bounded on $\ell^s(X,m)$ for all s between p and q and, in particular, for $s = 2$. Then, with $e_x = 1_x/\sqrt{m(x)}$ for $x \in X$, we deduce from the boundedness of $\mathcal{L}$ on $\ell^2(X,m)$ that

$$C \geq \langle \mathcal{L} e_x, e_x\rangle = \mathcal{L} 1_x(x) = \mathrm{Deg}(x)$$

for all $x \in X$. Thus, Deg is a bounded function. $\square$

We end this section by discussing the case of graphs with standard weights and counting measure.

Example 2.17 (Graphs with standard weights and counting measure). We let $m = 1$ and denote the Banach spaces $\ell^p(X,1)$ for $p \in [1,\infty]$ by $\ell^p(X)$. We consider a graph with standard weights, i.e., $b\colon X \times X \longrightarrow \{0,1\}$ and $c = 0$. As in Section 6, we denote the Laplacian $L^{(D)}_{b,0,1}$ associated to $Q^{(D)}_{b,0,1}$ by Δ. Then, Δ is a restriction of $\widetilde{\Delta}\colon C(X) \longrightarrow C(X)$ acting as

$$\widetilde{\Delta} f(x) = \sum_{y \in X, y\sim x} (f(x) - f(y)).$$

Furthermore, we denote the generators of the semigroup of Δ on $\ell^p(X)$ by $\Delta^{(p)}$. By Theorem 2.13, the operators $\Delta^{(p)}$ are also restrictions of $\widetilde{\Delta}$. Moreover, we recall that the combinatorial degree $\deg\colon X \longrightarrow \mathbb{N}_0$ is given by

$$\deg(x) = \#\{y \in X \mid y \sim x\},$$

which is equal to the weighted degree $\mathrm{Deg} = \deg$ in this case. Then, we obtain by Theorem 2.15 the equivalence of the following statements:

(i) The combinatorial degree deg is a bounded function on X.
(ii) The operator $\Delta^{(p)}$ is bounded for all $p \in [1,\infty]$.
(iii) The operator $\Delta^{(p)}$ is bounded for some $p \in [1,\infty]$.

2. Forms associated to graphs and restrictions to subsets

In this section we consider the restrictions of the energy form to subsets of X. We have already encountered one iteration of this idea in Section 3, where we restricted the form to finite sets. Here, we restrict to arbitrary sets, thus extending this theory. Furthermore, we also discuss the idea of extending a form.

Excavation Exercises 2.4 and 2.5, which recall facts about weakly convergent subsequences in a Hilbert space and the Banach–Saks theorem, will be used in this section.

Let (b, c) be a graph over (X, m) and let $\mathcal{Q} = \mathcal{Q}_{b,c}$ be the energy form over X. The main focus of our investigations is on $Q^{(D)}$. This form comes about as a restriction of $\mathcal{Q}$ to the smallest subspace of $\ell^2(X, m)$ containing $C_c(X)$ and giving a closed form. However, restrictions to other subspaces are also of interest. For example, we have already encountered the form $Q^{(N)}$. This is the restriction of $\mathcal{Q}$ to the largest subspace of $\ell^2(X, m)$ giving a closed form, that is, all functions of finite energy in $\ell^2(X, m)$. A study of various relevant restrictions is provided in this section.

We will first develop some pieces of the general theory of restricting forms and then apply this theory to our setting of graphs. Let Q be a form on $\ell^2(X, m)$, which we assume is positive and symmetric throughout, with domain $D(Q)$ such that $C_c(X) \subseteq D(Q)$. Let $U \subseteq X$ and let $C(U)$ denote the set of real-valued functions on U. Furthermore, let m_U denote the restriction of m to U. Then, we can define the restriction of Q to U, denoted by Q_U, as the restriction of Q to those functions which are supported in U. More specifically, we let

$$i_U : C(U) \longrightarrow C(X)$$

be the extension by zero of functions on U to X and define Q_U as a form with domain $D(Q_U)$ defined by

$$D(Q_U) = \{f \in \ell^2(U, m_U) \mid i_U f \in D(Q)\}$$

and

$$Q_U(f) = Q(i_U f)$$

for $f \in D(Q_U)$ and $Q_U(f) = \infty$ for $f \notin D(Q_U)$.

We now establish some properties of Q_U which follow from properties of Q.

Proposition 2.18 (What Q_U inherits from Q). *Let Q be a form on $\ell^2(X, m)$ with $C_c(X) \subseteq D(Q)$ and let $U \subseteq X$.*

(a) *If Q is a closed form, then Q_U is a closed form with $C_c(U) \subseteq D(Q_U)$.*
(b) *If Q is a Dirichlet form, then Q_U is a Dirichlet form.*
(c) *If Q is a regular Dirichlet form, then Q_U is a regular Dirichlet form.*

Proof. (a) From the definitions, it is clear that $C_c(U) \subseteq D(Q_U)$ if $C_c(X) \subseteq D(Q)$ and that Q_U is a form if Q is a form. We now show that Q_U is closed if Q is closed. This amounts to showing that Q_U is lower semi-continuous on $\ell^2(U, m_U)$, see Theorem B.9. Therefore, let $f_n \to f$ in $\ell^2(U, m_U)$ as $n \to \infty$. Then, $i_U f_n \to i_U f$ in $\ell^2(X, m)$ as $n \to \infty$ and from the assumption that Q is closed, we have

$$Q_U(f) = Q(i_U f) \leq \liminf_{n\to\infty} Q(i_U f_n) = \liminf_{n\to\infty} Q_U(f_n).$$

(b) Let $f \in D(Q_U)$ and let C be a normal contraction. As $f \in \ell^2(U, m_U)$, it follows that $C \circ f \in \ell^2(U, m_U)$. Furthermore, $i_U f \in D(Q)$ by definition and so $C \circ i_U f \in D(Q)$ as Q is a Dirichlet form. As $i_U(C \circ f) = C \circ i_U f$ since $C(0) = 0$, it follows that $C \circ f \in D(Q_U)$. Finally, as Q is compatible with normal contractions, we find

$$Q_U(C \circ f) = Q(i_U(C \circ f)) = Q(C \circ i_U f) \leq Q(i_U f) = Q_U(f).$$

This shows that Q_U is a Dirichlet form.

(c) In (b) we have already shown that Q_U is a Dirichlet form if Q is a Dirichlet form. To establish the regularity of Q_U under the assumption that Q is regular requires some work. Note, however, that we will later only need the case when U is finite or cofinite, i.e., the case when $X \setminus U$ is finite, and these two cases can be treated with substantially less work (Exercise 2.13).

Let $f \in D(Q_U)$ so that $i_U f \in D(Q)$. As Q is regular, there exists a sequence (φ_n) in $C_c(X)$ with

$$\varphi_n \to i_U f \text{ in } \ell^2(X, m) \quad \text{and} \quad Q(i_U f - \varphi_n) \to 0$$

as $n \to \infty$. We will modify φ_n to become a sequence with support in U and which still satisfies the above properties. This will prove the statement.

To this end, we define

$$\psi_n = (|i_U f| \wedge \varphi_n) \vee (-|i_U f|).$$

As $i_U f$ has support in U, it follows that ψ_n has support in U. Furthermore, $|f - \psi_n| \leq |f - \varphi_n|$ and, hence, $\psi_n \to f$ in $\ell^2(U, m)$ and $\psi_n \to i_U f$ in $\ell^2(X, m)$.

For functions $g, h \in C(X)$ it is easy to see that

$$g \wedge h = \frac{g + h - |g - h|}{2} \quad \text{and} \quad g \vee h = \frac{g + h + |g - h|}{2}.$$

As Q is positive, it follows that $Q(g \pm h) \leq 2(Q(g) + Q(h))$ for $g, h \in D(Q)$. Therefore, if $g, h \in D(Q)$ we find, using that Q is a Dirichlet form,

$$\begin{aligned} Q(g \wedge h) &\leq \frac{1}{2}\left(Q(g + h) + Q(|g - h|)\right) \\ &\leq \frac{1}{2}\left(Q(g + h) + Q(g - h)\right) \\ &= Q(g) + Q(h), \end{aligned}$$

and similarly,

$$Q(g \vee h) \leq Q(g) + Q(h).$$

Given this, it is immediate that the sequence $(Q(\psi_n))$ is bounded since $(Q(\varphi_n))$ is bounded.

Note that $\langle g, h\rangle_Q = Q(g,h) + \langle g, h\rangle$ is an inner product on $D(Q)$ with associated norm $\| \cdot \|_Q$. Moreover, as Q is a closed form, $D(Q)$ is complete with respect to this inner product. Hence, by Theorem B.9, $H_Q = (D(Q), \langle \cdot, \cdot \rangle_Q)$ is a Hilbert space.

The boundedness of $(Q(\psi_n))$ and the fact that (ψ_n) converges in $\ell^2(X,m)$ then gives that we can consider (ψ_n) as a bounded sequence in the Hilbert space H_Q. Hence, it contains a weakly convergent subsequence. Without loss of generality, we assume that the sequence itself converges weakly to some $g \in D(Q)$. Now, by the Banach–Saks theorem we can find a subsequence (ψ_{n_k}) such that

$$\widetilde{\psi}_N = \frac{1}{N}\sum_{k=1}^{N} \psi_{n_k} \to g$$

in the Hilbert space H_Q as $N \to \infty$. Then, $\widetilde{\psi}_N$ must also converge to g in $\ell^2(X,m)$. As ψ_n converges to $i_U f$ in $\ell^2(X,m)$, we conclude that $i_U f = g$. So, we conclude that $\widetilde{\psi}_N$ converges in $\| \cdot \|_Q$ to $i_U f$. Clearly, $\widetilde{\psi}_N$ are still supported on U and this finishes the proof. $\square$

Remark. The considerations of the preceding proposition can be adapted to treat restrictions to substantially more general subspaces than $\ell^2(U, m_U)$ (Exercise 2.14).

Remark. Subspaces of the form $\ell^2(U, m_U)$ are clearly invariant under normal contractions. In particular, they are closed under taking the modulus. In fact, they may be characterized by the order ideal or the multiplicative ideal properties (Exercise 2.15).

If Q is a closed form, then Q_U is closed by the previous proposition for $U \subseteq X$. Hence, the restriction of Q to $C_c(U) \times C_c(U)$ is closable and we denote its closure by $Q_U^{(D)}$.

As follows by Lemma 1.15, part (c) of Proposition 2.18 says that the form Q_U can also be defined as a closure when Q is a regular Dirichlet form. In this sense, restriction to subsets and taking closures commute. We note that when Q is a regular Dirichlet form, Q must come from a graph by Theorem 1.18.

Corollary 2.19 (Closure of restriction equals restriction of closure). *Let Q be a regular Dirichlet form on $\ell^2(X,m)$ and $U \subseteq X$. Then,*

$$Q_U = Q_U^{(D)}.$$

Proof. By Lemma 1.15 and the proposition above we have $C_c(U) \subseteq D(Q_U)$. Since $i_U C_c(U) = C_c(U)$, the restriction of Q and Q_U coincide on $C_c(U)$ and the result follows as Q_U is closed by the previous proposition. $\square$

The preceding results give not only information on restrictions of regular forms but also on restrictions of any form associated to a graph. Recall that a form Q on $\ell^2(X,m)$ is associated to a graph (b,c) if Q is closed, $D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)})$ and Q is a restriction of $Q^{(N)}$. Equivalently, Q is a closed restriction of $\mathcal{Q}_{b,c}$ with $C_c(X) \subseteq D(Q)$.

Corollary 2.20. *Let (b,c) be a graph over (X,m). Let Q be a form associated to (b,c) and let $U \subseteq X$. Then, Q_U is an extension of $Q_U^{(D)}$.*

Proof. As Q is associated to (b,c), the domain $D(Q)$ of Q contains $C_c(X)$ and Q is a restriction of $Q^{(N)}$. So, the domain of Q_U must contain $C_c(U)$ from Proposition 2.18 (a) and Q_U is a restriction of Q.

We now show that Q_U must be an extension of $Q_U^{(D)}$. Let $f \in D(Q_U^{(D)})$. Then $f \in \ell^2(U,m_U)$ and $i_U f \in D(Q^{(D)}) \subseteq D(Q)$ as Q is associated to the graph. In particular, $f \in D(Q_U)$. Furthermore,

$$Q_U^{(D)}(f) = Q^{(D)}(i_U f) = Q(i_U f) = Q_U(f).$$

Therefore, Q_U is an extension of $Q_U^{(D)}$. $\square$

The preceding result naturally raises the question if the form Q_U is associated to a graph whenever Q is associated to a graph. This is indeed the case. We now give the details on this connection.

Let (b,c) be a graph and let $U \subseteq X$. We define the graph $(b_U, c_U + d_U)$ over (U,m_U) by $b_U \colon U \times U \longrightarrow [0,\infty)$ via

$$b_U(x,y) = b(x,y),$$

c_U via the restriction of c to U and $d_U \colon U \longrightarrow [0,\infty)$ via

$$d_U(x) = \sum_{y \in X \setminus U} b(x,y).$$

Then we show next that $\mathcal{Q}_{b,c}(f) = \mathcal{Q}_{b_U,c_U+d_U}(f)$ for all $f \in \mathcal{D}$ with support contained in U.

Proposition 2.21 (Restricting energy forms to subsets). *Let (b,c) be a graph over (X,m) and let $U \subseteq X$. Then, the restriction of $\mathcal{Q}_{b,c}$ to the set of functions in $\mathcal{D}$ with support in U is given by $\mathcal{Q}_{b_U,c_U+d_U}$.*

PROOF. Let $f \in \mathcal{D}$ have support in U. Then, a short calculation gives that

$$\begin{aligned}\mathcal{Q}_{b,c}(f) &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))^2 + \sum_{x\in X} c(x)f^2(x)\\ &= \frac{1}{2}\sum_{x,y\in U} b(x,y)(f(x)-f(y))^2 + \frac{1}{2}\sum_{y\in U}\sum_{x\in X\setminus U} b(x,y)f^2(y)\\ &\qquad + \frac{1}{2}\sum_{x\in U}\sum_{y\in X\setminus U} b(x,y)f^2(x) + \sum_{x\in U} c(x)f^2(x)\\ &= \frac{1}{2}\sum_{x,y\in U} b(x,y)(f(x)-f(y))^2 + \sum_{x\in U}(c(x)+d_U(x))f^2(x),\end{aligned}$$

which proves the statement. □

Denote the formal operator associated to $\mathcal{Q}_{b_U,c_U+d_U}$ by $\mathcal{L}_U$ and its domain by $\mathcal{F}_U$. Specifically,

$$\mathcal{F}_U = \{f \in C(U) \mid \sum_{y\in U} b_U(x,y)|f(y)| < \infty \text{ for all } x \in U\}$$

with

$$\mathcal{L}_U f(x) = \frac{1}{m(x)}\sum_{y\in U} b_U(x,y)(f(x)-f(y)) + \frac{c_U(x)+d_U(x)}{m(x)}f(x)$$

for all $f \in \mathcal{F}_U$ and $x \in U$.

Applying the above to forms associated to graphs gives the following result.

COROLLARY 2.22 (Restricting forms associated to graphs). *Let (b,c) be a graph over (X,m). Let Q be a form associated to (b,c) and let $U \subseteq X$. Then, Q_U is associated to the graph $(b_U, c_U + d_U)$ and the self-adjoint operator associated to Q_U is a restriction of $\mathcal{L}_U$.*

It is worth noting that the action of $\mathcal{L}_U$ on a function f can essentially be though of as the action of $\mathcal{L}$ on $i_U f$. More specifically, the following is true.

PROPOSITION 2.23 (Action of the restriction of $\mathcal{L}$). *Let (b,c) be a graph over (X,m) and let $U \subseteq X$. Then,*

$$\mathcal{L}_U f(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(i_U f(x) - i_U f(y)) + \frac{c(x)+d_U(x)}{m(x)} i_U f(x)$$

for all $x \in U$ and $f \in \mathcal{F}_U$.

PROOF. Due to $i_U f = 0$ on $X \setminus U$ and, by the definition of d_U, we have

$$d_U(x)f(x) = \sum_{y\in X\setminus U} b(x,y) i_U f(x) = \sum_{y\in X\setminus U} b(x,y)(i_U f(x) - i_U f(y))$$

for $x \in U$. Now, a direct computation gives

$$\begin{aligned}\mathcal{L}_U f(x) &= \frac{1}{m(x)} \sum_{y \in U} b_U(x,y)(f(x) - f(y)) + \frac{c_U(x) + d_U(x)}{m(x)} f(x) \\ &= \frac{1}{m(x)} \sum_{y \in X} b(x,y)(i_U f(x) - i_U f(y)) + c(x) i_U f(x)\end{aligned}$$

for $x \in U$, which proves the proposition. $\square$

Remark. It is very tempting to write the equality in the proposition above as

$$\mathcal{L}_U f = (\mathcal{L} i_U f)|_U.$$

Indeed, this is just the statement of the proposition whenever $i_U f$ belongs to $\mathcal{F}$. However, in general, $i_U f$ will not belong to $\mathcal{F}$ when $f \in \mathcal{F}_U$ and $\mathcal{L} i_U f$ is not defined in this case.

Whenever Q is a form on $\ell^2(X,m)$ and U is a proper subset of X, the form Q_U is defined on a different Hilbert space than Q. This poses a problem if we want to compare Q and Q_U. For this reason, it is sometimes desirable to extend Q_U to a form on $\ell^2(X,m)$. Here, the natural extension is by setting the form to be zero on $\ell^2(X \setminus U, m_{X\setminus U}) \subset \ell^2(X,m)$. We finish this section by discussing some details of this extension process.

For $U \subseteq X$, we let $\pi_U \colon C(X) \longrightarrow C(U)$, $\pi_U f(x) = f|_U(x) = f(x)$ for $x \in U$ and $f|_U$ is the restriction of $f \in C(X)$ to U. We define the extension $\widehat{Q}_U$ of Q_U by

$$D(\widehat{Q}_U) = \{f \in \ell^2(X,m) \mid \pi_U f \in D(Q_U)\}$$

and, for $f \in D(\widehat{Q}_U)$,

$$\widehat{Q}_U(f) = Q_U(\pi_U f).$$

Now, clearly $i_U \pi_U f = 1_U f$, where 1_U is the characteristic function of U. So, we arrive at the following representation of $\widehat{Q}_U$

$$D(\widehat{Q}_U) = \{f \in \ell^2(X,m) \mid 1_U f \in D(Q)\}$$

and, for $f \in D(\widehat{Q}_U)$,

$$\widehat{Q}_U(f) = Q(1_U f).$$

Proposition 2.24. *Let Q be a closed form on $\ell^2(X,m)$ and let $U \subseteq X$. Then, $\widehat{Q}_U$ is a closed form on $\ell^2(X,m)$. Moreover, identifying $\ell^2(X,m)$ with $\ell^2(U,m_U) \oplus \ell^2(X \setminus U, m_{X\setminus U})$, we have*

$$D(\widehat{Q}_U) = D(Q_U) \oplus \ell^2(X \setminus U, m_{X\setminus U}) \quad \text{and} \quad \widehat{Q}_U = Q_U \oplus 0.$$

In particular, we have

$$\widehat{L}_U = L_U \oplus 0$$

for the operators $\widehat{L}_U$ and L_U associated to $\widehat{Q}_U$ and Q_U.

PROOF. We first show that $\widehat{Q}_U$ is closed. As usual, we extend all forms by ∞ outside of their domain. Let now $f_n \to f$ in $\ell^2(X,m)$ as $n \to \infty$. Then, clearly $1_U f_n \to 1_U f$ in $\ell^2(U, m_U)$ as $n \to \infty$. Hence, by the fact that Q is closed, we obtain that

$$\widehat{Q}_U(f) = Q(1_U f) \leq \liminf_{n\to\infty} Q(1_U f_n) = \liminf_{n\to\infty} \widehat{Q}_U(f_n).$$

This shows that $\widehat{Q}_U$ is closed. The other statements follow easily. □

3. The curse of non-locality: Leibniz and chain rules

A major difficulty in applying methods from analysis on manifolds and partial differential equations to discrete settings is the absence of a pointwise Leibniz rule and the absence of a chain rule. In this section we collect several estimates which allow us to circumvent this absence.

We first briefly discuss what we mean by non-locality. A form Q is called *local* if $Q(f,g) = 0$ whenever f and g have disjoint supports. Unlike in the case of energy forms appearing in the context of manifolds, this property clearly fails for the energy form on graphs whenever the supports are disjoint but are connected by an edge. This has consequences for local rules such as the Leibniz and the chain rule.

For the Leibniz rule, there exist three alternative formulas which follow from basic algebraic manipulations. Furthermore, we have an integrated Leibniz rule. We discuss this in Subsection 3.1.

For the missing chain rule, a first remedy is provided by the mean value theorem. More specifically, the mean value theorem states that for a given differentiable function $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $f\colon X \longrightarrow \mathbb{R}$ and $x, y \in X$ there exists a $\xi \in (f(x) \wedge f(y), f(x) \vee f(y))$ such that

$$\varphi(f(x)) - \varphi(f(y)) = \varphi'(\xi)(f(x) - f(y)).$$

As ξ is, for the most part, not given explicitly, $\varphi'(\xi)$ has to be estimated, for example, by $\varphi'(f(x) \wedge f(y))$ or $\varphi'(f(x) \vee f(y))$ if φ is monotone. However, this is often not sufficient for the purpose at hand. Therefore, we give more explicit estimates for functions φ that will find application in the chapters that follow. We address this in Subsection 3.2.

3.1. The Leibniz rule.

We first discuss variants of the Leibniz rule on graphs.

In the continuous setting, there is a canonical Leibniz rule. In the discrete setting, this is not the case. Instead there are several options for the Leibniz rule. We list three such options in the lemma below. For a function $f\colon X \longrightarrow \mathbb{R}$ and $x, y \in X$, to shorten notation we write

$$\nabla_{x,y} f = f(x) - f(y).$$

Lemma 2.25 (Pointwise Leibniz rule). *Let $f, g \in C(X)$ and let $x, y \in X$. Then,*

$$\begin{aligned}\nabla_{x,y}(fg) &= f(x)\nabla_{x,y}g + g(y)\nabla_{x,y}f \\ &= f(y)\nabla_{x,y}g + g(x)\nabla_{x,y}f \\ &= f(x)\nabla_{x,y}g + g(x)\nabla_{x,y}f - \nabla_{x,y}f \cdot \nabla_{x,y}g.\end{aligned}$$

Proof. The statement follows by direct computation. $\square$

We next present an integrated form of the Leibniz rule.

Lemma 2.26 (Integrated Leibniz rule). *Let $w \colon X \times X \longrightarrow [0, \infty)$ be symmetric and let $f, g, h \in C(X)$. Then,*

$$\begin{aligned}&\sum_{x,y \in X} w(x,y)\nabla_{x,y}(fg) \cdot \nabla_{x,y}h \\ &\quad = \sum_{x,y \in X} w(x,y)f(x)\nabla_{x,y}g \cdot \nabla_{x,y}h + \sum_{x,y \in X} w(x,y)g(x)\nabla_{x,y}f \cdot \nabla_{x,y}h\end{aligned}$$

whenever any two of the above sums converge absolutely.

Proof. The statement follows from the first equality in the lemma above and symmetry. $\square$

3.2. Alternatives for the chain rule. We now give some alternatives for the chain rule. We first use the mean value theorem to give an estimate. Afterwards, we present more elaborate inequalities for powers and exponentials.

We start with a mean value theorem estimate.

Lemma 2.27 (Mean value estimate). *Let $\varphi \colon \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable, $f \in C(X)$, $x, y \in X$ and $I = (f(x) \wedge f(y), f(x) \vee f(y))$. Then,*

$$\inf_{\xi \in I} \varphi'(\xi)|\nabla_{x,y}f| \leq |\nabla_{x,y}(\varphi \circ f)| \leq \sup_{\xi \in I} \varphi'(\xi)|\nabla_{x,y}f|.$$

Proof. By the mean value theorem,

$$\nabla_{x,y}(\varphi \circ f) = \varphi'(\xi)\nabla_{x,y}f$$

for some $\xi \in I$. Thus, the statement follows. $\square$

The next lemma deals with differences of powers of a function.

Lemma 2.28 (Estimates for differences of f^p). *Let $f \in C(X)$ with $f \geq 0$ and let $x, y \in X$. Furthermore, assume that $f > 0$ whenever $p \in [0, 1]$ in the statements below.*

(a) *For all $p \in [0, \infty)$,*

$$|\nabla_{x,y}f^p| \leq C_p \left(f^{p-1}(x) + f^{p-1}(y)\right)|\nabla_{x,y}f|,$$

where $C_p = p/2$ for $p \in [0, 1] \cup [2, \infty)$ and $C_p = 1$ for $p \in (1, 2)$.

(b) *For all $p \in [1, \infty)$,*

$$|\nabla_{x,y}f^p| \geq (1 \wedge p/2)\left(f^{p-1}(x) + f^{p-1}(y)\right)|\nabla_{x,y}f|.$$

(c) *For all* $p \in [0, \infty)$,

$$|\nabla_{x,y} f^p| \geq (1 \wedge p) \left(f(x) \vee f(y)\right)^{p-1} |\nabla_{x,y} f|.$$

PROOF. We assume, without loss of generality, that $f(y) \leq f(x)$ and let $a = f(y)$ and $b = f(x)$. Furthermore, recall that we assume $a > 0$ if $p \in [0, 1]$ and note that the only non-trivial cases are $0 < a < b$ and $p \neq 0, 1, 2$, which we assume from now on.

The key identity for the proofs of (a) and (b) is

$$b^p - a^p = (b - a) \left(b^{p-1} + a^{p-1}\right) + ab \left(b^{p-2} - a^{p-2}\right).$$

Therefore, the main goal is to estimate $ab(b^{p-2} - a^{p-2})$, which we do by cases below.

For $p \in (1, \infty)$, the function $t \mapsto t^{1-p}$ is convex, i.e., concave upwards, on $(0, \infty)$. Thus, its image lies below the line segment connecting the points (b^{-1}, b^{p-1}) and (a^{-1}, a^{p-1}). So the integral of the function can be estimated from above by the sum of the area of a triangle and the area of a rectangle. Therefore, we estimate, for $p > 1, p \neq 2$,

$$\begin{aligned}
\frac{1}{p-2} \left(b^{p-2} - a^{p-2}\right) &= \int_{b^{-1}}^{a^{-1}} t^{1-p} dt \\
&\leq \left(a^{-1} - b^{-1}\right) \left(\frac{b^{p-1} - a^{p-1}}{2} + a^{p-1}\right) \\
&= \frac{1}{2ab} (b - a) \left(b^{p-1} + a^{p-1}\right).
\end{aligned}$$

For $p \in (0, 1)$, the function $t \mapsto t^{1-p}$ is concave, i.e., concave downwards, on $(0, \infty)$ and, therefore, by the same arguments as above, we can estimate the integral from below by an area so that

$$\frac{1}{p-2} \left(b^{p-2} - a^{p-2}\right) = \int_{b^{-1}}^{a^{-1}} t^{1-p} dt \geq \frac{1}{2ab} (b - a) \left(b^{p-1} + a^{p-1}\right).$$

To prove (a), observe that the case $p > 2$ is given by the first inequality combined with the equality given at the beginning of the proof. The case $p \in (0, 1)$ is given by the second inequality since $p - 2 < 0$ in this case. Finally, for $p \in (1, 2)$, we observe that $b^{p-2} - a^{p-2} < 0$, which immediately implies the statement of (a) in this case. This finishes the proof of (a).

For (b), we note that the statement for $p \in (1, 2)$ follows from the first inequality and the equality in the beginning of the proof since $p - 2 < 0$ in this case. For $p > 2$ we observe that $b^{p-2} - a^{p-2} > 0$ which immediately implies the statement of (b) in this case from the equality.

For (c), note that the case $p > 1$ follows directly as $b^p - a^p > b^p - ab^{p-1} = b^{p-1}(b - a)$. The case $p \in [0, 1]$ follows from the mean value theorem. $\square$

Finally, we turn to the exponential function. The proof uses the lemma above.

LEMMA 2.29 (Estimates for differences of e^f). *Let $f \in C(X)$ and let $x, y \in X$.*

(a) *Then,*

$$\left|\nabla_{x,y} e^f\right| \leq \frac{1}{2}\left(e^{f(x)} + e^{f(y)}\right)|\nabla_{x,y} f|.$$

(b) *If $|\nabla_{x,y} f| \leq 1$ and $\beta > 0$, then*

$$\left|\nabla_{x,y} e^{\beta f}\right| \leq \frac{(e^\beta - 1)\left(e^{2\beta f(x)} + e^{2\beta f(y)}\right)^{1/2}}{(1 + e^{2\beta}|\nabla_{x,y} f|^2)^{1/2}}|\nabla_{x,y} f|.$$

PROOF. For part (a) assume, without loss of generality, that $f(x) > f(y)$. Secondly, we may also assume without loss of generality that $f(y) \geq 0$, since, if $f(y) < 0$, then we multiply the inequality by $e^{-f(y)}$ and we estimate $|\nabla_{x,y} e^g|$ with $g(x) = \nabla_{x,y} f \geq 0$ and $g(y) = 0$.

(a) Noting that the inequality

$$|\nabla_{x,y} f^p| \leq \frac{p}{2}\left(f^{p-1}(x) + f^{p-1}(y)\right)|\nabla_{x,y} f|$$

from Lemma 2.28 (a) for $p \geq 2$ is also true for $p = 1$ we get

$$\begin{aligned}\nabla_{x,y} e^f &= \sum_{p=1}^{\infty} \frac{\nabla_{x,y} f^p}{p!} \leq \frac{1}{2}\nabla_{x,y} f \sum_{p=1}^{\infty} \frac{f^{p-1}(x) + f^{p-1}(y)}{(p-1)!} \\ &= \frac{1}{2}(e^{f(x)} + e^{f(y)})\nabla_{x,y} f.\end{aligned}$$

This gives the desired inequality.

(b) Assume as in (a) that $f(x) > f(y)$. Let $t \in [0, 1]$ and $\beta > 0$. First, observe that

$$e^{\beta t} - 1 = \sum_{k=1}^{\infty} \frac{(\beta t)^k}{k!} \leq t \sum_{k=1}^{\infty} \frac{\beta^k}{k!} = t(e^\beta - 1).$$

Secondly, the function $r \mapsto r^2/(1+(r+1)^2)$ is monotonically increasing on $[0, \infty)$. Applying this with $r = e^{\beta t} - 1 \leq t(e^\beta - 1)$ we conclude

$$\frac{(e^{\beta t} - 1)^2}{1 + e^{2\beta t}} = \frac{(e^{\beta t} - 1)^2}{1 + (e^{\beta t} - 1 + 1)^2} \leq \frac{t^2(e^\beta - 1)^2}{1 + (t(e^\beta - 1) + 1)^2} \leq \frac{t^2(e^\beta - 1)^2}{1 + t^2 e^{2\beta}}.$$

Letting $t = f(x) - f(y)$ and multiplying both sides of the inequality by $e^{2\beta f(y)}$ and taking square roots we obtain the statement. $\square$

4. Creatures from the abyss*

The main focus of our investigation is on the spectral geometry of graphs over measure spaces. Indeed, we need a measure on the underlying set in order to have a Hilbert space and to define self-adjoint operators. It turns out, however, that certain parts of the basic theory can be set up without reference to a measure. We discuss this approach in this section. We will start to work in a slightly more general setting than is needed for graphs and only return to graphs at the very end. Neither the results nor the notation used in this section are necessary to understand the remaining parts of the book.

Consider a discrete topological space X. Let $C(X)$ denote the set of all real-valued function on X and let $C_c(X)$ be the set of real-valued functions with finite support. For any finite subset $K \subseteq X$, the space $C(K)$ of real-valued functions on K can naturally be embedded into $C_c(X)$ by setting the functions equal to zero outside of K. This embedding is denoted by

$$i_K \colon C(K) \longrightarrow C_c(X).$$

Let $C(K)$ have the topology arising from the supremum norm. The embeddings i_K then induce on $C_c(X)$ the *inductive limit topology*. By definition, this is the largest topology making the embedding i_K continuous for each $K \subseteq X$ finite. This topology can be understood in a number of ways. In particular, a set U is open in $C_c(X)$ with the inductive limit topology if and only if $i_K^{-1}(U)$ is open in $C(K)$ for all K finite. Furthermore, a map T from $C_c(X)$ into a topological space is continuous if and only if $T \circ i_K$ is continuous for any finite $K \subseteq X$. Finally, a sequence (φ_n) converges to φ if and only if $\varphi_n \to \varphi$ pointwise and there exists a finite $K \subseteq X$ which contains the supports of φ_n and φ for all n (Exercise 2.16).

By the Riesz–Markov theorem, the dual space of $C_c(X)$, i.e., the space of all linear continuous mappings from $C_c(X)$ into $\mathbb{R}$, is the space $\mathcal{M}(X)$ of all *signed Radon measures* on X. Thus, any element μ in the dual space can be uniquely written as $\mu_+ - \mu_-$ and we have

$$\mu(\varphi) = \mu_+(\varphi) - \mu_-(\varphi)$$

for all $\varphi \in C_c(X)$, where $\mu_\pm$ are positive measures on X assigning finite mass to finite sets of points and satisfying $\mu(\varphi) = \sum_{x\in X} \varphi(x)\mu(\{x\})$. Of course, any such measure can naturally be identified with a function $f_\mu \in C(X)$ with $f_\mu(x) = \mu_+(\{x\}) - \mu_-(\{x\})$ for all $x \in X$. In this sense, $\mathcal{M}(X)$ is naturally isomorphic to $C(X)$. In fact, it is easy to see directly that $C(X)$ can be seen as the dual of $C_c(X)$ in a natural way. For a structural understanding of the subsequent considerations, however, it will be useful to rather think of the dual of $C_c(X)$ as a space of measures.

Consider now a bilinear form

$$\mathcal{Q}\colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

such that the domain of definition $\mathcal{D}$ of $\mathcal{Q}$ contains the space $C_c(X)$. Then, for any $f \in \mathcal{D}$, we can consider the map

$$C_c(X) \longrightarrow \mathbb{R}, \quad \varphi \mapsto \mathcal{Q}(f, \varphi).$$

The restriction of this map to any $C(K)$ with $K \subseteq X$ finite is a linear map on a finite-dimensional space and, hence, continuous. Thus, this map is continuous. Hence, by the Riesz–Markov theorem, there exists a unique measure $\mu_f \in \mathcal{M}(X)$ with

$$\mu_f(\varphi) = \mathcal{Q}(f, \varphi)$$

for all $\varphi \in C_c(X)$. Clearly, the map $f \mapsto \mu_f$ is linear as $\mathcal{Q}$ is bilinear. Thus, we can define a linear operator $\mathcal{L}_{\mathcal{D}}\colon \mathcal{D} \longrightarrow \mathcal{M}(X)$ via

$$\mathcal{L}_{\mathcal{D}} f = \mu_f$$

with

$$\mathcal{Q}(f, \varphi) = (\mathcal{L}_{\mathcal{D}} f)(\varphi)$$

for all $\varphi \in C_c(X)$. Letting $\varphi = 1_x$ be the characteristic function of $x \in X$, we then obtain

$$(\mathcal{L}_{\mathcal{D}} f)(\{x\}) = \mathcal{Q}(f, 1_x).$$

For any measure $\mu \in \mathcal{M}(X)$, we let $|\mu|$ be the absolute value of μ, i.e., $|\mu|$ is the positive measure with $|\mu|(\{x\}) = |\mu(\{x\})|$ for all $x \in X$. Thus, for any $\varphi \in C_c(X)$, we have

$$\ell^1(X, |(\mathcal{L}_{\mathcal{D}}\varphi)|) = \{f \in C(X) \mid \sum_{x \in X} |f(x)(\mathcal{L}_{\mathcal{D}}\varphi)(\{x\})| < \infty\}.$$

We then define

$$\mathcal{F} = \bigcap_{\varphi \in C_c(X)} \ell^1(X, |\mathcal{L}_{\mathcal{D}}\varphi|).$$

For any $f \in \mathcal{F}$ and $\varphi \in C_c(X)$, we define

$$(\mathcal{L}_{\mathcal{D}}\varphi)(f) = \sum_{x \in X} (\mathcal{L}_{\mathcal{D}}\varphi)(\{x\}) f(x),$$

where the sum exists by the definition of $\mathcal{F}$. Clearly, the map

$$C_c(X) \longrightarrow \mathbb{R}, \quad \varphi \mapsto (\mathcal{L}_{\mathcal{D}}\varphi)(f)$$

is continuous for each fixed $f \in \mathcal{F}$ as its restriction to $C(K)$ for $K \subseteq X$ finite is continuous by the same reasoning as given above. Thus, there exists a unique operator

$$\mathcal{L}_{\mathcal{F}}\colon \mathcal{F} \longrightarrow \mathcal{M}(X)$$

with

$$(\mathcal{L}_{\mathcal{F}} f)(\varphi) = (\mathcal{L}_{\mathcal{D}}\varphi)(f)$$

for all $f \in \mathcal{F}$ and $\varphi \in C_c(X)$.

To develop the theory further we now make the following two additional assumptions:

(A1) $\mathcal{Q}$ is symmetric, i.e., $\mathcal{Q}(f,g) = \mathcal{Q}(g,f)$ for all $f, g \in \mathcal{D}$.

(A2) For any $f \in \mathcal{D}$ and $x \in X$,

$$\sum_{y \in X} f(y)\mathcal{Q}(1_x, 1_y) = Q(1_x, f),$$

where the sum is absolutely convergent.

The second assumption is a form of continuity. It implies, in particular, that

$$\mathcal{Q}(1_x, f) = \lim_{n\to\infty} \mathcal{Q}(1_x, f_n)$$

whenever (f_n) is a sequence in $C_c(X)$ satisfying

- $f_n(x) \to f(x)$ for all $x \in X$
- $|f_n| \le |f|$.

Indeed, this is a direct consequence of the dominated convergence theorem. In fact, it turns out that (A2) is equivalent to this form of continuity (Exercise 2.17).

Remark. It is not hard to see that both assumptions are satisfied by the form $\mathcal{Q}_{b,c}$ arising from a graph (b,c) over X on its domain $\mathcal{D}_{b,c}$. We will discuss this at the end of this section.

We now give some consequences of the additional assumptions.

Theorem 2.30. *Assume* (A1) *and* (A2). *Then, the following statements hold:*

(a) $\mathcal{F} = \{f \in C(X) \mid \sum_{y\in X} |f(y)\mathcal{Q}(1_x, 1_y)| < \infty$ *for all* $x \in X\}$.

(b) *We have* $\mathcal{D} \subseteq \mathcal{F}$ *and* $\mathcal{L}_{\mathcal{F}}$ *is an extension of* $\mathcal{L}_{\mathcal{D}}$.

(c) *"Green's formula"*

$$(\mathcal{L}_{\mathcal{F}} f)(\varphi) = (\mathcal{L}_{\mathcal{F}}\varphi)(f)$$

holds for all $f \in \mathcal{F}$ *and all* $\varphi \in C_c(X)$. *If* f *belongs to* $\mathcal{D}$, *then*

$$\mathcal{Q}(f, \varphi) = (\mathcal{L}_{\mathcal{F}} f)(\varphi).$$

(d) *If* $f_n, f \in \mathcal{F}$ *with* $|f_n| \le |f|$ *and* $f_n \to f$ *pointwise for* $n \to \infty$, *then*

$$\lim_{n\to\infty} (\mathcal{L}_{\mathcal{F}} f_n)(\{x\}) = (\mathcal{L}_{\mathcal{F}} f)(\{x\})$$

for all $x \in X$.

Proof. (a) We clearly have

$$\mathcal{F} = \bigcap_{x\in X} \ell^1(X, |\mathcal{L}_{\mathcal{D}} 1_x|).$$

Now, as shown above,

$$\mathcal{L}_{\mathcal{D}} 1_x(\{y\}) = \mathcal{Q}(1_x, 1_y).$$

Combining these observations, we easily obtain the desired statement for $\mathcal{F}$.

(b) By (A2) and (a) we have $\mathcal{D} \subseteq \mathcal{F}$. Moreover, for $f \in \mathcal{D}$ we find that for all $\varphi \in C_c(X)$,

$$\begin{array}{rcl} \mathcal{L}_{\mathcal{F}} f(\varphi) & = & \mathcal{L}_{\mathcal{D}}\varphi(f) \\ \text{(definition of } \mathcal{L}_{\mathcal{D}}\text{)} & = & \displaystyle\sum_{x\in X} (\mathcal{L}_{\mathcal{D}}\varphi)(\{x\}) f(x) \\ & = & \displaystyle\sum_{x\in X} \mathcal{Q}(\varphi, 1_x) f(x) \\ \text{(A2)} & = & \mathcal{Q}(\varphi, f) \\ \text{(A1)} & = & \mathcal{Q}(f, \varphi) \\ & = & \mathcal{L}_{\mathcal{D}} f(\varphi). \end{array}$$

This shows that $\mathcal{L}_{\mathcal{F}}$ and $\mathcal{L}_{\mathcal{D}}$ agree on $\mathcal{D}$.

(c) From the definition of $\mathcal{L}_{\mathcal{F}}$ and (b) we obtain

$$\mathcal{L}_{\mathcal{F}} f(\varphi) = \mathcal{L}_{\mathcal{D}}\varphi(f) = \mathcal{L}_{\mathcal{F}}\varphi(f)$$

for all $f \in \mathcal{F}$ and $\varphi \in C_c(X)$. Similarly, from the definition of $\mathcal{L}_{\mathcal{D}}$ and (b) we obtain

$$Q(f, \varphi) = \mathcal{L}_{\mathcal{D}} f(\varphi) = \mathcal{L}_{\mathcal{F}} f(\varphi)$$

for all $f \in \mathcal{D}$ and $\varphi \in C_c(X)$.

(d) This follows from the definitions and Lebesgue's dominated convergence theorem. Namely, since $f_n \to f$ pointwise, $|f_n| \leq |f|$ and $f \in \mathcal{F}$ so that $\sum_{y\in X} |(\mathcal{L}_{\mathcal{D}} 1_x)(y) f(y)| < \infty$ for all $x \in X$, we get

$$\begin{array}{rcl} \mathcal{L}_{\mathcal{F}} f_n(\{x\}) & = & \mathcal{L}_{\mathcal{D}} 1_x(f_n) \\ \text{(definition of } \mathcal{L}_{\mathcal{D}}\text{)} & = & \displaystyle\sum_{y\in X} (\mathcal{L}_{\mathcal{D}} 1_x)(\{y\}) f_n(y) \\ & \to & \displaystyle\sum_{y\in X} (\mathcal{L}_{\mathcal{D}} 1_x)(y) f(y) \\ & = & (\mathcal{L}_{\mathcal{D}} 1_x)(f) \\ & = & \mathcal{L}_{\mathcal{F}} f(x). \end{array}$$

This completes the proof. □

If (b, c) is a graph over X, then the previous theorem can be applied to $\mathcal{Q} = \mathcal{Q}_{b,c}$ on $\mathcal{D} = \mathcal{D}_{b,c}$. Indeed, in this case we have

$$\mathcal{Q}(1_x, 1_y) = -b(x, y)$$

for $x \neq y$ and

$$\mathcal{Q}(1_x) = \sum_{z\in X} b(x, z) + c(x)$$

for all $x \in X$. From these equations we easily find that (A1) and (A2) are satisfied. Moreover, from these equations we can also directly infer

$$\mathcal{F} = \mathcal{F}_{b,c} \quad \text{and} \quad \mathcal{L}_{\mathcal{F}} = \mathcal{L}_{b,c}.$$

This gives a structural understanding of how $\mathcal{F}_{b,c}$ and $\mathcal{L}_{b,c}$ come about in our theory. Along the way, we also obtain that the form $\mathcal{Q}_{b,c}$ has the continuity property (A2).

Furthermore, the local finiteness of the graph is equivalent to the fact that $\mathcal{L}_{\mathcal{D}}(C_c(X)) \subseteq C_c(X)$, which is equivalent to $\mathcal{F} = C(X)$ (Exercise 2.18). Also, it is possible to elaborate a theory of the dual of an operator in this context (Exercise 2.19).

Finally, we mention that the question of when a form $\mathcal{Q}$ arises from a graph can be addressed via the associated operator satisfying a maximum principle (Exercise 2.20).

5. Markov processes and the Feynman–Kac formula redux*

In this section we establish a connection between Dirichlet forms and the corresponding Markov processes. In particular, we prove a Feynman–Kac formula. Although the focus of the book is analytic rather than probabilistic, this connection is one of the major historical motivations for the theory and, therefore, of great conceptual importance. The intention of this section is to give a glimpse of these probabilistic aspects. However, in most of the book, we will not refer to this section, so it can safely be skipped by the reader only interested in the analytic aspects.

The proof of the Feynman–Kac formula is mainly an approximation argument that uses considerations for finite graphs. We briefly recall the construction of the process and refer to Section 10 for background and further details.

Let (b, c) be a graph over the measure space (X, m). Let $Q = Q^{(D)}_{b,c,m}$ be the associated form and $L = L^{(D)}_{b,c,m}$ be the associated Dirichlet Laplacian. We first construct the Markov process $\mathbb{X} = \mathbb{X}^b$ associated to b.

Let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be a discrete time Markov chain on X over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{P}\left(Y_n = y \,|Y_{n-1} = x\right) = \frac{b(x, y)}{\deg(x)}$$

for $n \in \mathbb{N}$ and $x, y \in X$, where the degree is given by $\deg(x) = \sum_{y\in X} b(x, y)$ since we work with b only.

To define the sequence of *holding times* S_n for $n \in \mathbb{N}$ and *jumping times* J_n for $n \in \mathbb{N}_0$, we let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables of parameter 1 which are also independent of Y and let

$$S_n = \frac{1}{\mathrm{Deg}(Y_{n-1})}\xi_n, \quad J_n = S_1 + \ldots + S_n$$

with the convention that $J_0 = 0$ where $\mathrm{Deg}(x) = \deg(x)/m(x)$.

Since X is assumed to be infinite, the random variable

$$\zeta = \sup_{n \in \mathbb{N}_0} J_n,$$

which is called the *lifetime* or *explosion time* of the process, may take a finite value with positive probability. Characterizations of this phenomenon will be discussed in the context of stochastic completeness of graphs in Section 9 of Chapter 7.

The Markov process $\mathbb{X} = \mathbb{X}^b \colon [0,\infty) \times \Omega \longrightarrow X$ is defined via

$$\mathbb{X}_t = Y_n \quad \text{ if } t \in [J_n, J_{n+1}).$$

For an event A, we define $\mathbb{E}_x(A) = \mathbb{E}(A \mid \mathbb{X}_0 = x)$.

The Feynman–Kac formula on infinite graphs reads as follows.

Theorem 2.31 (Feynman–Kac formula). *Let (b,c) be a graph over (X,m) with associated Laplacian $L = L^{(D)}$ and let $\mathbb{X} = \mathbb{X}^b$ be the process associated to b. Then,*

$$e^{-tL} f(x) = \mathbb{E}_x\Big(1_{\{t<\zeta\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\Big)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$.

We need the following lemma to transfer the Feynman–Kac formula proven in Section 10 to infinite graphs. In particular, Theorem 0.72 is shown for processes on finite graphs which are restricted to subgraphs. In what follows we consider restrictions to finite graphs first, however, the background process is defined on an infinite graph. We show that each process on an infinite graph restricted to a finite subgraph can be replaced by a process on a finite graph.

We recall the definition of the Dirichlet Laplacian on subsets. For a finite subset $K \subseteq X$, we let $\pi_K \colon \ell^2(X,m) \longrightarrow \ell^2(K,m_K)$ be the canonical projection, $i_K \colon \ell^2(K,m_K) \longrightarrow \ell^2(X,m)$ be the canonical embedding which is continuation by zero on $X \setminus K$ and let

$$L_K^{(D)} = i_K L \pi_K.$$

In particular,

$$L_K^{(D)} f(x) = \frac{1}{m(x)} \left(\sum_{y \in K} b(x,y)(f(x) - f(y)) + \big(d_K(x) + c(x)\big) f(x) \right)$$

for all $f \in \ell^2(K,m_K)$ and $x \in K$ where $d_K(x) = \sum_{y \in X \setminus K} b(x,y)$. See Section 3 for further details.

For the process $\mathbb{X} = \mathbb{X}^b$ we define the first exit time for $K \subseteq X$ to be the random variable τ_K given by

$$\tau_K = \inf\{t \geq 0 \mid \mathbb{X}_t \in X \setminus K\}.$$

LEMMA 2.32. *Let (b, c) be a graph over (X, m) and let $\mathbb{X} = \mathbb{X}^b$ be the process associated to b. For a subset $K \subseteq X$, let $L_K^{(D)}$ be the Dirichlet Laplacian and let τ_K be the first exit time for K. Then,*

$$e^{-tL_K^{(D)}} f(x) = \mathbb{E}_x\Big(1_{\{t<\tau_K \wedge \zeta\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\Big)$$

for all $f \in \ell^2(K, m)$, $x \in K$ and $t \geq 0$.

PROOF. For a finite set $K \subseteq X$, let $\widetilde{K} = K \cup \{\infty\}$. Let $\widetilde{m}$ be defined by $\widetilde{m} = m$ on K and with $\widetilde{m}(\infty)$ arbitrary, say $\widetilde{m}(\infty) = 1$. We define a graph $(\widetilde{b}, \widetilde{c})$ over the measure space $(\widetilde{K}, \widetilde{m})$ by letting $\widetilde{c}$ be the extension of c to $\widetilde{K}$ by zero and letting $\widetilde{b} = b$ on $K \times K$ with

$$\widetilde{b}(x, \infty) = \widetilde{b}(\infty, x) = \sum_{y \in X \setminus K} b(x, y).$$

We denote the Laplacian for the graph $(\widetilde{b}, \widetilde{c})$ over the finite measure space $(\widetilde{K}, \widetilde{m})$ by $\widetilde{L}$ and the restriction of $\widetilde{L}$ to K with Dirichlet boundary conditions by $\widetilde{L}_K^{(D)}$. By construction, we have for the restriction of L to K with Dirichlet boundary conditions,

$$\widetilde{L}_K^{(D)} = L_K^{(D)}$$

on $\ell^2(K, m_K)$ and, in particular,

$$e^{-t\widetilde{L}_K^{(D)}} = e^{-tL_K^{(D)}}$$

for $t \geq 0$.

Furthermore, let $\widetilde{\mathbb{X}}$ be the process associated to $\widetilde{b}$ over the finite measure space $(\widetilde{K}, \widetilde{m})$ and let $\mathbb{X}$ be the process associated to b over (X, m). Conditioning the processes $\widetilde{\mathbb{X}}$ and $\mathbb{X}$ on not leaving K, these processes are equivalent. More specifically, these conditioned processes are Markov processes associated to graphs on the finite set K, however, this time with a non-vanishing killing term, see Subsection 10.3. Furthermore, $c = \widetilde{c}$ on K and, therefore,

$$\widetilde{\mathbb{E}}_x\Big(1_{\{t<\widetilde{\tau}_K\}} e^{-\int_0^t (c/m)(\widetilde{\mathbb{X}}_s)ds} f(\widetilde{\mathbb{X}}_t)\Big) = \mathbb{E}_x\Big(1_{\{t<\tau_K\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\Big),$$

where $\widetilde{\tau}_K$ is the exit time of the process $\widetilde{\mathbb{X}}$ for $\widetilde{K}$ and $\widetilde{\mathbb{E}}_x$ is the expectation with regard to the probability measure of $\widetilde{\mathbb{X}}$ conditioned on $\mathbb{X}$ starting at x.

Thus, we obtain from Lemma 0.72

$$\begin{aligned} e^{-tL_K^{(D)}} f(x) &= e^{-t\widetilde{L}_K^{(D)}} f(x) \\ &= \widetilde{\mathbb{E}}_x\Big(1_{\{t<\widetilde{\tau}_K\}} e^{-\int_0^t (c/m)(\widetilde{\mathbb{X}}_s)ds} f(\widetilde{\mathbb{X}}_t)\Big) \\ &= \mathbb{E}_x\Big(1_{\{t<\tau_K\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\Big). \end{aligned}$$

Since the event $\{\zeta < \tau_K\}$ has probability zero, the events $\{t < \tau_K\}$ and $\{t < \tau_K \wedge \zeta\}$ have the same measure. This completes the proof of the statement. □

Proof of Theorem 2.31. Let X_k for $k \in \mathbb{N}_0$ be an exhausting sequence of X, i.e., $X_k \subseteq X$ are finite subsets with $X_k \subseteq X_{k+1}$ for $k \in \mathbb{N}_0$ and $X = \bigcup_k X_k$. Let $\pi_k\colon \ell^2(X,m) \longrightarrow \ell^2(X_k, m_{X_k})$ be the canonical projection and i_k be its dual, i.e., the canonical embedding.

Then, the Laplacians $L_k^{(D)} = i_k L \pi_k$ with Dirichlet boundary conditions form a sequence of operators on the finite dimensional Hilbert spaces $\ell^2(X_k, m_{X_k})$. By Lemma 1.21, we have

$$\lim_{k\to\infty} e^{-tL_k^{(D)}}\varphi = e^{-tL}\varphi$$

for all $\varphi \in C_c(X)$. By the uniform boundedness in t of the semigroups and the density of $C_c(X)$ in $\ell^2(X,m)$, we get

$$\lim_{k\to\infty} e^{-tL_k^{(D)}} f_k = e^{-tL} f$$

for all $f \in \ell^2(X,m)$ and $f_k = 1_{X_k} f$. By Lemma 2.32, the Feynman–Kac formula holds on X_k, so, we are left to show the convergence

$$\mathbb{E}_x\Big(1_{\{t<\tau_{X_k}\wedge\zeta\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f_k(\mathbb{X}_t)\Big) \to \mathbb{E}_x\Big(1_{\{t<\zeta\}} e^{-\int_0^t (c/m)(\mathbb{X}_s)ds} f(\mathbb{X}_t)\Big)$$

as $k \to \infty$ for all $f \in \ell^2(X,m)$ and $x \in X$.

Assume first that $f \geq 0$. Since the sequence $\tau_{X_k} \wedge \zeta$ converges monotonically increasingly to ζ and $f_k(\mathbb{X}_t)$ converges monotonically increasingly to $f(\mathbb{X}_t)$, the statement follows by monotone convergence.

Now, let $f \in \ell^2(X,m)$ be arbitrary. We then split f into positive and negative parts and apply the argument above. This completes the proof. □

Exercises

Excavation exercises.

EXERCISE 2.1 (The Banach spaces ℓ^p). Let (X, m) be a discrete measure space. Define, for $p \in [1, \infty)$,

$$\ell^p(X, m) = \{f\colon X \longrightarrow \mathbb{R} \mid \sum_{x \in X} |f(x)|^p m(x) < \infty\}$$

with

$$\|f\|_p = \left(\sum_{x \in X} |f(x)|^p m(x)\right)^{1/p}.$$

For $p = \infty$, let

$$\ell^\infty(X, m) = \{f\colon X \longrightarrow \mathbb{R} \mid \ f \text{ is bounded } \}$$

with

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

(a) Show that the $\ell^p(X, m)$ are subspaces of $C(X)$ for any $p \in [1, \infty]$.
(b) Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ (where the cases $p = 1, q = \infty$ and $p = \infty, q = 1$ are allowed). Show that for any $f \in \ell^p(X, m)$ and $g \in \ell^q(X, m)$ the product fg belongs to $\ell^1(X, m)$ and satisfies

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

(Hint: You may (why?) restrict attention to f, g with $\|f\|_p = 1 = \|g\|_q$ and use the inequality $ab \leq a^p/p + b^q/q$ which is valid for all $a, b \geq 0$.)
(c) Show that $\|\cdot\|_p$ is a norm on $\ell^p(X, m)$ for any $p \in [1, \infty]$ which makes $\ell^p(X, m)$ into a Banach space, i.e., a complete normed space.

EXERCISE 2.2 (Dual spaces of the ℓ^p-spaces). Let (X, m) be a discrete measure space. Consider $p \in [1, \infty)$ and $q \in (1, \infty]$ with $1/p + 1/q = 1$ (where the case $p = 1, q = \infty$ is allowed). Show that $\ell^q(X, m)$ is the dual space of $\ell^p(X, m)$ in the sense that the map $J\colon \ell^q(X, m) \longrightarrow (\ell^p(X, m))^*$ defined by

$$(Jf)(g) = \sum_{x \in X} f(x) g(x) m(x)$$

is bijective and isometric.

EXERCISE 2.3 (Inclusions among the ℓ^p-spaces). Let (X, m) be a discrete measure space. Set

$$I = \inf_{x \in X} m(x) \quad \text{and} \quad S = \sum_{x \in X}^{\infty} m(x).$$

Show the following statements for $1 \leq p < q < \infty$:

(a) Assume that $I > 0$. Then, $\ell^p(X,m) \subseteq \ell^q(X,m)$ and
$$\sup\{\|f\|_q \mid f \in \ell^p(X,m), \|f\|_p \le 1\} = I^{1/q-1/p}.$$
(b) Assume that $S < \infty$. Then, $\ell^q(X,m) \subseteq \ell^p(X,m)$ and
$$\sup\{\|f\|_p \mid f \in \ell^q(X,m), \|f\|_q \le 1\} = S^{1/p-1/q}.$$
(c) For $I = 0$ or $S = \infty$, the inclusions given in (a) and (b) do not hold.

Exercise 2.4 (Weakly convergent subsequences). Let H be a Hilbert space. Show that any bounded sequence in H has a weakly convergent subsequence.

Exercise 2.5 (Banach–Saks theorem). Let H be a Hilbert space. Let (f_n) be a sequence in H which converges weakly to f. Show that there exists a subsequence (f_{n_k}) of (f_n) such that the Cesàro means
$$\widetilde{f}_N = \frac{1}{N}\sum_{k=1}^{N} f_{n_k}$$
converge in norm to f.

Example exercises.

Exercise 2.6 ($\mathcal{L} \circ i_U \neq \mathcal{L}_U$). Give an example of a graph (b,c) over (X,m) with $U \subseteq X$ and $f \in \mathcal{F}_U$ such that $i_U f$ does not belong to $\mathcal{F}$. In particular, this shows that $\mathcal{L} i_U f = \mathcal{L}_U f$ does not hold.

(Hint: Consider the infinite star graph and let U consist of all vertices other than the center of the star.)

Extension exercises.

Exercise 2.7 (Solving the heat equation for bounded generators). Let E be a Banach space, $B(E)$ denote the space of bounded linear operators on E and $A \in B(E)$.

(a) Show that for every $t \in \mathbb{R}$, the series
$$S_A(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A^n$$
is absolutely convergent with $\|S_A(t)\| \le e^{-t\|A\|}$.

(b) Show that the map $S_A \colon [0,\infty) \longrightarrow B(E)$ given by (a) is a strongly continuous semigroup which is continuously differentiable with
$$\partial_t S_A(t) = -A S_A(t) = -S_A(t) A$$
for any $t \in [0,\infty)$.

Exercise 2.8 (Uniqueness of semigroups). Let E be a Banach space, $B(E)$ denote the space of bounded linear operators on E and $A \in B(E)$ with $S_A(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A^n$ for $t \geq 0$. Show the following statements:

(a) If $w\colon [0,\infty) \longrightarrow E$ is a solution of
$$\partial_t w = -Aw$$
with $w(0) = u$, then $w(t) = S_A(t)u$ for all $t \geq 0$.

(b) If T is a semigroup with generator $B \in B(E)$ and $B \subseteq A$, then $T = S_A$ and $B = A$.

Exercise 2.9 (Characterizing generators of semigroups). Let A be a closed operator on a Banach space E with dense domain of definition such that $A+\alpha$ is bijective with $\|(A+\alpha)^{-1}\| \leq 1/\alpha$ for all $\alpha > 0$. Show that A is the generator of a uniquely determined strongly continuous semigroup.

Exercise 2.10 (Strongly continuous resolvents). Let (X,m) be a discrete measure space. Let G be a strongly continuous resolvent on $\ell^p(X,m)$. Show that strong continuity of the resolvent implies that the map $(0,\infty) \longrightarrow [0,\infty)$
$$\alpha \mapsto \|G_\alpha f\|_p$$
is continuous for all $f \in \ell^p(X,m)$.

Exercise 2.11 (Weak* continuity and pointwise continuity). Let S be contraction semigroup on $\ell^\infty(X)$ for a discrete measure space (X,m). Show that the following statements are equivalent:

(i) S is weak* continuous, i.e., $t \mapsto (S(t)f,g)$ is continuous on $[0,\infty)$ for all $f \in \ell^\infty(X)$ and $g \in \ell^1(X,m)$.

(ii) S is pointwise continuous, i.e., $t \mapsto S(t)f(x)$ is continuous on $[0,\infty)$ for all $f \in \ell^\infty(X)$ and $x \in X$.

Exercise 2.12 (Direct proof of resolvent properties). Give a direct proof of Theorem 2.11 by using Lemma 2.8 and the properties of $G_{b,c}$ established in Proposition 2.10.

Exercise 2.13 (Restrictions to cofinite sets inherit regularity). Let (X,m) be a discrete measure space. Consider a closed form Q with domain $D(Q) \subseteq \ell^2(X,m)$ with $C_c(X) \subseteq D(Q)$. Let $U \subseteq X$ be such that $X \setminus U$ is finite. Show that the following statements hold:

(a) The restriction $f|_U$ of $f \in D(Q)$ belongs to $D(Q_U)$, so that the map $\pi_U\colon D(Q) \longrightarrow D(Q_U)$ given by $\pi_U f = f|_U$ is well-defined.

(b) The map π_U is continuous, where the form domains are equipped with the corresponding form norms.
(Hint: Use the closed graph theorem.)
(c) If $C_c(X)$ is dense in $D(Q)$ with respect to $\|\cdot\|_Q$, then $C_c(U)$ is dense in $D(Q_U)$ with respect to $\|\cdot\|_{Q_U}$.
(Hint: Use (b).)

Exercise 2.14 (Extending Proposition 2.18 to general subspaces). Let (X, m) be a discrete measure space and let Q be a closed form on $\ell^2(X, m)$. Let W be a not necessarily closed subspace of $\ell^2(X, m)$. Show the following:
(a) The restriction of Q to $W \cap D(Q)$ admits closed extensions. Denote the smallest such extension by Q_W.
(b) Consider the Hilbert space $\overline{W \cap D(Q)}$, where the closure is taken in $\ell^2(X, m)$. Show that Q_W is a closed form on this Hilbert space.
(c) If Q is a Dirichlet form and W is invariant under normal contractions, then Q_W is a Dirichlet form.
(Hint: For (c) you can mimic the reasoning in the proof of (c) of Proposition 2.18.)

Exercise 2.15 (Characterizing subspaces of the form $\ell^2(U, m_U)$). Let (X, m) be a discrete measure space.
(a) Show that the following three assertions for a closed subspace V of $\ell^2(X, m)$ are equivalent:
(i.a) There exists a $U \subseteq X$ with $V = \ell^2(U, m_U)$.
(ii.a) The subspace V is invariant under taking the absolute value $|\cdot|$ and if $g \in \ell^2(X, m)$ with $0 \leq g \leq f$ for some $f \in V$, then $g \in V$. (*"Order ideal property"*)
(iii.a) For any $f \in V$ and $g \in \ell^2(X, m) \cap \ell^\infty(X)$ the product fg also belongs to V. (*"Multiplicative ideal property"*).
(Hint: Define

$$U = \{x \in X \mid \text{there exists an } f \in V \text{ with } f(x) \neq 0\}.$$

Show that (ii.a)/(iii.a) imply that V contains $C_c(U)$.)
(b) Show by counterexamples that a closed subspace:
(i.b) May be invariant under taking modulus without satisfying (ii.a) or (iii.a).
(ii.b) May satisfy that $C_c(X) \cap V$ is dense in V without being of the form $\ell^2(U, m_U)$.
(Hint: $V = \{f \in \ell^2(X, m) \mid f(x_1) = f(x_2) \text{ for } x_1 \neq x_2\}$.)

Exercise 2.16 (Inductive limit topology). Let X be a discrete set. Let $C_c(X)$ have the inductive limit topology induced by the embeddings $i_K \colon C(K) \longrightarrow C_c(X)$ for finite $K \subseteq X$. Here, i_K extends a function by 0 and $C(K)$ is given the topology arising from the supremum norm.

(a) Show that $U \subseteq C_c(X)$ is open if and only if $i_K^{-1}(U)$ is open in $C(K)$ for every finite $K \subseteq X$.
(b) If Y is a topological space, show that any mapping $T\colon C_c(X) \longrightarrow Y$ is continuous if and only if $T \circ i_K$ is continuous for every finite $K \subseteq X$.
(c) Show that $\varphi_n \to \varphi$ in the inductive limit topology if and only if $\varphi_n \to \varphi$ pointwise and there exists a finite $K \subseteq X$ such that the supports of φ_n and φ are contained in K for all n.

Exercise 2.17 (Characterizing (A2)). Let $\mathcal{Q}\colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$ be a bilinear form over a discrete set X with $C_c(X) \subseteq \mathcal{D}$. Show that the following statements are equivalent:

(i) For any $f \in \mathcal{D}$ and $x \in X$,
$$\sum_{y\in X} f(y)\mathcal{Q}(1_x, 1_y) = Q(1_x, f),$$
where the sum is absolutely convergent.

(ii) For any $f \in \mathcal{D}$ and $x \in X$,
$$\mathcal{Q}(1_x, f) = \lim_{n\to\infty} \mathcal{Q}(1_x, \varphi_n)$$
whenever (φ_n) is a sequence in $C_c(X)$ satisfying $\varphi_n(x) \to f(x)$ for all $x \in X$ and $|\varphi_n| \le |f|$ for all n.

Exercise 2.18 (Characterizing local finiteness). Let (b, c) be a graph over X. Show that the following statements are equivalent:

(i) The graph (b, c) over X is locally finite.
(ii) $\mathcal{L}_{\mathcal{D}}(C_c(X)) \subseteq C_c(X)$.
(iii) $\mathcal{F} = C(X)$.

(Hint: Show that $\ell^1(X, g) \neq C(X)$ if and only if the support of g is infinite.)

Exercise 2.19 (Domains of dual operators). Let X be a discrete set. For any subset V of $C(X)$ define V^* by
$$V^* = \{f \in C(X) \mid \sum_{x\in X} |f(x)\varphi(x)| < \infty \text{ for all } \varphi \in V\}.$$
Show that:

(a) $(C_c(X))^* = C(X)$ and $C(X)^* = C_c(X)$.
(b) $(\ell^2(X, m))^* = \ell^2(X, 1/m)$ for any measure m on X with full support.

Furthermore, for a linear operator $\mathcal{L}\colon C_c(X) \longrightarrow C(X)$ define the dual operator $\mathcal{L}^*$ as having domain
$$D(\mathcal{L}^*) = (\mathcal{L}C_c(X))^*$$

and $\mathcal{L}^* f$ as the unique element of $C(X)$ with

$$\sum_{x\in X} \mathcal{L}^* f(x)\varphi(x) = \sum_{x\in X} f(x)\mathcal{L}\varphi(x)$$

for all $\varphi \in C_c(X)$. Show that:

(c) $V^* \subseteq D(\mathcal{L}^*)$ for any subspace V of $C(X)$ with $\mathcal{L}(C_c(X)) \subseteq V$.
(d) $\mathcal{L}(C_c(X)) \subseteq V$ for any subspace V of $C(X)$ with $V^* \subseteq D(\mathcal{L}^*)$ and $(V^*)^* = V$.
(e) $\mathcal{L}(C_c(X)) \subseteq C_c(X)$ if and only if $D(\mathcal{L}^*) = C(X)$.
(f) $\mathcal{L}(C_c(X)) \subseteq \ell^2(X,m)$ if and only if $\ell^2(X,m) \subseteq D(\mathcal{L}^*)$.

EXERCISE 2.20 (Maximum principle). Let X be a discrete set and let $\mathcal{Q}$ be a bilinear form on $C_c(X)$. Let $\mathcal{L}$ be a linear operator acting as

$$\mathcal{L}\varphi(x) = \mathcal{Q}(f, 1_x)$$

for $\varphi \in C_c(X)$ and $x \in X$. Show the following statements:

(a) There exists a graph (b,c) over X such that

$$\mathcal{Q}(\varphi) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi(x)-\varphi(y))^2 + \sum_{x\in X} c(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$ if and only if $\mathcal{L}\varphi(x) \geq 0$ at every non-negative maximum x of φ with $\varphi \in C_c(X)$.

(b) There exists a graph $(b,0)$ over X such that

$$\mathcal{Q}(\varphi) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(\varphi(x)-\varphi(y))^2$$

for all $\varphi \in C_c(X)$ if and only if $\mathcal{L}\varphi(x) \geq 0$ at every maximum x of φ with $\varphi \in C_c(X)$.

Notes

With the exception of Section 4, the material found in this chapter is certainly known to experts.

The theory of semigroups and their generators as well as the applications to Dirichlet forms and Markov processes is standard, see e.g. the book [**HP57**] for a discussion of semigroups and the work [**FŌT11**] for a treatment of semigroups in the context of Dirichlet forms. In Section 1 we apply this general theory to extend semigroups and resolvent to all ℓ^p spaces.

Restriction to compact subsets play a prominent role in the investigation of regular Dirichlet forms. Accordingly, as discussed in the notes to Section 3, restrictions to finite subsets of graphs appear in many places in the literature. As for restrictions to arbitrary subsets, we have not been able to locate a source covering the material presented in Section 2.

The Laplace–Beltrami operator on a manifold leads to a local Dirichlet form. Non-locality is the crucial feature of the Dirichlet form associated to a graph. Various tools and concepts have been developed for different applications in order to deal with this non-locality. In Section 3 we bring them together in a systematic way. The non-local Leibniz rules are standard. We also present various estimates that can be used instead of a chain rule for specific functions such as powers and the exponential function. While these estimates are certainly known in some context or other, we were particular inspired by calculations found in [**HS97**] and [**Amg03**].

The material presented in Section 4 is new.

Feynman–Kac type formulae, as discussed in Section 5, are valid for rather general Dirichlet forms [**FŌT11**]. Specific treatments for graphs can be found in the Diploma thesis of Metzger [**Met98**] and the article by Güneysu/Keller/Schmidt [**GKS16**] where the main focus lies on a more general model including magnetic fields. Moreover, on graphs there is also a path integral formula for the unitary group which can famously be formulated only heuristically in the continuum setting [**GK20**].

For complementary textbooks on infinite graphs we refer the reader to the corresponding comments at the end of the notes of Chapter 0.

CHAMBER 3

Markov Uniqueness and Essential Self-Adjointness

... let it be applied, Unique drop that science ...
ODB.

The uniqueness of self-adjoint and Markov extensions of a symmetric operator on a Hilbert space is a classical topic in operator theory. In this chapter we consider these problems from the viewpoint of solutions to equations involving the Laplacian and the viewpoint of the domains of both the operators and the forms.

In Section 1 we characterize the equality of the Dirichlet and Neumann form domains in terms of the Dirichlet Laplacian domain, a Green's formula and the triviality of α-harmonic functions in the Neumann form domain for positive α. In Section 2 we study essential self-adjointness via the Dirichlet Laplacian domain and the triviality of α-harmonic functions in $\ell^2(X, m)$ for positive α. Finally, Section 3 addresses the question of when the form arising from a self-adjoint positive restriction of the formal Laplacian is a Dirichlet form. In particular, we show that the case when such a form is unique, a property which we call Markov uniqueness, is equivalent to the equality of the Dirichlet and Neumann form domains discussed in Section 1.

We note that $\ell^p(X, m)$ theory, as developed in Section 1, appears in some places in Sections 1 and 2. The reader who has skipped Section 1 can safely let $p = 2$ for all statements presented in these sections as then the statements do not require the material on $\ell^p(X, m)$ spaces.

1. Uniqueness of associated forms

In this section we characterize the equality of the Dirichlet and Neumann forms. One characterization involves the absence of nontrivial α-harmonic functions in the Neumann form domain for $\alpha > 0$. Further characterizations involve explicitly describing the domain of the Dirichlet Laplacian and the validity of a Green's formula.

Throughout this section various basic facts about self-adjoint operators are used. Some of these facts are recalled in Excavation Exercise 3.1 which is used in the proof of Theorem 3.2. For basic definitions of operator theory, see Appendix A.

We start by recalling the definition of α-harmonic functions and variants of this notion called α-super(sub)harmonic. These notions will play a prominent role in the forthcoming considerations in this chapter and beyond. For a graph (b, c) over (X, m) and $\alpha \in \mathbb{R}$, a function $u \in \mathcal{F}$ is called α-harmonic (α-superharmonic or α-subharmonic, respectively) if

$$(\mathcal{L} + \alpha)u = 0 \qquad ((\mathcal{L} + \alpha)u \geq 0 \text{ or } (\mathcal{L} + \alpha)u \leq 0, \text{ respectively}).$$

We will show that the triviality of α-harmonic functions in the Neumann form domain for $\alpha > 0$ is equivalent to $Q^{(D)} = Q^{(N)}$. We recall that $Q^{(D)}$ is the minimal closed restriction of $\mathcal{Q}$ which has domain

$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$$

and that $Q^{(N)}$ is the maximal closed restriction of $\mathcal{Q}$ with domain

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m).$$

Furthermore, we recall that a form Q with domain $D(Q)$ is associated to a graph if Q is a closed restriction of $\mathcal{Q}$ such that

$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)}).$$

Hence, if $Q^{(D)} = Q^{(N)}$, then there is a unique form associated to a graph.

We have shown in Proposition 1.4 (b) that the space of functions of finite energy $\mathcal{D}$ is included in the formal domain of the Laplacian $\mathcal{F}$, i.e., $\mathcal{D} \subseteq \mathcal{F}$. Therefore, as $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$ by definition, it follows that $D(Q) \subseteq \mathcal{F}$ for any form associated to the graph. This allows us to apply Green's formula, Proposition 1.5, in various places below.

For the sake of contrast with the definition of $D(Q^{(N)})$, we recall that by Theorem 1.19 we have $D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X, m)$, where $\mathcal{D}_0$ is the space of functions in $\mathcal{D}$ which can be approximated by finitely supported functions pointwise and with respect to $\mathcal{Q}$. Hence, in the case that $Q^{(D)} = Q^{(N)}$ we see that $\mathcal{D}$ and $\mathcal{D}_0$ give the same spaces when intersected with $\ell^2(X, m)$. The question of when they are actually equivalent, that is, when $\mathcal{D} = \mathcal{D}_0$, will be taken up in the study of recurrence in Chapter 6.

We recall that an operator L is associated to a graph or just associated if it arises from a form which is associated to a graph and that any such operator L is a restriction of $\mathcal{L}$ by Theorem 1.12. Furthermore, by general theory,

$$D(L) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \text{there exists a } g \in \ell^2(X, m) \text{ such that} \\ Q(h, f) = \langle h, g \rangle \text{ for all } h \in D(Q) \end{array} \right\}$$

in which case $Lf = g$, see Theorem B.11 in Appendix B for more details. Therefore, $Q(h, f) = \langle h, Lf \rangle$ for all $f \in D(L)$ and $h \in D(Q)$. In particular, these statements apply to $L^{(D)}$ and $L^{(N)}$.

Let us also highlight the following immediate statement, which will be used several times below.

LEMMA 3.1. *Let (b, c) be a graph over (X, m). Let L be an operator associated to the graph with domain $D(L)$. Then,*

$$D(L) \subseteq \{f \in D(Q) \mid \mathcal{L}f \in \ell^2(X, m)\}.$$

PROOF. This follows from the facts that L maps $D(L) \subseteq D(Q)$ into $\ell^2(X, m)$ and that L is a restriction of $\mathcal{L}$ by Theorem 1.12. □

With these preparations, we now state and prove our characterizations of form equality.

THEOREM 3.2 (Characterization of $Q^{(D)} = Q^{(N)}$). *Let (b, c) be a graph over (X, m). Then, the following statements are equivalent:*

(i) $D(Q^{(D)}) = D(Q^{(N)})$.

(ii) $D(L^{(D)}) = \{f \in D(Q^{(N)}) \mid \mathcal{L}f \in \ell^2(X, m)\}$.

(iii) *For all $f, g \in D(Q^{(N)})$ such that $\mathcal{L}f \in \ell^2(X, m)$ we have*

$$Q^{(N)}(f, g) = \langle \mathcal{L}f, g\rangle. \qquad (\text{“Green's formula”})$$

(iv) *If $u \in D(Q^{(N)})$ is α-harmonic for $\alpha > 0$, then $u = 0$.*

(iv.a) *If $u \in D(Q^{(N)}) \cap \ell^p(X, m)$ for all $p \in [1, \infty]$ is α-harmonic for $\alpha > 0$ and $u \geq 0$, then $u = 0$.*

PROOF. (i) $\Longrightarrow$ (ii): It is immediate that $D(L^{(D)}) = D(L^{(N)})$ if $D(Q^{(D)}) = D(Q^{(N)})$. As $L^{(N)}$ is an associated operator, it follows that $D(L^{(N)}) \subseteq \{f \in D(Q^{(N)}) \mid \mathcal{L}f \in \ell^2(X, m)\}$ by Lemma 3.1. Therefore, it suffices to show

$$\{f \in D(Q^{(D)}) \mid \mathcal{L}f \in \ell^2(X, m)\} \subseteq D(L^{(D)}).$$

Now, for $f \in D(Q^{(D)})$ with $\mathcal{L}f \in \ell^2(X, m)$ from Green's formula, Proposition 1.5, we have

$$Q^{(D)}(\varphi, f) = \langle \varphi, \mathcal{L}f\rangle$$

for all $\varphi \in C_c(X)$. As $D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_Q}$ it follows that

$$Q^{(D)}(g, f) = \langle g, \mathcal{L}f\rangle$$

for all $g \in D(Q^{(D)})$. We thus conclude $f \in D(L^{(D)})$, which completes the proof.

(ii) $\Longrightarrow$ (iii): As $D(L^{(N)}) \subseteq \{f \in D(Q^{(N)}) \mid \mathcal{L}f \in \ell^2(X, m)\}$ by Lemma 3.1, we have $D(L^{(N)}) \subseteq D(L^{(D)})$ by assumption. As both $L^{(D)}$ and $L^{(N)}$ are restrictions of $\mathcal{L}$ by Theorem 1.12, the operator $L^{(N)}$ is a restriction of $L^{(D)}$ and thus $L^{(D)} = L^{(N)}$ as both operators are self-adjoint. Hence, for all $f, g \in D(Q^{(N)})$ with $\mathcal{L}f \in \ell^2(X, m)$, we have $f \in D(L^{(D)}) = D(L^{(N)})$ and, therefore,

$$Q^{(N)}(f, g) = \langle L^{(N)}f, g\rangle = \langle \mathcal{L}f, g\rangle.$$

(iii) $\Longrightarrow$ (iv): If $u \in D(Q^{(N)})$ is α-harmonic for $\alpha > 0$, then $\mathcal{L}u = -\alpha u \in \ell^2(X, m)$ since $u \in \ell^2(X, m)$. Therefore, by (iii), we get

$$0 \leq Q^{(N)}(u) = \langle \mathcal{L}u, u\rangle = -\alpha\|u\|^2 \leq 0$$

since $\alpha > 0$. Hence, $u = 0$.

(iv) $\Longrightarrow$ (iv.a): This is immediate.

(iv.a) $\Longrightarrow$ (i): Assume that $Q^{(D)} \neq Q^{(N)}$. It follows that $L^{(D)} \neq L^{(N)}$ and, therefore, $(L^{(D)} + \alpha)^{-1} \neq (L^{(N)} + \alpha)^{-1}$ for $\alpha > 0$. Since the set consisting of functions 1_x for $x \in X$ is total in $\ell^2(X, m)$, there exists an $x \in X$ such that

$$u = \left((L^{(N)} + \alpha)^{-1} - (L^{(D)} + \alpha)^{-1}\right) 1_x \neq 0.$$

To finish the proof we argue that u is a positive α-harmonic function in $D(Q^{(N)}) \cap \ell^p(X, m)$ for all $p \in [1, \infty]$: Since $L^{(D)}$ and $L^{(N)}$ are restrictions of $\mathcal{L}$ by Theorem 1.12 and the resolvents map into the corresponding domains of the operators, we infer that u is α-harmonic. Furthermore, as $Q^{(D)}$ and $Q^{(N)}$ are Dirichlet forms, both $(L^{(D)}+\alpha)^{-1}1_x$ and $(L^{(N)} + \alpha)^{-1}1_x$ are positive as both resolvents are positivity preserving by Proposition 2.10. This also follows by the general theory of Dirichlet forms, see Theorem C.4 in Appendix C. Thus, both of these functions are positive solutions of the equation $(\mathcal{L} + \alpha)v = 1_x$. However, $(L^{(D)} + \alpha)^{-1}1_x$ is the smallest such solution by Lemma 1.23. Thus, we have $u \geq 0$. Furthermore, both resolvent map into $D(Q^{(N)})$ as $D(Q^{(D)}) \subseteq D(Q^{(N)})$ so that $u \in D(Q^{(N)})$. Finally, both resolvents extend to Markov resolvents on $\ell^p(X, m)$ for $p \in [1, \infty]$ by Theorem 2.11. Thus, $u \in \ell^p(X, m)$ for all $p \in [1, \infty]$. $\Box$

Remark. In the proof of the implication (iv.a) $\Longrightarrow$ (i) presented directly above, we constructed a positive non-trivial α-harmonic function whenever $Q^{(D)} \neq Q^{(N)}$. It turns out that such functions are automatically strictly positive when they exist and the graph is connected. This theme will be taken up in the next chapter as a consequence of the local Harnack inequality, see Corollary 4.2.

Remark. We can also characterize $D(Q^{(D)}) = D(Q^{(N)})$ in terms of α-subharmonic functions with additional properties (Exercise 3.8).

We note the following immediate corollary which gives the domain of all operators associated to graphs in the case that the Dirichlet and Neumann restrictions agree.

Corollary 3.3. *Let (b, c) be a graph over (X, m). Let L be an operator associated to the graph. Then, $Q^{(D)} = Q^{(N)}$ if and only if*

$$D(L) = \{f \in \mathcal{D} \cap \ell^2(X, m) \mid \mathcal{L}f \in \ell^2(X, m)\}.$$

Proof. If L is an operator associated to the graph and Q is the associated form, then $D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)})$ and all forms are restrictions of $\mathcal{Q}$. Hence, if $D(Q^{(D)}) = D(Q^{(N)})$, then all forms agree so that $D(L) = D(L^{(D)})$ and the statement follows by Theorem 3.2.

On the other hand, assume $D(L) = \{f \in \mathcal{D} \cap \ell^2(X, m) \mid \mathcal{L}f \in \ell^2(X, m)\}$ for an operator L associated to the graph. Now, let $u \in$

$D(Q^{(N)})$ be α-harmonic for $\alpha > 0$. Then, $\mathcal{L}u = -\alpha u \in \ell^2(X, m)$ since $u \in \ell^2(X, m)$ and, therefore, $u \in D(L) \subseteq D(Q)$ for the form Q associated to L. We thus obtain

$$0 \leq Q(u) = \langle Lu, u \rangle = -\alpha \|u\|^2 \leq 0$$

since $\alpha > 0$. Hence, $u = 0$. Thus, we have shown statement (iv) of Theorem 3.2 and, therefore, $D(Q^{(D)}) = D(Q^{(N)})$. This completes the proof. □

2. Essential self-adjointness

In this section we consider the question of the uniqueness of self-adjoint extensions of the restriction of the formal Laplacian to the finitely supported functions. In order for this question to make sense, we have to make an additional assumption on our graphs. Our characterization will then be in terms of the Dirichlet Laplacian domain and the triviality of square summable α-harmonic functions for $\alpha > 0$.

The reader may wish to consult Excavation Exercises 3.1 and 3.2 for some general facts about adjoint operators and essential self-adjointness which will be used in the proof of Theorem 3.6.

A symmetric operator defined on a dense subspace of a Hilbert space is called *essentially self-adjoint* if the operator has a unique self-adjoint extension. For further details on the general theory of adjoints and self-adjointness, see Appendix A.

In our situation, it is natural to consider the question of whether the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint. Of course, this question only makes sense if $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$ and, in this case, this restriction is indeed symmetric by Green's formula, Proposition 1.5. Another natural question in our situation is if the Dirichlet Laplacian is the "maximal" restriction of $\mathcal{L}$ to an operator on $\ell^2(X, m)$, i.e., if

$$D(L^{(D)}) = \{f \in \ell^2(X, m) \mid \mathcal{L}f \in \ell^2(X, m)\}.$$

It turns out that the essential self-adjointness of the restriction of $\mathcal{L}$ to $C_c(X)$ is equivalent to the maximality of the Dirichlet Laplacian. Furthermore, both of these questions are equivalent to the absence of non-trivial α-harmonic functions in $\ell^2(X, m)$ for $\alpha > 0$. This is the content of Theorem 3.6.

The characterization of the maximality of the Laplacian domain in terms of α-harmonic functions is not restricted to the $\ell^2(X, m)$ setting but rather works for the generators of semigroups on all $\ell^p(X, m)$. This is the content of Theorem 3.8.

After this summary of results, we start by discussing some of the properties of the restriction of $\mathcal{L}$ to $C_c(X)$. Specifically, we let $L_{\min}$ denote the restriction of $\mathcal{L}$ to

$$D(L_{\min}) = C_c(X)$$

whenever $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. By Green's formula, Proposition 1.5, $L_{\min}$ is then a symmetric operator.

We recall that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ is characterized in Theorem 1.29. In particular, some conditions equivalent to $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ are that $\ell^2(X,m) \subseteq \mathcal{F}$ or that $C_c(X) \subseteq D(L^{(D)})$. In particular, this condition is always satisfied if the graph is locally finite or if $\inf_{x\in X} m(x) > 0$. As a consequence, if $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then there exists at least one self-adjoint extension of $L_{\min}$, namely, $L^{(D)}$. The question of essential self-adjointness then boils down to if there exist other self-adjoint extensions of $L_{\min}$.

By the definition of the adjoint operator and $L_{\min} = \mathcal{L}$ on $C_c(X)$, the domain of the adjoint of $L_{\min}$ is given by

$$D(L_{\min}^*) = \left\{ f \in \ell^2(X,m) \;\middle|\; \begin{array}{c} \text{there exists a } g \in \ell^2(X,m) \text{ such that} \\ \langle \mathcal{L}\varphi, f\rangle = \langle \varphi, g\rangle \text{ for all } \varphi \in C_c(X) \end{array} \right\}$$

in which case $L_{\min}^* f = g$. As $L_{\min}$ is symmetric, it follows that $L_{\min}^*$ is an extension of $L_{\min}$. We now give an explicit description of $D(L_{\min}^*)$ and the action of $L_{\min}^*$.

Lemma 3.4 (Domain and action of $L_{\min}^*$). *Let (b,c) be a graph over (X,m) such that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. Let $L_{\min}$ be the restriction of $\mathcal{L}$ to $D(L_{\min}) = C_c(X)$. Then,*

$$D(L_{\min}^*) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$$

and $L_{\min}^*$ is a restriction of $\mathcal{L}$.

Proof. Let $f \in D(L_{\min}^*)$, which is a subspace of $\ell^2(X,m)$ by definition. As $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, it follows that $\ell^2(X,m) \subseteq \mathcal{F}$ by Theorem 1.29. Therefore, $f \in \mathcal{F}$.

Furthermore, by the definition of $D(L_{\min}^*)$ it follows that $\langle \mathcal{L}\varphi, f\rangle = \langle \varphi, g\rangle$ for some $g \in \ell^2(X,m)$ and all $\varphi \in C_c(X)$. Using $f \in \mathcal{F}$ and invoking Green's formula, Proposition 1.5, we then obtain

$$\begin{aligned}\sum_{x\in X} \varphi(x)g(x)m(x) = \langle \varphi, g\rangle &= \langle \mathcal{L}\varphi, f\rangle \\ &= \sum_{x\in X} \mathcal{L}\varphi(x)f(x)m(x) \\ &= \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x).\end{aligned}$$

As this holds for all $\varphi \in C_c(X)$ we infer $\mathcal{L}f = g \in \ell^2(X,m)$. Hence, we have $f \in \ell^2(X,m) \cap \mathcal{F}$ and $\mathcal{L}f \in \ell^2(X,m)$. Furthermore, $L_{\min}^* f = g = \mathcal{L}f$ and thus $L_{\min}^*$ is a restriction of $\mathcal{L}$.

On the other hand, if $f \in \ell^2(X,m) \subseteq \mathcal{F}$ is such that $\mathcal{L}f \in \ell^2(X,m)$, then

$$\langle \mathcal{L}\varphi, f\rangle = \langle \varphi, \mathcal{L}f\rangle$$

for all $\varphi \in C_c(X)$ by Green's formula, Proposition 1.5. Therefore, $f \in D(L^*_{\min})$ with $L^*_{\min} f = \mathcal{L} f$ by definition. $\square$

As an immediate corollary of the above, we can explicitly determine the self-adjoint extensions of $L_{\min}$. It turns out that these are exactly the self-adjoint restrictions of $\mathcal{L}$.

Corollary 3.5 (Self-adjoint extensions of $L_{\min}$). *Let (b,c) be a graph over (X,m) such that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. Let $L_{\min}$ be the restriction of $\mathcal{L}$ to $D(L_{\min}) = C_c(X)$ and let L be an operator with domain $D(L) \subseteq \ell^2(X,m)$. Then, L is a self-adjoint extension of $L_{\min}$ if and only if L is a self-adjoint restriction of $\mathcal{L}$.*

Proof. Any self-adjoint extension of $L_{\min}$ is a restriction of $L^*_{\min}$ by general properties of adjoint operators. Therefore, if L is a self-adjoint extension of $L_{\min}$, then L is a restriction of $\mathcal{L}$ by Lemma 3.4.

On the other hand, if L is a self-adjoint restriction of $\mathcal{L}$, then clearly $D(L) \subseteq D(L^*_{\min})$ as $D(L^*_{\min}) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$ by Lemma 3.4. Thus, $D(L^{**}_{\min}) \subseteq D(L^*)$. Furthermore, it is easy to see from the definition of the adjoint and Green's formula that $C_c(X) \subseteq D(L^{**}_{\min})$. In summary, we obtain

$$C_c(X) \subseteq D(L^{**}_{\min}) \subseteq D(L^*) = D(L).$$

So that $C_c(X) \subseteq D(L)$ and L is an extension of $L_{\min}$. $\square$

With this preliminary discussion of the domain and action of $L^*_{\min}$ we can now state and prove our characterization of the essential self-adjointness of $L_{\min}$. As $L^*_{\min}$ is an extension of $L_{\min}$, by general theory the essential self-adjointness of $L_{\min}$ is equivalent to the self-adjointness of $L^*_{\min}$. This will be used in the proof below.

Theorem 3.6 (Characterization of essential self-adjointness). *Let (b,c) be a graph over (X,m) such that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. Then, the following statements are equivalent:*

(i) *The restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint.*
(ii) $D(L^{(D)}) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$.
(iii) *If $u \in \ell^2(X,m)$ is α-harmonic for $\alpha > 0$, then $u = 0$.*

Remark. We remark that in contrast to Theorem 3.2 (iv.a), we cannot assume that $u \geq 0$ in statement (iii) above. The reason is that we do not know if forms associated to self-adjoint extensions of $L_{\min}$ are necessarily Dirichlet forms and, as such, have positivity preserving resolvents. A necessary condition for the arising form to be a Dirichlet form will be given in Theorem 3.11 in Section 3.

Proof. (i) $\Longrightarrow$ (ii): Let $L_{\min}$ denote the restriction of $\mathcal{L}$ to $C_c(X)$. Since $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, it follows that $C_c(X) \subseteq D(L^{(D)})$ by Theorem 1.29, so that $L^{(D)}$ is a self-adjoint extension of $L_{\min}$. As we assume that $L_{\min}$ is essentially self-adjoint, $L_{\min}^*$ is also a self-adjoint extension of $L_{\min}$ by general theory so that $L^{(D)} = L_{\min}^*$ and, therefore,

$$D(L^{(D)}) = D(L_{\min}^*) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$$

by Lemma 3.4.

(ii) $\Longrightarrow$ (iii): If $u \in \ell^2(X,m)$ is α-harmonic, then $\mathcal{L}u = -\alpha u \in \ell^2(X,m)$ so that $u \in D(L^{(D)})$ by assumption. As $D(L^{(D)}) \subseteq D(Q^{(D)})$, we get

$$0 \leq Q^{(D)}(u) = \langle L^{(D)}u, u\rangle = -\alpha\|u\|^2 \leq 0$$

since $\alpha > 0$. Therefore, $u = 0$.

(iii) $\Longrightarrow$ (i): Assume that there exist two distinct self-adjoint extensions L_1 and L_2 of $L_{\min}$. Then $(L_1+\alpha)^{-1} \neq (L_2+\alpha)^{-1}$ on $\ell^2(X,m)$ for $\alpha > 0$. As the set of functions 1_x for $x \in X$ is total in $\ell^2(X,m)$, there exists an $x \in X$ such that

$$u = \left((L_1+\alpha)^{-1} - (L_2+\alpha)^{-1}\right) 1_x \neq 0.$$

Clearly, $u \in \ell^2(X,m)$. Furthermore, since both L_1 and L_2 are restrictions of $\mathcal{L}$ by Corollary 3.5 we infer that

$$(\mathcal{L}+\alpha)u = (L_1+\alpha)(L_1+\alpha)^{-1}1_x - (L_2+\alpha)(L_2+\alpha)^{-1}1_x = 0.$$

Hence u is a non-trivial, α-harmonic function in $\ell^2(X,m)$. □

Remark. The result above can also be formulated in terms of associated operators. More specifically, essential self-adjointness is equivalent to

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$$

for some (all) associated operators L (Exercise 3.9).

Combining the characterizations above with Theorem 3.2 gives the following immediate corollary.

Corollary 3.7 (Essential self-adjointness implies form equality). *Let (b,c) be a graph over (X,m) such that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. If the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint, then*

$$D(Q^{(D)}) = D(Q^{(N)}).$$

Remark. It can be shown by example that the opposite implication does not hold, see Exercise 3.6.

Part (ii) of Theorem 3.6 above determines the domain of the Dirichlet Laplacian explicitly when α-harmonic functions in $\ell^2(X,m)$ are trivial for $\alpha > 0$. We now prove a similar statement for the generators of semigroups and resolvents on $\ell^p(X,m)$ for $p \in [1,\infty)$. These generators $L^{(p)}$ were obtained by extending the semigroups e^{-tL} of $L = L^{(D)}$ to $\ell^p(X,m)$ and then taking the generator of each semigroup, see Section 1. The domain of the operators is then defined via either the semigroup or the resolvent by general theory.

THEOREM 3.8 (Domain of $L^{(p)}$). *Let (b,c) be a graph over (X,m) and let $p \in [1,\infty)$. Then, the following statements are equivalent:*

(i) $D(L^{(p)}) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}$.

(ii) *If $u \in \ell^p(X,m)$ is α-harmonic for $\alpha > 0$, then $u = 0$.*

PROOF. (i) $\implies$ (ii): If $u \in \ell^p(X,m)$ is α-harmonic for $p \in [1,\infty)$ and $\alpha > 0$, then $\mathcal{L}u = -\alpha u \in \ell^p(X,m)$ so that $u \in D(L^{(p)})$ by assumption. Since $L^{(p)} = \mathcal{L}$ on $D(L^{(p)})$ by Theorem 2.13, we infer from the existence of resolvents, Theorem 2.11, that

$$u = (L^{(p)} + \alpha)^{-1}(L^{(p)} + \alpha)u = (L^{(p)} + \alpha)^{-1}(\mathcal{L} + \alpha)u = 0.$$

This gives the conclusion.

(ii) $\implies$ (i): Let

$$D_p = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}.$$

By Theorem 2.13, the generator satisfies $L^{(p)} = \mathcal{L}$ on $D(L^{(p)})$. This easily implies $D(L^{(p)}) \subseteq D_p$.

On the other hand, let $f \in D_p$ and let $\alpha > 0$. Then, $(\mathcal{L} + \alpha)f \in \ell^p(X,m)$ so that we may apply the resolvent to get

$$(L^{(p)} + \alpha)^{-1}(\mathcal{L} + \alpha)f \in D(L^{(p)}).$$

We let $g = (L^{(p)} + \alpha)^{-1}(\mathcal{L} + \alpha)f$. By Theorem 2.13 again, we see that

$$(\mathcal{L} + \alpha)g = (\mathcal{L} + \alpha)f.$$

Therefore, $f - g \in \ell^p(X,m)$ is α-harmonic and so $f - g = 0$ by assumption. This implies $f \in D(L^{(p)})$ and completes the proof. $\square$

3. Markov uniqueness

In this section we consider the uniqueness of Markov restrictions of the formal Laplacian. By definition, these restrictions are such that the arising forms are Dirichlet forms. We will see that there is a unique such Markov restriction if and only if the Dirichlet and Neumann forms agree.

We will draw heavily from Appendix C, which develops the theory of Dirichlet forms. We will also need Excavation Exercises 2.4 and 2.5, which recall a basic fact about the existence of weakly convergent sequences and the Banach–Saks theorem.

In Section 1 we considered the question of when there is a unique form associated to a graph. In Section 2 we similarly explored the question of when the restriction of the formal Laplacian to the finitely supported functions has a unique self-adjoint extension under the assumption that the formal Laplacian maps finitely supported functions to square summable functions. In this section we will consider the question of when restrictions of the formal Laplacian have a unique

Dirichlet form, i.e., a unique positive closed form which is compatible with normal contractions.

It is intuitively clear that when there is a unique positive self-adjoint operator, then both the associated form and the arising Dirichlet form should be unique and this is indeed the case. Let us highlight, however, that we do not know if the forms arising from restrictions of the formal Laplacian $\mathcal{L}$ are restrictions of the energy form $\mathcal{Q}$ in general. Hence, as we do not know the action of the form on all functions in the domain, we must take particular care throughout our considerations and rely on the general theory of Dirichlet forms.

On the other hand, it is relatively easy to establish that a Dirichlet form coming from a restriction of the formal Laplacian acts as the energy form on the finitely supported functions. Furthermore, let us note that we have already seen in Section 2 that there is a one-to-one correspondence between graphs and regular Dirichlet forms. In this section, we drop the regularity assumption, which makes the analysis significantly more difficult. However, we will use the results of Section 2 along the way to our understanding of more general forms arising from the formal Laplacian in that we use approximating forms, which are regular forms, and then pass to the limit.

After this preliminary discussion, we start by defining the restrictions which will be of interest.

Definition 3.9 (Markov realization and Markov uniqueness). Let (b, c) be a graph over (X, m). A positive operator L with form Q is called a *realization* of $\mathcal{L}$ if

$$L = \mathcal{L} \quad \text{on } D(L) \qquad \text{and} \qquad C_c(X) \subseteq D(Q).$$

An operator L is called a *Markov realization* of $\mathcal{L}$ if L is a realization of $\mathcal{L}$ and Q is a Dirichlet form. The operator $\mathcal{L}$ is said to satisfy *Markov uniqueness* if there exists a unique Markov realization of $\mathcal{L}$.

We start by pointing out that we have seen at least two Markov realizations thus far.

Example 3.10 ($L^{(D)}$ and $L^{(N)}$ are Markov realizations). An operator L is associated to a graph if L comes from a closed form Q which is a restriction of the energy form $\mathcal{Q}$ and whose domain $D(Q)$ contains $C_c(X)$. As Q is a symmetric positive closed form, it follows that L is positive. Furthermore, any such operator is a restriction of $\mathcal{L}$ by Theorem 1.12. Hence, any associated L is a realization of $\mathcal{L}$. If Q is additionally a Dirichlet form, it follows that L is a Markov realization of $\mathcal{L}$. In particular, as both $Q^{(D)}$ and $Q^{(N)}$ are Dirichlet forms, see Lemma 1.16 and Proposition 1.14, both $L^{(D)}$ and $L^{(N)}$ are Markov realizations.

REMARK. We observe that if L with form Q is a realization of $\mathcal{L}$, then for all $f \in D(L)$ and $\varphi \in C_c(X) \subseteq D(Q)$,

$$Q(f, \varphi) = \langle Lf, \varphi \rangle = \langle \mathcal{L}f, \varphi \rangle = \mathcal{Q}(f, \varphi),$$

where the first equality follows by the connection between Q and L, see Corollary B.12, the second equality holds since L is a restriction of $\mathcal{L}$ and the third equality is Green's formula, Proposition 1.5.

Using the reasoning of Theorem 1.29, if $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$, then $C_c(X) \subseteq D(L)$ (Exercise 3.10). In this case the formula above holds for all $f \in C_c(X)$. By approximation we can, therefore, show $Q = Q^{(D)}$ on $D(Q^{(D)})$ in this case. For the general case this is not clear but in the case when L is a Markov realization we will show this below. Furthermore, we give a lower bound on Q by $Q^{(N)}$ on $D(Q)$ in this case.

Forms on $\ell^2(X, m)$ can be naturally ordered as follows: If Q_1 and Q_2 are forms with domains $D(Q_1)$ and $D(Q_2)$ in $\ell^2(X, m)$, then we will write

$$Q_1 \leq Q_2$$

if

$$D(Q_2) \subseteq D(Q_1) \quad \text{and} \quad Q_1(f) \leq Q_2(f)$$

for all $f \in D(Q_2)$.

Having established the relevant concepts and notations, we now state the main result of this section.

THEOREM 3.11 (Characterization of Markov restrictions). *Let (b, c) be a graph over (X, m). If L is a Markov realization of $\mathcal{L}$ and Q is the associated Dirichlet form, then*

$$Q^{(N)} \leq Q \leq Q^{(D)}.$$

REMARK. We note that it follows from $Q^{(N)} \leq Q \leq Q^{(D)}$ that

$$Q^{(N)}(f) \leq Q(f) \leq Q^{(D)}(f) = Q^{(N)}(f) = \mathcal{Q}(f)$$

for all $f \in D(Q^{(D)})$. In particular, this determines the action of Q on $C_c(X)$.

REMARK. Naively, one might think that the form Q associated to a Markov realization of $\mathcal{L}$ is a restriction of $\mathcal{Q}$. However, the point of this section is that this naive view is not clear as Green's formula only allows us to test with functions in $C_c(X)$.

REMARK. We note that not all forms associated to realizations of $\mathcal{L}$ satisfy the inequalities in Theorem 3.11, as can be shown by example, see Exercise 3.7. In particular, not all operators which are realizations of $\mathcal{L}$ are Markov.

The proof of Theorem 3.11 will require some work. However, let us note the following immediate consequence, which states that Markov uniqueness is equivalent to uniqueness of associated forms.

THEOREM 3.12 (Characterization of Markov uniqueness). *Let (b, c) be a graph over (X, m). Then, $\mathcal{L}$ satisfies Markov uniqueness if and only if $Q^{(D)} = Q^{(N)}$.*

PROOF. If $\mathcal{L}$ satisfies Markov uniqueness, then $L^{(D)} = L^{(N)}$ as both are Markov realizations of $\mathcal{L}$ so that $Q^{(D)} = Q^{(N)}$. On the other hand, if $Q^{(D)} = Q^{(N)}$ and L is a Markov realization of $\mathcal{L}$ with associated form Q, then, by Theorem 3.11, we have

$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)}) = D(Q^{(D)})$$

and

$$Q^{(N)} \leq Q \leq Q^{(D)} = Q^{(N)}$$

so that $Q = Q^{(D)} = Q^{(N)}$. $\square$

Therefore, to characterize Markov uniqueness, we may use any of the equivalent statements found in Theorem 3.2. We also highlight the following immediate connection between essential self-adjointness and Markov uniqueness.

COROLLARY 3.13 (Essential self-adjointness and Markov uniqueness). *Let (b, c) be a graph over (X, m) with $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$. If the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint, then $\mathcal{L}$ satisfies Markov uniqueness.*

PROOF. Combine Theorem 3.12 with Corollary 3.7. $\square$

We now begin the proof of Theorem 3.11. We will have several occasions to use the following lemma, which gives a Green's formula for forms associated to Markov realizations. Here and throughout the section the space $\ell^\infty(X)$ of bounded functions on X plays an important role.

LEMMA 3.14 (Green's formula). *Let (b, c) be a graph over (X, m). Let L be a Markov realization of $\mathcal{L}$ and Q be the associated Dirichlet form with domain $D(Q)$. If $\varphi \in C_c(X)$ and $f \in D(Q) \cap \ell^\infty(X) \subseteq \mathcal{F}$, then*

$$Q(\varphi, f) = \sum_{x \in X} \varphi(x) \mathcal{L}f(x) m(x).$$

PROOF. This is essentially a consequence of Lebesgue's dominated convergence theorem and the functional calculus, see Proposition A.26 and Lemma B.7. More specifically, for $f \in D(Q)$ we have $(L+\alpha)^{-1} f \in D(L)$ and $\varphi \in C_c(X) \subseteq D(Q)$ by assumption. By the spectral calculus we get

$$\begin{aligned} Q(\varphi, f) &= \lim_{\alpha\to\infty} Q(\varphi, \alpha(L+\alpha)^{-1}f) \\ &= \lim_{\alpha\to\infty} \langle \varphi, L\alpha(L+\alpha)^{-1}f\rangle \\ &= \lim_{\alpha\to\infty} \sum_{x\in X} \varphi(x)\mathcal{L}(\alpha(L+\alpha)^{-1}f)(x)m(x), \end{aligned}$$

where we use that L is a restriction of $\mathcal{L}$ in the last line.

We now note that, as Q is a Dirichlet form, $\alpha(L+\alpha)^{-1}$ is contracting, i.e., $|\alpha(L+\alpha)^{-1}f(x)| \leq \|f\|_\infty$ for all $x \in X$ and $f \in \ell^2(X, m)$, see Theorem C.4 in Appendix C. Moreover, $\alpha(L+\alpha)^{-1}f \to f$ as $\alpha \to \infty$ by Theorem A.34 in Appendix A. Consequently, by Lebesgue's dominated convergence theorem, we get

$$\lim_{\alpha\to\infty} \sum_{y\in X} b(x,y)\alpha(L+\alpha)^{-1}f(y) = \sum_{y\in X} b(x,y)f(y)$$

and, therefore,

$$\lim_{\alpha\to\infty} \mathcal{L}(\alpha(L+\alpha)^{-1}f)(x) = \mathcal{L}f(x).$$

Putting all of this together, we obtain

$$Q(\varphi, f) = \lim_{\alpha\to\infty} \sum_{x\in X} \varphi(x)\mathcal{L}(\alpha(L+\alpha)^{-1}f)(x)m(x) = \sum_{x\in X} \varphi(x)\mathcal{L}f(x)m(x),$$

where the sum has finitely many non-zero terms since $\varphi \in C_c(X)$. This completes the proof. $\square$

We now prove the upper bound by $Q^{(D)}$ in Theorem 3.11 by using the previous lemma.

Proposition 3.15 ($Q \leq Q^{(D)}$). *Let (b, c) be a graph over (X, m). If L is a Markov realization of $\mathcal{L}$ and Q is the associated Dirichlet form with domain $D(Q)$, then $Q = Q^{(D)}$ on $C_c(X)$ and*

$$Q \leq Q^{(D)}.$$

Proof. By Lemma 3.14 we get

$$Q(\varphi) = \sum_{x\in X} \mathcal{L}\varphi(x)\varphi(x)m(x)$$

for all $\varphi \in C_c(X) \subseteq D(Q) \cap \ell^\infty(X)$. Furthermore, by Green's formula, Proposition 1.5, and the fact that $Q^{(D)}$ is a restriction of $\mathcal{Q}$ by definition we get

$$Q^{(D)}(\varphi) = \mathcal{Q}(\varphi) = \sum_{x\in X} \mathcal{L}\varphi(x)\varphi(x)m(x) = Q(\varphi).$$

As Q is a closed form and Q and $\mathcal{Q}$ agree on $C_c(X)$, we get that they agree on $D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}} \subseteq D(Q)$. Therefore, Q is an extension of $Q^{(D)}$ so that $Q \leq Q^{(D)}$. $\square$

In order to proceed further, we will need two general facts about Dirichlet forms. The first one states that the set of bounded functions in the domain of a Dirichlet form is an algebra, i.e.,

$$D(Q) \cap \ell^\infty(X)$$

satisfies $fg \in D(Q) \cap \ell^\infty(X)$ whenever $f, g \in D(Q) \cap \ell^\infty(X)$. For the space of functions of finite energy, this can be shown directly (Exercise 3.11). The general case follows from abstract theory, see Corollary C.6 in Appendix C.

The second general fact, which is proven next, states that the algebra $D(Q) \cap \ell^\infty(X)$ is dense in $D(Q)$ with respect to the form norm $\| \cdot \|_Q$ which arises from the scalar product

$$\langle f, g \rangle_Q = Q(f, g) + \langle f, g \rangle.$$

We denote this Hilbert space by

$$H_Q = (D(Q), \langle \cdot, \cdot \rangle_Q).$$

LEMMA 3.16. *Let (b, c) be a graph over (X, m). Let Q be a Dirichlet form with domain $D(Q) \subseteq \ell^2(X, m)$ and let $f \in D(Q)$.*

(a) *For $n \in \mathbb{N}$ the functions $f_n = (f \wedge n) \vee -n$ are in $D(Q) \cap \ell^\infty(X)$ and converge to f with respect to $\| \cdot \|_Q$ as $n \to \infty$.*
(b) *For $\alpha > 0$ the functions $f_\alpha = f - ((f \wedge \alpha) \vee -\alpha)$ are in $D(Q)$ and converge to f with respect to $\| \cdot \|_Q$ as $\alpha \to 0^+$.*

PROOF. (a) Let $f \in D(Q)$. For $n \in \mathbb{N}$, let $f_n = (f \wedge n) \vee -n$. Then, clearly each f_n is bounded and since Q is a Dirichlet form and cutting above by a positive number and below by a negative number are normal contractions we have $f_n \in D(Q)$. Thus, $f_n \in D(Q) \cap \ell^\infty(X)$.

Observe that since $f_n \to f$ pointwise and $|f_n(x)| \leq |f(x)|$ for all $x \in X$, it follows that $f_n \to f$ in $\ell^2(X, m)$ by Lebesgue's dominated convergence theorem.

We will show that every subsequence of (f_n) has a further subsequence that converges to f with respect to $\| \cdot \|_Q$, which will complete the proof. We first note that as Q is a Dirichlet form, it follows that $Q(f_n) \leq Q(f)$, so that (f_n) is a bounded sequence in the Hilbert space H_Q. Now, any subsequence of (f_n) is also bounded in H_Q, so it has a weakly convergent subsequence, say (g_k) with weak limit g. By the Banach–Saks theorem, there exists a subsequence (g_{l_k}) of (g_l) whose Cesàro means converge to g strongly, i.e.,

$$\frac{1}{N} \sum_{k=1}^{N} g_{l_k} \to g$$

strongly as $N \to \infty$. In particular, since $f_n \to f$ pointwise and (g_{l_k}) is a subsequence of (f_n), it follows that $f = g$, so that f is the weak limit of (g_l).

Therefore, using the fact that Q is a Dirichlet form and thus $\|g_l\|_Q \leq \|f\|_Q$, we obtain

$$\|g_l - f\|_Q^2 = \|g_l\|_Q^2 - 2\langle g_l, f\rangle_Q + \|f\|_Q^2 \leq 2\|f\|_Q^2 - 2\langle g_l, f\rangle_Q \to 0$$

as $l \to \infty$. So, if (f_n) does not converge to f in $\|\cdot\|_Q$ it has a subsequence where each element has a uniformly positive distance to f. However, this is not possible since every subsequence has a convergent subsequence.

(b) The proof follows exactly along the same lines as the proof of (a) by noting that the functions $(f \wedge \alpha) \vee -\alpha$ for $\alpha > 0$ are in $D(Q)$ and converge to 0 with respect to $\|\cdot\|_Q$. $\square$

Remark. The convergence above can also be obtained via other means (Exercise 3.12).

Corollary 3.17 (Bounded functions are dense in the form domain). *Let (b, c) be a graph over (X, m). Let Q be a Dirichlet form with domain $D(Q)$. Then $D(Q) \cap \ell^\infty(X) \subseteq D(Q)$ is dense with respect to the form norm $\|\cdot\|_Q$.*

Proof. Use the sequence constructed in (a) of Lemma 3.16, i.e., for any $f \in D(Q)$, $f_n = (f \wedge n) \vee -n \in D(Q) \cap \ell^\infty(X)$ converges to f with respect to $\|\cdot\|_Q$. $\square$

The corollary is significant as it allows us to reduce all arguments for $D(Q)$ to those for $D(Q) \cap \ell^\infty(X)$, which is an algebra by Corollary C.6. In particular, we will prove that $Q \geq Q^{(N)}$ for functions in $D(Q) \cap \ell^\infty(X)$ and then pass to $D(Q)$ by using the corollary.

By the fact that $D(Q) \cap \ell^\infty(X)$ is an algebra, we can make the following definition. For $f, g \in D(Q) \cap \ell^\infty(X)$ and $\psi \in D(Q)$ with $0 \leq \psi \leq 1$, we let

$$Q_\psi(f, g) = Q(\psi f, \psi g) - Q(\psi f g, \psi).$$

As Q is a symmetric form, it follows easily that Q_ψ is a symmetric form as well. In particular,

$$Q_\psi(f) = Q(\psi f) - Q(\psi f^2, \psi)$$

is a quadratic form.

We note that we do not assume that ψ is finitely supported. However, when ψ is finitely supported we get the following proposition for the energy form by a direct computation.

Proposition 3.18. *Let (b, c) be a graph over (X, m). Then,*

$$\mathcal{Q}(\psi f) - \mathcal{Q}(\psi f^2, \psi) = \frac{1}{2} \sum_{x,y \in X} b(x, y)\psi(x)\psi(y)(f(x) - f(y))^2$$

for all $\psi \in C_c(X)$ and $f \in C(X)$.

PROOF. We have

$$\mathcal{Q}(\psi f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)((\psi f)(x) - (\psi f)(y))^2 + \sum_{x \in X} c(x)(\psi f)^2(x)$$

and

$$\mathcal{Q}(\psi f^2, \psi)$$
$$= \frac{1}{2} \sum_{x,y \in X} b(x,y)((\psi f^2)(x) - (\psi f^2)(y))(\psi(x) - \psi(y)) + \sum_{x \in X} c(x)(\psi f)^2(x).$$

Now, a direct computation shows

$$((\psi f)(x) - (\psi f)(y))^2 - \left((\psi f^2)(x) - (\psi f^2)(y)\right)(\psi(x) - \psi(y))$$
$$= \psi(x)\psi(y)(f(x) - f(y))^2.$$

Putting this together, we arrive at the statement of the proposition. □

REMARK. Letting

$$\mathcal{Q}_\psi(f) = \sum_{x,y \in X} b(x,y)\psi(x)\psi(y)(f(x) - f(y))^2$$

and invoking the Green's formula we obtain from the proposition

$$\mathcal{Q}(\psi f) = \mathcal{Q}_\psi(f) + \langle f, (\psi \mathcal{L} \psi) f \rangle$$

for all $\psi, f \in C_c(X)$. Now, it is not hard to see that this can be extended by simple limiting procedures to an arbitrary $\psi \in \mathcal{F}$ with $\psi \geq 0$ and $f \in C_c(X)$. In particular, if $\mathcal{L}\psi = \lambda\psi$ for some non-negative $\psi \in \mathcal{F}$ and $\lambda \in \mathbb{R}$, we obtain

$$\mathcal{Q}(\psi f) = \mathcal{Q}_\psi(f) + \lambda \|\psi f\|^2.$$

This is the starting point of the technique of the ground state transform, to be investigated in the next chapter.

We can apply the preceding proposition to calculate Q_ψ for ψ equal to the characteristic function of a finite set.

LEMMA 3.19. *Let (b, c) be a graph over (X, m). Let $W \subseteq X$ be a finite set. If L is a Markov realization of $\mathcal{L}$ and Q is the associated Dirichlet form with domain $D(Q)$, then*

$$Q_{1_W}(f) = \frac{1}{2} \sum_{x,y \in W} b(x,y)(f(x) - f(y))^2$$

for all $f \in D(Q) \cap \ell^\infty(X)$.

PROOF. We note that by definition

$$Q_{1_W}(f) = Q(1_W f) - Q(1_W f^2, 1_W).$$

As W is a finite set, $1_W f \in C_c(X)$. Therefore, as $Q = Q^{(D)}$ on $C_c(X)$ by Proposition 3.15, we get

$$Q(1_W f) = Q^{(D)}(1_W f) = \mathcal{Q}(1_W f) \text{ and } Q(1_W f^2, 1_W) = \mathcal{Q}(1_W f^2, 1_W).$$

Now, the statement follows from the previous proposition. □

We now decompose our Dirichlet form into two parts. For $f \in D(Q) \cap \ell^\infty(X)$, we define

$$Q_M(f) = \sup_{\substack{\psi \in D(Q) \\ 0 \leq \psi \leq 1}} Q_\psi(f)$$

as the *main part* of Q and let

$$Q_K(f) = Q(f) - Q_M(f)$$

denote the *killing part* of Q. Therefore,

$$Q(f) = Q_M(f) + Q_K(f).$$

Before we justify these definitions, let us mention the main idea. As already seen in Lemma 3.19, when we let $\psi = 1_W$ for a finite set $W \subseteq X$, we get that $Q_{1_W}(f)$ gives the energy of f coming from b over W. Thus, taking the supremum over all such functions shows that $Q_M(f)$ bounds $\mathcal{Q}_{b,0}(f)$ from above. We will show later that the killing part $Q_K(f)$ controls the part of the energy coming from the killing term c, that is, $Q_K(f) \geq \mathcal{Q}_{0,c}(f)$. Combining these two estimates gives that $Q(f) \geq \mathcal{Q}(f)$, which will finish the proof of our main result.

These estimates will be proven after some preliminary technicalities. In particular, we first have to justify that Q_M and Q_K take finite values and establish several properties listed in Lemma 3.21. In order to carry out the proof, we will use the general theory of approximating forms for a quadratic form. Specifically, for a quadratic form Q with operator L and $\alpha > 0$, we let

$$Q^\alpha(f, g) = \alpha \langle f, (I - \alpha(L + \alpha)^{-1})g \rangle$$

denote the *approximating form*. These are bounded forms which satisfy

$$\lim_{\alpha \to \infty} Q^\alpha(f, g) = Q(f, g)$$

for all $f, g \in D(Q)$, which is a consequence of the spectral calculus and is proven in Corollary B.14 in Appendix B. Furthermore, when Q is a Dirichlet form, it follows that Q^α is a Dirichlet form for every $\alpha > 0$, see Corollary C.5 in Appendix C for details.

As Q^α are bounded, they are defined on all of $\ell^2(X, m)$ and, as such, they are regular Dirichlet forms whenever Q is a Dirichlet form. Therefore, we may apply the theory developed in Section 2, which says that every such form on a discrete space is given by a graph, see

Theorem 1.18. In particular, to every Q^α there exists a graph (b^α, c^α) over (X,m). By the proof of Lemma 1.17, this graph satisfies

$$b^\alpha(x,y) = -Q^\alpha(1_x, 1_y)$$

for $x \neq y$ with $b^\alpha(x,x) = 0$ and

$$c^\alpha(x) = Q^\alpha(1_x) + \sum_{y \neq x} Q^\alpha(1_x, 1_y),$$

where the sum is absolutely convergent.

We now calculate the action of $(Q^\alpha)_\psi$ on functions of finite support and compare the results to those for Q^α. In particular, for $\varphi \in C_c(X)$ with $W = \operatorname{supp} \varphi$ being the finite support of φ, we get by a direct calculation that

$$(Q^\alpha)_\psi(\varphi) = \frac{1}{2} \sum_{x,y \in W} b^\alpha_\psi(x,y)(\varphi(x) - \varphi(y))^2 + \sum_{x \in W} c^\alpha_\psi(x)\varphi^2(x),$$

where

$$b^\alpha_\psi(x,y) = -(Q^\alpha)_\psi(1_x, 1_y)$$

for $x \neq y$ and $b^\alpha_\psi(x,x) = 0$ and

$$c^\alpha_\psi(x) = \sum_{y \in W} (Q^\alpha)_\psi(1_x, 1_y) = (Q^\alpha)_\psi(1_x, 1_W).$$

We will compare the coefficients $b^\alpha(x,y)$ with $b^\alpha_\psi(x,y)$ and $c^\alpha(x)$ with $c^\alpha_\psi(x)$ over W, which gives the core of the argument in the lemma below.

To this end, we observe that resolvents associated to operators coming from Dirichlet forms are both positivity preserving and contracting, see Theorem C.4. In particular,

$$0 \leq \alpha(L+\alpha)^{-1}\psi \leq 1$$

for all $\psi \in \ell^2(X,m)$ with $0 \leq \psi \leq 1$, where the lower bound comes from the positivity preserving property and the upper bound from the contracting property.

We next calculate the action of $(Q^\alpha)_\psi$ explicitly. In particular, we show that $(Q^\alpha)_\psi$ is bounded.

LEMMA 3.20. *Let (b,c) be a graph over (X,m). Let L be a Markov realization of $\mathcal{L}$ and Q be the associated Dirichlet form with domain $D(Q)$. Let $f, g, \psi \in \ell^2(X,m) \cap \ell^\infty(X)$ with $0 \leq \psi \leq 1$. Then*

$$(Q^\alpha)_\psi(f,g) = \alpha\left(\left\langle \psi f g, \alpha(L+\alpha)^{-1}\psi\right\rangle - \left\langle \psi f, \alpha(L+\alpha)^{-1}\psi g\right\rangle\right).$$

In particular, $(Q^\alpha)_\psi$ gives rise to a bounded form on $\ell^2(X,m)$.

PROOF. As $(Q^\alpha)_\psi(f,g) = Q^\alpha(\psi f, \psi g) - Q^\alpha(\psi f g, \psi)$, we calculate directly that

$$\begin{aligned}(Q^\alpha)_\psi(f,g)\\ &= \alpha\left\langle \psi f, \left(I-\alpha(L+\alpha)^{-1}\right)\psi g\right\rangle - \alpha\left\langle \psi f g, \left(I-\alpha(L+\alpha)^{-1}\right)\psi\right\rangle\\ &= \alpha\left(\left\langle \psi f g, \alpha(L+\alpha)^{-1}\psi\right\rangle - \left\langle \psi f, \alpha(L+\alpha)^{-1}\psi g\right\rangle\right)\end{aligned}$$

for $f, g, \psi \in \ell^2(X,m)\cap\ell^\infty(X)$ with $0 \leq \psi \leq 1$.

Setting $g = f$, for $f, \psi \in \ell^2(X,m)\cap\ell^\infty(X)$ with $0 \leq \psi \leq 1$ we obtain

$$(Q^\alpha)_\psi(f) = \alpha\left(\left\langle \psi f, f\alpha(L+\alpha)^{-1}\psi\right\rangle - \left\langle \psi f, \alpha(L+\alpha)^{-1}\psi f\right\rangle\right).$$

Therefore, we estimate, using the Cauchy–Schwarz inequality and the fact that the operator norm of $\alpha(L+\alpha)^{-1}$ is bounded by 1,

$$\begin{aligned}(Q^\alpha)_\psi(f) &\leq \alpha\left(\|\psi f\|\|f\alpha(L+\alpha)^{-1}\psi\| + \|\psi f\|\|\alpha(L+\alpha)^{-1}\psi f\|\right)\\ &\leq \alpha\left(\|\psi\|_\infty\|\alpha(L+\alpha)^{-1}\psi\|_\infty\|f\|^2 + \|\alpha(L+\alpha)^{-1}\|\|\psi\|_\infty^2\|f\|^2\right)\\ &\leq 2\alpha\|f\|^2.\end{aligned}$$

The boundedness of $(Q^\alpha)_\psi$ follows directly as $\ell^2(X,m)\cap\ell^\infty(X)$ is dense in $\ell^2(X,m)$. $\square$

With these preparations, we can now state and prove our main technical lemma concerning Q_ψ, Q_M and Q_K.

LEMMA 3.21 (Basic properties of Q_M and Q_K). *Let (b,c) be a graph over (X,m). Let L be a Markov realization of $\mathcal{L}$ and Q be the associated Dirichlet form with domain $D(Q)$. Let $f, g \in D(Q)\cap\ell^\infty(X)$.*

(a) *If $\psi_1, \psi_2 \in D(Q)$ with $0 \leq \psi_1 \leq \psi_2 \leq 1$, then*

$$0 \leq Q_{\psi_1}(f) \leq Q_{\psi_2}(f).$$

(b) $0 \leq Q_M(f) \leq Q(f)$ *and* $0 \leq Q_K(f) \leq Q(f)$.

(c) *Q_M and Q_K are quadratic forms.*

(d) *If $|f(x)| \leq |g(x)|$ for all $x \in X$, then*

$$Q_K(f) \leq Q_K(g).$$

PROOF. (a) From the discussion above for the form $(Q^\alpha)_\psi$ and $\varphi \in C_c(X)$ with finite support W we get

$$(Q^\alpha)_\psi(\varphi) = \frac{1}{2}\sum_{x,y\in W} b_\psi^\alpha(x,y)(\varphi(x)-\varphi(y))^2 + \sum_{x\in W} c_\psi^\alpha(x)\varphi^2(x),$$

where $b_\psi^\alpha(x,y) = -(Q^\alpha)_\psi(1_x, 1_y)$ for $x \neq y$, $b_\psi^\alpha(x,x) = 0$ and $c_\psi^\alpha(x) = (Q^\alpha)_\psi(1_x, 1_W)$ for all $\psi \in D(Q)$ with $0 \leq \psi \leq 1$.

Applying Lemma 3.20 with $f = 1_x$ and $g = 1_y$ for $x \neq y$, we see that

$$(Q^\alpha)_\psi(1_x, 1_y) = -\alpha\left\langle \alpha(L+\alpha)^{-1}\psi 1_x, \psi 1_y\right\rangle.$$

Therefore,

$$b_{\psi}^{\alpha}(x,y) = \alpha \left\langle \alpha(L+\alpha)^{-1}\psi 1_x, \psi 1_y \right\rangle$$

for all $x \neq y$. From this it follows that if $0 \leq \psi_1 \leq \psi_2 \leq 1$ are in $D(Q)$, then

$$0 \leq b_{\psi_1}^{\alpha}(x,y) \leq b_{\psi_2}^{\alpha}(x,y)$$

as resolvents associated to Dirichlet forms are positivity preserving.

We now calculate $c_{\psi}^{\alpha,W}(x) = \sum_{y\in W}(Q^{\alpha})_{\psi}(1_x, 1_y) = (Q^{\alpha})_{\psi}(1_x, 1_W)$ for $x \in W$, where we emphasize the dependence on W in the notation. Using the symmetry of the resolvent, we see from the general calculation of $(Q^{\alpha})_{\psi}(f,g)$ above that for $x \in W$

$$\begin{aligned}(Q^{\alpha})_{\psi}(1_x, 1_W) &= \alpha \left(\left\langle \psi 1_x 1_W, \alpha(L+\alpha)^{-1}\psi \right\rangle - \left\langle \psi 1_x, \alpha(L+\alpha)^{-1}\psi 1_W \right\rangle \right) \\ &= \alpha \left\langle \psi 1_x, \alpha(L+\alpha)^{-1}(\psi - \psi 1_W) \right\rangle \\ &= \alpha \left\langle \psi 1_x, \alpha(L+\alpha)^{-1}\psi 1_{X\setminus W} \right\rangle .\end{aligned}$$

From this, it follows that if $0 \leq \psi_1 \leq \psi_2 \leq 1$ for $\psi_1, \psi_2 \in D(Q)$, then

$$0 \leq c_{\psi_1}^{\alpha,W}(x) \leq c_{\psi_2}^{\alpha,W}(x)$$

for all $x \in W$ as $(L+\alpha)^{-1}$ is positivity preserving.

Combining all of the above, if $0 \leq \psi_1 \leq \psi_2 \leq 1$ for $\psi_1, \psi_2 \in D(Q)$ and $\varphi \in C_c(X)$ with support in W, then

$$\begin{aligned}0 \leq (Q^{\alpha})_{\psi_1}(\varphi) &= \frac{1}{2}\sum_{x,y\in W} b_{\psi_1}^{\alpha}(x,y)(\varphi(x)-\varphi(y))^2 + \sum_{x\in W} c_{\psi_1}^{\alpha,W}(x)\varphi^2(x) \\ &\leq \frac{1}{2}\sum_{x,y\in W} b_{\psi_2}^{\alpha}(x,y)(\varphi(x)-\varphi(y))^2 + \sum_{x\in W} c_{\psi_2}^{\alpha,W}(x)\varphi^2(x) \\ &= (Q^{\alpha})_{\psi_2}(\varphi).\end{aligned}$$

As $(Q^{\alpha})_{\psi}$ are bounded by Lemma 3.20, we obtain $0 \leq (Q^{\alpha})_{\psi_1}(f) \leq (Q^{\alpha})_{\psi_2}(f)$ for all $f \in \ell^2(X,m)$. Finally, $0 \leq Q_{\psi_1} \leq Q_{\psi_2}$ by letting $\alpha \to \infty$.

(b) Similar to the proof of (a) above, we can calculate for $x \neq y$

$$b^{\alpha}(x,y) = -Q^{\alpha}(1_x, 1_y) = \alpha \left\langle 1_x, \alpha(L+\alpha)^{-1}1_y \right\rangle .$$

Since

$$b_{\psi}^{\alpha}(x,y) = \alpha \left\langle \psi 1_x, \alpha(L+\alpha)^{-1}\psi 1_y \right\rangle$$

we get

$$b_{\psi}^{\alpha}(x,y) \leq b^{\alpha}(x,y)$$

for all $0 \leq \psi \leq 1$ with $\psi \in D(Q)$ and all $x \neq y$ as resolvents associated to Dirichlet forms are positivity preserving.

Similarly, letting $W \subseteq X$ be any finite set such that $x \in W$ and using the symmetry of the resolvent we obtain

$$c^{\alpha,W}(x) = Q^{\alpha}(1_x, 1_W) = \alpha \left\langle 1_x, 1_W - \alpha(L+\alpha)^{-1}1_W \right\rangle .$$

Furthermore, for $\psi \in D(Q)$ with $0 \leq \psi \leq 1$ we get from the above

$$\begin{aligned} c_\psi^{\alpha,W}(x) &= Q_\psi^\alpha(1_x, 1_W) = \alpha \left\langle \psi 1_x, \alpha(L+\alpha)^{-1}\psi 1_{X\setminus W} \right\rangle \\ &\leq \alpha \left\langle 1_x, \alpha(L+\alpha)^{-1}\psi 1_{X\setminus W} \right\rangle \end{aligned}$$

as resolvents are positivity preserving. Combining these two calculations, we obtain

$$\frac{1}{\alpha}\left(c^{\alpha,W}(x) - c_\psi^{\alpha,W}(x)\right) \geq \left\langle 1_x, 1_W - \alpha(L+\alpha)^{-1}(1_W - \psi 1_{X\setminus W})\right\rangle \geq 0$$

since $1_W - \psi 1_{X\setminus W} \leq 1$ implies that $\alpha(L+\alpha)^{-1}(1_W - \psi 1_{X\setminus W}) \leq 1$.

From the above, we see that $b_\psi^\alpha(x,y) \leq b^\alpha(x,y)$ and that $c_\psi^{\alpha,W}(x) \leq c^{\alpha,W}(x)$ for all $x, y \in W$ so that

$$(Q^\alpha)_\psi(\varphi) \leq Q^\alpha(\varphi)$$

for $\varphi \in C_c(X)$ with support in W and $\psi \in D(Q)$ with $0 \leq \psi \leq 1$. Therefore, since $(Q^\alpha)_\psi$ are bounded by Lemma 3.20, we get $(Q^\alpha)_\psi(f) \leq Q^\alpha(f)$ for all $f \in \ell^2(X,m) \cap \ell^\infty(X)$ and letting $\alpha \to \infty$ we obtain

$$Q_\psi(f) \leq Q(f)$$

for all $f \in D(Q) \cap \ell^\infty(X)$ and $\psi \in D(Q)$ such that $0 \leq \psi \leq 1$.

This shows

$$Q_M(f) = \sup_{\substack{\psi \in D(Q) \\ 0 \leq \psi \leq 1}} Q_\psi(f) \leq Q(f)$$

for all $f \in D(Q) \cap \ell^\infty(X)$. As we have shown that $Q_\psi(f) \geq 0$ in part (a), it follows that $Q_M(f) \geq 0$. Therefore, we obtain $0 \leq Q_M(f) \leq Q(f)$. As $Q_K(f) = Q(f) - Q_M(f)$ it follows that $0 \leq Q_K(f) \leq Q(f)$ as well.

(c) We now show that Q_M and Q_K are quadratic forms. As Q is a quadratic form and $Q_K = Q - Q_M$, it suffices to show that Q_M is a quadratic form, that is,

$$Q_M(af) = a^2 Q_M(f)$$

for all $a \in \mathbb{R}$ and $f \in D(Q) \cap \ell^\infty(X)$ and

$$Q_M(f+g) + Q_M(f-g) = 2\left(Q_M(f) + Q_M(g)\right)$$

for all $f, g \in D(Q) \cap \ell^\infty(X)$.

Since Q is a quadratic form, it follows that Q_ψ is a quadratic form for all $\psi \in D(Q)$ with $0 \leq \psi \leq 1$. Therefore,

$$Q_M(af) = \sup_{\substack{\psi \in D(Q) \\ 0 \leq \psi \leq 1}} Q_\psi(af) = \sup_{\substack{\psi \in D(Q) \\ 0 \leq \psi \leq 1}} a^2 Q_\psi(f) = a^2 Q_M(f).$$

Furthermore, using that Q_ψ is a quadratic form again, we get

$$\begin{aligned}
Q_M(f+g)+Q_M(f-g) &= \sup_{\substack{\psi\in D(Q)\\ 0\le\psi\le 1}} Q_\psi(f+g) + \sup_{\substack{\psi\in D(Q)\\ 0\le\psi\le 1}} Q_\psi(f-g)\\
&\geq \sup_{\substack{\psi\in D(Q)\\ 0\le\psi\le 1}} \left(Q_\psi(f+g)+Q_\psi(f-g)\right)\\
&= 2\sup_{\substack{\psi\in D(Q)\\ 0\le\psi\le 1}} \left(Q_\psi(f)+Q_\psi(g)\right)\\
&= 2(Q_M(f)+Q_M(g)).
\end{aligned}$$

This gives half of the required equality. The other half is obtained by what we have already shown as follows

$$\begin{aligned}
4\left(Q_M(f)+Q_M(g)\right) &= Q_M(2f)+Q_M(2g)\\
&= Q_M((f+g)+(f-g)) + Q_M((f+g)-(f-g))\\
&\geq 2\left(Q_M(f+g)+Q_M(f-g)\right).
\end{aligned}$$

Combining the two inequalities we obtain

$$Q_M(f+g)+Q_M(f-g) = 2\left(Q_M(f)+Q_M(g)\right),$$

which completes the proof.

(d) Let $f,g\in D(Q)\cap\ell^\infty(X)$ be such that $|f|\le|g|$. We have to show that $Q_K(f)\le Q_K(g)$. We break down the proof into two steps. We first assume that there exists a $\psi_0\in D(Q)$ such that $1_{\operatorname{supp} g}\le\psi_0\le 1$.

Let $\varepsilon>0$. As we have already shown that Q_ψ is monotone in ψ in part (a) and since Q_M is the supremum over all $\psi\in D(Q)$ such that $0\le\psi\le1$ by definition, for all $\psi\in D(Q)$ with $\psi_0\le\psi\le 1$ large enough, we have

$$Q_M(g)-Q_\psi(g)<\varepsilon.$$

Using $Q_K=Q-Q_M$ as well as $Q-Q_M\le Q-Q_\psi$ and the definition of Q_ψ, we get for all $\psi\in D(Q)$ with $\psi_0\le\psi\le 1$ large enough that

$$\begin{aligned}
Q_K(g)-Q_K(f) &= Q(g)-Q_M(g)-(Q(f)-Q_M(f))\\
&\geq Q(g)-Q_\psi(g)-(Q(f)-Q_\psi(f))-\varepsilon\\
&= Q(g)-Q(\psi g)+Q(\psi g^2,\psi)\\
&\quad -\big(Q(f)-Q(\psi f)+Q(\psi f^2,\psi)\big)-\varepsilon.
\end{aligned}$$

Since the support of f is included in the support of g, we note that $\psi=1$ on both the support of f and the support of g. Therefore, $f=\psi f$ and $g=\psi g$. Hence, we conclude

$$Q_K(g)-Q_K(f)\geq Q(g^2,\psi)-Q(f^2,\psi)-\varepsilon = Q(g^2-f^2,\psi)-\varepsilon.$$

As $g^2-f^2\geq 0$, $0\le\psi\le 1$ with $\psi=1$ on the support of g^2, we get for any $s>0$ that $(\psi+s(g^2-f^2))\wedge 1=\psi$. Since Q is a Dirichlet form, it follows that, for $s > 0$,

$$\begin{aligned} Q(\psi) &= Q\left(\left(\psi + s(g^2 - f^2)\right) \wedge 1\right) \\ &\leq Q\left(\psi + s(g^2 - f^2)\right) \\ &= Q(\psi) + 2sQ\left(\psi, g^2 - f^2\right) + s^2 Q\left(g^2 - f^2\right). \end{aligned}$$

Therefore,

$$-sQ\left(g^2 - f^2\right) \leq 2Q\left(\psi, g^2 - f^2\right) = 2Q\left(g^2 - f^2, \psi\right)$$

for all $s > 0$ and letting $s \to 0$, we get that

$$0 \leq Q\left(g^2 - f^2, \psi\right).$$

Putting everything together, we get

$$Q_K(g) - Q_K(f) \geq -\varepsilon$$

and, thus,

$$Q_K(g) \geq Q_K(f)$$

as $\varepsilon > 0$ was arbitrary. This completes the proof in the case that there exists a $\psi_0 \in D(Q)$ such that $1_{\operatorname{supp} g} \leq \psi_0 \leq 1$.

In the general case, we argue as follows. We let

$$f_\alpha = f - ((f \wedge \alpha) \vee -\alpha) \qquad \text{and} \qquad g_\alpha = g - ((g \wedge \alpha) \vee -\alpha)$$

for $\alpha > 0$. By Lemma 3.16 (b) we get $f_\alpha \to f$ and $g_\alpha \to g$ as $\alpha \to 0^+$ with respect to $\|\cdot\|_Q$. Now, as $|f| \leq |g|$, we get $|f_\alpha| \leq |g_\alpha|$. Furthermore, we let

$$\psi_\alpha = \left(\frac{|g_\alpha|}{\alpha}\right) \wedge 1.$$

Clearly $0 \leq \psi_\alpha \leq 1$. Furthermore, we observe that x is in the support of g_α if and only if $((g \wedge \alpha) \vee -\alpha)(x) \neq g(x)$, that is, if and only if $|g(x)| > \alpha$ and for all such x we get that $\psi_\alpha(x) = 1$. Hence, we have $1_{\operatorname{supp} g_\alpha} \leq \psi_\alpha \leq 1$. Finally, as Q is a Dirichlet form, we get $\psi_\alpha \in D(Q)$. Therefore, we infer

$$Q_K(g_\alpha) \geq Q_K(f_\alpha)$$

by what we have already shown above. Now, since $Q_K \leq Q$ as we have already shown in part (b) and since $Q(g - g_\alpha) \to 0$ and $Q(f - f_\alpha) \to 0$ as $\alpha \to 0$, we get $Q_K(g - g_\alpha) \to 0$ and $Q_K(f - f_\alpha) \to 0$ as $\alpha \to 0$. By (c) we know that Q_K is a quadratic form, so, we obtain

$$Q_K(g_\alpha) \to Q_K(g) \qquad \text{and} \qquad Q_K(f_\alpha) \to Q_K(f)$$

as $\alpha \to 0$. Therefore, we conclude

$$Q_K(g) \geq Q_K(f),$$

which completes the proof. □

Remark. In the proof of (c) above we see that in order for a form to be quadratic it suffices that one of the equalities only has to be an inequality. This is, in fact, true for both inequalities (Exercise 3.13).

Given the lemma above, we can now give the lower estimate on Q as follows.

Proposition 3.22 ($Q^{(N)} \leq Q$). *Let (b, c) be a graph over (X, m). If L is a Markov realization of $\mathcal{L}$ with associated Dirichlet form Q, then*

$$Q^{(N)} \leq Q.$$

Proof. We will show

$$Q_M(f) \geq \mathcal{Q}_{b,0}(f) \quad \text{and} \quad Q_K(f) \geq \mathcal{Q}_{0,c}(f)$$

for all $f \in D(Q) \cap \ell^\infty(X)$. Assuming we have shown this, we get, as both $Q_M(f)$ and $Q_K(f)$ are finite by Lemma 3.21 (b) and (c),

$$\mathcal{Q}_{b,c}(f) = \mathcal{Q}_{b,0}(f) + \mathcal{Q}_{0,c}(f) < \infty,$$

so that $f \in \mathcal{D} \cap \ell^2(X, m) = D(Q^{(N)})$ and

$$Q^{(N)}(f) = \mathcal{Q}(f) \leq Q_M(f) + Q_K(f) = Q(f)$$

for all $f \in D(Q) \cap \ell^\infty(X)$.

Now, for $f \in D(Q)$ it follows from Corollary 3.17 that there exists a sequence $f_n \in D(Q) \cap \ell^\infty(X)$ such that $f_n \to f$ in $\|\cdot\|_Q$. In particular, by what we have already shown above, $f_n \in D(Q^{(N)})$ and using the lower semi-continuity of $Q^{(N)}$ we infer

$$Q^{(N)}(f) \leq \liminf_{n\to\infty} Q^{(N)}(f_n) \leq \liminf_{n\to\infty} Q(f_n) = Q(f) < \infty,$$

which shows $f \in D(Q^{(N)})$ and $Q^{(N)}(f) \leq Q(f)$ for all $f \in D(Q)$.

We now show the two required inequalities.

$Q_M(f) \geq \mathcal{Q}_{b,0}(f)$: Let $W_n \subseteq X$ be finite with $X = \bigcup_n W_n$. Then, by Lemma 3.19, we get

$$Q_M(f) = \sup_{\substack{\psi \in D(Q) \\ 0 \leq \psi \leq 1}} Q_\psi(f) \geq Q_{1_{W_n}}(f) = \frac{1}{2} \sum_{x,y \in W_n} b(x, y)(f(x) - f(y))^2.$$

Now, letting $n \to \infty$ gives the required inequality.

$Q_K(f) \geq \mathcal{Q}_{0,c}(f)$: Let $\varepsilon > 0$. Since Q_ψ is monotone in ψ by Lemma 3.21 (a), we get that there exists a $\psi_0 \in D(Q)$ with $0 \leq \psi_0 \leq 1$ such that

$$Q_M(f) \leq Q_\psi(f) + \varepsilon$$

for all $\psi \in D(Q)$ with $\psi_0 \leq \psi \leq 1$. Let $W \subseteq X$ be finite and choose ψ_0 such that $\psi_0 \geq 1_W$. As $|f| \geq |1_W f|$, it follows from Lemma 3.21 (d) that

$$Q_K(f) \geq Q_K(1_W f) = Q(1_W f) - Q_M(1_W f) \geq Q(1_W f) - Q_\psi(1_W f) - \varepsilon$$

for all $\psi \in D(Q)$ with $\psi_0 \leq \psi \leq 1$. Now, as $\psi 1_W = 1_W$ for all ψ with $1_W \leq \psi \leq 1$, we see that

$$Q_\psi(1_W f) = Q(\psi 1_W f) - Q(\psi(1_W f)^2, \psi) = Q(1_W f) - Q(1_W f^2, \psi).$$

Therefore,

$$Q_K(f) \geq Q(1_W f) - Q_\psi(1_W f) - \varepsilon = Q(1_W f^2, \psi) - \varepsilon$$

for all $\psi \in D(Q)$ with $\psi_0 \leq \psi \leq 1$.

As $1_W f^2 \in C_c(X)$ and $\psi \in D(Q) \cap \ell^\infty(X)$, applying Lemma 3.14, we get

$$Q(1_W f^2, \psi) = \sum_{x \in X} 1_W(x) f^2(x) \mathcal{L}\psi(x) m(x) = \sum_{x \in W} f^2(x) \mathcal{L}\psi(x) m(x).$$

Now, we can apply the above estimate to a sequence ψ_n which satisfies $\psi_0 \leq \psi_n \leq 1$ and such that $\psi_n \to 1$ pointwise as $n \to \infty$. By applying the Lebesgue dominated convergence theorem to such a sequence, it follows that

$$\mathcal{L}\psi_n(x) m(x) \to c(x)$$

as $n \to \infty$. This gives

$$Q(1_W f^2, \psi_n) = \sum_{x \in W} f^2(x) \mathcal{L}\psi_n(x) m(x) \to \sum_{x \in W} c(x) f^2(x)$$

as $n \to \infty$. Therefore,

$$Q_K(f) \geq Q(1_W f^2, \psi_n) - \varepsilon \to \sum_{x \in W} c(x) f^2(x) - \varepsilon,$$

which implies

$$Q_K(f) \geq \sum_{x \in W} c(x) f^2(x).$$

Finally, as $W \subseteq X$ is an arbitrary finite set, we conclude

$$Q_K(f) \geq \sum_{x \in X} c(x) f^2(x) = \mathcal{Q}_{0,c}(f).$$

This completes the proof. □

Proof of Theorem 3.11. To show that $Q^{(N)} \leq Q \leq Q^{(D)}$ simply combine Proposition 3.15, which gives the upper bound, and Proposition 3.22, which gives the lower bound. □

Exercises

Excavation exercises.

EXERCISE 3.1 (Adjoint operator inclusions). Let H be a Hilbert space and let A_1 and A_2 be densely defined operators with domains $D(A_1)$ and $D(A_2)$, respectively. We say that A_2 is an extension of A_1 if $D(A_1) \subseteq D(A_2)$ and $A_2 f = A_1 f$ for all $f \in D(A_1)$. We write $A_1 \subseteq A_2$ in this case.

(a) Show that if $A_1 \subseteq A_2$, then $A_2^* \subseteq A_1^*$.
(b) Show that if A_1 and A_2 are self-adjoint and $A_1 \subseteq A_2$, then $A_1 = A_2$.

EXERCISE 3.2 (Adjoint operators and essential self-adjointness). Let H be a Hilbert space and let A be a densely defined symmetric operator with domain $D(A) \subseteq H$. Let $\overline{A}$ denote the closure of A, that is, the smallest closed extension of A. Show that:

(a) $\overline{A} = A^{**}$.
(b) A is essentially self-adjoint if and only if $\overline{A}$ is self-adjoint.
(c) A is essentially self-adjoint if and only if A^* is self-adjoint.

EXERCISE 3.3. Let H be a Hilbert space and L be a positive operator with associated form Q. Let (ψ_n) be a sequence of bounded real-valued functions which converges pointwise to a bounded function ψ. Show that $\psi_n(L)f \to \psi(L)f$ for all $f \in D(Q)$ with respect to $\|\cdot\|_Q$, where $\|f\|_Q^2 = Q(f) + \|f\|^2$.

Example exercises.

EXERCISE 3.4 (Infinite star graphs). Let (b, c) be an infinite star graph over (X, m). That is, let $X = \mathbb{N}_0$ with $b(0, n) = b(n, 0) > 0$ for all $n \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} b(0, n) < \infty$ and $b = 0$ otherwise.

(a) Show that $D(Q^{(D)}) = D(Q^{(N)})$.
(b) Characterize the condition $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$.
(c) Show that the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint when $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$.

EXERCISE 3.5 ($Q^{(D)} \neq Q^{(N)}$). Give an example of a graph which satisfies $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$ but for which $Q^{(D)} \neq Q^{(N)}$ so that, in particular, $L_{\min}$ is not essentially self-adjoint and $\mathcal{L}$ does not satisfy Markov uniqueness.

(Hint 1: Consider α-harmonic functions on an infinite path graph.)

(Hint 2: If you have tried for a sufficiently long time without success, then try to work out the following: Let $(b, 0)$ be an infinite path graph over (X, m). That is, let $X = \mathbb{N}_0$ with $b(n, n+1) = b(n+1, n) > 0$ for all $n \in \mathbb{N}$ and $b = 0$ otherwise. Assume, additionally, that $m(X) < \infty$ and that

$$\sum_{n=0}^{\infty} \frac{1}{b(n, n+1)} < \infty.$$

Show by induction that any $u \in \mathcal{F} = C(X)$ such that $(\mathcal{L} + \alpha)u = 0$ for $\alpha > 0$ satisfies

$$u(n+1) - u(n) = \frac{\alpha}{b(n, n+1)} \sum_{k=0}^{n} u(k)m(k).$$

Use this to show that, under the additional assumptions on b and m, u has finite energy and is bounded.)

Exercise 3.6 (Equality of form domains does not imply essential self-adjointness). Give an example of a graph for which $Q^{(D)} = Q^{(N)}$ and $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$ and for which $L_{\min}$ is not essentially self-adjoint.

(Hint 1: Try a two-sided path graph.)

(Hint 2: If you have tried for a sufficiently long time without success, then try to work out the following: Let $(b, 0)$ be the two-sided path graph over (X, m) with standard weights. That is, let $X = \mathbb{Z}$ with $b(n, n+1) = b(n+1, n) = 1$ for all $n \in \mathbb{Z}$ and $b = 0$ otherwise. First, show that there do not exist any non-trivial α-harmonic functions with finite energy on this graph for $\alpha > 0$. Then, consider the function $u(x) = x$, which is α-harmonic for $\alpha = 0$, and show that for an appropriate measure, $u \in D(L_{\min}^*)$. However, u does not have finite energy.)

Exercise 3.7 (Non-Markov operator). Give an example of a self-adjoint operator which is a realization of $\mathcal{L}$ but is not Markov.

(Hint 1: Use Exercise 3.6 above.)

(Hint 2: If you have tried for a sufficiently long time without success, then try to work out the following: Take the example above and let L_0 be the restriction of $\mathcal{L}$ to $D(L_0) = C_c(X) + \mathrm{Lin}\{u\}$. Show that L_0 is symmetric and take the Friedrichs extension of L_0.)

Extension exercises.

Exercise 3.8 (Form uniqueness and α-subharmonic functions). Let (b, c) be a graph over (X, m). Show that $D(Q^{(D)}) = D(Q^{(N)})$ if and only if every $u \in \mathcal{D}$ with $u \geq 0$ such that $\mathcal{L}u \in \ell^2(X, m)$ and $(\mathcal{L} + \alpha)u \leq 0$ for $\alpha > 0$ satisfies $u = 0$.

EXERCISE 3.9 (Essential self-adjointness and associated operators). Let (b,c) be a graph over (X,m) with $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. Show that the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint if and only if

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$$

for some (all) associated operators L.

EXERCISE 3.10 (Realizations and finitely supported functions). Let (b,c) be a graph over (X,m). Let L be a realization of $\mathcal{L}$. Show that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ if and only if $C_c(X) \subseteq D(L)$.

EXERCISE 3.11 (Bounded functions of finite energy form an algebra). Let (b,c) be a graph over X. Let $\mathcal{D}$ denote the functions of finite energy and let $\ell^\infty(X)$ denote the bounded functions on X. Show that $\mathcal{D}\cap\ell^\infty(X)$ is an algebra, i.e., $fg \in \mathcal{D}\cap\ell^\infty(X)$ for all $f,g \in \mathcal{D}\cap\ell^\infty(X)$.

EXERCISE 3.12 (Convergence in $\|\cdot\|_Q$). Let (b,c) be a graph over (X,m). Let Q be a Dirichlet form with domain $D(Q) \subseteq \ell^2(X,m)$ and let $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$ for all $f \in D(Q)$. Show that $f_n \to f$ in $\|\cdot\|_Q$ if and only if

$$\|f_n - f\| \to 0 \qquad \text{and} \qquad \limsup_{n\to\infty} Q(f_n) \le Q(f).$$

EXERCISE 3.13 (Quadratic forms). Let q be a form on $\ell^2(X,m)$ with domain $D(q)$. Show that q is a quadratic form, i.e., q satisfies $q(af) = a^2q(f)$ and $q(f+g) + q(f-g) = 2(q(f)+q(g))$ for all $a \in \mathbb{R}$ and $f,g \in D(q)$ if and only if

$$q(af) \le a^2 q(f) \qquad \text{and} \qquad q(f+g) + q(f-g) \le 2(q(f)+q(g))$$

if and only if

$$q(af) \le a^2 q(f) \qquad \text{and} \qquad q(f+g) + q(f-g) \ge 2(q(f)+q(g))$$

for all $a \in \mathbb{R}$ and $f,g \in D(q)$.

Notes

In large part, the material in this chapter can be understood as working out the general abstract theory of form uniqueness as well as the uniqueness of self-adjoint and Markov extensions in the concrete setting of weighted graphs. For the general theory, we merely reference some standard textbooks such as [**FŌT11, RS75, RS80, Wei80**] as well as the historical works [**Fri34, vN30**].

For the concrete setting that we consider, the equivalence of (i) and (ii) in Theorem 3.2 can be inferred from [**Sch17b**]. The equivalence of (i) and (iv) in Theorem 3.2 can be found as Corollary 4.3 in [**HKLW12**]. The equivalences found in Theorem 3.6 are worked out in the proof of Theorem 6 in [**KL12**]. The characterization in Theorem 3.8 is used in the proof of Theorem 5 in [**KL12**]. The characterization of Markov realizations as presented in Theorem 3.11 is proven for locally finite graphs as Theorem 5.2 in [**HKLW12**]. The general case, that is, for not necessarily locally finite graphs, is shown as Theorem 11.6.5 in [**Sch20b**] found within [**KLW20**]. In particular, the theory of the main and killing part of a Dirichlet form is developed more generally in Chapter 3 of [**Sch17a**], see [**Sch20a**] as well. We follow the presentation in [**Sch20b**]. Some further general connections between essential self-adjointness and Liouville properties can be found in [**HMW21**].

CHAMBER 4

Agmon–Allegretto–Piepenbrink and Persson Theorems

Shake the ground while my beats just break you down.
Raekwon.

In this chapter we take a first step towards studying the spectral theory of the Laplacian $L = L_{b,c,m}^{(D)}$ associated to the regular form $Q = Q_{b,c,m}^{(D)}$ of a graph (b, c) over (X, m). We will characterize the bottom of the spectrum and the bottom of the essential spectrum of L via strictly positive generalized eigenfunctions. The corresponding results are known as Agmon–Allegretto–Piepenbrink theorems. Along the way, we will also show a Persson theorem relating the essential spectrum of L to the spectra of restrictions of L to complements of finite sets.

More specifically, we let $\sigma(L)$ denote the spectrum of L and let

$$\lambda_0(L) = \inf \sigma(L)$$

denote the bottom of the spectrum of L. We will characterize $\lambda_0(L)$ in terms of the existence of positive α-superharmonic functions for $\alpha \geq -\lambda_0(L)$ on X. We recall that for $\alpha \in \mathbb{R}$ a function u is called α-harmonic if $u \in \mathcal{F}$ and $(\mathcal{L} + \alpha)u = 0$ (and α-superharmonic if $(\mathcal{L} + \alpha)u \geq 0$), where $\mathcal{L} = \mathcal{L}_{b,c,m}$. Furthermore, u is called positive if $u \geq 0$. On connected graphs, we will show that any positive non-trivial α-harmonic function u is automatically strictly positive, i.e., satisfies $u > 0$. We then present the characterization of the bottom of the spectrum in terms of such functions in Section 3.

The proof uses techniques that are interesting on their own. On one hand, the ground state transform shows that if a positive α-harmonic function exists, then $\alpha \geq -\lambda_0(L)$. We establish this in Section 2. On the other hand, in Section 1 we prove a Harnack inequality which allows us to construct strictly positive α-superharmonic functions for $\alpha \geq -\lambda_0(L)$ via a limiting procedure. Combining these two results yields our characterization of $\lambda_0(L)$ in terms of strictly positive α-superharmonic functions.

In Section 4 we study the bottom of the essential spectrum of L. The essential spectrum is the complement in the spectrum of the isolated eigenvalues of finite multiplicity. We will denote the essential spectrum by $\sigma_{\mathrm{ess}}(L)$ and the bottom of the essential spectrum by

$$\lambda_0^{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L).$$

By general theory, see Theorem E.7, the essential spectrum is stable under compact perturbations. Now, under a suitable condition on the graph, the removal of a compact, i.e., finite, set is a compact perturbation and this allows us to prove a Persson theorem which gives

$$\lambda_0^{\mathrm{ess}}(L) = \sup_{K \subseteq X, K \text{ finite}} \lambda_0(L_{X\setminus K}^{(D)}).$$

Here, $L_{X\setminus K}^{(D)}$ denotes the Laplacian associated to the closure of the restriction of Q to $C_c(X \setminus K)$, denoted by $Q_{X\setminus K}^{(D)}$. Combining the Persson result with the characterization of the infimum of the spectrum shown in Section 3, we can then characterize the infimum of the essential spectrum via functions which are strictly positive and α-superharmonic outside of a finite set.

1. A local Harnack inequality and consequences

In this section we first present a local Harnack inequality for positive α-superharmonic functions. A slight extension of this statement allows us to then prove a Harnack principle which yields a procedure for constructing α-superharmonic functions for $\alpha \geq -\lambda_0(L)$ via two approximation procedures. These procedures involve approximating on both the level of α and on the level of geometry as we exhaust the vertex set by finite connected subsets. Finally, we show that α-superharmonicity can be improved to α-harmonicity under certain additional assumptions such as local finiteness.

The reader may consult Excavation Exercise 4.1 to recall the diagonal subsequence trick which will be used in the proof of Theorem 4.4.

We start with a local Harnack inequality that allows us to estimate the maximum of a positive α-superharmonic function u on a finite connected set by the minimum.

THEOREM 4.1 (Local Harnack inequality). *Let (b, c) be a graph over (X, m) and let $W \subseteq X$ be finite and connected. Then, there exists a monotonically increasing function $C_W : \mathbb{R} \longrightarrow [0, \infty)$ such that for every $\alpha \in \mathbb{R}$ and every $u \in \mathcal{F}$ with $u \geq 0$ and*

$$(\mathcal{L} + \alpha)u \geq 0$$

on W we have

$$\max_{x \in W} u(x) \leq C_W(\alpha) \min_{x \in W} u(x).$$

In particular, $u > 0$ whenever $u \neq 0$ on W.

Proof. Let $\alpha \in \mathbb{R}$ and let $u \geq 0$ satisfy $(\mathcal{L}+\alpha)u \geq 0$ on $W \subseteq X$. Rewriting $(\mathcal{L}+\alpha)u(x) \geq 0$ we arrive at the inequality

$$(\deg + \alpha m)(x)u(x) \geq \sum_{y \in X} b(x,y)u(y) \geq b(x,z)u(z)$$

for all $x \in W$ and $z \sim x$, where $\deg(x) = \sum_{y \in X} b(x,y) + c(x)$. Let $x, y \in W$ and let $x = x_0 \sim x_1 \sim \ldots \sim x_n = y$ be a path in W. Iterating the above gives

$$u(x_0) \leq \frac{(\deg + \alpha m)(x_1)}{b(x_0,x_1)} u(x_1) \leq \left(\prod_{j=0}^{n-1} \frac{(\deg + \alpha m)(x_{j+1})}{b(x_j, x_{j+1})} \right) u(x_n).$$

Using the finiteness and connectedness of W, we can then define $c_W(\alpha)$ for $\alpha \in \mathbb{R}$ via

$$c_W(\alpha) = \max_{x,y \in W} \min_{x=x_0 \sim \ldots \sim x_n = y} \prod_{j=0}^{n-1} \frac{(\deg + \alpha m)(x_{j+1})}{b(x_j, x_{j+1})}.$$

Choosing x to be the vertex where u attains its maximum on W and y to be the vertex where u attains its minimum on W we find

$$\max_W u \leq c_W(\alpha) \min_W u.$$

The function c_W is clearly monotonically increasing with respect to α. If $c_W(\alpha) \leq 0$, then there are no non-trivial positive functions with $(\mathcal{L}+\alpha)u \geq 0$ on W by what we have shown. Hence, we can set $C_W = c_W \vee 0$. The "in particular" statement is clear since if u is non-trivial on W, then $C_W > 0$. $\square$

The local Harnack inequality has some immediate consequences. The first states that positive non-trivial α-superharmonic functions are immediately strictly positive whenever the underlying graph is connected.

Corollary 4.2. *Let (b,c) be a connected graph over (X,m). Let $\alpha \in \mathbb{R}$ and let $u \geq 0$ be a non-trivial α-superharmonic function. Then, $u > 0$.*

Proof. Since $u \geq 0$ is non-trivial there exists an $x \in X$ such that $u(x) > 0$. Let $y \in X$. By connectedness, there exists a path $x = x_0 \sim \ldots \sim x_n = y$. Let $W = \{x_0, \ldots, x_n\}$. By Theorem 4.1 we obtain $u > 0$ on W and, in particular, $u(y) > 0$. As $y \in X$ was arbitrary, $u > 0$. $\square$

We recall that an operator is called positivity improving if the operator maps positive non-trivial functions to strictly positive functions. As a second consequence of the local Harnack inequality we show that $(L+\alpha)^{-1}$ is positivity improving for $\alpha > -\lambda_0(L)$. This extends one of the implications of connectedness found in Theorem 1.26.

We recall by the variational characterization of the bottom of the spectrum that

$$\lambda_0(L) = \inf_{f \in D(Q), \|f\|=1} Q(f),$$

see Theorem E.8. In particular, $Q \geq \lambda_0(L)$, i.e., $Q(f) \geq \lambda_0(L)\|f\|^2$ for all $f \in D(Q)$.

COROLLARY 4.3. *Let (b, c) be a connected graph over (X, m). Let $\alpha > -\lambda_0(L)$ and let $f \in \ell^2(X, m)$ with $f \geq 0$ be non-trivial. Then,*

$$(L + \alpha)^{-1} f > 0.$$

PROOF. As $\alpha > -\lambda_0(L)$, it follows that $-\alpha$ is not in the spectrum of L so that $(L + \alpha)^{-1}$ exists. Furthermore, as $Q \geq \lambda_0(L)$, $(L + \alpha)^{-1}$ is positivity preserving for $\alpha > -\lambda_0(L)$ by Corollary 1.25. Thus, $(L + \alpha)^{-1} f \geq 0$. As L is a restriction of $\mathcal{L}$ by Theorem 1.6 we obtain

$$(\mathcal{L} + \alpha)(L + \alpha)^{-1} f = (L + \alpha)(L + \alpha)^{-1} f = f \geq 0.$$

Therefore, $(L + \alpha)^{-1} f$ is positive α-superharmonic and non-trivial and thus
$(L + \alpha)^{-1} f$ is strictly positive by Corollary 4.2. □

From the local Harnack inequality we now deduce the Harnack principle. This principle gives a procedure for creating strictly positive α-(super)harmonic functions on X from a sequence of non-trivial positive, and, thus, strictly positive by Corollary 4.2, α_n-(super)harmonic functions on connected increasing sets K_n with $X = \bigcup_n K_n$ and $\alpha = \lim_{n\to\infty} \alpha_n$. This allows us to pass from local properties of solutions to global properties.

In what follows, we call any sequence of increasing connected sets K_n such that $X = \bigcup_n K_n$ an *exhaustion sequence* of X. In particular, we note that we do not require the K_n to be finite sets for the Harnack principle.

THEOREM 4.4 (Harnack principle). *Let (b, c) be a connected graph over (X, m). Let $o \in X$ and let (K_n) be an exhaustion sequence of X with $o \in K_n$ for all n. Let (α_n) be a sequence in $\mathbb{R}$ with $\alpha = \lim_{n\to\infty} \alpha_n$. Let (u_n) be a sequence of positive functions in $\mathcal{F}$ satisfying*

$$(\mathcal{L} + \alpha_n) u_n \geq 0$$

on K_n with $u_n(o) = 1$ for all $n \in \mathbb{N}$. Then, there exists a subsequence (u_{n_k}) of (u_n) that converges pointwise to a strictly positive α-superharmonic function u on X.

Furthermore, assume that one of the following properties holds:

- *The graph is locally finite.*
- *The subsequence (u_{n_k}) is monotonically increasing in k.*
- *There exists an $f \in \mathcal{F}$ such that $u_{n_k} \leq f$ on K_{n_k} for all k.*

Then,

$$\lim_{k\to\infty} \mathcal{L}u_{n_k} = \mathcal{L}u.$$

In particular, under any of the additional assumption above, if u_{n_k} are α_{n_k}-harmonic on K_{n_k}, then u is α-harmonic on X.

Proof. Let $x \in X$. We claim that there exists a constant $C_x > 0$ such that $u_n(x) \leq C_x$ for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be the smallest index such that $x \in K_{n_0}$. Then, by the connectedness of K_{n_0}, there exists a path $o = x_0 \sim \ldots \sim x_k = x$ in K_{n_0} connecting o and x. Let $W = \{x_0, \ldots, x_k\}$. As (K_n) is an increasing sequence of subsets of X, it follows that $W \subseteq K_n$ for all $n \geq n_0$.

Now, as $(\mathcal{L}+\alpha_n)u_n \geq 0$ on W, $u_n \geq 0$ and $u_n(o) = 1$, it follows that $u_n > 0$ on W by the last statement of Theorem 4.1. Thus, applying the rest of Theorem 4.1, we get

$$\frac{1}{C_W(\alpha_n)} \leq \frac{\min_{y\in W} u_n(y)}{\max_{y\in W} u_n(y)} \leq \frac{u_n(x)}{u_n(o)} \leq \frac{\max_{y\in W} u_n(y)}{\min_{y\in W} u_n(y)} \leq C_W(\alpha_n),$$

where

$$C_W(\alpha_n) = \prod_{i=0}^{k-1} \frac{(\deg + \alpha_n m)(x_i)}{b(x_i, x_{i+1})}.$$

Since $C_W(\alpha_n) \to C_W(\alpha)$ as $n \to \infty$, it follows that $(C_W(\alpha_n))$ is a bounded sequence and letting

$$C_x = \sup_n C_W(\alpha_n) \vee \max\{u_1(x), u_2(x), \ldots, u_{n_0-1}(x)\}$$

we get

$$\frac{1}{C_x} \leq u_n(x) \leq C_x$$

for all $n \in N$ as $u_n(o) = 1$ for all $n \in \mathbb{N}$. Therefore, by a diagonal subsequence argument, it follows that there exists a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \to u$ pointwise as $k \to \infty$.

We are left to check that $u > 0$ and that u is α-superharmonic. The functions u_{n_k} satisfy $(\mathcal{L} + \alpha_{n_k})u_{n_k} \geq 0$ on K_{n_k}. Let $x \in X$. There exists an $N \in \mathbb{N}$ such that $x \in K_{n_k}$ for all $n_k \geq N$ so that

$$\sum_{y\in X} b(x,y)u_{n_k}(y) \leq (\deg + \alpha_{n_k} m)(x)u_{n_k}(x)$$

for all $n_k \geq N$. Therefore, by Fatou's lemma, we infer

$$\begin{aligned}\sum_{y\in X} b(x,y)u(y) &\leq \liminf_{k\to\infty} \sum_{y\in X} b(x,y)u_{n_k}(y) \\ &\leq \lim_{k\to\infty} (\deg + \alpha_{n_k} m)(x)u_{n_k}(x) = (\deg + \alpha m)(x)u(x).\end{aligned}$$

Thus, $u \in \mathcal{F}$ and $(\mathcal{L}+\alpha)u \geq 0$. Since $u_{n_k} \geq 0$ and $u_{n_k}(o) = 1$, we have $u \geq 0$ and $u(o) = 1$. Hence, $u > 0$ by Corollary 4.2. This finishes the proof of the first part of the theorem.

We now prove the convergence statements, that is, $\mathcal{L}u_{n_k} \to \mathcal{L}u$ as $k \to \infty$ under the additional assumptions. If the graph is locally finite, then all sums involve only finitely many terms. Therefore, we can interchange the sum with the limit by Fatou's lemma. If (u_{n_k}) is monotonically increasing in k (respectively, $u_{n_k} \leq f \in \mathcal{F}$ for all k), then we can apply the monotone convergence theorem of Beppo Levi (respectively, the dominated convergence theorem of Lebesgue) to get the convergence $\mathcal{L}u_{n_k} \to \mathcal{L}u$ as $k \to \infty$. This completes the proof. □

We now present two ways to apply the Harnack principle to construct positive α-superharmonic functions for $\alpha \geq -\lambda_0(L)$. Both constructions involve resolvents. In the first construction, we use the resolvent of the Laplacian on the entire space and in the second construction, we use the resolvent of the Dirichlet Laplacian associated to a finite subset of X.

For the first construction we recall that for $\alpha > -\lambda_0(L)$ the resolvent $(L+\alpha)^{-1}$ is positivity improving by Corollary 4.3. This is relevant for the definition of the sequence.

COROLLARY 4.5. *Let (b, c) be a connected graph over (X, m). Let $\alpha_n > -\lambda_0(L)$, $n \in \mathbb{N}$, be a sequence which converges to α. Let $x_n \in X$ for $n \in \mathbb{N}_0$. Then, the sequence*

$$u_n = \frac{1}{(L+\alpha_n)^{-1}1_{x_n}(x_0)}(L+\alpha_n)^{-1}1_{x_n}$$

for $n \in \mathbb{N}$ has a subsequence which converges pointwise to a strictly positive α-superharmonic function u.

Furthermore, if the graph is locally finite and (x_n) is chosen to leave every finite set, then u is α-harmonic.

PROOF. For $\alpha_n > -\lambda_0(L)$, the resolvent $(L+\alpha_n)^{-1}$ is positivity improving by Corollary 4.3. Thus, the definition of u_n makes sense. As L is a restriction of $\mathcal{L}$ by Theorem 1.6, it follows that u_n satisfies $(\mathcal{L}+\alpha_n)u_n \geq 0$ on X. As $u_n(x_0) = 1$ for all $n \in \mathbb{N}$, we may apply the Harnack principle, Theorem 4.4, with $K_n = X$ for all $n \in \mathbb{N}$ to obtain the required subsequence converging to a strictly positive superharmonic function u.

Furthermore, we note that, $(\mathcal{L}+\alpha_n)u_n = 0$ on $X \setminus \{x_n\}$. Hence, if (x_n) eventually leaves every finite set, we can take an exhaustion sequence (K_n) such that $x_n \notin K_n$ for all $n \in \mathbb{N}$. In this case, the functions u_n are α_n-harmonic on K_n so that u is α-harmonic when the graph is locally finite by the additional statements in Theorem 4.4. □

For the second construction we will use the resolvent of the Dirichlet Laplacian on a finite subset of X. We now briefly recall the basic properties of this operator. We recall that for a finite set $K \subseteq X$, we let $Q_K^{(D)}$ be the restriction of Q to $C_c(K) = \ell^2(K, m_K)$ where m_K

denotes the restriction of m to K. It follows that $Q_K^{(D)}$ is a Dirichlet form and the associated operator $L_K^{(D)}$ is the restriction of $\mathcal{L}$ to $C_c(K)$. See Section 3 for a thorough discussion.

COROLLARY 4.6. *Let (b, c) be a connected graph over an infinite measure space (X, m). Let (K_n) be an exhaustion sequence of X consisting of finite sets and let $x_n \in K_n$ for $n \in \mathbb{N}$. Then, $L_{K_n}^{(D)} + \alpha$ is invertible for $\alpha \geq -\lambda_0(L)$ and the sequence*

$$u_n^{(D)} = \frac{1}{(L_{K_n}^{(D)} + \alpha)^{-1} 1_{x_n}(x_0)} (L_{K_n}^{(D)} + \alpha)^{-1} 1_{x_n}$$

for $n \in \mathbb{N}$ has a subsequence which converges pointwise to a strictly positive α-superharmonic function u.

Furthermore, if the graph is locally finite and (x_n) is chosen to leave every finite set, then u is α-harmonic.

PROOF. Let $K \subseteq X$ be finite. Since $Q_K^{(D)}$ is a restriction of Q, it follows directly from the variational characterization of the bottom of the spectrum, Theorem E.8, that $\lambda_0(L) \leq \lambda_0(L_K^{(D)})$. In fact, we can even show

$$\lambda_0(L) < \lambda_0(L_K^{(D)})$$

as follows: Suppose that $\lambda_0(L) = \lambda_0(L_K^{(D)})$. As K is finite, there exists a normalized eigenfunction $f_K \in \ell^2(K, m_K)$ corresponding to $\lambda_0(L_K^{(D)})$. Now, considering $|f_K|$ we get

$$\lambda_0(L_K^{(D)}) \leq Q_K^{(D)}(|f_K|) \leq Q_K^{(D)}(f_K) = \lambda_0(L_K^{(D)}),$$

where we used the variational characterization for the first inequality and that $Q_K^{(D)}$ is a Dirichlet form for the second. Thus, $|f_K|$ is also an eigenfunction corresponding to $\lambda_0(L_K^{(D)})$ by Theorem E.8. Hence, by replacing f_K by $|f_K|$, we can assume that $f_K \geq 0$. We can then extend $f_K \geq 0$ to be zero outside of K and, as $Q_K^{(D)}$ and Q agree on K, we obtain

$$\lambda_0(L) \leq Q(f_K) = Q_K^{(D)}(f_K) = \lambda_0(L_K^{(D)}) = \lambda_0(L).$$

Thus, by Theorem E.8 again, f_K is an eigenfunction for L corresponding to $\lambda_0(L)$. As f_K is positive and the graph is connected, it follows by Corollary 4.2 that f_K is strictly positive on X, which contradicts that f_K is zero outside of K. This implies $\lambda_0(L) < \lambda_0(L_K^{(D)})$.

Hence, $L_K^{(D)} + \alpha$ is invertible and the inverse is positivity improving for $\alpha \geq -\lambda_0(L) > -\lambda_0(L_K^{(D)})$ by Corollary 4.3. In particular, this shows that the definition of $u_n^{(D)}$ makes sense for all $n \in \mathbb{N}$ and that $u_n^{(D)} \geq 0$. Since $L_{K_n}^{(D)}$ and $\mathcal{L}$ agree when applied to $u_n^{(D)}$, it follows that $(\mathcal{L} + \alpha) u_n^{(D)} \geq 0$ on K_n. Furthermore, $u_n^{(D)}(x_0) = 1$ for all $n \in \mathbb{N}$. Therefore, by the Harnack principle, Theorem 4.4, we obtain a subsequence of $(u_n^{(D)})$ converging pointwise to a strictly positive α-superharmonic function u.

In the locally finite case, when (x_n) leaves every finite subset, we can consider $u_n^{(D)}$ as defined above and restrict $u_n^{(D)}$ to an exhaustion sequence $K_n' \subseteq K_n$ such that $x_n \notin K_n'$. In this case, $u_n^{(D)}$ satisfies $(\mathcal{L}+\alpha)u_n^{(D)} = 0$ on K_n' so that u is α-harmonic by the additional statements in Theorem 4.4. □

Remark. We note that if X is finite and $\alpha > -\lambda_0(L)$, then the limiting functions u in the corollaries above are only superharmonic and not harmonic for $\alpha > -\lambda_0(L)$ (Exercise 4.5). This shows one of the contrasts between finite and infinite graphs, as from Corollaries 4.5 and 4.6 above, for infinite locally finite graphs, there always exist α-harmonic functions for all $\alpha \geq -\lambda_0(L)$.

Remark. We will show later in Lemma 6.27 in Section 4 that the sequences (u_n) and $(u_n^{(D)})$ constructed above converge for $\alpha = 0$ without choosing subsequences.

2. The ground state transform

In this section we prove variants of a ground state transform. A generalized ground state is a strictly positive generalized eigenfunction for the bottom of the spectrum. Such a generalized eigenfunction yields a transform of the operator. However, this transform not only works for a generalized ground state but also for more general functions. This is how we present it in this section.

Excavation Exercise 4.2 introduces the form associated to multiplication by a function and recalls the fact that this form is closed if the function is bounded below.

The starting point for the material in this section is the following observation: Whenever $\mathcal{L} = \mathcal{L}_{b,c,m}$ is the formal Laplacian and $\mathcal{U}_f$ is the formal operator of multiplication by a function $f \in C(X)$, then for any $u > 0$ with $u \in \mathcal{F}$, the operator $\mathcal{U}_u^{-1}\mathcal{L}\mathcal{U}_u - \mathcal{U}_w$ with $w = \mathcal{L}u/u$ is also a formal Laplacian arising from a graph which does not have a killing term. A precise formulation of this in the context of forms is given by the equality

$$\mathcal{Q}_{b,c}(u\varphi) - \langle wu\varphi, u\varphi\rangle = \mathcal{Q}_{b_u,0}(\varphi)$$

for all $\varphi \in C_c(X)$ with a suitable modification b_u of b.

This formula is most often applied with u being a generalized ground state, i.e., $u \in \mathcal{F}$ with $u > 0$ and $\mathcal{L}u = \lambda_0 u$ for the bottom of the spectrum $\lambda_0 = \lambda_0(L)$. Accordingly, this formula is often referred to as a ground state transform. As $\mathcal{Q}_{b_u,0}$ is an energy form arising from a graph, $\mathcal{Q}_{b_u,0}(\varphi) \geq 0$ for all $\varphi \in C_c(X)$. Thus, as a corollary of the ground state transform, we obtain the lower bound

$$Q_{b,c,m}(\varphi) \geq \langle w\varphi, \varphi \rangle$$

for any $u > 0$ and any $\varphi \in C_c(X)$ where $w = \mathcal{L}u/u$. In particular, if $u > 0$ is α-harmonic, i.e., $w = \mathcal{L}u/u = -\alpha$, we find

$$Q_{b,c,m} \geq -\alpha$$

and this gives that the spectrum of the Laplacian associated to $Q_{b,c,m}$ is bounded below by $-\alpha$. A closer inspection of the underlying considerations gives that this inequality persists for α-superharmonic functions and we can also bound the bottom of the spectrum in terms of such functions. A converse to this was discussed in the last section and the combination of these two results then gives a version of the Agmon–Allegretto–Piepenbrink theorem in the next section.

As usual, we will denote $\mathcal{L}_{b,c,m}$ by $\mathcal{L}$, $\mathcal{Q}_{b,c}$ by $\mathcal{Q}$ and $Q_{b,c,m}$ by Q. However, we will write the subscripts when they are not the standard ones or when we want to emphasize the dependence on the graph, as will often happen throughout this section.

After this preliminary discussion, we now work towards making the concepts introduced above precise. In order to state our results we will need two ingredients. One ingredient is the operator of multiplication by a function. The other is the modification of the original graph.

For a function $u \in C(X)$, we let $\mathcal{U}_u$ denote the formal operator of multiplication by u. That is, we define $\mathcal{U}_u \colon C(X) \longrightarrow C(X)$ by

$$\mathcal{U}_u f = uf.$$

The restriction of $\mathcal{U}_u$ to $\ell^2(X, u^2 m)$ will be denoted by U_u and can be seen to map into $\ell^2(X, m)$, i.e.,

$$U_u \colon \ell^2(X, u^2 m) \longrightarrow \ell^2(X, m)$$

via $U_u f = uf$. By direct calculation, if $u(x) \neq 0$ for all $x \in X$, then U_u is unitary and

$$U_u^{-1} = U_{1/u} = U_u^*.$$

This gives the first ingredient. For the second ingredient, let $u \in C(X)$ and define

$$b_u(x, y) = b(x, y)u(x)u(y)$$

for all $x, y \in X$. Clearly, b_u is symmetric and has vanishing diagonal. Moreover, $b_u \geq 0$ if $u \geq 0$ or $u \leq 0$. Furthermore, if $u \in \mathcal{F}$, then it follows directly that $\sum_{y \in X} b_u(x, y) < \infty$ for all $x \in X$. Hence, b_u is a graph over X whenever $u \in \mathcal{F}$ and $u \geq 0$ or $u \leq 0$. Finally, $u^2 m$ is a measure of full support whenever $u > 0$ or $u < 0$.

Summarizing the above considerations, b_u is a graph over $(X, u^2 m)$ and $\ell^2(X, u^2 m)$ is unitarily equivalent to $\ell^2(X, m)$ whenever $u \in \mathcal{F}$ and

$u > 0$ or $u < 0$. This is the required modification of the underlying graph.

We now show that the operator $\mathcal{U}_u^{-1}\mathcal{L}\mathcal{U}_u - \mathcal{U}_w$ acts like the formal Laplacian of the graph b_u on $C_c(X)$ when $w = \mathcal{L}u/u$ for $u \in \mathcal{F}$ with $u > 0$. This can be seen as a first version of the ground state transform in terms of formal Laplacians.

LEMMA 4.7. *Let (b, c) be a graph over (X, m), $\mathcal{L} = \mathcal{L}_{b,c,m}$ and let $u \in \mathcal{F}$. Then, for all $f \in C(X)$ such that $\mathcal{U}_u f \in \mathcal{F}$,*

$$\mathcal{L}\mathcal{U}_u f(x) = f(x)\mathcal{L}u(x) + \frac{1}{m(x)}\sum_{y\in X} b(x,y)u(y)(f(x) - f(y))$$

for all $x \in X$. If, additionally, $u > 0$ or $u < 0$ and $w = \mathcal{L}u/u$, i.e., $\mathcal{L}u = wu$, then $\mathcal{L}_u = \mathcal{L}_{b_u,0,u^2m}$ satisfies

$$\mathcal{L}_u = \mathcal{U}_u^{-1}\mathcal{L}\mathcal{U}_u - \mathcal{U}_w$$

on $C_c(X)$.

PROOF. The first statement follows by a direct computation using $(uf)(x) - (uf)(y) = (u(x) - u(y))f(x) + u(y)(f(x) - f(y))$, which gives

$$\begin{aligned}\mathcal{L}\mathcal{U}_u f(x) &= \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)((uf)(x) - (uf)(y)) + c(x)(uf)(x)\right)\\ &= f(x)\mathcal{L}u(x) + \frac{1}{m(x)}\sum_{y\in X} b(x,y)u(y)(f(x) - f(y)).\end{aligned}$$

If $u > 0$ or $u < 0$ and $w = \mathcal{L}u/u$, then dividing this formula by $u(x)$, applying $\mathcal{L}u = wu$ and rearranging the terms gives the second statement. $\square$

The preceding lemma deals with formal operators. Using Green's formula we can then easily derive a variant in terms of forms. This is the content of the next lemma.

LEMMA 4.8. *Let (b, c) be a graph over (X, m). Let $u \in \mathcal{F}$ with $u \geq 0$ or $u \leq 0$ and let $w \in C(X)$ satisfy $\mathcal{L}u \geq wu$. Then, for all $\varphi \in C_c(X)$,*

$$\mathcal{Q}_{b_u,0}(\varphi) \leq \mathcal{Q}(\mathcal{U}_u\varphi) - \langle w\mathcal{U}_u\varphi, \mathcal{U}_u\varphi\rangle.$$

If $\mathcal{L}u = wu$, then we get equality in the above equation.

PROOF. If $\varphi \in C_c(X)$, then $\mathcal{U}_u\varphi = u\varphi \in C_c(X) \subseteq \mathcal{F} \cap \ell^2(X, m)$. By Green's formula, Proposition 1.5, we obtain

$$\mathcal{Q}(\mathcal{U}_u\varphi) - \langle w\mathcal{U}_u\varphi, \mathcal{U}_u\varphi\rangle = \sum_{x\in X}\mathcal{U}_u\varphi(x)(\mathcal{L} - w)(\mathcal{U}_u\varphi)(x)m(x).$$

Now, by the first statement in Lemma 4.7 and the assumption that $\mathcal{L}u \geq wu$, we obtain

$$\mathcal{U}_u\varphi(x)(\mathcal{L}\mathcal{U}_u\varphi)(x) \geq (w\varphi^2u^2)(x)+\frac{\varphi(x)}{m(x)}\sum_{y\in X} b(x,y)u(x)u(y)(\varphi(x)-\varphi(y)).$$

Putting these together, we get

$$\begin{aligned}\mathcal{Q}(\mathcal{U}_u\varphi) - \langle w\mathcal{U}_u\varphi,\mathcal{U}_u\varphi\rangle &\geq \sum_{x\in X}\varphi(x)\sum_{y\in X} b(x,y)u(x)u(y)(\varphi(x)-\varphi(y))\\ &= \mathcal{Q}_{b_u,0}(\varphi).\end{aligned}$$

The statement for equality in the case $\mathcal{L}u = wu$ follows analogously. This proves the lemma. $\square$

For later use we state the following convenient reformulation of the second statement of the lemma.

Corollary 4.9. *Let (b, c) be a graph over (X, m). Let $u \in \mathcal{F}$ with $u > 0$ or $u < 0$ and set $w = \mathcal{L}u/u$. Then, for all $\varphi \in C_c(X)$,*

$$\mathcal{Q}_{b_u,0}\left(\frac{\varphi}{u}\right) = \mathcal{Q}(\varphi) - \langle w\varphi, \varphi\rangle.$$

We now turn to consequences for Laplacians and forms on Hilbert spaces. We first discuss a direct consequence of the previous lemma. We note that when u is α-harmonic, i.e., $u \in \mathcal{F}$ and $(\mathcal{L}+\alpha)u = 0$, then $-\alpha = \mathcal{L}u/u$ whenever $u > 0$. We denote by

$$L_u = L_{b_u,0,u^2m}$$

the operator associated to the form

$$Q_u = Q_{b_u,0,u^2m}$$

acting on $\ell^2(X, u^2m)$.

Corollary 4.10 (Ground state transform – preliminary version). *Let (b, c) be a graph over (X, m) and let $\alpha \in \mathbb{R}$. Let $u \in \mathcal{F}$ with $u > 0$ be α-harmonic. Then, the forms $Q_u = Q_{b_u,0,u^2m}$ and $Q+\alpha$ are unitarily equivalent via U_u and so are the associated operators. Specifically,*

$$D(Q_u) = U_u^{-1}D(Q), \qquad Q_u(f) = Q(U_uf) + \alpha\|U_uf\|^2$$

for all $f \in D(Q_u)$ and for the operator $L_u = L_{b_u,0,u^2m}$

$$D(L_u) = U_u^{-1}D(L), \qquad L_u = U_u^{-1}LU_u + \alpha.$$

Proof. As unitary equivalence is passed on from forms to operators, it suffices to show the statement for the forms. As U_u is a unitary operator mapping $C_c(X) \subseteq \ell^2(X, u^2m)$ onto $C_c(X) \subseteq \ell^2(X, m)$ and $C_c(X)$ is dense in the form domains, it suffices to show the statement about forms for $\varphi \in C_c(X)$. This is a direct consequence of the equality statement in Lemma 4.8, that is,

$$Q_u(\varphi) = Q(U_u\varphi) + \alpha\|U_u\varphi\|^2$$

for all $\varphi \in C_c(X)$. $\square$

REMARK. We note that in the corollary above, we trade a graph with an arbitrary killing term c for a graph without a killing term. This is achieved by changing the edge weight and the measure using the α-harmonic function. This is particularly convenient for $\alpha = 0$.

In fact, Lemma 4.8 allows us to derive a substantial generalization of Corollary 4.10. This generalization gives a more complete version of the ground state transform.

In order to state this generalization, we need one more piece of notation. For $w \in C(X)$ bounded below, we denote by q_w the form associated to the operator of multiplication by w. That is,

$$D(q_w) = \{f \in \ell^2(X,m) \mid \sum_{x\in X} w(x)f^2(x)m(x) < \infty\}$$

with

$$q_w(f) = \langle f, wf\rangle$$

for all $f \in D(q_w)$. It is of the essence for the arguments below that q_w is closed and, therefore, lower semi-continuous, see Theorem B.9 for the equivalence between closed and semi-continuous forms.

THEOREM 4.11 (Generalized ground state transform). *Let (b,c) be a graph over (X,m). Let $u \in \mathcal{F}$ with $u > 0$ be such that $\mathcal{L}u/u$ is bounded below and let $Q_u = Q_{b_u,0,u^2m}$. Then, for any $w \in C(X)$ which is bounded below with $w \le \mathcal{L}u/u$, we obtain*

$$U_u^{-1}D(Q) \subseteq D(Q_u)$$

and for all $f \in D(Q)$,

$$Q_u(U_u^{-1}f) \le Q(f) - q_w(f).$$

Furthermore, if w is bounded and $w = \mathcal{L}u/u$, then we get equality of the form domains and equality in the estimate above.

REMARK. The assumption that $\mathcal{L}u/u$ is bounded below is non-trivial. The most relevant situation in which it is true is when $u > 0$ is α-superharmonic for some $\alpha \in \mathbb{R}$, in which case $\mathcal{L}u/u \ge -\alpha$. This includes the case $\alpha = 0$, i.e., $\mathcal{L}u \ge 0$.

PROOF. Let $u \in \mathcal{F}$, $u > 0$ and $w \in C(X)$ bounded below be such that $w \le \mathcal{L}u/u$.

From Lemma 4.8 we immediately obtain the following:

Fact 1: For all $\varphi \in C_c(X)$,

$$Q_u(U_u^{-1}\varphi) \le Q(\varphi) - q_w(\varphi).$$

We will extend this to all functions in $D(Q)$ using the regularity of Q, i.e., the density of $C_c(X)$ in $D(Q)$. Some care has to be exercised as both the behavior of Q_u and of q_w has to be controlled.

Let C be a lower bound for w. The control on q_w which we need is stated in the next fact.

Fact 2: For all $\varphi \in C_c(X)$,

$$C\|\varphi\|^2 \leq q_w(\varphi) \leq Q(\varphi).$$

Here, the first estimate follows as w is bounded below by C and the second estimate is a direct consequence of the inequality above and $Q_u \geq 0$.

Consider now an arbitrary $f \in D(Q)$. By regularity, there exists a sequence (φ_n) in $C_c(X)$ with $\|f - \varphi_n\| \to 0$ and $Q(f - \varphi_n) \to 0$ as $n \to \infty$. As q_w is a closed form, we obtain from Fact 2 that $q_w(f) < \infty$ since q_w is lower semi-continuous. Therefore, $f \in D(q_w)$. Furthermore, using the fact that q_w is closed again, as well as that $\|f - \varphi_n\| \to 0$ and $q_w(\varphi_n - \varphi_k) \to 0$ as $n, k \to \infty$, it follows that $q_w(f - \varphi_n) \to 0$ as $n \to \infty$.

As Q_u is a closed form and $\varphi_n \to f$ in $\ell^2(X, m)$, so that $U_u^{-1}\varphi_n \to U_u^{-1} f$ in $\ell^2(X, u^2 m)$ as $n \to \infty$, we then find using Fact 1

$$\begin{aligned} Q_u(U_u^{-1} f) &\leq \liminf_{n\to\infty} Q_u(U_u^{-1}\varphi_n) \\ &\leq \liminf_{n\to\infty} (Q(\varphi_n) - q_w(\varphi_n)) \\ &= Q(f) - q_w(f). \end{aligned}$$

As $Q(f) - q_w(f) < \infty$ this implies $Q_u(U_u^{-1} f) < \infty$ and since $Q(f - \varphi_n) \to 0$ and $q_w(f - \varphi_n) \to 0$ as $n \to \infty$, it follows that $U_u^{-1}\varphi_n \to U_u^{-1} f$ in $\|\cdot\|_{Q_u}$. Therefore, $U_u^{-1} f \in D(Q_u)$ as well as

$$Q_u(U_u^{-1} f) \leq Q(f) - q_w(f).$$

This proves the first statement.

The last statement can be proven along very similar lines using the additional assumptions as follows: By $w = \mathcal{L}u/u$ and Lemma 4.8 we obtain equality in the statement above, i.e.,

$$Q_u(U_u^{-1}\varphi) = Q(\varphi) - q_w(\varphi)$$

for all $\varphi \in C_c(X)$. As w is both bounded below and bounded above there exists a $C > 0$ with

$$-C\|f\|^2 \leq q_w(f) \leq C\|f\|^2$$

for all $f \in \ell^2(X, m)$. Given this, the equality above easily gives that a sequence (φ_n) in $C_c(X)$ converges to f with respect to $\|\cdot\|_Q$ if and only if $(U_{u^{-1}}\varphi_n)$ converges to $U_{u^{-1}} f$ with respect to $\|\cdot\|_{Q_u}$. This implies the desired statement. $\square$

Remark (Note on the name of the ground state transform). As mentioned in the introduction, a generalized ground state is a generalized strictly positive eigenfunction for the bottom of the spectrum, i.e., a function $u \in \mathcal{F}$ with $u > 0$ and $\mathcal{L}u = \lambda_0 u$ where $\lambda_0 = \lambda_0(L)$ is the bottom of the spectrum of L. Note that, in contrast to other contexts, we do not assume any minimality properties of u.

In general, such a generalized ground state may or may not exist. For locally finite connected graphs, it exists by Corollaries 4.5 and 4.6. In this case, we can use Corollary 4.10 as u is $-\lambda_0$-harmonic and strictly positive. More generally, if we assume that the graph is connected but not necessarily locally finite, then there still exists a strictly positive superharmonic function for the bottom of the spectrum by Corollaries 4.5 and 4.6. We think of such functions as generalized ground states and use them in Theorem 4.11. However, let us reiterate that both results hold for much more general functions.

Theorem 4.11 has two immediate corollaries. The first one will be used in our study of recurrence later.

Corollary 4.12. *Let (b, c) be a graph over (X, m). Let $u \in \mathcal{F}$ with $u > 0$ be such that $\mathcal{L}u/u$ is bounded below. Then, for any $w \in C(X)$ which is bounded below and satisfies $w \leq \mathcal{L}u/u$, we get*

$$Q(f) \geq q_w(f)$$

for all $f \in D(Q)$.

Proof. This is an immediate consequence of Theorem 4.11 as $Q_u = Q_{b_u,0,u^2m} \geq 0$. □

The second corollary will be used in the proof of the Agmon–Allegretto–Piepenbrink theorem found in the next section.

Corollary 4.13. *Let (b, c) be a graph over (X, m). Let $u \in \mathcal{F}$ with $u > 0$ be α-superharmonic. Then,*

$$Q(f) \geq -\alpha \|f\|^2$$

for all $f \in D(Q)$.

Proof. This is a direct consequence of the previous corollary with $w = -\alpha \leq \mathcal{L}u/u$. □

3. The bottom of the spectrum

In this section we provide a characterization of the bottom of the spectrum in terms of strictly positive α-superharmonic functions. More specifically, we show that energies $\alpha \geq -\lambda_0(L)$ are characterized by the existence of strictly positive α-superharmonic functions.

Excavation Exercise 4.3 recalls a basic fact about the spectrum of an orthogonal sum of operators which will be used in the proof below.

In this section we prove the Agmon–Allegretto–Piepenbrink theorem for the bottom of the spectrum. This theorem characterizes $\lambda_0(L)$

in terms of α-superharmonic functions, i.e., $u \in \mathcal{F}$ with $u > 0$ and $(\mathcal{L} + \alpha)u \geq 0$ for $\alpha \in \mathbb{R}$, where $\mathcal{L} = \mathcal{L}_{b,c,m}$. The proof naturally reduces to:

- Proving the existence of strictly positive α-superharmonic functions for $\alpha \geq -\lambda_0(L)$.
- Showing that $-\lambda_0(L)$ is bounded above by α whenever there exists a strictly α-superharmonic function.

For connected graphs, this characterization is a direct consequence of the results of the preceding two sections with Section 1 giving the first point and Section 2 giving the second. In fact, the results in Section 2 do not even require connectedness of the graph, in contrast to the existence statements in Section 1.

Now, for the applications in the next section, we need a version for graphs without a connectedness assumption. For this reason, we state and prove a statement without such an assumption. The only remaining task in the proof is to reduce the general case to the connected case by restricting attention to connected components.

Theorem 4.14 (Agmon–Allegretto–Piepenbrink – spectrum). *Let (b, c) be a graph over (X, m) and let $\alpha \in \mathbb{R}$. Then, the following statements are equivalent:*

(i) $\alpha \geq -\lambda_0(L)$.

(ii) *There exists a strictly positive α-superharmonic function.*

Furthermore, if the graph is infinite and locally finite, then the above are also equivalent to the following statement:

(iii) *There exists a strictly positive α-harmonic function.*

Proof. As the operator L decomposes into an orthogonal sum of restrictions of L to the connected components of the graph, we can assume that the graph is connected.

(i) $\Longrightarrow$ (ii)/(iii): We apply Corollary 4.5 (or Corollary 4.6) to conclude the existence of a strictly positive α-(super)harmonic function. This finishes the proof of this direction.

(ii)/(iii) $\Longrightarrow$ (i): Assume that there exists a strictly positive α-superharmonic function u. Then, the desired statement follows from Corollary 4.13, i.e., $Q(f) \geq -\alpha\|f\|^2$ and the variational characterization of the bottom of the spectrum, i.e., $\lambda_0(L) = \inf Q(f)$, where the infimum is taken over all normalized $f \in D(Q)$, see Theorem E.8. $\square$

Remark. If the graph is connected, then we can replace the assumption of strict positivity of the α-(super)harmonic function in (ii) and (iii) by non-triviality together with positivity. This follows by Corollary 4.2.

Remark. We can deduce from the theorem above that if the graph is connected, then the only possible eigenvalue of L with positive eigenfunction is $\lambda_0(L)$ (Exercise 4.6).

REMARK. Clearly, (iii) fails in the case of finite graphs for $\alpha > -\lambda_0(L)$ as α-harmonic functions are eigenfunctions for finite graphs. We next give an example that shows that also in the non-locally finite case there do not always exist non-trivial α-harmonic functions for all $\alpha \geq -\lambda_0(L)$.

EXAMPLE 4.15 (No α-harmonic function for $\alpha > 0$). Let $X = \mathbb{N}_0$, $m = 1$ and let b be an infinite star graph with center 0, i.e., $b(0,k) = b(k,0) > 0$ for $k \in \mathbb{N}$ satisfying $\sum_{k=1}^{\infty} b(0,k) < \infty$ and $b(k,n) = 0$ otherwise and $c = 0$. Let $u \geq 0$ be α-harmonic for $\alpha > 0$. Then,

$$(\mathcal{L} + \alpha)u(0) = \sum_{k=1}^{\infty} b(0,k)(u(0) - u(k)) + \alpha u(0) = 0$$

and

$$(\mathcal{L} + \alpha)u(k) = b(k,0)(u(k) - u(0)) + \alpha u(k) = 0$$

for $k \in \mathbb{N}$. Hence, as $b(0,k)(u(0) - u(k)) = \alpha u(k)$ from the second set of equations above, we get in the first equation that

$$\alpha \left(u(0) + \sum_{k=1}^{\infty} b(0,k) u(k) \right) = 0,$$

which implies that $u = 0$ as $u \geq 0$ and $\alpha > 0$.

4. The bottom of the essential spectrum

In this final section we turn to characterizations of the bottom of the essential spectrum. We will present two such characterizations. One is given in terms of the spectra of restrictions of the operator to complements of finite sets. Combined with the Agmon–Allegretto–Piepenbrink theorem for the bottom of the spectrum, this then gives the second characterization via functions which are both α-superharmonic and strictly positive outside of a finite set.

We recall that the essential spectrum is the complement in the spectrum of the isolated eigenvalues of finite multiplicity. We will denote the essential spectrum by $\sigma_{\mathrm{ess}}(L)$ and denote the bottom of the essential spectrum of L by

$$\lambda_0^{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L).$$

A basic fact about the essential spectrum is that it is not altered by compact perturbations of the operator. In fact, this can even be shown to characterize the essential spectrum, see Theorem E.7 and the subsequent remark in Appendix 2. Hence, compactness in one form or another is always crucial when dealing with the essential spectrum. In our considerations below this is reflected in the fact that changes on compact, i.e., finite sets lead to compact perturbations, see Lemma 4.18. As a consequence, such changes do not alter the essential spectrum, which is proven in Corollary 4.19. Given this, we can easily obtain a Persson result which characterizes the bottom of the essential spectrum in terms of the bottom of the spectrum of the perturbed operators. Combining this theorem with the Agmon–Allegretto–Piepenbrink result of the previous section, we then obtain a characterization of the bottom of the essential spectrum via positive α-superharmonic functions outside of a finite set.

Indeed, we need to specify what we mean by α-superharmonic outside of a finite set. Given a finite set $K \subseteq X$, the most intuitive way to say $u \geq 0$ is α-superharmonic on $X \setminus K$ is to assume u is in $\mathcal{F}$ and satisfies

$$(\mathcal{L} + \alpha)u \geq 0 \quad \text{on } X \setminus K.$$

There are, however, two modifications we have to make in order to define α-superharmonicity outside of a set K. First we have to assume that $u = 0$ on K. Secondly, we do not need that u is in $\mathcal{F}$, which imposes the assumption $\sum_{y \in X} b(x,y)|u(y)| < \infty$ for all $x \in X$. We only need to assume that $\sum_{y \in X \setminus K} b(x,y)|u(y)| < \infty$ for all $x \in X \setminus K$ since we evaluate $\mathcal{L}u$ only outside of K and $u = 0$ on K.

This space of functions already has appeared in Section 2 as $\mathcal{F}_{X \setminus K}$ on which the operator $\mathcal{L}_{X \setminus K}$ acts similarly to $\mathcal{L}$. Now, we say that a function u is *α-superharmonic outside of K* if $u = 0$ on K, the restriction $u|_{X \setminus K}$ is in $\mathcal{F}_{X \setminus K}$ and

$$(\mathcal{L}_{X \setminus K} + \alpha)u|_{X \setminus K} \geq 0.$$

Similarly, we say that u is *α-harmonic outside of K* if u satisfies the above and $(\mathcal{L}_{X \setminus K} + \alpha)u|_{X \setminus K} = 0$. In summary we say that there *exists an α-(super)harmonic function outside of a finite set* whenever there is a finite $K \subseteq X$ and a function u such that u is α-(super)harmonic outside of K.

The basic argument connecting compactness in space with compactness of the operators can be seen as establishing the compatibility of geometry with operator theory. This requires an additional assumption, namely that

$$\mathcal{L}C_c(X) \subseteq \ell^2(X,m).$$

This assumption is characterized in Theorem 1.29 and is for example satisfied if the graph is locally finite or if $\inf_{x \in X} m(x) > 0$.

Following this preliminary discussion and these definitions, we now state the theorem, which we will ultimately prove in this section, giving a characterization of the bottom of the essential spectrum in terms of functions which are superharmonic outside of a finite set.

THEOREM 4.16 (Agmon–Allegretto–Piepenbrink – essential spectrum). *Let (b,c) be a connected graph over (X,m) such that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$.*

(a) *If* $\alpha > -\lambda_0^{\mathrm{ess}}(L)$*, then there exists a strictly positive* α*-superharmonic function outside of a finite set. If the graph is locally finite, then this function can be chosen to be* α*-harmonic.*
(b) *If there exists a strictly positive* α*-superharmonic function outside of a finite set, then* $\alpha \geq -\lambda_0^{\mathrm{ess}}(L)$*.*

In particular,

$$\lambda_0^{\mathrm{ess}}(L) = \sup\left\{ -\alpha \in \mathbb{R} \;\middle|\; \begin{array}{c} \textit{there exists a strictly positive function} \\ \alpha\textit{-superharmonic outside of a finite set} \end{array} \right\}.$$

Remark (Idea of the proof). Before we turn to the actual proof of the theorem we present the core of the argument in a nutshell: As is clear from the statement, the theorem deals with restrictions of operators to complements of finite sets. Thus, the proof of the theorem will rely on a closer look at such restrictions. Such restrictions can be seen as the difference between the operator acting on the entire space X and the operator on a finite subset. The assumption that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ allows us to show the smallness, in the sense of compactness of the operator, of this difference from the finiteness of the set. Compactness, in turn, is the crucial property underlying the stability of the essential spectrum. To make all of this precise we need some further notation and concepts to deal with restrictions to complements of sets.

We will need the restriction of the form Q to subsets U of X with $X \setminus U$ being finite. An extensive discussion of restrictions of forms to arbitrary subsets of X was given in Section 2. Here, we briefly discuss the essential points and the simplifications arising from the finiteness of $X \setminus U$. In particular, we also include a proof of regularity in our situation as this is substantially easier than the proof given in Section 2 for the case of general subsets.

For $U \subseteq X$, we recall the definition of Q_U with domain

$$D(Q_U) = \{g \in \ell^2(U, m_U) \mid i_U g \in D(Q)\}$$

and acting as

$$Q_U(g) = Q(i_U g).$$

In the definition

$$i_U \colon C(U) \longrightarrow C(X)$$

is extension by zero. The associated self-adjoint operator to Q_U is denoted by L_U. Since $Q = Q_{b,c,m}^{(D)}$ is a regular Dirichlet form, we have by Corollary 2.19

$$Q_U = Q_U^{(D)}$$

where $Q^{(D)}$ is the closure of the restriction of Q to $C_c(U) \times C_c(U)$ and, therefore,

$$L_U = L_U^{(D)},$$

where $L_U^{(D)}$ is the associated operator to $Q_U^{(D)}$.

For $f \in C(X)$ the restriction of f to U is denoted by $f|_U$. We will also need the extension $\widehat{Q}_U$ of Q_U to $\ell^2(X,m)$. This extension is defined on the set

$$D(\widehat{Q}_U) = \{f \in D(Q) \mid f|_U \in D(Q_U)\}$$

via

$$\widehat{Q}_U(f) = Q_U(f|_U).$$

By construction, $\widehat{Q}_U$ may be viewed as an orthogonal sum of Q_U on $\ell^2(U, m_U)$ and 0 on $\ell^2(X \setminus U, m_{X\setminus U})$ and this implies that the self-adjoint operator $\widehat{L}_U$ associated to $\widehat{Q}_U$ is the orthogonal sum of L_U and the zero operator on $\ell^2(X \setminus U, m_{X\setminus U})$.

If $X \setminus U$ is finite, then we can say much more. We collect some facts next. Although parts of this result are contained already in Section 2 we give a short proof here as well.

LEMMA 4.17. *Let (b, c) be a graph over (X, m). Let $U \subseteq X$ be such that $X \setminus U$ is finite. Then, Q_U is a regular Dirichlet form with*

$$D(Q_U) = \{f|_U \mid f \in D(Q)\}$$

as well as

$$D(\widehat{Q}_U) = D(Q) \quad \text{and} \quad \widehat{Q}_U(f) = Q(1_U f).$$

PROOF. Note that whenever g belongs to $D(Q_U)$ we have $g = (i_U g)|_U$ with $i_U g \in D(Q)$ and, conversely, whenever f belongs to $D(Q)$ we find

$$i_U(f|_U) = 1_U f = f - 1_{X\setminus U} f \in D(Q)$$

as $1_{X\setminus U} f \in C_c(X) \subseteq D(Q)$. This implies the first statement and, in particular, that the map $\pi_U \colon D(Q) \longrightarrow D(Q_U)$ given by

$$\pi_U f = f|_U$$

is well-defined. Since both i_U and π_U commute with normal contractions, Q_U is a Dirichlet form.

If we equip $D(Q)$ with $\|\cdot\|_Q$ and $D(Q_U)$ with $\|\cdot\|_{Q_U}$, we see that π_U is closed. Hence, the closed graph theorem implies that π_U is continuous. This allows us to show the regularity of Q_U in the following simple manner: Let $g \in D(Q_U)$ be arbitrary and set $f = i_U f$ for $f \in D(Q_U)$. Then, by the regularity of Q there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ with respect to $\|\cdot\|_Q$. As π_U is continuous, this implies convergence of $\pi_U \varphi_n \to \pi_U f$ in $\|\cdot\|_{Q_U}$ for $n \to \infty$ and this gives the desired regularity statement. Thus, Q_U is a regular Dirichlet form. Since $f|_U$ belongs to $D(Q)$ for any $f \in D(Q)$ and $i_u(f|_U) = 1_U f$ we then conclude the equality of the domains of $\widehat{Q}_U$ and Q. $\square$

Having this description of $\widehat{Q}_U$ at our disposal, we can now provide the necessary perturbation theory.

LEMMA 4.18. *Let (b,c) be a graph over (X,m). Let $U \subseteq X$ be such that $X \setminus U$ is finite. If $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then there exists a unique self-adjoint compact operator A with*

$$(Q - \widehat{Q}_U)(f,g) = \langle f, Ag\rangle$$

for all $f, g \in D(Q) = D(\widehat{Q}_U)$. In particular, $L = \widehat{L}_U + A$.

PROOF. By the preceding lemma, $D(\widehat{Q}_U) = D(Q)$. Hence, $Q - \widehat{Q}_U$ is also defined on $D(Q)$. By $C_c(X) \subseteq D(Q)$ we then directly infer the uniqueness of A.

We now turn to showing existence. Set $K = X \setminus U$. Then, K is a finite set. Hence, $1_K h$ belongs to $D(Q)$ for any $h \in \ell^2(X,m)$ as $1_K h \in C_c(X)$. Moreover, by the previous lemma again, we have that $\widehat{Q}_U(f,g) = Q(1_U f, 1_U g)$ for all $f, g \in D(Q)$. Thus, we obtain for all $f, g \in D(Q)$,

$$\begin{aligned} Q(f,g) - \widehat{Q}_U(f,g) &= Q(1_U f + 1_K f, 1_U g + 1_K g) - Q(1_U f, 1_U g) \\ &= Q(1_U f, 1_K g) + Q(1_K f, 1_U g) + Q(1_K f, 1_K g). \end{aligned}$$

Now, as K is finite, the operator associated to the form $(f,g) \mapsto Q(1_K f, 1_K g)$ can easily be seen to be finite-dimensional and, hence, compact. So, it remains to consider the forms $(f,g) \mapsto Q(1_U f, 1_K g)$ and $(f,g) \mapsto Q(1_K f, 1_U g)$.

We first deal with the form $(f,g) \mapsto Q(1_U f, 1_K g)$. Using the assumption on $\mathcal{L}$ we can define for $x \in X$ the function

$$v_x = \mathcal{L}1_x \in \ell^2(X,m).$$

By Green's formula, Proposition 1.5, we then find that $Q(1_U f, 1_x) = \langle 1_U f, \mathcal{L}1_x\rangle = \langle 1_U f, v_x\rangle$. Therefore,

$$\begin{aligned} Q(1_U f, 1_K g) &= \sum_{x \in K} g(x) Q(1_U f, 1_x) \\ &= \sum_{x \in K} g(x) \langle 1_U f, v_x\rangle \\ &= \langle f, \sum_{x \in K} g(x) 1_U v_x\rangle \\ &= \langle f, A_{U,K} g\rangle. \end{aligned}$$

Here, $A_{U,K}$ is the finite-dimensional and, hence, compact operator defined by

$$A_{U,K} g = \sum_{x \in K} g(x) 1_U v_x.$$

As for the form $(f,g) \mapsto Q(1_K f, 1_U g)$, an analogous computation can be carried out yielding

$$Q(1_K f, 1_U g) = \langle f, A_{K,U} g\rangle$$

with the compact operator $A_{K,U} = A^*_{U,K}$.

The last statement follows from the preceding considerations by standard theory. For the convenience of the reader we include a proof: Let $f \in D(L)$. By the definition of L, there exists an $h = Lf \in \ell^2(X,m)$ with $Q(f,g) = \langle h, g\rangle$ for all $g \in D(Q)$. By what we have shown already this implies

$$\widehat{Q}_U(f,g) = \langle h, g\rangle + \langle f, Ag\rangle = \langle (h + Af), g\rangle$$

for all $g \in D(Q) = D(\widehat{Q}_U)$. By the definition of $\widehat{L}_U$ we infer that $f \in D(\widehat{L}_U)$ with $\widehat{L}_U f = Lf + Af$. Analogously, we can show that any $f \in D(\widehat{L}_U)$ belongs to $D(L)$ and $Lf = \widehat{L}_U + Af$. $\square$

By the last lemma and the fact that compact perturbations do not effect the essential spectrum we obtain the following immediate corollary.

Corollary 4.19 (Stability of essential spectrum under removing finite sets). *Let (b,c) be a graph over (X,m). Let $U \subseteq X$ be such that $X \setminus U$ is finite. If $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then*

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L_U).$$

Proof. By the previous lemma, the Laplacians associated to Q and $\widehat{Q}_U$ differ only by A, which is a compact operator, i.e., we have $L = \widehat{L}_U + A$. This implies

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(\widehat{L}_U)$$

as a compact operator does not change the essential spectrum, see Theorem E.7. By definition, we furthermore have that $\widehat{L}_U = L_U \oplus 0_{X\setminus U}$ and, hence, $\sigma_{\mathrm{ess}}(\widehat{L}_U) = \sigma_{\mathrm{ess}}(L_U) \cup \sigma_{\mathrm{ess}}(0_{X\setminus U})$. As $X \setminus U$ is finite, the essential spectrum of $0_{X\setminus U}$ is empty. Putting all of this together, we obtain the desired statement. $\square$

As alluded to above, we need one more ingredient in order to prove our main result. This ingredient is a Persson theorem.

Theorem 4.20 (Persson theorem). *Let (b,c) be a graph over (X,m). If $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then*

$$\lambda_0^{\mathrm{ess}}(L) = \sup_{K \subseteq X, K \text{ finite}} \lambda_0(L^{(D)}_{X\setminus K}).$$

Proof. Set

$$\alpha = \sup_{K \subseteq X, K \text{finite}} \lambda_0(L_{X\setminus K})$$

and recall that $L_{X\setminus K} = L^{(D)}_{X\setminus K}$. We show two inequalities:

$\alpha \leq \lambda_0^{\mathrm{ess}}(L)$: Invoking the preceding corollary, with $U = X \setminus K$ for K finite, we find that

$$\lambda_0(L_{X\setminus K}) \leq \lambda_0^{\mathrm{ess}}(L_{X\setminus K}) = \lambda_0^{\mathrm{ess}}(L).$$

Thus, the desired inequality follows.

$\lambda_0^{\mathrm{ess}}(L) \leq \alpha$: We first note the following monotonicity property of the infimum of the spectrum. For $K_1 \subseteq K_2$ we have

$$\lambda_0(L_{X\setminus K_1}) = \inf_{\varphi \in C_c(X\setminus K_1)} \frac{Q_{X\setminus K}(\varphi)}{\|\varphi\|^2} \leq \inf_{\varphi \in C_c(X\setminus K_2)} \frac{Q_{X\setminus K_2}(\varphi)}{\|\varphi\|^2} = \lambda_0(L_{X\setminus K_2})$$

as $Q_{X\setminus K_2}$ and $Q_{X\setminus K_1}$ agree on $C_c(X \setminus K_2)$. From this monotonicity we obtain that we can, without loss of generality, increase the finite sets K in question in our subsequent considerations.

Choose a sequence of finite sets (K_n) with

$$\alpha = \lim_{n\to\infty} \lambda_0(L_{X\setminus K_n}).$$

By increasing the sets if necessary, we can then assume without loss of generality that the sets K_n are an exhaustion, i.e., $K_n \subseteq K_{n+1}$ and $\bigcup_n K_n = X$. Choose φ_n in $C_c(X \setminus K_n)$ with $\|\varphi_n\| = 1$ and

$$Q_{X\setminus K_n}(\varphi_n) \leq \lambda_0(L_{X\setminus K_n}) + \frac{1}{n}$$

for all $n \in \mathbb{N}$.

As the sets K_n are increasing and their union covers the space we can assume, without loss of generality, that the support of φ_n is contained in K_{n+1} for all n as otherwise we can pass to a subsequence. Hence, the supports of the φ_n are pairwise disjoint, so that the φ_n themselves are pairwise orthogonal. Note that, by construction, the φ_n are normalized. Altogether we then find that the φ_n form an orthonormal sequence and this gives that they converge weakly to 0, i.e.,

$$\langle f, \varphi_n \rangle \to 0 \text{ as } n \to \infty$$

for all $f \in \ell^2(X, m)$. Given this, the desired inequality follows directly from Theorem E.12. $\square$

Proof of Theorem 4.16. (a) Let $\alpha > -\lambda_0^{\mathrm{ess}}(L)$. By the Persson theorem, Theorem 4.20, there exists a finite set $K \subseteq X$ such that

$$\alpha > -\lambda_0(L^{(D)}_{X\setminus K}).$$

By the Agmon–Allegretto–Piepenbrink theorem for the bottom of the spectrum, Theorem 4.14, there exists a $u > 0$ which is α-superharmonic on $X \setminus K$. Furthermore, by the additional statements in Theorem 4.14, if the graph is locally finite, then $u > 0$ is even α-harmonic.

(b) Let K be a finite set and let u be a function on X which vanishes on K and is strictly positive and α-superharmonic function on $X \setminus K$. Then, u is α-superharmonic for $L^{(D)}_{X\setminus K}$. Thus, the Agmon–Allegretto–Piepenbrink theorem for the bottom of the spectrum, Theorem 4.14, gives that

$$\alpha \geq -\lambda_0(L^{(D)}_{X\setminus K}).$$

Moreover, from the Persson theorem, Theorem 4.20, we find

$$\lambda_0(L^{(D)}_{X\setminus K}) \leq \lambda_0^{\mathrm{ess}}(L)$$

so the desired statement follows. □

Exercises

Excavation exercises.

EXERCISE 4.1 (Diagonal subsequence). Let (f_n) be a sequence with $f_n \in C(X)$ such that for every $x \in X$ there exists a constant C_x with $|f_n(x)| \leq C_x$ for all $n \in \mathbb{N}$. Show that there exists a subsequence of (f_n) which converges pointwise at all $x \in X$.

EXERCISE 4.2 (Multiplication by a lower bounded function gives a closed form). Let (X, m) be a measure space. Let $w \in C(X)$ be bounded below. Let Q_w be the form associated to multiplication by w on $\ell^2(X, m)$, i.e.,

$$D(Q_w) = \{f \in \ell^2(X, m) \mid \sum_{x \in X} w(x) f^2(x) m(x) < \infty\}$$

with

$$Q_w(f) = \langle f, wf \rangle$$

for $f \in D(Q_w)$. Show that Q_w is a closed form.

EXERCISE 4.3 (Spectrum and orthogonal sums). Let H be a Hilbert space and let A be an operator on H with domain $D(A)$. Suppose that A can be written as an orthogonal sum of operators, i.e., $A = \bigoplus_{n=0}^{\infty} A_n$. Show that

$$\sigma(A) = \overline{\bigcup_{n=0}^{\infty} \sigma(A_n)}.$$

Example exercises.

EXERCISE 4.4 (λ_0 for anti-trees). Let $(b, 0)$ be a graph with standard weights over $(X, 1)$. That is, $b(x, y) \in \{0, 1\}$ with $c = 0$ and $m = 1$. Let $X = \bigcup_{n=0}^{\infty} S_n$, where S_n are disjoint sets with $\#S_n = (n + 1)^2$. Suppose that $b(x, y) = 1$ for all $x \in S_n$ and $y \in S_{n+1}$ for $n \in \mathbb{N}_0$. Show that $\lambda_0(L) \geq 2$.

(Hint: Apply the Agmon–Allegretto–Piepenbrink theorem for the bottom of the spectrum to the function u which takes the value $(n+1)^2$ on S_n.)

Extension exercises.

EXERCISE 4.5 (α-harmonic functions for finite graphs). Let (b, c) be a connected graph over a finite measure space (X, m). Let L denote the Laplacian associated to (b, c) over (X, m). Show that for every $\alpha > -\lambda_0(L)$ any α-superharmonic function u is not harmonic.

Exercise 4.6 (Positive eigenfunctions). Let (b, c) be a connected graph over a measure space (X, m). Let L denote the Laplacian associated to (b, c) over (X, m). If there is an eigenvalue λ of L with a positive eigenfunction, then $\lambda = \lambda_0(L)$. Show that this statement is false if the graph is not connected.

Notes

There is a long history for the corresponding results on (subsets of) Euclidean space, manifolds and strongly local Dirichlet forms. As for graphs, various parts of the results in this chapter are scattered around the literature. Our main inspiration is [**HK11**]. We give a more specific discussion of each section below.

The Harnack inequality goes back to work of Harnack [**Har87**]. In the context of graphs, it appears already in the work of Dodziuk in [**Dod84**]. A Harnack principle can be found (in a special case) in the book of Woess [**Woe00**]. Our presentation in Section 1 mainly follows [**HK11**]. In the discrete setting such a Harnack inequality is surprisingly simple to obtain, while it requires a much more thorough analysis in the setting of strongly local Dirichlet forms, see e.g. [**BM95**].

The ground state transform is well known for Schrödinger operators and the Laplacian on manifolds. A recent treatment in the non-linear case along with various classical references is contained in [**FS08**]. In the discrete setting it appeared for the first time in [**FSW08**] in the context of Jacobi matrices. For general regular Dirichlet forms, the ground state transform is discussed in [**FLW14**]. In probability theory, the corresponding method is often discussed under the name of h-transform or Doob-transform. For functions with finite support on graphs, a treatment is given in [**HK11**]. Our considerations in Section 2 extend this discussion.

A characterization of the infimum of the spectrum via positive solutions for Schrödinger operators on Euclidean domains appears in work of Agmon in [**Agm83**]. For the Laplacian on manifolds, it is discussed in [**CY75, FCS80, Sul87**]. A generalization to strongly local Dirichlet forms can be found in [**LSV09**]. The result for connected graphs with standard weights and counting measure can be found in [**Woj08, Woj09**] and for general weights and measure in [**HK11**]. In the generality presented in Section 3, the results seem to be new. In the context of random walks or, more generally, positive matrices, a corresponding result is known as the Perron–Frobenius Theorem, see [**Pru64, VJ67, VJ68, Woe00**].

The investigation of the infimum of the essential spectrum via positive solutions outside of finite sets for Schrödinger operators on Euclidean domains goes back to the work of Allegretto [**All74**] and Piepenbrink [**Pie74**]. For Laplacians on locally finite graphs it can be found in [**BG15**] and for Schrödinger operators on graphs in [**KPP20**]. Our treatment in Section 4 is similar to these last two works. In the continuum setting the Persson theorem goes back to [**Per60**]. For local Dirichlet forms, see [**Gri98**], and for general regular Dirichlet forms, see [**LS19**]. On a different note, the existence of a positive supersolution at the bottom of the essential spectrum can be characterized by finiteness of the number of eigenvalues below the essential spectrum [**Sim11**], see also [**Dev12**] and [**FCS80, FC85**] for the case of manifolds.

A word about the name Agmon–Allegretto–Piepenbrink theorem may be in order: The original work of Allegretto [**All74**] and Piepenbrink [**Pie74**], see also [**MP78**], deals with Schrödinger operators on Euclidean space. It provides a characterization of the infimum of the essential spectrum in terms of superharmonic functions. Thus, it is basically a precursor of the results in Section 4. However, it seems that subsequent to their work, the name Allegretto–Piepenbrink theorem was often assigned to results like those in Section 3 dealing with the infimum of the spectrum. In fact, this is how the names are given in the influential monograph [**CFKS87**] and subsequent articles, e.g., [**LSV09**]. On the other hand, the work of Agmon [**Agm83**] treats the bottom of the spectrum of Schrödinger operators and provides a characterization in terms of superharmonic functions. Thus, it is a precursor of the results in Section 3. Clearly, the results in Section 3 and Section 4 and their proofs are related. For this reason, one may speak about Agmon–Allegretto–Piepenbrink theorems as we do in this chapter.

CHAMBER 5

Large Time Behavior of the Heat Kernel

We out for the Gusto, and we gon' keep it raw.
Ghostface Killah.

In this chapter we study the large time behavior of the semigroup. We will show two convergence results: In the first result, the limit is either the ℓ^2 ground state, i.e., the strictly positive normalized eigenfunction corresponding to the bottom of the spectrum, or zero, in the case that there is no ℓ^2 ground state. In the second result, the limit is the ground state energy, i.e., the bottom of the spectrum. To this end, we consider kernels of the semigroup. These convergence results are presented in Section 2

The statements on convergence hold not only for the Dirichlet and Neumann Laplacians $L^{(D)}$ and $L^{(N)}$ but also for all operators arising from Dirichlet forms associated to graphs. In order to carry out the proofs for all such operators, we extend certain previously established positivity properties of the semigroup associated to $L^{(D)}$. We also show that if the ground state exists for such operators, then it is unique. We carry this out in Section 1.

Finally, in Section 3 we turn our focus to the Neumann Laplacian $L^{(N)}$. We characterize when the bottom of the spectrum of $L^{(N)}$ is zero in terms of finiteness of the measure of the entire space. We combine this characterization with the large time convergence results to discuss when the heat kernel associated to $L^{(N)}$ converges to zero.

1. Positivity improving semigroups and the ground state

Any semigroup coming from an operator associated to a Dirichlet form is positivity preserving. In this section, we will show that if the operator comes from a Dirichlet form which is associated to a connected graph, then the semigroup is positivity improving. We will also show that for any such operator, if there exists a ground state, then there exists a unique strictly positive normalized ground state.

We start with some general facts. For any self-adjoint operator L associated to a positive symmetric closed form Q on $\ell^2(X,m)$, the semigroup e^{-tL} for $t \geq 0$ is a bounded self-adjoint positive operator, see Appendices A and B for more details. By the discreteness of the space X, the semigroup has a kernel, i.e., there exists a map

$$p\colon [0,\infty) \times X \times X \longrightarrow \mathbb{R}$$

such that

$$e^{-tL}f(x) = \sum_{y\in X} p_t(x,y)f(y)m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $t \geq 0$. We call p the *heat kernel* associated to L. An easy calculation gives that

$$p_t(x,y) = \frac{1}{m(x)m(y)}\langle 1_x, e^{-tL}1_y\rangle$$

for all $x, y \in X$ and $t \geq 0$.

These are general facts concerning all forms and operators on discrete spaces. We will now focus on the case of graphs. If (b,c) is a graph over (X,m), $Q^{(D)} = Q^{(D)}_{b,c,m}$ is the minimal form and $Q^{(N)} = Q^{(N)}_{b,c,m}$ is the maximal form, then recall that a form Q with domain $D(Q)$ is associated to (b,c) if Q is closed, $D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)})$ and $Q = Q^{(N)}$ on $D(Q)$. We say that the self-adjoint operator L arising from Q is also associated to (b,c) and note that by Theorem 1.12, L is a restriction of the formal Laplacian $\mathcal{L}$.

We recall that an operator is called positivity preserving if the operator maps positive functions to positive functions and positivity improving if the operator maps nontrivial positive functions to strictly positive functions. By the general theory of Dirichlet forms, see Theorem C.4 in Appendix C, the semigroup of an operator associated to a Dirichlet form is positivity preserving. Furthermore, we have previously shown that the semigroup associated to the Laplacian $L^{(D)}$ is even positivity improving if the graph is connected. Combining this with the fact that $L^{(D)}$ generates the minimal semigroup gives the following result.

LEMMA 5.1 (Positivity improving semigroups). *Let (b,c) be a connected graph over (X,m). Let Q be a Dirichlet form associated to (b,c) with operator L. Then, the semigroup e^{-tL} is positivity improving for all $t > 0$.*

PROOF. As L is an operator associated to a Dirichlet form, it follows that e^{-tL} is positivity preserving for all $t \geq 0$, see Theorem C.4. Therefore, $e^{-tL}g \geq 0$ for all $g \in \ell^2(X,m)$ with $g \geq 0$. As L is a restriction of $\mathcal{L}$ by Theorem 1.12, it follows that $v = e^{-tL}g$ is a positive solution of the heat equation with initial condition g, i.e., $-\mathcal{L}v = \partial_t v$ with $v_0 = g$, see Theorem A.33. Since $e^{-tL^{(D)}}g$ is the minimal positive solution by Lemma 1.24, we have

$$e^{-tL}g \geq e^{-tL^{(D)}}g$$

for all $g \in \ell^2(X,m)$ with $g \geq 0$ and $t > 0$. As the graph is connected, by Theorem 1.26 we get $e^{-tL^{(D)}}g > 0$ if $g \geq 0$ and $g \neq 0$. This shows that the semigroup e^{-tL} is positivity improving for all $t > 0$. $\square$

As we assume that the form Q is a Dirichlet form, it follows by general theory that the semigroup is also contracting, i.e., $e^{-tL}f \leq 1$ whenever $f \leq 1$, see Theorem C.4. That e^{-tL} is positivity improving and contracting for any L arising from a Dirichlet form which is associated to a graph and any $t > 0$ will be used repeatedly below.

We denote the spectrum of an operator L by $\sigma(L)$ and the bottom of the spectrum by

$$\lambda_0 = \inf \sigma(L).$$

By the variational characterization of the bottom of the spectrum, Theorem E.8, it follows that

$$\lambda_0 = \inf_{f \in D(Q), \|f\|=1} Q(f) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle.$$

The following lemma considers the bottom of the spectrum for connected graphs. Specifically, whenever the bottom of the spectrum is an eigenvalue, then there exists a unique strictly positive normalized eigenfunction.

LEMMA 5.2 (Uniqueness of eigenfunctions to λ_0). *Let (b,c) be a connected graph over (X,m). Let Q be a Dirichlet form associated to (b,c) with operator L such that the bottom of the spectrum $\lambda_0 = \inf \sigma(L)$ is an eigenvalue. Then, there exists a unique strictly positive normalized eigenfunction corresponding to λ_0.*

PROOF. Let $u \in D(L)$ be a normalized eigenfunction corresponding to λ_0. We will show that u must be strictly positive or strictly negative. Without loss of generality, we may assume that $u(x) > 0$ for some $x \in X$. Let $u_+ = u \vee 0$ and $u_- = -u \vee 0$ so that $u = u_+ - u_-$ and $|u| = u_+ + u_-$. From the variational characterization of the bottom of the spectrum, Theorem E.8, and the fact that Q is a Dirichlet form we get

$$\lambda_0 \leq Q(|u|) \leq Q(u) = \lambda_0$$

so that $Q(|u|) = Q(u)$. Therefore, $|u|$ is also a normalized eigenfunction corresponding to λ_0 by Theorem E.8. As both u and $|u|$ are eigenfunctions corresponding to λ_0, we get that

$$u_+ = \frac{u + |u|}{2}$$

is also an eigenfunction corresponding to λ_0. We note that u_+ is nonzero as we assumed that $u(x) > 0$ for some $x \in X$.

The semigroup e^{-tL} on a connected graph is positivity improving by Lemma 5.1. Therefore, as $u_+ \geq 0$ satisfies $Lu_+ = \lambda_0 u_+$ and is nonzero, by the functional calculus and the positivity improving property we obtain

$$0 < e^{-tL}u_+ = e^{-t\lambda_0}u_+$$

for any $t > 0$. Hence, $u_+ > 0$ so that $u = u_+ > 0$. Therefore, any eigenfunction corresponding to λ_0 which is positive at some vertex is strictly positive.

From the argument above, it follows that any eigenfunction corresponding to λ_0 has a strict sign, i.e., is strictly positive or strictly negative. It is clear that any two functions of strict sign are not orthogonal in $\ell^2(X, m)$. This gives the uniqueness of u. □

If L is a self-adjoint operator arising from a Dirichlet form associated to a connected graph and λ_0 is an eigenvalue, then we have a unique strictly positive eigenfunction which minimizes the energy by the lemma above. In this context, we will refer to this eigenfunction as the ground state and λ_0 as the ground state energy.

We now discuss the case when the ground state energy is zero.

EXAMPLE 5.3 (When $\lambda_0 = 0$ is an eigenvalue). Suppose that (b, c) is a connected graph over (X, m) and L is an operator coming from a Dirichlet form Q associated to (b, c). If $\lambda_0 = 0$ is an eigenvalue for L, then $c = 0$ and $m(X) < \infty$.

Indeed, this follows as if $u > 0$ is a ground state for $\lambda_0 = 0$ given by the lemma above, then

$$0 = \lambda_0 = Q(u) = \frac{1}{2}\sum_{x,y\in X} b(x, y)(u(x) - u(y))^2 + \sum_{x\in X} c(x)u^2(x).$$

This shows that u is constant and $c = 0$. As $u \in D(L) \subseteq \ell^2(X, m)$, it follows that $m(X) < \infty$. In particular, as u is normalized, we obtain $u = 1/\sqrt{m(X)}$.

We will see that $c = 0$ and $m(X) < \infty$ implies that $\lambda_0 = 0$ is an eigenvalue for the Neumann Laplacian, i.e., $L = L^{(N)}$ in Section 3.

Furthermore, in the next chapter we will see that $\lambda_0 = 0$ is an eigenvalue for $L^{(D)}$ if and only if $c = 0$, $m(X) < \infty$ and the underlying graph is recurrent.

REMARK. Recall that a function $u \in \mathcal{F}$ is called α-superharmonic for $\alpha \in \mathbb{R}$ if $(\mathcal{L} + \alpha)u \geq 0$. It is called α-harmonic if the above is an equality. In the case of locally finite connected graphs, by the spectral version of the Agmon–Allegretto–Piepenbrink theorem, Theorem 4.14, we get that there exists a $-\lambda_0(L^{(D)})$-harmonic function. This gives a generalized ground state at the bottom of the spectrum for $L^{(D)}$. To be a ground state, we additionally require that the function is in the domain of the operator.

Furthermore, Theorem 4.14 also gives the existence of a strictly positive α-superharmonic function if and only if $\alpha \geq -\lambda_0(L^{(D)})$. Combining this with Lemma 5.2 above we immediately obtain that the only eigenvalue of $L^{(D)}$ which has a positive eigenfunction is $\lambda_0(L^{(D)})$. In the next section, we will extend this to all operators arising from Dirichlet forms associated to graphs.

2. Theorems of Chavel–Karp and Li

In this section we prove two convergence results. The first result implies that the heat kernel decays at a certain rate. In particular, if the bottom of the spectrum of the operator is positive, then the heat kernel must decay exponentially. The second result gives the convergence of the logarithm of the heat kernel to the bottom of the spectrum.

Both convergence results hinge on the use of the spectral theorem, see Appendix A for the necessary background. Furthermore, Excavation Exercise 5.1 recalls a standard fact concerning superadditive functions which will be used in the proof of Theorem 5.6.

We now present the first of our convergence results. We recall that the heat kernel of an operator L on $\ell^2(X,m)$ is given by

$$p_t(x,y) = \frac{\langle 1_x, e^{-tL}1_y\rangle}{m(x)m(y)}.$$

The following result connects the heat kernel, the bottom of the spectrum and the ground state.

THEOREM 5.4 (Theorem of Chavel–Karp). *Let (b,c) be a connected graph over (X,m). Let Q be a Dirichlet form associated to (b,c) with operator L. Let $\lambda_0 = \inf \sigma(L)$. Then, there exists a function $u\colon X \longrightarrow [0,\infty)$ such that*

$$\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y)$$

for all $x, y \in X$. If λ_0 is not an eigenvalue, then $u = 0$. If λ_0 is an eigenvalue, then u is the ground state, i.e., the unique normalized positive eigenfunction corresponding to λ_0.

PROOF. The proof is a direct application of the spectral theorem. Let $E = 1_{\{\lambda_0\}}(L)$ be the spectral projection onto the eigenspace of λ_0. By Proposition E.2, $E = 0$ if λ_0 is not an eigenvalue and, if λ_0 is an eigenvalue, then $E = \langle u, \cdot\rangle u$, where u is the unique positive normalized eigenfunction corresponding to λ_0 given by Lemma 5.2.

Let μ be the signed spectral measure of L associated to $1_x, 1_y$ for $x, y \in X$. That is, μ is the unique signed measure which is characterized by

$$\langle 1_x, \psi(L)1_y\rangle = \int_{\lambda_0}^{\infty} \psi(s)d\mu(s)$$

for all bounded measurable functions on $[\lambda_0,\infty)$, see Proposition A.26. Assume that λ_0 is an eigenvalue so that $1_{\{\lambda_0\}}(L) = \langle u, \cdot\rangle u$. We then get

$$\begin{aligned} m(x)m(y)|e^{\lambda_0 t}p_t(x,y) - u(x)u(y)| &= |\langle 1_x, (e^{\lambda_0 t}e^{-tL} - 1_{\{\lambda_0\}}(L))1_y\rangle| \\ &= \left| \int_{\lambda_0}^{\infty} \left(e^{-t(s-\lambda_0)} - 1_{\{\lambda_0\}}(s) \right) d\mu(s) \right| \\ &\to 0 \end{aligned}$$

as $t \to \infty$ by Lebesgue's dominated convergence theorem. Note that μ is a finite measure so that the bounding function can be chosen as 1. If λ_0 is not an eigenvalue, then a similar argument gives the conclusion. $\square$

We highlight one immediate corollary of the theorem above which characterizes when there exists a ground state.

Corollary 5.5 (Characterization of existence of a ground state). *Let (b, c) be a connected graph over (X, m). Let Q be a Dirichlet form associated to (b, c) with operator L. Let $\lambda_0 = \inf \sigma(L)$. Then, λ_0 is an eigenvalue for L if and only if*

$$\lim_{t \to \infty} e^{\lambda_0 t} p_t(x, y) \neq 0$$

for any (all) $x, y \in X$.

We will now state and prove the second of our convergence statements, which gives that the logarithm of the heat kernel converges to the bottom of the spectrum.

Theorem 5.6 (Theorem of Li). *Let (b, c) be a connected graph over (X, m). Let Q be a Dirichlet form associated to (b, c) with operator L. Let $\lambda_0 = \inf \sigma(L)$. Then,*

$$\lim_{t \to \infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0$$

for all $x, y \in X$.

Proof. Let $e_x = 1_x / \sqrt{m(x)}$, $x \in X$ and observe that $\{e_x\}_{x \in X}$ is an orthonormal basis for $\ell^2(X, m)$. Let

$$a_t(x, y) = \langle e_x, e^{-tL} e_y \rangle$$

for $x, y \in X$, $t \geq 0$ and let $a_t(x) = a_t(x, x)$. We will show that the function $t \mapsto \log a_t(x)$ on $[0, \infty)$ is superadditive for all $x \in X$.

Note that, as L is an operator coming from a Dirichlet form, e^{-tL} is positivity improving for $t > 0$ by Lemma 5.1 above and clearly positivity preserving for $t = 0$. Therefore, for all $x \in X$, $s, t \geq 0$, we obtain

$$\begin{aligned}
a_{s+t}(x) &= \langle e_x, e^{-(s+t)L} e_x \rangle \\
&= \langle e^{-sL} e_x, e^{-tL} e_x \rangle \\
&= \sum_{y \in X} \langle e^{-sL} e_x, e_y \rangle \langle e_y, e^{-tL} e_x \rangle \\
&\geq \langle e^{-sL} e_x, e_x \rangle \langle e_x, e^{-tL} e_x \rangle \\
&= a_s(x) a_t(x).
\end{aligned}$$

Lemma 5.1 implies $a_t(x) > 0$ for all $t \geq 0$, thus, we may take the logarithm of $a_t(x)$ for all $x \in X$ and $t \geq 0$. The estimate above then shows that $t \mapsto \log a_t(x)$ is superadditive, i.e., satisfies

$$\log a_s(x) + \log a_t(x) \leq \log a_{s+t}(x)$$

for $s, t \geq 0$. Furthermore, $a_t(x) \leq 1$ since $a_t(x) = e^{-tL} 1_x(x)$ and semigroups associated to operators coming from Dirichlet forms are contracting by Theorem C.4. Therefore, $\log a_t(x) \leq 0$. Putting all of this together, we get that the following limit exists for every $x \in X$

$$\lim_{t \to \infty} \frac{1}{t} \log a_t(x) = \sup_{t \in (0,\infty)} \frac{1}{t} \log a_t(x).$$

Now, for $t \geq 1$ and $x, y \in X$, by a similar reasoning as above we obtain

$$\begin{aligned}
a_{t-1}(x) a_1(x,y) &= \langle e^{-(t-1)L} e_x, e_x \rangle \langle e_x, e^{-L} e_y \rangle \\
&\leq \sum_{z \in X} \langle e^{-(t-1)L} e_x, e_z \rangle \langle e_z, e^{-L} e_y \rangle \\
&= \langle e^{-(t-1)L} e_x, e^{-L} e_y \rangle \\
&= \langle e_x, e^{-tL} e_y \rangle \\
&= a_t(x,y).
\end{aligned}$$

By the same arguments for $t \geq 0$,

$$a_1(x,y) a_t(x,y) \leq \sum_{z \in X} \langle e^{-L} e_y, e_z \rangle \langle e_z, e^{-tL} e_y \rangle = a_{t+1}(y).$$

Hence, as $a_1(x,y) > 0$, we get

$$a_{t-1}(x) a_1(x,y) \leq a_t(x,y) \leq \frac{1}{a_1(x,y)} a_{t+1}(y).$$

Combining this line of inequalities with the fact that $\lim_{t \to \infty} \frac{1}{t} \log a_t(x)$ exists and $a_t(x,y) = a_t(y,x)$ gives that $\lim_{t \to \infty} \frac{1}{t} \log a_t(x,y)$ exists and is independent of $x, y \in X$.

Let

$$\lim_{t \to \infty} \frac{1}{t} \log a_t(x,y) = -\lambda.$$

Since

$$a_t(x,y) = \langle e_x, e^{-tL} e_y \rangle = \sqrt{m(x)m(y)}p_t(x,y)$$

we conclude that

$$-\lambda = \lim_{t\to\infty} \frac{1}{t} \log a_t(x,y) = \lim_{t\to\infty} \frac{1}{t} \log p_t(x,y).$$

We will now show that $\lambda = \lambda_0$, which will complete the proof. First, we note that

$$\lim_{t\to\infty} \frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) = \lambda_0 - \lambda$$

for all $x, y \in X$. If λ_0 is an eigenvalue for L, it follows from Theorem 5.4 that $\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y) > 0$ so that

$$\lim_{t\to\infty} \frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) = 0$$

and, hence, $\lambda = \lambda_0$ in this case.

If λ_0 is not an eigenvalue for L, then Theorem 5.4 states that $e^{\lambda_0 t} p_t(x,y) \to 0$ as $t \to \infty$. Therefore, $\log \left(e^{\lambda_0 t} p_t(x,y) \right) < 0$ for all t large enough and since $\frac{1}{t} \log \left(e^{\lambda_0 t} p_t(x,y) \right) \to \lambda_0 - \lambda$ as $t \to \infty$, it follows that $\lambda_0 \leq \lambda$.

We will now show that $\lambda_0 \geq \lambda$. Let $\varepsilon > 0$. From Proposition E.2 we get

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) \neq 0$$

since $\lambda_0 \in \sigma(L)$. As the set of functions 1_x for $x \in X$ is total in $\ell^2(X,m)$, it follows that there exists an $x \in X$ such that

$$1_{[\lambda_0, \lambda_0+\varepsilon]}(L) 1_x \neq 0.$$

Let μ_x be the spectral measure of L associated to 1_x. Proposition A.24 gives

$$\frac{p_t(x,x)}{m^2(x)} = \langle 1_x, e^{-tL} 1_x \rangle = \int_{\lambda_0}^{\lambda_0+\varepsilon} e^{-ts} d\mu_x(s) \geq e^{-t(\lambda_0+\varepsilon)} \mu_x([\lambda_0, \lambda_0+\varepsilon])$$

as the spectral measure μ_x is supported on $[\lambda_0, \lambda_0 + \varepsilon]$ by Proposition A.29. Therefore,

$$-\lambda = \lim_{t\to\infty} \frac{1}{t} \log p_t(x,x) \geq -(\lambda_0 + \varepsilon),$$

that is, $\lambda \leq \lambda_0 + \varepsilon$. As $\varepsilon > 0$ was arbitrary, it follows that $\lambda \leq \lambda_0$, which concludes the proof. $\square$

From the theorem above we immediately obtain the following corollary which states that the existence of a positive eigenfunction implies that the eigenvalue is the bottom of the spectrum.

COROLLARY 5.7 (Positive eigenfunctions are multiples of ground states). *Let (b, c) be a connected graph over (X, m). Let Q be a Dirichlet form associated to (b, c) with operator L. Let $\lambda_0 = \inf \sigma(L)$. If there exists a non-trivial $u \geq 0$ with $u \in D(L)$ such that*

$$Lu = \lambda u$$

i.e., u is a positive eigenfunction corresponding to λ, then $\lambda = \lambda_0$. Furthermore, $u > 0$ and u is the ground state.

PROOF. As λ is an eigenvalue, $\lambda \in \sigma(L)$ so that $\lambda_0 \leq \lambda$ by definition. Now, if $u \geq 0$ in $D(L)$ is non-trivial and satisfies $Lu = \lambda u$, then the functional calculus gives

$$e^{-tL}u = e^{-t\lambda}u.$$

Therefore, for an arbitrary $x \in X$, using the positivity of u we get

$$p_t(x,x)u(x)m(x) \leq \sum_{y\in X} p_t(x,y)u(y)m(y) = e^{-tL}u(x) = e^{-t\lambda}u(x).$$

Applying Theorem 5.6 and choosing $x \in X$ such that $u(x) \neq 0$, we get

$$-\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log\left(p_t(x,x)u(x)m(x)\right) \leq \lim_{t\to\infty} \frac{1}{t} \log\left(e^{-t\lambda}u(x)\right) = -\lambda$$

so that $\lambda_0 \geq \lambda$. Therefore, $\lambda_0 = \lambda$.

The strict positivity of u follows from the proof of Lemma 5.2, which shows that $u = u_+ > 0$ and also gives the uniqueness of u. □

REMARK. One criterion for the existence of such u is given in Corollary 5.5. In the next section, we discuss this question for the Neumann Laplacian $L^{(N)}$ when $c = 0$. Furthermore, such u exist whenever the spectrum of L is discrete. Conditions for the discreteness of the spectrum will be given in later parts of the book, see Chapters 9 and 10.

REMARK. There is an alternative approach to proving the corollary above using the results of Chapter 4 (Exercise 4.6).

3. The Neumann Laplacian and finite measure

To conclude this chapter we take a look at the case of graphs with finite measure, i.e., graphs with $m(X) < \infty$. We will show a characterization of finiteness of the measure in terms of the domain of the Neumann form $Q^{(N)}$, the domain of the Neumann Laplacian $L^{(N)}$ and $\lambda_0 = 0$ being an eigenvalue for $L^{(N)}$. We will then combine this characterization with the convergence results presented in the previous section to discuss when the heat kernel associated to the Neumann Laplacian converges to 0.

We start by recalling some generalities about $Q^{(N)}$ and $L^{(N)}$. By definition, $Q^{(N)}$ is a restriction of $\mathcal{Q}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$ where

$\mathcal{D}$ denotes the functions of finite energy. Furthermore, by standard theory,

$$D(L^{(N)}) = \left\{ f \in D(Q^{(N)}) \;\middle|\; \begin{array}{c} \text{there exists a } g \in D(Q^{(N)}) \text{ with} \\ Q^{(N)}(h,f) = \langle h, g\rangle \text{ for all } h \in D(Q^{(N)}) \end{array} \right\},$$

in which case $L^{(N)} f = g$, see Theorem B.11.

We will write $\ell^1(X)$ for the space $\ell^1(X,1)$ in what follows and $1 \in C(X)$ for the function which is 1 on all vertices. We note that as soon as 1 is in a subspace of $\ell^2(X,m)$, then all constant functions are in that subspace. Hence, our result below can also be phrased in terms of all constant functions being in the corresponding domains.

We now characterize graphs over finite measure spaces.

THEOREM 5.8 (Characterization of finite measure). *Let (X,m) be a discrete measure space. Then, the following statements are equivalent:*

(i) $m(X) < \infty$.

(ii) $1 \in D(Q^{(N)}_{b,0,m})$ *for all graphs $(b,0)$ over (X,m).*

(ii′) $1 \in D(Q^{(N)}_{b,c,m})$ *for all graphs (b,c) over (X,m) with $c \in \ell^1(X)$.*

(iii) $1 \in D(L^{(N)}_{b,0,m})$ *for all graphs $(b,0)$ over (X,m).*

(iii′) $1 \in D(L^{(N)}_{b,c,m})$ *for all graphs (b,c) over (X,m) with $c/m \in \ell^2(X,m)$.*

(iv) $\lambda_0 = 0$ *is an eigenvalue for $L^{(N)}_{b,0,m}$ for all graphs $(b,0)$ over (X,m).*

REMARK (Reason for considering the Neumann Laplacian). We briefly discuss the reason why the theorem above concerns $L^{(N)}$. As already mentioned, $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$. Therefore, if $c = 0$, then $1 \in \mathcal{D}$ so that $1 \in D(Q^{(N)})$ if and only if $1 \in \ell^2(X,m)$ which can be characterized by the finiteness of the measure. In contrast, $D(Q^{(D)}) = \mathcal{D}_0 \cap \ell^2(X,m)$, where $\mathcal{D}_0$ denotes those functions in $\mathcal{D}$ which can be approximated by finitely supported functions pointwise and with respect to energy. If $c = 0$, it turns out that $1 \in \mathcal{D}_0$ is equivalent to recurrence. The question of when a graph is recurrent will be taken up in Chapter 6. However, aspects of the above result can still be recovered for general operators associated to graphs (Exercise 5.4).

PROOF. Throughout the proof, we use the fact that $m(X) < \infty$ is equivalent to $1 \in \ell^p(X,m)$ for some (all) $p \in [1,\infty)$.

(i) $\Longrightarrow$ (ii): As noted above, $m(X) < \infty$ implies $1 \in \ell^2(X,m)$ and, as $\mathcal{Q}_{b,0}(1) = 0$, it follows that $1 \in D(Q^{(N)}_{b,0,m})$.

(ii) $\Longrightarrow$ (ii′): If $1 \in D(Q^{(N)}_{b,0})$, then $1 \in \ell^2(X,m)$. As $\mathcal{Q}_{b,c}(1) = \sum_{x\in X} c(x) < \infty$ if $c \in \ell^1(X)$, it follows that $1 \in D(Q^{(N)}_{b,c,m})$.

(ii′) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (i): These are obvious.

(ii) $\Longrightarrow$ (iii): If $1 \in D(Q^{(N)}_{b,0,m})$, then $Q^{(N)}_{b,0,m}(h,1) = 0 = \langle h, 0\rangle$ for all $h \in D(Q^{(N)}_{b,0,m})$. This implies $1 \in D(L^{(N)}_{b,0,m})$.

(iii) $\implies$ (ii): This is obvious since $D(L^{(N)}_{b,0,m}) \subseteq D(Q^{(N)}_{b,0,m})$.

(iii) $\implies$ (iii′): If $1 \in D(L^{(N)}_{b,0,m})$, then $m(X) < \infty$. Therefore, if $c/m \in \ell^2(X,m)$, then by the Cauchy–Schwarz inequality

$$\sum_{x\in X} c(x) \leq \left(\sum_{x\in X} \frac{c^2(x)}{m(x)}\right)^{1/2} \left(\sum_{x\in X} m(x)\right)^{1/2} < \infty,$$

which gives that $c \in \ell^1(X)$. Therefore, as (iii) implies (ii), which is equivalent to (ii′) by what we have already shown, we get $1 \in D(Q^{(N)}_{b,c,m})$. Hence,

$$Q^{(N)}_{b,c,m}(h,1) = \sum_{x\in X} c(x)h(x) = \left\langle h, \frac{c}{m}\right\rangle$$

for all $h \in D(Q^{(N)})$ since $c/m \in \ell^2(X,m)$ by assumption. Therefore, $1 \in D(L^{(N)}_{b,c,m})$ for all c with $c/m \in \ell^2(X,m)$.

(iii′) $\implies$ (iii): This is obvious.

(iii) $\implies$ (iv): If $1 \in D(L^{(N)}_{b,0,m})$, then $L^{(N)}_{b,0,m}1 = 0$ so that 1 is an eigenfunction corresponding to $\lambda_0 = 0$.

(iv) $\implies$ (i): If u is an eigenfunction corresponding to $\lambda_0 = 0$ for $L^{(N)}_{b,0,m}$ where $(b,0)$ is a connected graph over (X,m), then

$$Q^{(N)}_{b,0,m}(u) = \langle L^{(N)}_{b,0,m}u, u\rangle = 0.$$

In particular, u is a nontrivial constant function in $D(Q^{(N)}_{b,0,m})$. As $u \in \ell^2(X,m)$ is then constant, it follows that $m(X) < \infty$. □

Remark. In the case of $c = 0$ and $m(X) < \infty$, it is possible to characterize the dimension of the eigenspace of $\lambda_0 = 0$ for $L^{(N)}_{b,0,m}$ geometrically by the number of connected components of the graph (Exercise 5.5).

The following immediate corollary gives another way of thinking of finiteness of the measure for the Neumann kernel in the case of no killing term. Namely, the Neumann kernel goes to zero in the long term if and only if the measure of the entire graph is infinite. This makes precise the limit found in Corollary 5.5 for the case of the Neumann Laplacian.

Corollary 5.9. *Let $(b,0)$ be a connected graph over (X,m). Let p be the heat kernel associated to $L^{(N)}$. If $m(X) < \infty$, then for all $x,y \in X$*

$$\lim_{t\to\infty} p_t(x,y) = \frac{1}{m(X)}.$$

Furthermore, $\lim_{t\to\infty} p_t(x,y) = 0$ for all $x,y \in X$ if and only if $m(X) = \infty$.

PROOF. By Theorem 5.8 above, if $m(X) < \infty$, then the constant functions for are eigenfunctions corresponding to the eigenvalue $\lambda_0 = 0$ for $L^{(N)}$. The positive normalized eigenfunction u corresponding to this eigenvalue is then $u = 1/\sqrt{m(X)}$. Hence, by Theorem 5.4, we infer

$$\lim_{t\to\infty} p_t(x,y) = \lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y) = u(x)u(y) = \frac{1}{m(X)}.$$

This gives the conclusion in the case of finite measure.

On the other hand, if $m(X) = \infty$, then 0 is not an eigenvalue for $L^{(N)}$ by Theorem 5.8. Now, if $\lambda_0 > 0$, then as $\lim_{t\to\infty} e^{\lambda_0 t} p_t(x,y)$ exists by Theorem 5.4, it follows that $\lim_{t\to\infty} p_t(x,y) = 0$. If $\lambda_0 = 0$, then $\lim_{t\to\infty} p_t(x,y) = 0$ by Corollary 5.5 since 0 is not an eigenvalue. $\square$

REMARK. Parts of the corollary above hold for more general operators coming from Dirichlet forms associated to graphs (Exercise 5.6).

Exercises

Excavation exercises.

Exercise 5.1 (Superadditive functions). Let $f\colon (0,\infty) \longrightarrow (-\infty, 0]$ be a continuous function such that $f(s) + f(t) \leq f(s+t)$. Show that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \sup_{t>0} \frac{f(t)}{t}.$$

Example exercises.

Exercise 5.2 (Two-sided path graph). Let $X = \mathbb{Z}$ with $b(x,y) = 1$ if $|x-y| = 1$ and 0 otherwise. Let $c = 0$ and choose m such that $m(X) < \infty$. Show that:

(a) $\lambda_0(L^{(N)}) = 0$ is an eigenvalue for $L^{(N)}$.
(b) $\lambda_0(L^{(D)}) = 0$ is an eigenvalue for $L^{(D)}$.
   (Hint: Construct a sequence $\varphi_n \in C_c(X)$ such that $\varphi_n \to 1$ pointwise and $\mathcal{Q}(\varphi_n) \to 0$, i.e., show that $1 \in \mathcal{D}_0$.)
(c) The function $u(x) = x$ satisfies $\mathcal{L}u = 0$ but is not an eigenfunction for $L^{(D)}$ or $L^{(N)}$ for any choice of m.
   (Hint: Calculate the energy of u.)
(d) The function $u(x) = x$ is an eigenfunction for the operator $L^*_{\min}$ which is a restriction of $\mathcal{L}$ to the set $D(L^*_{\min}) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}$ whenever $\sum_{x\in\mathbb{Z}} x^2 m(x) < \infty$.

Extension exercises.

Exercise 5.3 (Theorem of Chavel–Karp for resolvents). Let Q be a Dirichlet form associated to (b,c) with operator L and $\lambda_0 = \inf \sigma(L)$. Let

$$g\colon (0,\infty) \times X \times X \longrightarrow \mathbb{R}$$

be such that

$$(L+\alpha)^{-1} f(x) = \sum_{y\in X} g_\alpha(x,y) f(y) m(y)$$

for all $f \in \ell^2(X,m)$, $x \in X$ and $\alpha > 0$. Show that $g_\alpha > 0$ and that there exists a $u\colon X \longrightarrow [0,\infty)$ such that

$$\lim_{\alpha\to 0^+} \alpha g_\alpha(x,y) = u(x)u(y)$$

for all $x, y \in X$. Show furthermore that $\lambda_0 = 0$ is an eigenvalue of L if and only if $u \neq 0$, in which case u is the ground state, i.e., the unique normalized positive eigenfunction for $\lambda_0 = 0$.

Exercise 5.4 ($\lambda_0 = 0$ for general L). Let (b,c) be a graph over (X,m) and let Q be an associated form with operator L. Show that $1 \in D(Q)$ and $Q(1) = 0$ if and only if 0 is an eigenvalue for L.

EXERCISE 5.5 (Eigenspace of $\lambda_0 = 0$ for the Neumann Laplacian). Let b be a graph over (X, m) with $m(X) < \infty$. Show that the dimension of the eigenspace associated to the eigenvalue $\lambda_0 = 0$ for $L^{(N)}_{b,0,m}$ is equal to the number of connected components of b.

EXERCISE 5.6 (Vanishing of the heat kernel). Let (b, c) be a connected graph over (X, m) with $m(X) = \infty$. Let Q be a Dirichlet form associated to (b, c) with operator L. Let p be the heat kernel associated to L. Show that

$$p_t(x, y) \to 0$$

as $t \to \infty$ for all $x, y \in X$.

Notes

The convergence results in this section are directly inspired by the work of Chavel/Karp on Riemannian manifolds [**CK91**]. In particular, Theorem 5.4 above is a counterpart to the Theorem in [**CK91**] while Theorem 5.6 is a counterpart to Corollary 1 in [**CK91**]. The second result is attributed to a paper of Li [**Li86**] which contains the statement for compact manifolds. The argument for compact manifolds, however, only involves eigenvalues and eigenfunctions of the Laplace–Beltrami operator. This type of argument was already given for finite graphs in Section 7. Let us also note that the argument of Chavel/Karp uses exhaustion by compact sets, hence, their proof only carries over to operators arising from regular Dirichlet forms.

Subsequently, Simon [**Sim93**] gave an argument for the result of Chavel/Karp which only uses the spectral theorem and elliptic regularity. The proof of Simon was adapted to the discrete setting in [**HKLW12**] and [**KLVW15**] and the proofs of Theorems 5.4 and 5.6 are adapted from the proof of Theorem 8.1 in [**HKLW12**]. The paper [**KLVW15**] covers an even more general setting which only requires a positivity improving self-adjoint semigroup which has a kernel. This setting contains both Riemannian manifolds and infinite graphs. For related results concerning differential operators which are not necessarily self-adjoint, we refer to the review article of Pinchover [**Pin13**].

The uniqueness of the ground state when it exists, as presented in Lemma 5.2, is a rather general phenomenon. It is well known in the manifold case, see, for example, Theorem 2.8 in [**Sul87**] and can also be found in textbooks such as [**RS78**]. For finite-dimensional spaces, this is sometimes referred to as the Perron–Frobenius theorem, see the notes to Section 7 for the relevant discussion in this case.

The characterization of finite measure found in Theorem 5.8 is an extension of Theorem 6.1 in [**GHK$^+$15**]. It is inspired by a result of Yau which states that all positive harmonic functions on a complete Riemannian manifold which are in L^p are constant, see Theorem 3 in [**Yau76**]. Corollary 5.9 is adapted from Corollary 8.2 in [**HKLW12**], which in turn was inspired by Corollary 2 in [**CK91**].

CHAMBER 6

Recurrence

... but the sun will still come out tomorrow and shine shine shine like a gold mine.
GZA.

The topic presented in this chapter is recurrence. This concept can be studied via probability, potential theory and operator theory and has interpretations in each context. Classically, recurrence has been studied for graphs b over X, i.e., graphs with $c = 0$, and is a measure-independent property. However, the measure independence can also be formulated by stating that certain properties hold for all measures of full support on X. Furthermore, some implications also hold in the case of non-vanishing c in the sense that the properties in question already imply that $c = 0$. We will indicate this in the proofs.

We let $\mathcal{Q} = \mathcal{Q}_{b,c}$ denote the energy form. As usual, $\mathcal{D}$ denotes the set of functions of finite energy, i.e.,

$$\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}.$$

Furthermore, we let $\mathcal{D}_0$ be the vector space of all $f \in \mathcal{D}$ such that there exists a sequence of finitely supported functions (φ_n) with $\mathcal{Q}(f - \varphi_n) \to 0$ and $\varphi_n \to f$ pointwise as $n \to \infty$.

For an arbitrary vertex $o \in X$, we introduce the map $\langle \cdot, \cdot \rangle_o \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$ via

$$\langle f, g \rangle_o = \mathcal{Q}(f, g) + f(o)g(o).$$

If (b, c) is connected, then this map is easily seen to be an inner product. This inner product then gives a norm on the space $\mathcal{D}$ by

$$\|f\|_o = \left(\mathcal{Q}(f) + f^2(o)\right)^{1/2}.$$

We will show that $\mathcal{D}$ is a Hilbert space with respect to this inner product and that $\mathcal{D}_0$ is the closure of $C_c(X)$ with respect to the norm $\|\cdot\|_o$, i.e.,

$$\mathcal{D}_0 = \overline{C_c(X)}^{\|\cdot\|_o}.$$

By definition, $\mathcal{D}_0$ is a subspace of $\mathcal{D}$. We will see that recurrence is equivalent to these two spaces being equal.

Whenever we consider a measure m of full support, we write

$$Q_m^{(D)} = Q_{b,c,m}^{(D)}, \quad Q_m^{(N)} = Q_{b,c,m}^{(N)},$$

$$L_m = L_m^{(D)} = L_{b,c,m}^{(D)}, \quad \mathcal{L}_m = \mathcal{L}_{b,c,m}.$$

We note that $\mathcal{L}_m$ is usually denoted by $\mathcal{L}$. At various points where the measure does not play a role, we drop the subscript and write $\mathcal{L}$ to denote the operator $\mathcal{L}_m$ with $m = 1$. We recall that

$$D(Q_m^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$$

by definition while

$$D(Q_m^{(D)}) = \mathcal{D}_0 \cap \ell^2(X, m)$$

by Theorem 1.19. We denote by $\sigma(L_m^{(D)})$ the spectrum of $L_m^{(D)}$ and by $\lambda_0(L_m^{(D)})$ the bottom of the spectrum, i.e.,

$$\lambda_0(L_m^{(D)}) = \inf \sigma(L_m^{(D)}).$$

We will need to restrict forms and operators to subsets, as discussed in Section 3. Specifically, for a finite set K, we denote the operator associated to the restriction of the form $Q_m^{(D)}$ to $C_c(K)$ by $L_K^{(D)}$ and note that $L_K^{(D)}$ is a restriction of $\mathcal{L}_m$. Since $L_K^{(D)}$ is a Laplacian with a killing term which does not vanish at any vertex in K that has a neighbor in $X \setminus K$, the operator $L_K^{(D)}$ is invertible by Proposition 1.20 whenever X is infinite and the graph is connected. As usual, we understand $C_c(K)$ as a subspace of $\ell^2(X, m)$ by extending functions by 0.

We now introduce some new quantities which will play a central role in our main characterization of recurrence. We first define the *Green's function* G_m, which is given by

$$G_m(x, y) = \int_0^\infty e^{-tL_m} 1_y(x) dt$$

for $x, y \in X$. Note that this function takes values in $[0, \infty]$ as the semigroup e^{-tL_m} is positivity preserving by Corollary 1.22 and even positivity improving whenever b is connected by Theorem 1.26. We will be interested in the question if the value the Green's function takes is infinite or finite in what follows below.

We will show that the Green's function can also be constructed by approximating via resolvents either on the level of energy or on the level of geometry. More specifically, we will show that

$$G_m(x, y) = \lim_{\alpha \to 0^+} (L_m + \alpha)^{-1} 1_y(x) = \lim_{n \to \infty} (L_{K_n}^{(D)})^{-1} 1_y(x),$$

where (K_n) is any increasing sequence of finite connected sets with $X = \bigcup_n K_n$.

Furthermore, for any $x \in X$ we let the *capacity* of x be given by

$$\text{cap}(x) = \inf\{\mathcal{Q}(\varphi) \mid \varphi \in C_c(X), \varphi(x) = 1\}.$$

It is not hard to see that the capacity can also be defined by taking the infimum over $\mathcal{D}_0$ instead of $C_c(X)$. We will show that there exists a unique minimizer u of $\mathcal{Q}$ over the set of $f \in \mathcal{D}_0$ with $f(x) = 1$ and that this minimizer satisfies $0 \leq u \leq 1$ and $Q(u) = \text{cap}(x)$. This minimizer is called the *equilibrium potential* for $x \in X$.

Recall that a function $u \in C(X)$ is called harmonic (superharmonic, respectively) if $u \in \mathcal{F}$ and

$$\mathcal{L}u = 0 \qquad (\mathcal{L}u \geq 0, \text{ respectively})$$

that is, if u is α-harmonic (α-superharmonic, respectively) for $\alpha = 0$. We will be particularly interested in superharmonic functions u with

$$\mathcal{L}u = 1_x$$

for some $x \in X$. Such a function is called a *monopole* at $x \in X$. It turns out that the existence of monopoles is a remarkable property for a graph, whereas the existence of a *dipole*, i.e., a function $u \in \mathcal{F}$ with $\mathcal{L}u = 1_x - 1_y$ for $x, y \in X$ and $\mathcal{L}$ is always true.

We now state our characterization of recurrence. This is followed by an informal discussion of the contents of the theorem and a description of how the proof is carried out in the remaining parts of this chapter.

THEOREM 6.1 (Characterization of recurrence). *Let b be a connected graph over X. Then, the following statements are equivalent:*

(i) $D(Q_m^{(D)}) = D(Q_m^{(N)})$ *for all measures m.*
- (i.a) $\mathcal{D}_0 = \mathcal{D}$.
- (i.b) $1 \in \mathcal{D}_0$.
- (i.c) *There exists a $u \in \mathcal{D}_0$ and a finite set $K \subseteq X$ with*
$$\inf_{x \in X \setminus K} u(x) > 0.$$
- (i.d) *There exists a sequence of functions (e_n) in $C_c(X)$ with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and $\mathcal{Q}(e_n) \to 0$ as $n \to \infty$.*
- (i.e) *There exists a sequence of functions (e_n) in $C_c(X)$ with $e_n \to 1$ pointwise as $n \to \infty$ and $\sup_{n \in \mathbb{N}} \mathcal{Q}(e_n) < \infty$.*

(ii) $D(L_m^{(D)}) = \{f \in D(Q_m^{(N)}) \mid \mathcal{L}_m f \in \ell^2(X, m)\}$ *for all measures m.*

(iii) *If $u \in \mathcal{D}$ satisfies $\mathcal{L}_m u \in \ell^1(X, m)$ and $v \in \mathcal{D} \cap \ell^\infty(X)$, then*
$$\mathcal{Q}(u, v) = \sum_{x \in X} \mathcal{L}_m u(x) v(x) m(x)$$
for some (all) measure(s) m. (*"Green's formula"*)
- (iii.a) *If $u \in \mathcal{D}$ satisfies $\mathcal{L}_m u \in \ell^1(X, m)$, then*
$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) = 0$$
for some (all) measure(s) m.
- (iii.b) *If $u \in \ell^\infty(X)$ satisfies $\mathcal{L}_m u \in \ell^1(X, m)$, then*
$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) = 0$$
for some (all) measure(s) m.

(iv) *All superharmonic functions $u \geq 0$ are constant.*

(iv.a) *All superharmonic functions* $u \in \mathcal{D}_0$ *are constant.*
(iv.b) *All superharmonic functions* $u \in \mathcal{D}$ *are constant.*
(iv.c) *All superharmonic functions* $u \in \ell^\infty(X)$ *are constant.*

(v) $(\mathcal{D}_0, \mathcal{Q})$ *is not a Hilbert space.*
(v.a) $\mathcal{Q}$ *is degenerate on* $\mathcal{D}_0$.
(v.b) $\mathcal{Q}^{1/2}$ *and* $\|\cdot\|_o$ *are not equivalent norms on* $C_c(X)$ *for some (all)* $o \in X$.

(vi) *The point evaluation map*

$$\delta_x \colon (\mathcal{D}_0, \mathcal{Q}) \longrightarrow \mathbb{R}, \quad \delta_x(f) = f(x)$$

is not continuous for some (all) $x \in X$.

(vii) $\operatorname{cap}(x) = 0$ *for some (all)* $x \in X$.
(vii.a) *The equilibrium potential for some (all)* $x \in X$ *is given by the constant function* 1.

(viii) *There does not exist a non-trivial positive function* $w \in C(X)$ *such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x \in X} w(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$. *("Hardy's inequality")*
(viii.a) *There does not exist a strictly positive function* $w \in C(X)$ *such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x \in X} w(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$.
(viii.b) $\lambda_0(L_m^{(D)}) = 0$ *for all measures* m *on* X.

(ix) *There exists a non-trivial harmonic function* $u \in \mathcal{D}_0$, *i.e.,* $\mathcal{L}_m$ *is not injective on* $\mathcal{D}_0$ *for some (all) measure(s)* m.

(x) *For some (all)* $x \in X$ *there does not exist a monopole in* $\mathcal{D}_0$ *at* x.

(xi) *For some (all)* $x, y \in X$ *and some (all) measure(s)* m,

$$G_m(x, y) = \infty.$$

("Green's function")
(xi.a) *For some (all)* $x, y \in X$ *and some (all) measure(s)* m,

$$\lim_{\alpha \to 0^+} (L_m + \alpha)^{-1} 1_y(x) = \infty.$$

(xi.b) *For some (all)* $x, y \in X$ *and some (all) sequence(s)* (K_n) *of increasing finite sets such that* $\bigcup_n K_n = X$ *and some (all) measure(s)* m,

$$\lim_{n \to \infty} (L_{K_n}^{(D)})^{-1} 1_y(x) = \infty.$$

Definition 6.2. A connected graph b over X that satisfies any of the conditions in the theorem above is called *recurrent*. Otherwise, a connected graph is called *transient*.

Remark. We note that the order of the first four properties given above mirrors the order of the properties listed for the equivalence of $Q^{(D)}$ and $Q^{(N)}$ presented in Theorem 3.2.

In the subsequent sections of this chapter, we elaborate on various aspects of recurrence. Along the way, we will prove the main theorem above. Before diving into the details, we will pause for a moment to discuss the intuitive meaning of the above conditions and the connections between them. At this stage, this discussion must be somewhat vague. Still, we feel it will provide a valuable perspective on the considerations below which, in some parts, become rather technical. All points discussed next will be taken up and made precise in later proofs.

The main theme in (i) is that all elements of the space $\mathcal{D}$ can be approximated pointwise and in terms of energy by functions with compact support, i.e., that $\mathcal{D} = \mathcal{D}_0$. This equality implies, in particular, that the constant function 1 belongs to $\mathcal{D}_0$. This is quite remarkable as, intuitively, we might expect some form of decay for functions in $\mathcal{D}_0$. In fact, it turns out that the absence of this decay is the crucial ingredient for the equality of $\mathcal{D}$ and $\mathcal{D}_0$. More specifically, this equality is valid if and only if there exists a uniformly positive function in $\mathcal{D}_0$ which, in turn, holds if and only if the constant function 1 belongs to $\mathcal{D}_0$. Roughly speaking, these approximation properties mean that there is nothing happening at infinity that we cannot already see on finite sets.

A precise version of already being able to see things happening at infinity on finite sets is provided in the context of boundary terms. In fact, it is natural to expect that all sorts of boundary terms in partial integrations vanish when dealing with functions in $C_c(X)$. So, the approximability given in (i) should imply vanishing boundary terms for functions in $\mathcal{D}$. It turns out that this approximability is even equivalent to vanishing boundary terms in various settings. This is the content of (ii) and (iii). Specifically, (ii) characterizes the domain of the generators for all measures and (iii) can be understood as a version of Green's formula.

Another way of understanding the equality of $\mathcal{D}$ and $\mathcal{D}_0$ is via superharmonic functions. To make this precise, we consider $\mathcal{D}$ equipped with the inner product $\langle \cdot, \cdot \rangle_o$ for $o \in X$. Then, the orthogonal complement of $\mathcal{D}_0$ in $\mathcal{D}$ is given by functions $u \in \mathcal{D}$ with

$$\mathcal{L}u(x) = 0 \quad \text{for } x \neq o \qquad \text{and} \qquad \mathcal{L}u(o) = -u(o).$$

This can be used to show that equality of $\mathcal{D}$ and $\mathcal{D}_0$ is equivalent to the absence of superharmonic functions $u \in \mathcal{D}$ with $\mathcal{L}u = 1_o$, i.e., of monopoles at $o \in X$, which is (x). It turns out that the absence of such superharmonic functions is equivalent to the absence of positive superharmonic functions as well as bounded superharmonic functions.

This extension requires quite some care and attention. This is the content of (iv).

We give a different aspect of the equality of $\mathcal{D}$ and $\mathcal{D}_0$ in (v). This aspect concerns (non-)degeneracy properties of $\mathcal{Q}$. In the setting of infinite graphs, $\mathcal{Q}$ is an inner product on $C_c(X)$ but is not an inner product on $\mathcal{D}$ since 1 belongs to $\mathcal{D}$. Given this situation, a natural question is whether $\mathcal{Q}$ is an inner product on $\mathcal{D}_0$. Obviously, $\mathcal{Q}$ cannot be an inner product if $\mathcal{D}_0 = \mathcal{D}$. It turns out that the converse also holds, i.e., $\mathcal{Q}$ is an inner product if $\mathcal{D}_0 \neq \mathcal{D}$. In this case, $(\mathcal{D}_0, \mathcal{Q})$ is even a Hilbert space.

We give meaning to the (non-)degeneracy of $\mathcal{Q}$ on $\mathcal{D}_0$ by pointwise estimates as follows: If $\mathcal{Q}$ is not degenerate, then point evaluation is continuous. Therefore, for every $x \in X$, there exists a $c_x > 0$ with $c_x f^2(x) \leq \mathcal{Q}(f)$ for all $f \in \mathcal{D}_0$. This is the basic connection between (v), (vii) and (viii). A short argument then shows that the constants c_x are nothing but the capacities of the vertices $x \in X$. This connects (vi) and (vii). To actually prove the full equivalence between (v), (vi), (vii) and (viii) we still have to argue that the c_x are either all zero or all non-zero. This is a consequence of the connectedness of the graph. In this context, we also encounter the equilibrium potential for $x \in X$, i.e., the unique minimizer of $\mathcal{Q}$ on the set of functions $f \in \mathcal{D}_0$ with $f(x) = 1$. It turns out that this equilibrium potential is given by $1 \in \mathcal{D}_0$ if $\operatorname{cap}(x) = 0$ and by a multiple of a monopole at x, otherwise. This ultimately gives the equivalence of (vii) and (i). We note in passing that when the inequality which is excluded in (viii) is valid, it is known under the name of Hardy's inequality.

In the considerations above we have focused on understanding the (non-)degeneracy of the form $\mathcal{Q}$. We now turn to operators. Here, the (non-)degeneracy of $\mathcal{Q}$ on $\mathcal{D}_0$ is mirrored by the injectivity of $\mathcal{L}_m$ on $\mathcal{D}_0$. Indeed, as all constant functions are harmonic in the case of a connected graph b, the operator $\mathcal{L}_m$ is clearly injective on $C_c(X)$ and clearly not injective on $\mathcal{D}$. It turns out that the equivalence between the invertibility of $\mathcal{L}_m$ and the non-degeneracy of $\mathcal{Q}$ on $\mathcal{D}$ extends to $\mathcal{D}_0$. This connects (v) and (ix).

Quite remarkably, we are also able to phrase the degeneracy of $\mathcal{L}$ on $\mathcal{D}_0$ as a type of failure of surjectivity, i.e., the non-existence of monopoles, see (x). The property (x) can then be understood by taking a closer look at the failure of invertibility of $\mathcal{L}_m$, as discussed in (xi): We view the Green's function G as the kernel of the inverse of $\mathcal{L}_m$, where the value ∞ arises if and only if $\mathcal{L}_m$ is not invertible. The values of G can therefore be determined by solving the equality $\mathcal{L}_m u = 1_x$ for all $x \in X$ and by setting the solution to be ∞ if the equation is not solvable. Now, there are two further natural ways to compute the inverse. For one way, we take the limit $\alpha \to 0^+$ in $(L_m + \alpha)^{-1}$ and, for the other way, we take the limit $n \to \infty$ in $(L_{K_n}^{(D)})^{-1}$ for an exhaustion

K_n of X via finite sets. It turns out that all three procedures give the same result. This has nothing to do with recurrence but is generally true. Recurrence then is the phenomenon that all of these values limits explode. Of course, this is a strong version of non-invertibility.

We point out that the preceding discussion shows a close connection between monopoles, equilibrium potentials and the columns of the Green's function, i.e., the values when one variable is fixed. Indeed, the existence of monopoles, the existence of non-trivial equilibrium potentials and the finiteness of the Green's functions are all equivalent. Specifically, if valid, then the monopole at $x \in X$, the equilibrium potential for $x \in X$ and the column of the Green's function in $x \in X$ all agree up to a multiple.

We also mention a stochastic perspective on recurrence. While this is not the core focus of the book, it is nevertheless revealing and we include a short discussion here as well as in the last section of the chapter. In the stochastic interpretation, we are concerned with a particle jumping from the vertices according to rules coming from the edge weights. Now, there is a basic dichotomy: Either the particle leaves any compact set for good or it returns to any such set again and again. The latter property is what is captured by recurrence and is also responsible for the name. By its definition, the Green's function G_m can then be seen as a summation or rather integration over all of the returns. In particular, it describes an equilibrium situation and the infinite number of returns to any fixed point is then also responsible for the explosion of $G_m(x, y)$ given in (xi).

The remaining sections of this chapter are organized as follows: We first present in Section 1 some general preliminaries which are used throughout this chapter. We then turn to a study of aspects of recurrence related to the form $\mathcal{Q}$ as a (pseudo) inner product on $\mathcal{D}$ and $\mathcal{D}_0$ in Section 2. In Theorem 6.17 we prove the equivalence between (i), (ii), (v), (vi), (vii) and (viii) of Theorem 6.1. In the subsequent Section 3 we then deal with superharmonic functions and provide the equivalence between (i.b), (iv), (ix), (x) in Theorem 6.25. In Section 4 we deal with the aspects of recurrence related to the Green's function and prove the equivalence between (i.b) and (xi) in Theorem 6.28. In the following Section 5 we look at boundary terms and establish the equivalence between (i.b) and (iii) in Theorem 6.33. Finally, in Section 6 we discuss the connection to the stochastic perspective on recurrence.

1. General preliminaries

In this section we provide some elementary properties of the norm $\|\cdot\|_o$ on the spaces $\mathcal{D}_0$ and $\mathcal{D}$. These results will be used freely in the remaining parts of this chapter.

Excavation Exercises 6.1 and 6.2 recall basic facts about the existence of weakly convergent subsequences and the existence of unique minimizers in Hilbert spaces which will be used in this section.

We recall that the space $\mathcal{D}$ of functions of finite energy is defined via the form $\mathcal{Q}$. This form is positive, i.e., $\mathcal{Q}$ takes non-negative values on the diagonal but may be degenerate even if (b, c) is connected. This follows since $\mathcal{Q}(f) = 0$ for $f \in \mathcal{D}$ only implies that f is constant whenever b is connected. On the other hand $\mathcal{Q}(f) = 0$ does imply that $f = 0$ whenever b is connected and $c \neq 0$. Hence, $\mathcal{Q}$ on its own does not necessarily define a norm on $\mathcal{D}$.

Given this situation, a natural notion of convergence of a sequence (f_n) to $f \in \mathcal{D}$ is that $f_n \to f$ pointwise and $\mathcal{Q}(f - f_n) \to 0$. Indeed, the subspace $\mathcal{D}_0 \subseteq \mathcal{D}$ is defined as the closure of $C_c(X)$ with respect to this convergence, i.e., $\mathcal{D}_0$ denotes those functions in $\mathcal{D}$ that can be approximated pointwise and in terms of energy by functions in $C_c(X)$.

We will now show that this type of convergence can be phrased using an inner product. In doing so, we encounter the following issue: The convergence above is defined without reference to any distinguished vertex. The inner product we are about to define, on the other hand, will distinguish a vertex. However, it turns out that the induced notion of convergence is independent of this vertex whenever the graph is connected.

More specifically, if (b, c) is a connected graph over X and $o \in X$, then we define a bilinear map

$$\langle \cdot, \cdot \rangle_o \colon \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{R}$$

by

$$\langle f, g \rangle_o = \mathcal{Q}(f, g) + f(o)g(o).$$

The connectedness of (b, c) easily implies that $\langle \cdot, \cdot \rangle_o$ is an inner product for any $o \in X$. The associated norm is then given by

$$\|f\|_o = \left(\mathcal{Q}(f) + f^2(o)\right)^{1/2}.$$

We now collect some basic properties of this norm.

LEMMA 6.3. *Let (b, c) be a connected graph over X.*

(a) *For all $x, y \in X$ and $f \in \mathcal{D}$ there exists a $C(x, y) \geq 0$ such that*

$$(f(x) - f(y))^2 \leq C(x, y)\mathcal{Q}(f).$$

(b) *The norms $\|\cdot\|_o$ and $\|\cdot\|_{o'}$ on $\mathcal{D}$ are equivalent for all $o, o' \in X$.*

(c) *$\mathcal{D}_0$ is the closure of $C_c(X)$ in $\mathcal{D}$ with respect to $\|\cdot\|_o$ for an arbitrary $o \in X$, i.e.,*

$$\mathcal{D}_0 = \overline{C_c(X)}^{\|\cdot\|_o}.$$

(d) *$(\mathcal{D}, \langle \cdot, \cdot \rangle_o)$ and $(\mathcal{D}_0, \langle \cdot, \cdot \rangle_o)$ are Hilbert spaces.*

(e) *The point evaluation maps given by*

$$\delta_x \colon (\mathcal{D}, \|\cdot\|_o) \longrightarrow \mathbb{R}, \quad \delta_x(f) = f(x)$$

are continuous for every $x \in X$.

PROOF. (a) Let $x, y \in X$ and let $x = x_0 \sim \ldots \sim x_n = y$ be a path from x to y. We estimate by a telescoping sum argument and the Cauchy–Schwarz inequality

$$\begin{aligned}
|f(x)-&f(y)| \\
&\leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \\
&= \sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})^{1/2}} b(x_i, x_{i+1})^{1/2} |f(x_i) - f(x_{i+1})| \\
&\leq \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})} \right)^{1/2} \left(\sum_{i=0}^{n-1} b(x_i, x_{i+1})(f(x_i) - f(x_{i+1}))^2 \right)^{1/2} \\
&\leq \left(\sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})} \right)^{1/2} \mathcal{Q}^{1/2}(f).
\end{aligned}$$

Hence, taking

$$C(x,y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{i=0}^{n-1} \frac{1}{b(x_i, x_{i+1})}$$

we conclude the statement.

(b) Let $o, o' \in X$. Then, by (a) and $s^2 \leq 2(s-t)^2 + 2t^2$, which follows from $(s-2t)^2 \geq 0$ for $s, t \in \mathbb{R}$, we get

$$f^2(o) \leq 2(f(o) - f(o'))^2 + 2f^2(o') \leq 2C(o,o')\mathcal{Q}(f) + 2f^2(o').$$

Hence,

$$\|f\|_o^2 = f^2(o) + \mathcal{Q}(f) \leq (2C(o,o') + 1)\mathcal{Q}(f) + 2f^2(o') \leq C' \|f\|_{o'}^2$$

for $C' = (2C(o,o') + 1) \vee 2$. The symmetry of the argument above yields the conclusion.

(c) Clearly $\mathcal{D}_0 \subseteq \overline{C_c(X)}^{\|\cdot\|_o}$ for an arbitrary $o \in X$ by the definition of $\mathcal{D}_0$. On the other hand, if $\|f - \varphi_n\|_o \to 0$ as $n \to \infty$ for some (φ_n) in $C_c(X)$ and $f \in \mathcal{D}$, then $\varphi_n(o) \to f(o)$ and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$. Since for arbitrary $x \in X$ the norms $\|\cdot\|_o$ and $\|\cdot\|_x$ are equivalent, we obtain that $\varphi_n(x) \to f(x)$ for all $x \in X$ and thus $f \in \mathcal{D}_0$. Hence, $D_0 = \overline{C_c(X)}^{\|\cdot\|_o}$.

(d) Clearly, the norm $\|\cdot\|_o$ is associated with the scalar product

$$\langle f, g \rangle_o = \mathcal{Q}(f,g) + f(o)g(o)$$

for $f, g \in \mathcal{D}$. The completeness of $\mathcal{D}$ with respect to $\|\cdot\|_o$ follows from the lower semi-continuity of $\mathcal{Q}$, Proposition 1.3. The completeness of $\mathcal{D}_0$ then follows by part (c).

(e) This follows directly from (b) which gives that convergence in $\|\cdot\|_o$ implies pointwise convergence. $\square$

Remark (Resistance metrics). From (a) of the previous lemma we conclude that

$$r(x, y) = \inf\{C \geq 0 \mid |f(x) - f(y)| \leq C\mathcal{Q}^{1/2}(f) \text{ for all } f \in \mathcal{D}\}$$

is finite for all $x, y \in X$. Now, clearly

$$r(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

and r is a metric. In fact, it is possible to show that r^2 is also a metric. Likewise we may define the metric r_0 by replacing $\mathcal{D}$ with $\mathcal{D}_0$ (or even $C_c(X)$) in the above formulae. The metrics r^2 and r_0^2 are known as resistance metrics (Exercise 6.9).

Remark (Decomposing $\mathcal{D}$ for $c = 0$). Lemma 6.3 shows that the norms $\|\cdot\|_o$ and $\|\cdot\|_{o'}$ are equivalent. One way to understand this for connected graphs b over X is the following: Let $o \in X$ and define $\mathcal{D}_o = \{f \in \mathcal{D} \mid f(o) = 0\}$, which should not be confused with $\mathcal{D}_0$. Then, $\mathcal{Q}$ is an inner product on $\mathcal{D}_o$ and $(\mathcal{D}_o, \mathcal{Q})$ is a Hilbert space. It is not hard to see that

$$\mathcal{D} = \mathcal{D}_o \oplus \mathrm{Lin}\{1\},$$

where Lin stands for the linear hull. On the other hand, the map

$$\mathcal{D}_o \longrightarrow \mathcal{D}/\mathrm{Lin}\{1\}, \quad f \mapsto [f],$$

is bijective and isometric if $\mathcal{D}_o$ is equipped with $\mathcal{Q}$ and $\mathcal{D}/\mathrm{Lin}\{1\}$ is equipped with $\mathcal{Q}([f]) = \mathcal{Q}(f)$. This shows that the inner product on $\mathcal{D}_o$ is in a certain sense independent of $o \in X$ (Exercise 6.10).

As a consequence of the lemma above we obtain the desired characterization of convergence with respect to $\|\cdot\|_o$.

Corollary 6.4 (Convergence with respect to $\|\cdot\|_o$). *Let (b, c) be a connected graph over X and let $o \in X$. Then, for functions f_n and f in $\mathcal{D}$, $f_n \to f$ with respect to $\|\cdot\|_o$ if and only if $f_n \to f$ pointwise and $\mathcal{Q}(f - f_n) \to 0$.*

It turns out that even the following holds.

Lemma 6.5. *Let (b, c) be a connected graph over X and let $o \in X$. Let $f \in \mathcal{D}$ and (f_n) be a sequence in $\mathcal{D}$. Then, $f_n \to f$ with respect to $\|\cdot\|_o$ if and only if $f_n \to f$ pointwise and*

$$\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f).$$

PROOF. The "only if" direction follows directly from Corollary 6.4 above. For the other implication, if (f_n) converges pointwise to f and $\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f)$, then (f_n) is a bounded sequence in $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$. As $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$ is a Hilbert space by Lemma 6.3 (d), every ball is weakly compact, so there exists a weakly convergent subsequence of (f_n). By the pointwise convergence of (f_n), we deduce that the limits of all possible weakly converging subsequences coincide, i.e., there exists only one accumulation point. Hence, (f_n) converges weakly. Now,

$$0 \leq \|f - f_n\|_o^2 = \mathcal{Q}(f) + \mathcal{Q}(f_n) + f^2(o) + f_n^2(o) - 2\langle f, f_n\rangle_o.$$

Thus, invoking the assumption $\limsup_{n\to\infty} \mathcal{Q}(f_n) \leq \mathcal{Q}(f)$ we find that the right-hand side converges to 0 and this gives $\|f - f_n\|_o \to 0$. $\square$

We now show that convergence in $\|\cdot\|_o$ respects taking maxima and minima. In particular, we show that we can approximate positive functions in $\mathcal{D}_0$ monotonically from below by functions in $C_c(X)$. We recall that

$$u_+ = u \vee 0 \quad \text{ and } \quad u_- = -u \vee 0$$

so that $u = u_+ - u_-$.

LEMMA 6.6 (Bounded and monotone approximation). *Let (b, c) be a graph over X. Let $u \in \mathcal{D}$ and (u_n) be a sequence in $\mathcal{D}$ such that $\|u - u_n\|_o \to 0$ as $n \to \infty$.*

(a) *Then, $v_n = -u_- \vee u_n \wedge u_+$ also satisfies $\|u - v_n\|_o \to 0$ as $n \to \infty$.*
(b) *Furthermore, if $u \geq 0$ and (u_n) consists of functions in $C_c(X)$, then there exists a sequence (φ_n) consisting of functions in $C_c(X)$ which is monotonically increasing such that $0 \leq \varphi_n \leq u$ and $\|u - \varphi_n\|_o \to 0$ as $n \to \infty$. On the set where $u > 0$, we can even choose φ_n such that $0 \leq \varphi_n < u$.*

PROOF. (a) We let $f_n = u - u_n$ and $g_n = u - v_n$. First, we note that $f_n \to 0$ pointwise. Thus,

$$g_n = -u_- \vee (u - u_n) \wedge u_+$$

also converges to 0 pointwise and $|g_n| \leq |f_n|$.

Next we show

$$|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)| \vee |u(x) - u(y)|$$

as follows: First, if $|g_n(x) - g_n(y)| > |u(x) - u(y)|$, then $u(x)$ and $u(y)$ must have the same sign since $|g_n| \leq |u|$, so without loss of generality we may assume $u(x) \geq u(y) \geq 0$.

Now, if $g_n(x) > g_n(y) \geq 0$, then $g_n(x) - g_n(y) > u(x) - u(y)$ yields $g_n(x) > u(x) - u(y) \geq 0$ and $g_n(y) < u(y)$. Therefore,

$$g_n(x) = f_n(x) \wedge u(x) \leq f_n(x)$$
$$g_n(y) = 0 \vee f_n(y) \wedge u(y) \geq f_n(y),$$

which imply $|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)|$.

The remaining case of $g_n(y) > g_n(x)$ similarly leads to $g_n(x) < 2u(y) - u(x)$ and $g_n(y) > u(x) - u(y)$. These imply $g_n(x) \geq f_n(x)$ and $g_n(y) \leq f_n(y)$, resulting in $|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)|$, which was to be shown.

Since $u \in \mathcal{D}$, we have

$$\frac{1}{2} \sum_{x,y \in X} b(x,y)(u(x) - u(y))^2 < \infty.$$

This means that for every $\varepsilon > 0$ there exists a finite set $E_\varepsilon \subseteq X \times X$ such that

$$\frac{1}{2} \sum_{(x,y) \notin E_\varepsilon} b(x,y)(u(x) - u(y))^2 < \varepsilon.$$

Using $|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)| \vee |u(x) - u(y)|$, we can now easily prove

$$\frac{1}{2} \sum_{(x,y) \notin E_\varepsilon} b(x,y)(g_n(x) - g_n(y))^2 \leq \frac{1}{2} \sum_{x,y \in X} b(x,y)(f_n(x) - f_n(y))^2 + \varepsilon.$$

Combining the above statements, we get since $|g_n| \leq |f_n|$

$$\mathcal{Q}(u - v_n) = \mathcal{Q}(g_n) \leq \frac{1}{2} \sum_{(x,y) \in E_\varepsilon} b(x,y)(g_n(x) - g_n(y))^2 + \mathcal{Q}(f_n) + \varepsilon.$$

Now, $\sum_{(x,y) \in E_\varepsilon} b(x,y)(g_n(x) - g_n(y))^2 \to 0$ because E_ε is finite and $g_n \to 0$ pointwise. Furthermore, $\mathcal{Q}(f_n) \to 0$ by assumption and since $\varepsilon > 0$ was chosen arbitrarily, we have $\mathcal{Q}(u - v_n) \to 0$ as $n \to \infty$.

(b) Let (v_n) be given as in (a), which consists of functions in $C_c(X)$ whenever (u_n) consists of functions in $C_c(X)$ and satisfies $v_n \geq 0$ as $u \geq 0$. We choose (η_n) in $C_c(X)$ with $\operatorname{supp} \eta_n = \operatorname{supp} v_n$, $0 \leq \eta_n \leq v_n$ on $\operatorname{supp} \eta_n$ with $\eta_n \to 0$ in $\|\cdot\|_o$ as $n \to \infty$. For example, η_n can be chosen as

$$\eta_n = \frac{\delta_n}{n(\mathcal{Q}(\delta_n) \vee 1)^{1/2}} \qquad \text{with} \qquad \delta_n = \left(\min_{x \in \operatorname{supp} v_n} \frac{v_n(x)}{2} \right) 1_{\operatorname{supp} v_n}.$$

Note that $0 < \eta_n < v_n$ if $v_n \neq 0$. Then, $v_n - \eta_n$ is such that $0 \leq v_n - \eta_n \leq u$ with a strict inequality on the set where $u > 0$ and when $v_n \neq 0$ and satisfies $v_n - \eta_n \to u$ in $\|\cdot\|_o$.

Furthermore, we extract a monotonically increasing subsequence (φ_{n_k}) of $(v_n - \eta_n)$ as follows. Let $n_0 = 0$. Given n_k let

$$n_{k+1} = \min\{l > n_k \mid (v_l - \eta_l)(x) \geq (v_{n_k} - \eta_{n_k})(x) \text{ for all } x \in X\}.$$

The minimum exists as $v_n - \eta_n < u$ on $\operatorname{supp} v_n$, $v_n - \eta_n \to u$ as $n \to \infty$ pointwise and $v_{n_k} \in C_c(X)$. By definition (φ_{n_k}) is monotonically increasing, $\varphi_{n_k} \to u$ and $\mathcal{Q}(\varphi_{n_k} - u) \to 0$ as $n \to \infty$. $\square$

Next we show that the space $\mathcal{D}_0$ is invariant under normal contractions.

LEMMA 6.7 ($\mathcal{D}_0$ is invariant under normal contractions). *Let (b,c) be a graph over X. If $f \in \mathcal{D}_0$, then $C \circ f \in \mathcal{D}_0$ for every normal contraction C.*

PROOF. Let $f \in \mathcal{D}_0$ and let m be a measure such that $f \in \ell^2(X,m)$. Then, $f \in \mathcal{D}_0 \cap \ell^2(X,m) = D(Q_m^{(D)})$ by Theorem 1.19. Since $Q_m^{(D)}$ is a Dirichlet form by Theorem 1.18, we have $C \circ f \in D(Q_m^{(D)}) \subseteq \mathcal{D}_0$ for every normal contraction. This proves the statement. $\square$

REMARK. Indeed, we can invoke the proof of Theorem 1.19 directly to prove the statement above.

The preceding lemmas allow us to easily prove a Green's formula on $\mathcal{D}$. In fact, as usual in measure theory, there are two versions of Green's formula. One can be thought of as an ℓ^1 version and the other is a version for positive functions.

LEMMA 6.8 (Green's formula on $\mathcal{D}$). *Let (b,c) be a graph over X. Let $v \in \mathcal{D}_0$ and $u \in \mathcal{D}$ with either*
- *$\mathcal{L}u \geq 0$ or*
- *$\sum_{x\in X} |\mathcal{L}u(x)| < \infty$ and $v \in \ell^\infty(X)$.*

Then, we have
$$\mathcal{Q}(u,v) = \sum_{x\in X} \mathcal{L}u(x)v(x)$$
with absolutely converging sum.

PROOF. For $v = \varphi \in C_c(X)$ this is clear from the Green's formula presented in Proposition 1.4. We note that this does not need any assumption on u except for $u \in \mathcal{D}$. Under the assumption $\mathcal{L}u \geq 0$ we can split v into v_+ and v_-, which are functions in $\mathcal{D}_0$ by Lemma 6.7. Thus, we can apply Lemma 6.6 (b) to approximate v_+ and v_- monotonically by functions in $C_c(X)$. Hence, we obtain
$$\mathcal{Q}(u,v_\pm) = \sum_{x\in X} \mathcal{L}u(x)v_\pm(x),$$
where the right-hand sides are finite since the left-hand sides are. Then, subtracting the terms yields the statement.

Under the assumptions $\sum_{x\in X} |\mathcal{L}u(x)| < \infty$ and $v \in \ell^\infty(X)$ we approximate v with functions in $C_c(X)$ which are smaller in modulus than v, which is possible by Lemma 6.6 (a). Then, the statement follows from Lebesgue's dominated convergence theorem. $\square$

As another direct consequence of the Hilbert space methods applied to $\langle\cdot,\cdot\rangle_o$ we give a construction and basic properties of the equilibrium potential at a vertex. We recall that the capacity is defined as an infimum over finitely supported functions. The result below shows that this infimum is achieved by a function in $\mathcal{D}_0$.

PROPOSITION 6.9 (Existence of equilibrium potentials). *Let* (b, c) *be a connected graph and let* $x \in X$. *Then,*

$$\operatorname{cap}(x) = \inf\{\mathcal{Q}(f) \mid f \in \mathcal{D}_0, f(x) = 1\}$$

and there exists a unique $u \in \mathcal{D}_0$ *with* $u(x) = 1$ *and* $\mathcal{Q}(u) = \operatorname{cap}(x)$. *Furthermore,* $0 \leq u \leq 1$.

PROOF. We start by showing the equality. By definition, the capacity of x is given as

$$\inf\{\mathcal{Q}(\varphi) \mid \varphi \in C_c(X), \varphi(x) = 1\}.$$

On the other hand, if $f \in \mathcal{D}_0$ with $f(x) = 1$, then there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\mathcal{Q}(f - \varphi_n) \to 0$ as $n \to \infty$. We can then assume without loss of generality that $\varphi_n(x) = 1$ for all $n \in \mathbb{N}$. Combining these statements gives the equality.

It remains to show the statement on the minimizer. Consider the set

$$\mathcal{A} = \{f \in \mathcal{D}_0 \mid f(x) = 1\}.$$

This is clearly a convex closed set in $(\mathcal{D}_0, \langle \cdot, \cdot \rangle_x)$. Hence, there is a unique minimizer of $\| \cdot \|_x$ on $\mathcal{A}$. Since $f(x) = 1$ for all $f \in \mathcal{A}$, this is then the unique minimizer of $\mathcal{Q}$ on $\mathcal{A}$ and the desired statement follows.

Finally, to show $0 \leq u \leq 1$ for the minimizer u we note that $\mathcal{Q}$ is compatible with normal contractions and $\mathcal{D}_0$ is closed under normal contraction by Lemma 6.7. Therefore, $(0 \vee u) \wedge 1$ is also a minimizer and we obtain $u = (0 \vee u) \wedge 1$ by uniqueness. $\square$

Given the previous proposition we can now provide the following definition of the equilibrium potential, which was already mentioned in the introduction to this chapter.

DEFINITION 6.10 (Equilibrium potential). The unique function $u \in \mathcal{D}_0$ with $u(x) = 1$ and $\mathcal{Q}(u) = \operatorname{cap}(x)$ is called the *equilibrium potential* for $x \in X$.

2. The form perspective

In this section we start our investigation of recurrence. In particular, we focus on the form perspective. This means we consider those properties which can be stated in terms of the space $\mathcal{D}_0$ equipped with semi-inner product $\mathcal{Q}$ and the associated semi-norm $\mathcal{Q}^{1/2}$. Our overall strategy is to show that certain assertions follow (rather easily) from $1 \in \mathcal{D}_0$ and that the opposite assertions follow (again rather easily) from $1 \notin \mathcal{D}_0$. Put together this establishes the desired equivalences for recurrence.

Before we deal with the finer properties of the space $\mathcal{D}_0$ equipped with $\mathcal{Q}$, we first address the question whether $\mathcal{D}_0$ and $\mathcal{D}$ agree. As, clearly, the constant function 1 belongs to $\mathcal{D}$, a necessary condition for equality of $\mathcal{D}_0$ and $\mathcal{D}$ is that 1 belongs to $\mathcal{D}_0$. Quite remarkably the converse is also true.

PROPOSITION 6.11 ($1 \in \mathcal{D}_0$ implies $\mathcal{D} = \mathcal{D}_0$). *Let b be a graph over X. If $1 \in \mathcal{D}_0$, then $\mathcal{D} = \mathcal{D}_0$.*

PROOF. It follows by definition and Lemma 6.6 (a) that if $1 \in \mathcal{D}_0$, then we can choose a sequence (e_n) in $C_c(X)$ with $e_n \to 1$ pointwise and $\mathcal{Q}(1 - e_n) \to 0$ as $n \to \infty$ such that $0 \leq e_n \leq 1$ for $n \in \mathbb{N}_0$.

Let $f \in \mathcal{D} \cap \ell^\infty(X)$. Then, $e_n f \in C_c(X)$ and by the simple algebraic manipulation found in Lemma 2.25 we get

$$\begin{aligned}
\mathcal{Q}(f - e_n f) &= \mathcal{Q}(f(1 - e_n)) \\
&\leq \sum_{x \in X} (1 - e_n(x))^2 \sum_{y \in X} b(x,y)(f(x) - f(y))^2 \\
&\qquad + \sum_{y \in X} f(y)^2 \sum_{x \in X} b(x,y)(e_n(x) - e_n(y))^2 \\
&\leq \sum_{x \in X} (1 - e_n(x))^2 \sum_{y \in X} b(x,y)(f(x) - f(y))^2 + 2\|f\|_\infty^2 \mathcal{Q}(e_n) \\
&\to 0,
\end{aligned}$$

where we use Lebesgue's dominated convergence theorem for the first term, which is applicable since $f \in \mathcal{D}$ and $\mathcal{Q}(e_n) \to 0$ as $n \to \infty$ for the second term. Therefore, $\mathcal{D} \cap \ell^\infty(X) \subseteq \mathcal{D}_0$.

Now, an arbitrary function $f \in \mathcal{D}$ can be approximated by the bounded functions $f_k = -k \vee f \wedge k$ for $k \in \mathbb{N}$. We show that $f_k \to f$ in $\|\cdot\|_o$ as $k \to \infty$. Clearly $f_k \to f$ pointwise as $k \to \infty$. By Fatou's lemma and the fact that $\mathcal{Q}$ is compatible with normal contractions, we have

$$\mathcal{Q}(f) \leq \liminf_{k \to \infty} \mathcal{Q}(f_k) \leq \limsup_{k \to \infty} \mathcal{Q}(f_k) \leq \mathcal{Q}(f).$$

Hence, Lemma 6.5 implies $f_k \to f$ in $\|\cdot\|_o$ as $k \to \infty$. Thus, by combining the two convergence arguments given above, we get that $\mathcal{D} = \mathcal{D}_0$. □

The previous result suggests that we have a closer look at the condition $1 \in \mathcal{D}_0$. By the definition of $\mathcal{D}_0$, one rather easily finds the following characterization.

LEMMA 6.12 (Approximating 1). *Let b be a graph over X. Then, the following assertions are equivalent:*

(i) $1 \in \mathcal{D}_0$.
(ii) *There exists a sequence (φ_n) in $C_c(X)$ with $0 \leq \varphi_n \leq 1$ for $n \in \mathbb{N}$ such that $\varphi_n \to 1$ pointwise and $\mathcal{Q}(\varphi_n) \to 0$ as $n \to \infty$.*

(iii) *There exists a sequence* (φ_n) *in* $C_c(X)$ *with* $\varphi_n \to 1$ *as* $n \to \infty$ *pointwise and* $\sup_{n\in\mathbb{N}} \mathcal{Q}(\varphi_n) < \infty$.

Proof. (i) $\Longrightarrow$ (ii): If 1 belongs to $\mathcal{D}_0$ there exists a sequence (ψ_n) in $C_c(X)$ with $\psi_n \to 1$ as $n \to \infty$ pointwise which is a Cauchy sequence with respect to $\mathcal{Q}^{1/2}$. Hence,

$$\mathcal{Q}(\psi_n) = \mathcal{Q}(\psi_n - 1) \leq \liminf_{k\to\infty} \mathcal{Q}(\psi_n - \psi_k) \to 0$$

as $n \to \infty$. The function $\varphi_n = (1 \wedge \psi_n) \vee 0$ satisfies $0 \leq \varphi_n \leq 1$ for $n \in \mathbb{N}$. Moreover, as (ψ_n) converges to 1 pointwise, so does (φ_n). Finally, as $\mathcal{Q}$ is compatible with normal contractions we find

$$\mathcal{Q}(\varphi_n) \leq \mathcal{Q}(\psi_n) \to 0$$

as $n \to \infty$. This gives (ii).

(ii) $\Longrightarrow$ (iii): This is clear.

(iii) $\Longrightarrow$ (i): Without loss of generality we can assume that the graph is connected as otherwise we work on each connected component separately. Fix an arbitrary $o \in X$. By assumption, the sequence (φ_n) is bounded in the Hilbert space $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$. Hence, without loss of generality, we can assume that it converges weakly to some $u \in \mathcal{D}$ as otherwise we could pass to a subsequence. Invoking the Banach–Saks theorem, we then obtain a sequence ψ_n in $C_c(X)$ consisting of finite convex combinations of the φ_n with $\psi_n \to u$ in the Hilbert space $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$. Note that the functions ψ_n must converge pointwise to 1 as the φ_n have this property. As point evaluation is continuous on the Hilbert space $(\mathcal{D}, \langle\cdot,\cdot\rangle_o)$ by Lemma 6.3 (e), we find $\psi_n(x) \to u(x)$ for $x \in X$. This gives $u = 1$ and finishes the proof. $\square$

Having given a characterization of $1 \in \mathcal{D}_0$, which implies $\mathcal{D}_0 = \mathcal{D}$, we now turn to some basic questions concerning the space $\mathcal{D}_0$ equipped with the semi-inner product $\mathcal{Q}$. As it may add a useful perspective to keep in mind, we first discuss which questions we consider.

A very natural question is whether $\mathcal{Q}$ is actually an inner product and, if so, whether $(\mathcal{D}_0, \mathcal{Q})$ is complete, i.e., a Hilbert space. This is clearly related to the question whether the seminorm $\mathcal{Q}^{1/2}$ and $\|\cdot\|_o$ are equivalent.

Another natural question concerns lower bounds for $\mathcal{Q}$. Here, we say that $w\colon X \longrightarrow [0,\infty)$ is a *lower bound* for $\mathcal{Q}$, written as $\mathcal{Q} \geq w$, if

$$\mathcal{Q}(\varphi) \geq \sum_{x\in X} w(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$. We can extend this inequality to $\mathcal{D}_0$. Indeed, by approximating an arbitrary $f \in \mathcal{D}_0$ with respect to $\mathcal{Q}$ by $\varphi_n \in C_c(X)$ we obtain from Fatou's lemma

$$\mathcal{Q}(f) = \lim_{n\to\infty} \mathcal{Q}(\varphi_n) \geq \liminf_{n\to\infty} \sum_{x\in X} w(x)\varphi_n^2(x) \geq \sum_{x\in X} w(x) f^2(x).$$

We note that $\operatorname{cap}(x)1_x$ is clearly a lower bound for $\mathcal{Q}$ as $\operatorname{cap}(x)\varphi^2(x) \leq \mathcal{Q}(\varphi)$ for all $\varphi \in C_c(X)$ and, conversely, whenever w is a lower bound for $\mathcal{Q}$ we must have $w(x) \leq \operatorname{cap}(x)$ for all $x \in X$.

Another very natural question is whether the *point evaluation map* $\delta_x \colon \mathcal{D}_0 \longrightarrow \mathbb{R}$ given by

$$\delta_x(f) = f(x)$$

is continuous for $x \in X$ with respect to $\mathcal{Q}^{1/2}$. We note that the answer to this question is given by positivity of the capacity, as we now show.

LEMMA 6.13 (Characterization of positive capacity). *Let b be a graph over X and let $x \in X$. Then, the point evaluation δ_x is continuous on $\mathcal{D}_0$ with respect to $\mathcal{Q}^{1/2}$ if and only if* $\operatorname{cap}(x) > 0$. *In this case,* $\|\delta_x\| = \operatorname{cap}(x)^{-1/2}$.

PROOF. Clearly, continuity of δ_x on $\mathcal{D}_0$ with respect to $\mathcal{Q}^{1/2}$ is equivalent to continuity of δ_x on $C_c(X)$ with respect to $\mathcal{Q}^{1/2}$ and this is equivalent to the finiteness of

$$\begin{aligned}\|\delta_x\| &= \sup\{|\varphi(x)| \mid \varphi \in C_c(X) \text{ with } \mathcal{Q}(\varphi) \leq 1\} \\ &= \sup\left\{\frac{|\varphi(x)|}{\mathcal{Q}^{1/2}(\varphi)} \mid 0 \neq \varphi \in C_c(X)\right\}.\end{aligned}$$

On the other hand, positivity of the capacity is equivalent to

$$\begin{aligned}0 < \operatorname{cap}(x) &= \inf\{\mathcal{Q}(\varphi) \mid \varphi \in C_c(X) \text{ with } \varphi(x) = 1\} \\ &= \inf\left\{\frac{\mathcal{Q}(\varphi)}{\varphi^2(x)} \mid \varphi(x) \neq 0\right\}.\end{aligned}$$

Now, the equivalence follows easily. □

Having discussed the questions we have in mind, we now gather some rather simple consequences for these questions if $1 \in \mathcal{D}_0$. As shown subsequently, each of these consequences is actually a characterization of $1 \in \mathcal{D}_0$ provided that the graph is connected.

PROPOSITION 6.14 (Consequences of $1 \in \mathcal{D}_0$). *Let b be a graph over X. If $1 \in \mathcal{D}_0$, then the following statements hold:*

(a) *The point evaluation map $\delta_x \colon \mathcal{D}_0 \longrightarrow \mathbb{R}$, $\delta_x(f) = f(x)$ is not continuous with respect to $\mathcal{Q}^{1/2}$ for all $x \in X$.*
(b) $\operatorname{cap}(x) = 0$ *for all $x \in X$.*
(c) *The norms $\mathcal{Q}^{1/2}$ and $\|\cdot\|_o$ are not equivalent on $C_c(X)$.*
(d) *If $\mathcal{Q} \geq w$ for some $w \geq 0$, then $w = 0$.*
(e) *$(\mathcal{D}_0, \mathcal{Q})$ is degenerate and, in particular, not a Hilbert space.*

PROOF. (a) If $1 \in \mathcal{D}_0$, then the point evaluation map on $\mathcal{D}_0$ cannot be continuous for each $x \in X$ since we have $1 = 1(x)$ whereas $0 = \mathcal{Q}(1)$.

(b) From Lemma 6.13 we know already that (a) and (b) are equivalent. Alternatively, it is not hard to argue directly as follows: By $1 \in \mathcal{D}_0$, there exists a sequence (φ_n) in $C_c(X)$ with $0 \leq \varphi_n \leq 1$ for

$n \in \mathbb{N}$ which converges pointwise to 1 and satisfies $\mathcal{Q}(\varphi_n) \to 0$ as $n \to \infty$. Assuming that $\varphi_n(x) \neq 0$ for all $n \in \mathbb{N}$, it follows that $\psi_n = \varphi_n/\varphi_n(x)$ belongs to $C_c(X)$ and satisfies $\psi_n(x) = 1$ for all $n \in \mathbb{N}$. Hence, from the definition of the capacity we obtain

$$\operatorname{cap}(x) \leq \mathcal{Q}(\psi_n) = \frac{1}{\varphi_n^2(x)}\mathcal{Q}(\varphi_n) \to 0$$

as $n \to \infty$, which completes the proof.

(c) If the norms are equivalent on $C_c(X)$, then they have to be equivalent on $\mathcal{D}_0$ as well. This, however, is not true, as can be seen by considering the function 1.

(d) This follows easily by plugging in 1.

(e) This is clear as $\mathcal{Q}(1) = 0$. □

To show that each of the preceding properties in fact characterizes that 1 belongs to $\mathcal{D}_0$ we need one more ingredient. This is given by the following lemma, which holds for connected graphs.

Lemma 6.15 (Consequence of zero capacity). *Let b be a connected graph over X. If there exists an $x \in X$ with $\operatorname{cap}(x) = 0$, then $1 \in \mathcal{D}_0$. In particular, $\operatorname{cap}(y) = 0$ for all $y \in X$.*

Proof. By the existence of an equilibrium potential for x, Proposition 6.9, there exists a unique $u \in \mathcal{D}_0$ with $u(x) = 1$ and $\operatorname{cap}(x) = \mathcal{Q}(u)$. Thus, u satisfies $\mathcal{Q}(u) = 0$. As b is connected this implies that u is constant and by $u(x) = 1$ we infer $1 = u \in \mathcal{D}_0$. That $\operatorname{cap}(y) = 0$ for all $y \in X$ follows from Proposition 6.14 (b) directly above. □

Remark. It is also possible to give a direct proof of the previous result without the use of equilibrium potentials (Exercise 6.11).

Proposition 6.16 (Consequences of $1 \notin \mathcal{D}_0$). *Let b be a connected graph over X. If $1 \notin \mathcal{D}_0$, then the following statements hold:*

(a) *The point evaluation map $\delta_x \colon \mathcal{D}_0 \longrightarrow \mathbb{R}$, $\delta_x(f) = f(x)$ is continuous with respect to $\mathcal{Q}^{1/2}$ for all $x \in X$.*
(b) $\operatorname{cap}(x) > 0$ *for any $x \in X$.*
(c) *The norms $\mathcal{Q}^{1/2}$ and $\|\cdot\|_o$ are equivalent.*
(d) *There exists a $w \geq 0$, $w \neq 0$, with $\mathcal{Q} \geq w$. In fact, there even exists a $w > 0$ with $\mathcal{Q} \geq w$.*
(e) $(\mathcal{D}_0, \mathcal{Q})$ *is a Hilbert space.*

Proof. (a)/(b) By Lemma 6.13 we know that (a) and (b) are equivalent. By Lemma 6.15, $1 \notin \mathcal{D}_0$ implies (b).

(c) This is clear from the fact that $\mathcal{Q}^{1/2} \leq \|\cdot\|_o$ and from (b) as

$$\operatorname{cap}(o)\|\varphi\|_o^2 = \operatorname{cap}(o)\varphi^2(o) + \operatorname{cap}(o)\mathcal{Q}(\varphi) \leq (1 + \operatorname{cap}(o))\mathcal{Q}(\varphi)$$

for all $\varphi \in C_c(X)$.

(d) This is also clear from (b). Indeed, $\mathrm{cap}(x)1_x$ is a possible w and so is then any sum of the form $\sum_{x\in X} a_x \mathrm{cap}(x)1_x$ with $a_x > 0$ for all $x \in X$ and $\sum_{x\in X} a_x = 1$ as $\mathrm{cap}(x) > 0$ for all $x \in X$ by Lemma 6.15.

(e) As point evaluation is continuous with respect to $\mathcal{Q}^{1/2}$ by (a), $\mathcal{Q}$ is non-degenerate, i.e., an inner product on $\mathcal{D}_0$. The completeness of $\mathcal{D}_0$ with respect to $\mathcal{Q}^{1/2}$ is clear from the equivalence of the norms $\mathcal{Q}^{1/2}$ and $\|\cdot\|_o$ from part (c) and the fact that $\mathcal{D}_0$ is complete with respect to $\|\cdot\|_o$ established in Lemma 6.3. □

As a consequence of the considerations so far we obtain the following list of equivalences, which form part of our main characterization, Theorem 6.1.

THEOREM 6.17 (Characterization of recurrence – forms). *Let b be a connected graph over X. Then, the following statements are equivalent:*

(i) $D(Q_m^{(D)}) = D(Q_m^{(N)})$ *for all measures m.*
  (i.a) $\mathcal{D}_0 = \mathcal{D}$.
  (i.b) $1 \in \mathcal{D}_0$.
  (i.c) *There exists $u \in \mathcal{D}_0$ and a finite set $K \subseteq X$ with*

$$\inf_{x\in X\setminus K} u(x) > 0.$$

  (i.d) *There exists a sequence of functions (e_n) in $C_c(X)$ with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $\mathcal{Q}(e_n) \to 0$ and $e_n \to 1$ pointwise as $n \to \infty$.*
  (i.e) *There exists a sequence of functions (e_n) in $C_c(X)$ with $e_n \to 1$ pointwise as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \mathcal{Q}(e_n) < \infty$.*

(ii) $D(L_m^{(D)}) = \{f \in D(Q_m^{(N)}) \mid \mathcal{L}_m f \in \ell^2(X,m)\}$ *for all measures m.*

(v) $(\mathcal{D}_0, \mathcal{Q})$ *is not a Hilbert space.*
  (v.a) $\mathcal{Q}$ *is degenerate on* $\mathcal{D}_0$.
  (v.b) $\mathcal{Q}^{1/2}$ *and $\|\cdot\|_o$ are not equivalent norms on $C_c(X)$ for some (all) $o \in X$.*

(vi) *The point evaluation map $\delta_x\colon (\mathcal{D}_0, \mathcal{Q}) \longrightarrow \mathbb{R}$ given by $\delta_x(f) = f(x)$ is not continuous for some (all) $x \in X$.*

(vii) $\mathrm{cap}(x) = 0$ *for some (all) $x \in X$.*
  (vii.a) *The equilibrium potential for some (all) $x \in X$ is given by the constant function* 1.

(viii) *There does not exist a non-trivial positive function $w \in C(X)$ such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x\in X} w(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$. (*"Hardy's inequality"*) (viii.a) *There does not exist a strictly positive function* $w \in C(X)$ *such that*

$$\mathcal{Q}(\varphi) \geq \sum_{x \in X} w(x)\varphi^2(x)$$

for all $\varphi \in C_c(X)$.

(viii.b) $\lambda_0(L_m^{(D)}) = 0$ *for all measures* m *on* X.

PROOF. (i) $\Longrightarrow$ (i.b): Let m be a finite measure, i.e., $m(X) < \infty$. Then, $1 \in \ell^2(X, m)$. Furthermore, $1 \in \mathcal{D}$ as $c = 0$ and, therefore, applying the definition of $D(Q_m^{(N)})$ and Theorem 1.19 gives

$$1 \in \mathcal{D} \cap \ell^2(X, m) = D(Q_m^{(N)}) = D(Q_m^{(D)}) = \mathcal{D}_0 \cap \ell^2(X, m) \subseteq \mathcal{D}_0.$$

(i.b) $\Longrightarrow$ (i.a): This is shown in Proposition 6.11.

(i.a) $\Longrightarrow$ (i): This is clear as

$$D(Q_m^{(D)}) = \mathcal{D}_0 \cap \ell^2(X, m) = \mathcal{D} \cap \ell^2(X, m) = D(Q_m^{(N)})$$

for all measures m if $\mathcal{D}_0 = \mathcal{D}$.

(i.b) $\Longleftrightarrow$ (i.d) $\Longleftrightarrow$ (i.e): This is shown in Lemma 6.12.

(i.b) $\Longrightarrow$ (i.c): This is clear.

(i.c) $\Longrightarrow$ (i.b): By (i.c) and as $C_c(X) \subseteq \mathcal{D}_0$, we infer that $v = u + 1_K$ satisfies $v \in \mathcal{D}_0$ with $v(x) \geq C$ for all $x \in X$ for a suitable $C > 0$. Hence $1 = (v/C) \wedge 1$ belongs to $\mathcal{D}_0$ by Lemma 6.7.

(i) $\Longleftrightarrow$ (ii): This follows from Theorem 3.2.

The remaining assertions follow easily from Propositions 6.14 and Proposition 6.16:

(i.b) $\Longleftrightarrow$ (vi): This follows from (a) of the mentioned propositions.

(i.b) $\Longleftrightarrow$ (vii): This follows from (b) of the mentioned propositions.

(i.b) $\Longleftrightarrow$ (v.b): This follows from (c) of the mentioned propositions.

(i.b) $\Longleftrightarrow$ (viii)/(viii.a): This follows from (d) of the mentioned propositions.

(viii.a) $\Longleftrightarrow$ (viii.b): Failure of (viii.a) is equivalent to the existence of a $w > 0$ with $\mathcal{Q} \geq w$ on $C_c(X)$. Failure of (viii.b) is equivalent to existence of a measure m and $\lambda > 0$ with $\mathcal{Q} \geq \lambda m$. In this case, we can assume without loss of generality that $\lambda = 1$ after replacing m by λm. Now, the desired equivalence is clear.

(i.b) $\Longleftrightarrow$ (v)/(v.a): This follows from (e) of the mentioned propositions. $\square$

We end this section with a series of remarks that extend the considerations of the theorem above via exercises.

REMARK. We first note that (i) is stated for all measures m. However, this is also equivalent to the condition on the forms for one finite measure (Exercise 6.12).

Remark. Conditions (i.d) and (i.e) show that various ways of approximating the function 1 are equivalent to recurrence. Such sequences can be used to show that recurrence implies the existence of a function of finite energy which goes to infinity (Exercise 6.13). Furthermore, with the help the material presented in the Green's function section, Section 4, we can actually show that b is recurrent if and only if there exists a sequence (e_n) in $C_c(X)$ such that $0 \leq e_n \leq 1$, $e_n(x) \to 1$ as $n \to \infty$ for every $x \in X$ and

$$\lim_{n\to\infty} \mathcal{Q}(e_n, f) = 0$$

for every $f \in \mathcal{D}_0$ (Exercise 6.14).

Remark. One useful consequence of the criteria for recurrence above is that transience is stable under the operation of taking subgraphs (Exercise 6.15).

Remark. In view of (viii.b) there is yet another characterization of recurrence. Specifically, the graph b is recurrent and the measure m is finite if and only if 0 is an eigenvalue of the Dirichlet Laplacian $L_m^{(D)}$ associated with b over (X, m) (Exercise 6.16).

3. The superharmonic function perspective

In this section we look at the spaces $\mathcal{D}_0$ and $\mathcal{D}$ from the perspective of Hilbert spaces and (super)harmonic functions. Large parts of this section can be understood as a study of (super)harmonic functions in $\mathcal{D}_0$ and the complement of $\mathcal{D}_0$ in $\mathcal{D}$. This question is already of interest on its own. In the context of the present chapter, we use these results to prove various parts of our main characterization of recurrence.

Excavation Exercise 6.3, which recalls the invertibility of the Dirichlet Laplacian on finite sets, and Excavation Exercise 6.4, which establishes some basic properties of superharmonic functions, will be used in this section.

We start with a description of harmonic functions in $\mathcal{D}_0$.

Lemma 6.18 (Harmonic functions in $\mathcal{D}_0$). *Let (b, c) be a connected graph over X. Then any harmonic function in $\mathcal{D}_0$ is constant. In particular, there exists a non-trivial harmonic function in $\mathcal{D}_0$ if and only if $1 \in \mathcal{D}_0$ and $c = 0$.*

Proof. Let $u \in \mathcal{D}_0$ with $\mathcal{L}u = 0$. Then, by the Green's formula given in Lemma 6.8 we find

$$\mathcal{Q}(u) = \sum_{x\in X} \mathcal{L}u(x)u(x) = 0.$$

As b is connected, we find that u must be constant. Furthermore, $c = 0$ if $u \neq 0$. The last statement is then clear. $\square$

Having dealt with harmonic functions in $\mathcal{D}_0$, we now turn to superharmonic functions. As we have already discussed the case of constant functions in the previous lemma, we focus on non-constant superharmonic functions next.

LEMMA 6.19 (Superharmonic functions in $\mathcal{D}_0$). *Let (b,c) be a connected graph over X. Any non-constant superharmonic function u in $\mathcal{D}_0$ satisfies $u \geq 0$.*

PROOF. Let u be a non-constant superharmonic function in $\mathcal{D}_0$. Then, $v = u \wedge 0$ is superharmonic by Lemma 1.9, $v \in \mathcal{D}_0$ by Lemma 6.7 and $v \leq 0$. It suffices to show $v = 0$. Assume the contrary. Then, as u is not constant, v cannot be constant and this implies

$$0 < \mathcal{Q}(v).$$

On the other hand, by the Green's formula given in Lemma 6.8, we clearly have

$$\mathcal{Q}(v) = \sum_{x \in X} \mathcal{L}v(x)v(x) \leq 0$$

as $\mathcal{L}v \geq 0$ and $v \leq 0$. This gives a contradiction. $\square$

The previous lemma provides a property of superharmonic functions in $\mathcal{D}_0$. It does not deal with the existence of such functions. We now study this existence. It turns out that equilibrium potentials, i.e., functions $f \in \mathcal{D}_0$ with $f(x) = 1$ and $\mathcal{Q}(f) = \operatorname{cap}(x)$ provide examples of such functions. In particular, we note that the following result gives the existence of non-constant superharmonic functions whenever there exists an $x \in X$ with $\operatorname{cap}(x) > 0$ which, by Theorem 6.17, is equivalent to $1 \notin \mathcal{D}_0$.

PROPOSITION 6.20 (Equilibrium potentials are superharmonic functions in $\mathcal{D}_0$). *Let (b,c) be a connected graph over X and let $x \in X$. Then, there exists a unique superharmonic $u \in \mathcal{D}_0$ with $u(x) = 1$ and $\mathcal{L}u(y) = 0$ for all $y \in X$ with $y \neq x$. Furthermore, the function u satisfies*

- $\mathcal{L}u(x) = \operatorname{cap}(x) = \mathcal{Q}(u)$.
- $0 \leq u \leq 1$.

In particular, u is the equilibrium potential for x.

PROOF. We first show uniqueness: Let u and v be two such functions. Consider $w = u - v$. Then, w belongs to $\mathcal{D}_0$ with $\mathcal{L}w(y) = 0$ for $y \neq x$ and $w(x) = 0$. Hence, we obtain by Green's formula, Lemma 6.8,

$$\mathcal{Q}(w) = \sum_{y \in X} \mathcal{L}w(y)w(y) = 0.$$

Thus, w must be constant. By $w(x) = 0$ we find $w = 0$ and this is the desired uniqueness statement.

We now discuss the existence of a function u with all of the stated properties. We will show that the equilibrium potential from Proposition 6.9 is the required function. Recall that the equilibrium potential for $x \in X$ is the unique minimizer u of $\|\cdot\|_x$ on the convex closed set

$$\{f \in \mathcal{D}_0 \mid f(x) = 1\}.$$

Furthermore, u satisfies $0 \leq u \leq 1$.

By the minimizing property of u we see that $\mathcal{Q}(u + s1_y) \geq \mathcal{Q}(u)$ for all $s \in \mathbb{R}$ and $y \neq x$ and for all $s \geq 0$ for $y = x$. This easily implies that $\mathcal{L}u(y) = 0$ for $y \neq x$ and $\mathcal{L}u(x) \geq 0$. Hence, u is superharmonic.

From Proposition 6.9 we find

$$\operatorname{cap}(x) = \mathcal{Q}(u).$$

Furthermore, by Green's formula, Lemma 6.8, we then obtain

$$\operatorname{cap}(x) = \mathcal{Q}(u) = \sum_{y \in X} \mathcal{L}u(y)u(y) = \mathcal{L}u(x),$$

where we used $\mathcal{L}u(y) = 0$ for $y \neq x$ as well as $u(x) = 1$ to obtain the last equality. This completes the proof □

Remark. Note that it may well be that the function u appearing in the previous proposition is a constant function. In fact, u is constant if and only if $\operatorname{cap}(x) = 0$: Indeed, if $\operatorname{cap}(x) > 0$, then u cannot be constant as $\mathcal{L}u(x) = \operatorname{cap}(x) > 0$ whereas $\mathcal{L}1 = 0$. Conversely, if $\operatorname{cap}(x) = 0$, then $\mathcal{L}u = 0$ so that u is harmonic and thus constant by Lemma 6.18.

As a consequence of the previous proposition we can set up a complete solution theory for existence of monopoles in $\mathcal{D}_0$, i.e., solutions of equations of the form

$$\mathcal{L}u = 1_x$$

for $x \in X$ and $u \in \mathcal{D}_0$. This is contained in the subsequent corollary. We will come back to it in the next section from a different perspective. This will show that the solutions in $\mathcal{D}_0$ we find here have a minimality property among all solutions.

Corollary 6.21 (Existence of monopoles). *Let (b, c) be a connected graph over X and let $x \in X$. Then, the following statements hold:*

(a) *If $\operatorname{cap}(x) > 0$, then there exists a unique function $g_x \in \mathcal{D}_0$ with $\mathcal{L}g_x = 1_x$. This function g_x satisfies $0 \leq g_x \leq 1/\operatorname{cap}(x)$.*
(b) *If $\operatorname{cap}(x) = 0$, then there does not exist a solution of $\mathcal{L}u = 1_x$ in $\mathcal{D}_0$.*

Proof. Both (a) and (b) follow from the previous proposition and Green's formula. We now give the details.

(a) Let $u \in \mathcal{D}_0$ be the unique solution given by Proposition 6.20 above. The function $g_x = u/\operatorname{cap}(x)$ clearly belongs to $\mathcal{D}_0$ and satisfies

$\mathcal{L}g_x = 1_x$. Furthermore, as $0 \le u \le 1$, we get $0 \le g_x \le 1/\mathrm{cap}(x)$. This shows the existence of such a function.

To show uniqueness we observe that whenever $g_x \in \mathcal{D}_0$ satisfies $\mathcal{L}g_x = 1_x$, the function g_x is non-constant and, hence, satisfies $\mathcal{Q}(g_x) > 0$. From Green's formula, Lemma 6.8, we find

$$0 < \mathcal{Q}(g_x) = g_x(x).$$

Then, $u = g_x/g_x(x)$ satisfies the statement of Proposition 6.20 and is, therefore, unique. Thus, g_x is a multiple of u. Clearly, there can be at most one multiple of u solving the equation in question. This gives that g_x is unique.

(b) Assume that there exists a $u \in \mathcal{D}_0$ with $\mathcal{L}u = 1_x$. Then, u is not constant and, hence, satisfies $\mathcal{Q}(u) > 0$. From Green's formula, Lemma 6.8, we then find

$$0 < \mathcal{Q}(u) = u(x).$$

Hence, we can consider the function $v = u/u(x)$ and this function is superharmonic and satisfies $v(x) = 1$ and $\mathcal{L}v(y) = 0$ for all $y \neq x$. So, by Proposition 6.20, v is the unique superharmonic function with these properties and

$$\mathrm{cap}(x) = \mathcal{L}v(x) = \frac{1}{u(x)} > 0.$$

By contraposition, this completes the proof. □

Remark (Alternative proof of the existence of monopoles). We note that an alternative reasoning for the existence of monopoles can be given using the Riesz representation theorem (Exercise 6.17).

So far, we have dealt with superharmonic functions in $\mathcal{D}_0$. We can also describe the orthogonal complement of $\mathcal{D}_0$ in $\mathcal{D}$ using superharmonic functions.

Lemma 6.22 (Orthogonal complement of $\mathcal{D}_0$ in $\mathcal{D}$). *Let (b, c) be a connected graph over X. Let $o \in X$. Then, $u \in \mathcal{D}$ satisfies $u \perp \mathcal{D}_0$ with respect to $\langle \cdot, \cdot \rangle_o$ if and only if*

$$\mathcal{L}u(y) = 0 \quad \text{for all } y \neq o \qquad \text{and} \qquad \mathcal{L}u(o) = -u(o).$$

In particular, $(\mathcal{L}u(o))u$ is superharmonic whenever u belongs to the orthogonal complement of $\mathcal{D}_0$ in $\mathcal{D}$.

Proof. As $C_c(X)$ is dense in $\mathcal{D}_0$ with respect to $\|\cdot\|_o$, we obtain that $u \in \mathcal{D}$ is orthogonal to $\mathcal{D}_0$ if and only if

$$\langle u, \varphi \rangle_o = \mathcal{Q}(u, \varphi) + u(o)\varphi(o) = 0$$

for all $\varphi \in C_c(X)$. As any $\varphi \in C_c(X)$ can be written as a sum of a $\psi \in C_c(X)$ with $\psi(o) = 0$ and a multiple of 1_o we see that $u \in \mathcal{D}$ is orthogonal to $\mathcal{D}_0$ if and only if both

$$\mathcal{Q}(u, \psi) = 0 \qquad \text{and} \qquad \mathcal{Q}(u, 1_o) + u(o) = 0$$

for all $\psi \in C_c(X)$ with $\psi(o) = 0$. From Green's formula, Lemma 6.8, we find that these statements are equivalent to

$$\mathcal{Q}(u, \psi) = \sum_{y \in X} \mathcal{L}u(y)\psi(y) = 0$$

and

$$-u(o) = \mathcal{Q}(u, 1_o) = \mathcal{L}u(o).$$

From these equivalences, the desired statements follow. □

So far our results have not made any assumptions on the graph. We now turn to study the case of $1 \in \mathcal{D}_0$ and assume that the graph is connected.

Proposition 6.23 (Superharmonic functions in $\mathcal{D}$ are constant). *Let b be a connected graph over X. If $1 \in \mathcal{D}_0$, then any superharmonic function in $\mathcal{D}$ is constant.*

Proof. We proceed in two steps. As a first step, we show that any superharmonic function in $\mathcal{D}$ must be harmonic. Thus, let u be a superharmonic function in $\mathcal{D}$. Then, from Green's formula, Lemma 6.8, we find

$$0 = \mathcal{Q}(u, 1) = \sum_{x \in X} \mathcal{L}u(x).$$

As $\mathcal{L}u(x) \geq 0$ for all $x \in X$ the desired harmonicity follows.

We now show that any harmonic function in $\mathcal{D}$ is constant: Let $u \in \mathcal{D}$ be harmonic. Let $o \in X$ and consider $v = u - u(o)1$. Then, clearly $v \in \mathcal{D}$ is harmonic and vanishes at o. It suffices to show $v = 0$.

As v is harmonic, the function $-|v|$ is superharmonic, see Lemma 1.9. Hence, $-|v|$ must be harmonic by what we have shown in the first step. Thus, $|v|$ is harmonic as well. As both $|v|$ and v are harmonic, we conclude that both $v_\pm = (\pm v) \vee 0 = (|v| \pm v)/2$ are harmonic. Clearly, both v_+ and v_- are non-negative. By connectedness, each of them must then either be strictly positive or vanish identically, see Corollary 4.2. By $v(o) = 0$ we have $v_+(o) = 0 = v_-(o)$ and both v_+ and v_- must vanish identically. This shows $v = 0$, which completes the proof. □

Remark (Approximation free proof of $\mathcal{D}_0 = \mathcal{D}$ for $1 \in \mathcal{D}_0$). As shown in Proposition 6.11 via approximation, we have $\mathcal{D} = \mathcal{D}_0$ whenever $1 \in \mathcal{D}_0$. Thus, it is interesting to note that the preceding two results give another, approximation free, proof of the equality of $\mathcal{D}_0$ and $\mathcal{D}$ under the condition $1 \in \mathcal{D}_0$: Let $u \in \mathcal{D}$. Then, considering the Hilbert space $\mathcal{D}$ with inner product $\langle \cdot, \cdot \rangle_o$ for $o \in X$, we can decompose u as $u = v + r$ with $v \in \mathcal{D}_0$ and $r \perp \mathcal{D}_0$. By Lemma 6.22, we can then conclude that either r or $-r$ is superharmonic. By Proposition 6.23 this gives that r is constant. Hence, r belongs to $\mathcal{D}_0$ as well and, therefore, has to be 0. Thus, $u = v$ belongs to $\mathcal{D}_0$.

In the preceding considerations we have dealt with (super)harmonic functions in $\mathcal{D}$ using methods from Hilbert space theory. It is remarkable that the conclusion of the previous proposition continues to hold for general superharmonic functions well outside any context of Hilbert spaces. This is the content of the next proposition.

PROPOSITION 6.24 (Positive superharmonic functions are constant). *Let b be a connected graph. Assume $1 \in \mathcal{D}_0$. If $u \geq 0$ is superharmonic, then u is constant.*

PROOF. If $u = 0$ there is nothing left to show. Thus, we can assume that there exists an $x \in X$ with $u(x) > 0$. By Lemma 4.2, we then have $u > 0$. As $1 \in \mathcal{D}_0$, there exists a sequence (e_n) in $C_c(X)$ approximating 1 with respect to $\| \cdot \|_o$. Hence, by the ground state transform, Corollary 4.9, we infer

$$0 \leq \frac{1}{2} \sum_{x,y \in X} b(x,y)u(x)u(y) \left(\frac{e_n}{u}(x) - \frac{e_n}{u}(y) \right)^2 \leq \mathcal{Q}(e_n).$$

As $e_n(x) \to 1$ for each $x \in X$ as $n \to \infty$ and $\mathcal{Q}(e_n) \to \mathcal{Q}(1) = 0$, we infer from Fatou's lemma that

$$0 \leq \frac{1}{2} \sum_{x,y \in X} b(x,y)u(x)u(y) \left(\frac{1}{u}(x) - \frac{1}{u}(y) \right)^2 \leq 0.$$

As b is connected this implies that u is constant. $\square$

Our study of superharmonic functions now easily allows us to prove the following result which forms part of Theorem 6.1.

THEOREM 6.25 (Characterization of recurrence – superharmonic functions). *Let b be a connected graph over X. Then, the following statements are equivalent:*

(i.b) $1 \in \mathcal{D}_0$.

(iv) *All superharmonic functions $u \geq 0$ are constant.*

  (iv.a) *All superharmonic functions $u \in \mathcal{D}_0$ are constant.*

  (iv.b) *All superharmonic functions $u \in \mathcal{D}$ are constant.*

  (iv.c) *All superharmonic functions $u \in \ell^\infty(X)$ are constant.*

(ix) *There exists a non-trivial harmonic function $u \in \mathcal{D}_0$, i.e., $\mathcal{L}_m$ is not injective on $\mathcal{D}_0$ for some (all) measure(s) m.*

(x) *For some (all) $x \in X$ there does not exist a monopole in $\mathcal{D}_0$ at x.*

PROOF. (i.b) $\Longrightarrow$ (iv.b): This is shown in Proposition 6.23.

(iv.b) $\Longrightarrow$ (iv.a): This is clear as $\mathcal{D}_0 \subseteq \mathcal{D}$.

(iv.a)/(iv) $\Longrightarrow$ (i.b): Let $x \in X$. By Proposition 6.20 there exists a positive superharmonic function $u \in \mathcal{D}_0$ with $u(x) = 1$. By (iv.a)/(iv) we must then have $u = 1$ and, hence, 1 belongs to $\mathcal{D}_0$.

(i.b) $\Longrightarrow$ (iv): This is shown in Proposition 6.24.

(iv) $\Longrightarrow$ (iv.c): Let $u \in \ell^\infty(X)$ be superharmonic. Then, $u + C1$ is a superharmonic positive function for sufficiently large $C > 0$. Hence, $u + C1$ is constant by (iv) and thus u is constant as well.

(iv.c) $\Longrightarrow$ (iv): This follows easily as superharmonicity is stable under cutoff. More specifically, whenever u is superharmonic so is $u \wedge k$ for any number $k \geq 0$ by Lemma 1.9.

(i.b) $\Longleftrightarrow$ (ix): This is immediate from Lemma 6.18.

(i.b) $\Longleftrightarrow$ (x): We already know by Theorem 6.17 that (i.b) is equivalent to $\mathrm{cap}(x) = 0$ for some (all) $x \in X$. Now, the desired statement follows directly from Corollary 6.21. $\square$

Remark. One use of non-trivial positive superharmonic harmonic functions is to derive a Hardy inequality (Exercise 6.18). Now, any Hardy inequality can always be extended from $C_c(X)$ to $\mathcal{D}_0$. In the recurrent case, one can even extend a Hardy inequality to $\mathcal{D}$ (Exercise 6.19).

4. The Green's function perspective

In this section we study properties of the Green's function. We first show that there are three possible ways of introducing the Green's function. More specifically, the Green's function can be defined via the semigroup, as the limit of resolvents or as the limit of the inverses of the Dirichlet Laplacians. We will then show that a graph is recurrent if and only if the Green's functions is finite at one (equivalently, all) pair(s) of vertices. We will also provide alternative approaches to various topics discussed in preceding sections, including the existence of monopoles as well as approximating the constant function 1 by a sequence in $C_c(X)$.

Excavation Exercises 6.3 and 6.4 will be used in this section.

Let (b, c) be a connected graph over (X, m). We recall that the Green's function $G = G_m \colon X \times X \longrightarrow [0, \infty]$ is given by

$$G_m(x, y) = \int_0^\infty e^{-tL_m} 1_y(x) dt$$

where $L_m = L_m^{(D)}$. The theorem below gives some basic properties of the Green's function. In particular, we show that the Green's function can also be defined via resolvents or via the inverse of the Dirichlet Laplacians for any exhaustion sequence of the graph. Furthermore, we show that if the Green's function is finite at some pair of vertices, then it is finite for all pairs of vertices.

Theorem 6.26 (Basic properties of G_m). *Let (b, c) be a connected graph over (X, m). Then, for all $\varphi \in C_c(X)$ and $x \in X$, we have*

$$\lim_{\alpha \to 0^+} (L_m + \alpha)^{-1} \varphi(x) = \int_0^\infty e^{-tL_m} \varphi(x) dt = \lim_{n \to \infty} (L_{K_n}^{(D)})^{-1} \varphi(x),$$

where (K_n) is an arbitrary sequence of increasing finite sets such that $\bigcup_n K_n = X$. In particular, for all $x, y \in X$,

$$G_m(x,y) = \lim_{\alpha \to 0^+} (L_m + \alpha)^{-1} 1_y(x) = \lim_{n\to\infty} (L_{K_n}^{(D)})^{-1} 1_y(x).$$

Furthermore,

(a) *$G_m(x,y)m(x) = G_m(y,x)m(y)$ for all $x,y \in X$.*
(b) *$G_m > 0$.*
(c) *If $G_m(x,y) = \infty$ for some $x,y \in X$, then $G_m(x,y) = \infty$ for all $x, y \in X$.*
(d) *If $G_m(x,y) < \infty$ for some $x,y \in X$, then $G_m(x,y) < \infty$ for all $x,y \in X$ and, for an arbitrary $o \in X$, the function $G_m(\cdot, o)$ is superharmonic with*

$$\mathcal{L}_m G_m(\cdot, o) = 1_o.$$

Furthermore, $G_m(\cdot,o)$ is the smallest $u \in \mathcal{F}$ with $u \geq 0$ such that $\mathcal{L}_m u \geq 1_o$.
(e) *If $G_m(x,y) < \infty$ for some $x,y \in X$ and $c(o) = 0$ for some $o \in X$, then $G_m(\cdot, o)$ is not constant. Furthermore, if $G_m(\cdot,o)$ is constant, then $G_m(\cdot, o')$ is not constant for all $o' \neq o$.*

Remark. We note that it is possible that $G_m(\cdot,o)$ is actually constant for one $o \in X$ (Exercise 6.20).

Remark. There is yet another way to introduce the Green's function $G_m(\cdot, o)$ for a connected transient graph. Consider the smallest solution $u_\lambda \geq 0$ to $(\mathcal{L}_m + \lambda w)u_\lambda = 1_o$ for $\lambda > 0$ and $w \geq 0$ non-trivial. Then, $G_m(x,o) = \lim_{\lambda\to 0} u_\lambda(x)$ (Exercise 6.21).

Proof. By decomposing into positive and negative parts, we can assume that $\varphi \geq 0$. The spectral theorem gives

$$(L_m + \alpha)^{-1} = \int_0^\infty e^{-\alpha t} e^{-tL_m} dt,$$

see Theorem A.35 in Appendix A. Therefore, the first formula follows by monotone convergence since e^{-tL_m} for $t \geq 0$ is positivity preserving by Corollary 1.22. Furthermore, for any K_n such that $\varphi \in C_c(K_n)$ we have by the spectral theorem

$$\left(L_{K_n}^{(D)}\right)^{-1} \varphi = \int_0^\infty e^{-tL_{K_n}^{(D)}} \varphi dt.$$

By decomposing into negative and positive parts and using Lemma 1.21 it follows that $e^{-tL_{K_n}^{(D)}}\varphi \nearrow e^{-tL_m}\varphi$ pointwise so that the right-hand side converges to $\int_0^\infty e^{-tL_m}\varphi dt$ by monotone convergence arguments. Hence, the second equality also holds and the main statement follows.

We now turn to the additional statements.

(a) The symmetry follows directly from the equality

$$e^{-tL_m} 1_x(y) m(y) = \langle e^{-tL_m} 1_x, 1_y \rangle = \langle 1_x, e^{-tL_m} 1_y \rangle = e^{-tL_m} 1_y(x) m(x)$$

for all $x, y \in X$.

(b) The strict positivity $G_m > 0$ follows directly from the definition of G_m and the fact that the semigroup is positivity improving for $t > 0$ by Theorem 1.26 as we assume connectedness.

(c) To show that $G_m(x, y) = \infty$ for all $x, y \in X$ if $G_m(x, y) = \infty$ for some $x, y \in X$ let $e_x = 1_x/\sqrt{m(x)}$ for $x \in X$. Let $x, y, x_0 \in X$. We calculate for $t > 1$,

$$\begin{aligned}
e^{-tL_m}1_y(x) &= e^{-L_m}e^{-(t-1)L_m}1_y(x) \\
&= \frac{1}{m(x)}\langle e^{-L_m}e^{-(t-1)L_m}1_y, 1_x\rangle \\
&= \frac{1}{m(x)}\sum_{z\in X}\langle e^{-(t-1)L_m}1_y, e_z\rangle\langle e^{-L_m}1_x, e_z\rangle \\
&\geq \frac{1}{m(x)}\langle e^{-(t-1)L_m}1_y, e_{x_0}\rangle\langle e^{-L_m}1_x, e_{x_0}\rangle \\
&= \frac{m(x_0)}{m(x)}e^{-(t-1)L_m}1_y(x_0)e^{-L_m}1_x(x_0).
\end{aligned}$$

Since the semigroups are positivity improving on a connected graph by Theorem 1.26, we infer that $C = e^{-L_m}1_x(x_0)m(x_0)/m(x) > 0$. Then,

$$\begin{aligned}
G_m(x, y) &= \int_0^\infty e^{-tL_m}1_y(x)dt \\
&\geq \int_1^\infty e^{-tL_m}1_y(x)dt \\
&\geq C\int_0^\infty e^{-tL_m}1_y(x_0)dt \\
&= CG_m(x_0, y).
\end{aligned}$$

By the symmetry shown in (a) and repeating the calculation allows us to estimate $G_m(x_0, y)$ by $G_m(x_0, y_0)$ for any $y_0 \in X$. As $x, y, x_0, y_0 \in X$ were chosen arbitrarily, the statement follows.

(d) That $G_m(x, y) < \infty$ for all $x, y \in X$ if $G_m(x, y) < \infty$ for some $x, y \in X$ follows from part (c) directly shown above.

Now, the function $\alpha \mapsto (L_m + \alpha)^{-1}1_y(x)$ can be seen to be monotonically decreasing by the resolvent identity

$$(L_m + \alpha)^{-1} - (L_m + \beta)^{-1} = (\beta - \alpha)(L_m + \alpha)^{-1}(L_m + \beta)^{-1}$$

for $\alpha, \beta > 0$ as the resolvents are positivity preserving, Corollary 1.22. We calculate for $z, o \in X$,

$$\begin{aligned}
\mathcal{L}_m(L_m + \alpha)^{-1}1_o(z) &= L_m(L_m + \alpha)^{-1}1_o(z) \\
&= 1_o(z) - \alpha(L_m + \alpha)^{-1}1_o(z).
\end{aligned}$$

Taking the limit $\alpha \to 0^+$ we see that the right-hand side converges to $1_o(z)$ from the spectral theorem, see Proposition A.24 in Appendix A. Moreover, for the left-hand side, we have $\lim_{\alpha\to 0^+}(L_m+\alpha)^{-1}1_o(z) = G_m(z,o) < \infty$ as we assume that $G_m(x,y) < \infty$ for some $x, y \in X$. Since the limit on the right-hand side exists and

$$\lim_{\alpha\to 0^+} \mathrm{Deg}(z)(L_m+\alpha)^{-1}1_o(z) = \mathrm{Deg}(z)G_m(z,o)$$

we infer that $G_m(\cdot,o) \in \mathcal{F}$ and

$$\mathcal{L}_m G_m(\cdot,o)(z) = 1_o(z)$$

by monotone convergence, cf. Lemma 1.8.

We now show that $G_m(\cdot,o)$ is the smallest function $u \in \mathcal{F}$ such that $u \geq 0$ and $\mathcal{L}_m u \geq 1_0$. Let $o \in X$, let (K_n) be an arbitrary sequence of increasing finite sets such that $\bigcup_n K_n = X$ and $o \in K_n$ for all $n \in \mathbb{N}$. Let $g_n = (L_{K_n}^{(D)})^{-1}1_o$ and let $u \geq 0$ satisfy $\mathcal{L}_m u \geq 1_o$. Then, $v_n = u - g_n$ is superharmonic on K_n, satisfies $v_n \geq 0$ outside of K_n and $v_n \wedge 0$ assumes its minimum on the finite set K_n. Hence, by the minimum principle, Theorem 1.7, we infer that $v_n \geq 0$ and, therefore, $u \geq g_n$. Since g_n converges to $G_m(\cdot,o)$, it follows that $u \geq G_m(\cdot,o)$.

(e) By (d) we have $\mathcal{L}_m G_m(\cdot,o) = 1_o$. So, if $c(o) = 0$, then it is clear that $G_m(\cdot,o)$ is not constant.

We now prove the remaining statement. Assume that $G_m(\cdot,x)$ are constant for $x = o, o' \in X$. Then, this implies

$$\mathcal{L}_m G_m(\cdot,x) = \frac{c}{m}G_m(\cdot,x).$$

We will show that $o = o'$. We deduce from (a), the constancy and from $\mathcal{L}_m G_m(\cdot,x) = 1_x$ for $x = o, o'$ that

$$\begin{aligned}
\frac{c(o)}{m(o')}G_m(o,o') &= \frac{c(o)}{m(o)}G_m(o',o) \\
&= \frac{c(o)}{m(o)}G_m(o,o) \\
&= \mathcal{L}_m G_m(\cdot,o)(o) \\
&= 1 \\
&= \mathcal{L}_m G_m(\cdot,o')(o') \\
&= \frac{c(o')}{m(o')}G_m(o,o').
\end{aligned}$$

We infer $c(o) = c(o') \neq 0$ and $1 = c(o')G_m(o,o')/m(o')$ from this calculation. Using this, (a) and that $G_m(\cdot,o)$ is constant we obtain

$$1 = \frac{c(o')}{m(o')}G_m(o,o') = \frac{c(o)}{m(o)}G_m(o',o) = \mathcal{L}_m G_m(\cdot,o)(o') = 1_o(o').$$

Hence, $1_o(o') = 1$, and, therefore, $o = o'$. $\square$

To study further properties of the Green's function associated to a graph, we normalize the approximating sequences of Theorem 6.26 at one vertex. It is clear that, in the case of $G_m(x,y) < \infty$, normalizing the approximating sequences of $G_m(\cdot, o)$ at a vertex $o \in X$ yields a limiting function $g = G_m(\cdot, o)/G_m(o,o)$. It turns out that even in the case of $G_m(x,y) = \infty$, these normalizing sequences yield a finite limit.

Indeed, we have seen this phenomenon before. In Corollary 4.5 and Corollary 4.6 from Chapter 4, we obtained superharmonic functions g and $g^{(D)}$ via pointwise limits of subsequences of the functions

$$g_n = \frac{1}{(L_m + \alpha_n)^{-1} 1_o(o)} (L_m + \alpha_n)^{-1} 1_o$$

and

$$g_n^{(D)} = \frac{1}{(L_{K_n}^{(D)})^{-1} 1_o(o)} (L_{K_n}^{(D)})^{-1} 1_o,$$

where $\alpha_n > 0$ for $n \in \mathbb{N}_0$ is a sequence with $\alpha_n \to 0$ as $n \to \infty$ and (K_n) is an increasing sequence of finite sets with $\bigcup_n K_n = X$ and $o \in X$ is such that $o \in K_n$ for $n \in \mathbb{N}_0$.

Clearly, in the case $G_m(x,y) < \infty$ for all $x, y \in X$, we have $g = g^{(D)} = G_m(\cdot, o)/G_m(o,o)$ by Theorem 6.26 above and the limits are independent of the choice of (α_n) and (K_n). The next lemma shows that also in the case that $G_m(x,y) = \infty$ for all $x, y \in X$, these limits also exist, coincide and are independent of the choice of (α_n) and (K_n). Furthermore, we show that this pointwise convergence is even convergence with respect to $\|\cdot\|_o$. As a fundamental consequence, the limit $g = g^{(D)}$ is in $\mathcal{D}_0$ as $g_n^{(D)} \in C_c(X)$ so that

$$G_m(\cdot, o) \in \mathcal{D}_0$$

in the case of $G_m(x,y) < \infty$ for all $x, y \in X$. Furthermore, we compute $\mathcal{Q}(g)$ and we will later show that $\mathcal{Q}(g) = \operatorname{cap}(o)$. In particular, g is the equilibrium potential at o as discussed in Propositions 6.9 and 6.20.

Lemma 6.27. *Let (b,c) be a connected graph over (X,m) and let $o \in X$. Let $g_n, g_n^{(D)}$ be given as above. Then, there exists a $g \in \mathcal{D}_0$ with $g_n^{(D)} \to g$ and $g_n \to g$ with respect to $\|\cdot\|_o$ as $n \to \infty$. Moreover,*

$$\mathcal{Q}(g) = \frac{m(o)}{G_m(o,o)}.$$

Furthermore, we have the following case distinction:

(a) *If $G_m(x,y) = \infty$ for some (all) $x, y \in X$, then $c = 0$ and $g = 1$ is harmonic.*
(b) *If $G_m(x,y) < \infty$ for some (all) $x, y \in X$, then the function*

$$g = G_m(\cdot, o)/G_m(o,o)$$

is superharmonic and satisfies

$$\mathcal{L}_m g = \frac{1_o}{G_m(o,o)}.$$

In particular, $G_m(\cdot, o) \in \mathcal{D}_0$ *in this case.*

As a consequence, the limit g *is independent of the choice of the sequences* (α_n) *and* (K_n)*.*

Proof. By Corollaries 4.5 and 4.6 there exist subsequences of the g_n and the $g_n^{(D)}$ that converge pointwise irrespective of the choice of α_n and K_n. By the local Harnack inequality, Theorem 4.1, any such subsequence is pointwise bounded. Our subsequent considerations will show that any pointwise convergent subsequence actually converges with respect to $\|\cdot\|_o$ and that all possible limits agree. This will establish the existence of the limit along all sequences $\alpha_n \to 0$ and increasing finite $K_n \subseteq X$ with $\bigcup_n K_n = X$.

To avoid cumbersome index notation we assume without loss of generality that

$$g = \lim_{n\to\infty} g_n, \qquad g^{(D)} = \lim_{n\to\infty} g_n^{(D)}$$

exist pointwise.

We first show that $g^{(D)}, g \in \mathcal{D}$. Let $g_n^{(D)}$ be given as above. Since $g_n^{(D)} \in C_c(X)$, we obtain by the use of Green's formula, Proposition 1.5, and the facts that $L_{K_n}^{(D)}$ is a restriction of $\mathcal{L}_m$ and $g_n^{(D)}(o) = 1$ that

$$\mathcal{Q}(g_n^{(D)}) = \sum_{x\in K_n} \left(L_{K_n}^{(D)} g_n^{(D)}\right)(x) g_n^{(D)}(x) m(x) = \frac{m(o)}{(L_{K_n}^{(D)})^{-1} 1_o(o)}.$$

By lower semi-continuity and Theorem 6.26 we get

$$\mathcal{Q}(g^{(D)}) \le \liminf_{n\to\infty} \mathcal{Q}(g_n^{(D)}) = \frac{m(o)}{G_m(o,o)},$$

which implies that $g^{(D)} \in \mathcal{D}$.

By a similar argument, as $g_n \in D(L_m)$ and $g_n(o) = 1$, we have

$$\mathcal{Q}(g_n) = \langle L_m g_n, g_n\rangle = \frac{\langle 1_o, g_n\rangle - \alpha_n \|g_n\|^2}{(L_m+\alpha)^{-1} 1_o(o)} \le \frac{m(o)}{(L_m+\alpha_n)^{-1} 1_o(o)}.$$

Again, by lower semi-continuity and Theorem 6.26, as $\alpha_n \to 0$ when $n \to \infty$, we get

$$\mathcal{Q}(g) \le \liminf_{n\to\infty} \mathcal{Q}(g_n) \le \frac{m(o)}{G_m(o,o)},$$

which implies that $g \in \mathcal{D}$.

If $G_m(o,o) = \infty$, the estimates above give that $\mathcal{Q}(g^{(D)}) = \mathcal{Q}(g) = 0$. As $g^{(D)}, g > 0$ by Corollaries 4.5 and 4.6, it follows that $c = 0$ in this case and hence, g and $g^{(D)}$ are constant so that $g = g^{(D)} = 1$ since

$g_n(o) = g_n^{(D)}(o) = 1$. Furthermore, the convergence with respect to $\|\cdot\|_o$ follows as we have pointwise convergence and

$$\mathcal{Q}(g) = \lim_{n\to\infty} \mathcal{Q}(g_n) = \lim_{n\to\infty} \mathcal{Q}(g_n^{(D)}) = \mathcal{Q}(g^{(D)}) = 0.$$

If $G_m(o,o) < \infty$, we obtain $g = g^{(D)} = G_m(\cdot,o)/G_m(o,o)$ directly from Theorem 6.26 above. Moreover, by Theorem 6.26 (d) we have

$$\mathcal{L}_m g = \frac{1}{G_m(o,o)} \mathcal{L}_m G_m(\cdot,o) = \frac{1_o}{G_m(o,o)}.$$

Since $g_n^{(D)} \in C_c(X)$ and $g \in \mathcal{D} \subseteq \mathcal{F}$ we have by Green's formula, Proposition 1.5, and $g_n^{(D)}(o) = 1$ that

$$\begin{aligned}
\mathcal{Q}(g - g_n^{(D)}) &= \mathcal{Q}(g) - 2\mathcal{Q}(g, g_n^{(D)}) + \mathcal{Q}(g_n^{(D)}) \\
&= \mathcal{Q}(g) - 2\sum_{x\in X} (g_n^{(D)} \mathcal{L}_m g)(x) m(x) + \sum_{x\in X} (g_n^{(D)} \mathcal{L}_m g_n^{(D)})(x) m(x) \\
&= \mathcal{Q}(g) - 2\frac{m(o)}{G_m(o,o)} + \frac{m(o)}{(L_{K_n}^{(D)})^{-1} 1_o(o)} \\
&\to \mathcal{Q}(g) - \frac{m(o)}{G_m(o,o)} \leq 0
\end{aligned}$$

as $n \to \infty$, where the last inequality follows by the estimate on $\mathcal{Q}(g)$ established above. Hence, $g_n^{(D)} \to g = g^{(D)}$ with respect to $\|\cdot\|_o$ and we deduce that $g \in \mathcal{D}_0$. Therefore, as $g = G_m(\cdot,o)/G_m(o,o)$ in this case, we get that $G_m(\cdot,o) \in \mathcal{D}_0$ as a consequence.

The independence of this construction on the choice of sequences follows directly from the considerations above. $\square$

The preceding lemma ties in with various further considerations. In particular, (a) is connected to $1 \in \mathcal{D}_0$, as we will see below, and (b) is connected to solutions of $\mathcal{L}u = 1_x$, i.e., to monopoles.

So, after these discussions it is not hard to connect (xi) with (i.b) in our main characterization of recurrence, Theorem 6.1. This is done next.

THEOREM 6.28 (Characterization of recurrence – Green's function). *Let b be a connected graph over X. Then, the following statements are equivalent:*

(i.b) $1 \in \mathcal{D}_0$.

(xi) $G_m(x,y) = \infty$ *for some (all) $x, y \in X$ and some (all) measure(s) m.*

- (xi.a) $\lim_{\alpha\to 0^+} (L_m + \alpha)^{-1} 1_y(x) = \infty$ *for some (all) $x, y \in X$ and some (all) measure(s) m.*
- (xi.b) $\lim_{n\to\infty} (L_{K_n}^{(D)})^{-1} 1_y(x) = \infty$ *for some (all) $x, y \in X$ and some (all) sequence(s) (K_n) of increasing sets such that $\bigcup_n K_n = X$ and some (all) measure(s) m.*

PROOF. By Theorem 6.26 the statements (xi), (xi.a) and (xi.b) are all equivalent. So, to prove the theorem it suffices to show that (i.b) is equivalent to (xi). This is carried out next, based on Lemma 6.27. Note that we show that (i.b) implies (xi) for all measures m and that the validity of (xi) for some measure implies (i.b). This gives that the validity of (xi) for some measure is equivalent to the validity of (xi) for all measures.

(i.b) $\Longrightarrow$ (xi): Assume $G_m(x,y) < \infty$ for some measure m and some $x,y \in X$. Then, we obtain from (b) of Lemma 6.27 that there exists a superharmonic $g \in \mathcal{D}_0$ with $\mathcal{L}g \neq 0$. This, however, implies $1 \notin \mathcal{D}_0$, as otherwise we have from Green's formula, Lemma 6.8,

$$0 < \sum_{x \in X} \mathcal{L}g(x) = \mathcal{Q}(g,1) = 0,$$

which is clearly a contradiction.

(xi) $\Longrightarrow$ (i.b): From (xi) and (a) of Lemma 6.27 we immediately obtain (i.b). $\square$

We end this section by proving three properties of the Green's function and show that each of them indeed is a characterization of the Green's function. While these are not used in the subsequent considerations they are of interest in their own right.

THEOREM 6.29 (Characterizations of the Green's function). *Let (b,c) be a connected graph over (X,m). Assume that $G_m(x,y) < \infty$ for some $x,y \in X$ and let $o \in X$.*

(a) *The function $G_m(\cdot,o)$ is the unique function $u \in \mathcal{D}_0$ such that*
$$\mathcal{L}_m u = 1_o.$$
(b) *The function $G_m(\cdot,o)$ is the unique function $u \in \mathcal{D}_0$ such that for all $f \in \mathcal{D}_0$,*
$$\mathcal{Q}(u,f) = f(o)m(o).$$
(c) *The function $G_m(\cdot,o)$ is the unique minimizer of $\mathcal{Q}$ on*
$$\{u \in \mathcal{D}_0 \mid u(o) = G_m(o,o)\}.$$

PROOF. (a) Existence is a direct consequence of (b) of Lemma 6.27. Uniqueness is easy to show: Let u, v be two such functions in $\mathcal{D}_0$. Then, $u-v$ is a harmonic function in $\mathcal{D}_0$ and, hence, constant by Lemma 6.18. By Theorem 6.28 this constant must be zero.

(b) This follows directly from (a) and Green's formula, Lemma 6.8.

(c) Recall that for an exhaustion sequence (K_n) with $o \in K_n$ we defined
$$g_n^{(D)} = \frac{1}{(L_{K_n}^{(D)})^{-1}1_o(o)}(L_{K_n}^{(D)})^{-1}1_o$$

and in Lemma 6.27 (b) we have shown that $g = \lim_{n\to\infty} g_n^{(D)}$ satisfies $g = G_m(\cdot, o)/G_m(o, o)$ in the case that $G_m(x, y) < \infty$.

Hence, it suffices to show that g is the unique minimizer of $\mathcal{Q}$ on $\{u \in \mathcal{D}_0 \mid u(o) = 1\}$. Note that $g_n^{(D)}$ solves the Dirichlet problem

$$\begin{aligned} L_{K_n}^{(D)} u &= 0 \qquad \text{on } A = K_n \setminus \{o\} \\ u &= 1 \qquad \text{on } B = \{o\}. \end{aligned}$$

So, by standard arguments, cf. Theorem 0.41, the function $g_n^{(D)}$ is the unique minimizer of the restriction $Q_{K_n}^{(D)}$ of $\mathcal{Q}$ to $\{u \in C(K_n) \mid u(o) = 1\}$. Furthermore, by the convergence $g_n^{(D)} \to g$ with respect to $\|\cdot\|_o$, we have for all $\psi \in C_c(X)$ with $\psi(o) = 1$,

$$\mathcal{Q}(g) = \lim_{n\to\infty} \mathcal{Q}(g_n^{(D)}) = \lim_{n\to\infty} Q_{K_n}^{(D)}(g_n^{(D)}) \leq \lim_{n\to\infty} Q_{K_n}^{(D)}(\psi) = \mathcal{Q}(\psi).$$

Since $C_c(X)$ is dense in $\mathcal{D}_0$, the function g is a minimizer of $\mathcal{Q}$ on $\{u \in \mathcal{D}_0 \mid u(o) = 1\}$.

Finally, we show that g is the unique such minimizer. So, let $v \in \mathcal{D}_0$ be another minimizer and let $v_n \in C_c(X)$ be such that $v_n \to v$ with respect to $\|\cdot\|_o$. Then, applying Green's formula gives

$$\begin{aligned} \mathcal{Q}(v - g) &= \lim_{n\to\infty} \mathcal{Q}(v_n - g) \\ &= \mathcal{Q}(v) - 2 \lim_{n\to\infty} \mathcal{Q}(v_n, g) + \mathcal{Q}(g) \\ &\leq 2\mathcal{Q}(g) - 2 \lim_{n\to\infty} \sum_{x\in X} \mathcal{L}g(x) v_n(x) \\ &= 2\mathcal{Q}(g) - 2\frac{m(o)}{G_m(o, o)} \\ &\leq 0, \end{aligned}$$

where the last equality follows from $\mathcal{Q}(g) = m(o)/G_m(o, o)$ shown in Lemma 6.27. As b is connected this shows that $v - g$ is constant. As v and g agree on o, they must then be equal. $\square$

By statement (c) of the preceding theorem we get an immediate consequence for the function $g = \lim_{n\to\infty} g_n^{(D)} = \lim_{n\to\infty} g_n$ which appears in the proof of the theorem. More specifically, we obtain that g is the equilibrium potential for o first constructed in Proposition 6.9. In particular, this allows us to connect the capacity of points with the Green's function.

Corollary 6.30 (Green's function and equilibrium potentials). *Let (b, c) be a connected graph over (X, m) and let $o \in X$. Let $g = \lim_{n\to\infty} g_n^{(D)} = \lim_{n\to\infty} g_n$. Then g is the equilibrium potential for o, i.e., the unique minimizer of $\mathcal{Q}$ on $\{u \in \mathcal{D}_0 \mid u(o) = 1\}$ which satisfies $0 \leq g \leq 1$ and*

$$\mathcal{Q}(g) = \operatorname{cap}(o).$$

Furthermore, $\operatorname{cap}(o) = 0$ *if and only if* $G_m(x, y) = \infty$ *for some (all)* $x, y \in X$.

PROOF. The fact that $\operatorname{cap}(o)$ is given by the energy of the minimizer follows from Proposition 6.9. Now, if $G_m(x, y) < \infty$, for some (all) $x, y \in X$, then $\mathcal{Q}(g) = \operatorname{cap}(o) > 0$ follows by Lemma 6.27 (b) and Theorem 6.29 (c) above as $g(\cdot) = G_m(\cdot, o)/G_m(o, o) \in \mathcal{D}_0$ is superharmonic in this case. If $G_m(x, y) = \infty$, for some (all) $x, y \in X$, Lemma 6.27 (a) yields $c = 0$ and $g = 1 \in \mathcal{D}_0$. This immediately gives $\mathcal{Q}(g) = \operatorname{cap}(o) = 0$. □

5. The Green's formula perspective

In this section we study recurrence from the point of view of Green's formulas. In particular, we will see that a variant of Green's formula which allows us to pair functions of finite energy whose Laplacian is in ℓ^1 with bounded functions of finite energy is equivalent to recurrence.

In Lemma 6.8 we have already encountered a Green's formula in the form

$$\mathcal{Q}(u, v) = \sum_{x \in X} \mathcal{L}u(x)v(x)$$

with absolutely converging sum for $v \in \mathcal{D}_0$ and $u \in \mathcal{D}$ with either $\mathcal{L}u \geq 0$ or $\sum_{x \in X} |\mathcal{L}u(x)| < \infty$ and $v \in \ell^\infty(X)$. Indeed, this formula has been used in the previous sections in various places. The main message of this section is that the validity of this formula with $v \in \mathcal{D} \cap \ell^\infty(X)$ instead of $v \in \mathcal{D}_0$ is a characterization of recurrence. In fact, it even suffices to consider $v = 1$, in which case the formula simplifies to

$$0 = \sum_{x \in X} \mathcal{L}u(x).$$

A basic insight behind the reasoning in this section is that this type of formula actually excludes the existence of superharmonic functions which are not harmonic.

We will need certain consequences of the previous sections in order to provide a proof of the remaining parts of our main theorem. These are discussed next.

PROPOSITION 6.31. *Let b be a connected graph over (X, m). If*

$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) = 0$$

for all bounded $u \in \mathcal{D}_0$ with $u \geq 0$ and $\mathcal{L}_m u \in \ell^1(X, m)$, then $1 \in \mathcal{D}_0$.

PROOF. By Proposition 6.20 there exists for any $x \in X$ a superharmonic function $u \in \mathcal{D}_0$ with $u(x) = 1$, $0 \leq u \leq 1$, $\mathcal{L}u(y) = 0$ for all

$y \neq x$ and $\mathcal{L}u(x) = \mathcal{Q}(u)$. From our assumption we find

$$0 = \sum_{z \in X} \mathcal{L}u(z) = \mathcal{L}u(x) = \mathcal{Q}(u).$$

By connectedness we infer that u is constant. As $u(x) = 1$, we obtain $1 = u \in \mathcal{D}_0$. □

From $1 \in \mathcal{D}_0$ and the definition of $\mathcal{D}_0$ we obtain the existence of a sequence in $C_c(X)$ converging to 1 with respect to $\|\cdot\|_o$. The considerations on the Green's functions as the limit of restrictions from the preceding section actually provide a specific such sequence with additional features. This is the content of the next proposition.

PROPOSITION 6.32. *Let (b, c) be a connected graph over (X, m). Assume that $1 \in \mathcal{D}_0$ and let (K_n) be an arbitrary sequence of finite subsets of X with $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_n K_n = X$. Then, there exists a sequence (e_n) in $C_c(X)$ with $0 \leq e_n \leq 1$ and $\mathcal{L}_m e_n \geq 0$ on K_n for each $n \in \mathbb{N}$ and $e_n \to 1$ as $n \to \infty$ with respect to $\|\cdot\|_o$ for any $o \in K_1$.*

PROOF. From $1 \in \mathcal{D}_0$ and Theorem 6.28 we obtain $G_m(x, y) = \infty$ for all $x, y \in X$. Consider

$$g_n^{(D)} = \frac{1}{(L_{K_n}^{(D)})^{-1} 1_o(o)} (L_{K_n}^{(D)})^{-1} 1_o$$

for $n \in \mathbb{N}$. By $G_m(x, y) = \infty$ it follows that $g_n^{(D)} \to g = 1$ in $\|\cdot\|_o$ by Lemma 6.27 (a). Now define

$$e_n = g_n^{(D)} \wedge 1,$$

which satisfies $e_n \in C_c(X)$ and $0 \leq e_n \leq 1$ for $n \in \mathbb{N}$. Moreover, $e_n \to 1$ as $n \to \infty$ with respect to $\|\cdot\|_o$ for an arbitrary $o \in X$ and we have $\mathcal{L}_m e_n \geq 0$ on K_n for all $n \in \mathbb{N}$ as the minimum of two superharmonic functions is superharmonic by Lemma 1.9. □

After these preparations we conclude the proof of the main result, Theorem 6.1.

THEOREM 6.33 (Characterization of recurrence – Green's formula). *Let b be a graph over (X, m). Then, the following statements are equivalent:*

(i.b) $1 \in \mathcal{D}_0$.

(iii) *If $u \in \mathcal{D}$ satisfies $\mathcal{L}_m u \in \ell^1(X, m)$ and $v \in \mathcal{D} \cap \ell^\infty(X)$, then*

$$\mathcal{Q}(u, v) = \sum_{x \in X} \mathcal{L}_m u(x) v(x) m(x)$$

for some (all) measure(s) m. (*"Green's formula"*) (iii.a) *If $u \in \mathcal{D}$ satisfies $\mathcal{L}_m u \in \ell^1(X,m)$, then*

$$\sum_{x\in X} \mathcal{L}_m u(x) m(x) = 0$$

for some (all) measure(s) m.

(iii.b) *If $u \in \ell^\infty(X)$ satisfies $\mathcal{L}_m u \in \ell^1(X,m)$, then*

$$\sum_{x\in X} \mathcal{L}_m u(x) m(x) = 0$$

for some (all) measure(s) m.

Proof. The fact that statements (iii), (iii.a), (iii.b) hold for all measures m if they hold for one measure m is clear from the definition of $\mathcal{L}_m$.

(iii)/(iii.a)/(iii.b) $\Longrightarrow$ (i.b): Each of (iii), (iii.a) and (iii.b) clearly implies

$$\sum_{x\in X} \mathcal{L}_m u(x) m(x) = 0$$

for all bounded $u \in \mathcal{D}$ with $u \geq 0$ and $\mathcal{L}_m u \in \ell^1(X,m)$. That $1 \in \mathcal{D}_0$ now follows from Proposition 6.31.

(i.b) $\Longrightarrow$ (iii)/(iii.a): The assumption $1 \in \mathcal{D}_0$ implies $\mathcal{D}_0 = \mathcal{D}$ by Proposition 6.11 and now (iii) is a direct consequence of the Green's formula, Lemma 6.8. Furthermore, (iii.a) clearly follows from (iii).

(i.b) $\Longrightarrow$ (iii.b): Let (K_n) be a sequence of finite subsets of X with $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_n K_n = X$. By Proposition 6.32 there exists a sequence (e_n) in $C_c(X)$ such that $e_n \to 1$ as $n \to \infty$ with respect to $\|\cdot\|_o$ for arbitrary $o \in X$ and $\mathcal{L}_m e_n \geq 0$ on K_n for all $n \in \mathbb{N}$.

Let $u \in \ell^\infty(X)$ with $\mathcal{L}_m u \in \ell^1(X,m)$ and assume, without loss of generality, that $0 \leq u \leq 1$. We infer by Green's formula, Proposition 1.5, and the third pointwise Leibniz rule in Lemma 2.25 applied to $f = g = e_n$, that

$$\begin{aligned}\sum_{x\in X} e_n^2(x)\mathcal{L}_m u(x)m(x) &= \sum_{x\in X} \mathcal{L}_m e_n^2(x)u(x)m(x)\\ &= 2\sum_{x\in K_n} e_n(x)\mathcal{L}_m e_n(x)u(x)m(x) - \sum_{x,y\in X} b(x,y)u(x)(e_n(x)-e_n(y))^2.\end{aligned}$$

By Lebesgue's dominated convergence theorem and the fact that $e_n \leq 1$ we conclude, for the first term above, that

$$\lim_{n\to\infty} \sum_{x\in X} e_n^2(x)\mathcal{L}_m u(x)m(x) = \sum_{x\in X} \mathcal{L}_m u(x)m(x).$$

Furthermore, as $u \leq 1$, the last term above satisfies

$$0 \leq \sum_{x,y\in X} b(x,y)u(x)(e_n(x)-e_n(y))^2 \leq 2\mathcal{Q}(e_n) \to 0$$

as $n \to \infty$. These consideration together with $\mathcal{L}_m e_n \geq 0$ on K_n and $e_n, u \geq 0$ for $n \in \mathbb{N}$ give

$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) = 2 \lim_{n \to \infty} \sum_{x \in K_n} e_n(x) \mathcal{L}_m e_n(x) u(x) m(x) \geq 0.$$

The same argument applied to the function $1 - u$ gives

$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) \leq 0$$

and, hence, (iii.b) follows. $\square$

Remark. We note that the equivalence of (iii.a) and degeneracy of the form $\mathcal{Q}$ on $\mathcal{D}_0$, condition (v) in Theorem 6.1, also holds for general c (Exercise 6.22).

Remark (Existence of dipoles). In the two previous sections we have discussed how recurrence is equivalent to the existence of monopoles. The material of this section allows us to easily conclude the existence of dipoles for an arbitrary graph. Indeed, whenever b is a connected graph over X and $x, y \in X$ then there exists a solution of $\mathcal{L}u = 1_x - 1_y$ in $\mathcal{D}_0$.

We now prove this statement. To avoid trivialities we only consider the case $x \neq y$. If $1 \notin \mathcal{D}_0$, then we have $\operatorname{cap}(x), \operatorname{cap}(y) > 0$ from Lemma 6.15 and from Corollary 6.21 we obtain the existence of monopoles g_x and g_y for x and y, respectively. Therefore, $u = g_x - g_y$ has the desired property.

If $1 \in \mathcal{D}_0$, then we consider the minimizer u of $\mathcal{Q}$ on $\mathcal{A} = \{f \in \mathcal{D}_0 \mid f(x) \geq 1, f(y) \leq -1\}$. As in the proof of Proposition 6.20, we infer that $\mathcal{L}u(z) = 0$ for $z \neq x, y$ as well as $\mathcal{L}u(x) \geq 0$ and $\mathcal{L}u(y) \leq 0$. As $u(x) \geq 1$ and $u(y) \leq -1$ we obtain $u(x) \neq u(y)$ and thus u is not constant. As all harmonic functions in $\mathcal{D}_0$ are constant due to $1 \in \mathcal{D}_0$ by Lemma 6.18, we infer that u cannot be harmonic. So, it is not possible that both $\mathcal{L}u(x) = 0$ and $\mathcal{L}u(y) = 0$ hold. Moreover, by Green's formula, Lemma 6.8, we have

$$0 = \mathcal{Q}(u, 1) = \sum_{z \in X} \mathcal{L}u(z)$$

and this shows $\mathcal{L}u(x) = -\mathcal{L}u(y) \neq 0$. Hence, $v = u/\mathcal{L}u(x)$ has the desired properties.

6. A probabilistic point of view*

In this section we connect recurrence and random walks. For this purpose, we focus on the normalizing measure in this section.

Let (b, c) be a graph over X. In Section 5, we introduced a Markov chain (Y_n) with transition probabilities

$$p(x, y) = \frac{b(x, y)}{\sum_{z \in X} b(x, z) + c(x)}$$

for $x, y \in X$. To these probabilities we associate the transition operator

$$P_0 \varphi(x) = \sum_{y \in X} p(x, y) \varphi(y)$$

for $\varphi \in C_c(X)$ and $x \in X$. The next lemma shows that this operator extends to a bounded operator on $\ell^2(X, n)$, where n is the normalizing measure $n(x) = \sum_{x \in X} b(x, y) + c(x)$ for $x \in X$.

Lemma 6.34 (Transition operator). *Let (b, c) be a graph over X. Then, the operator P_0 extends to a bounded self-adjoint operator $P \colon \ell^2(X, n) \longrightarrow \ell^2(X, n)$ with operator norm bounded by 1.*

Proof. A direct calculation shows that P_0 is symmetric. Furthermore, for $\varphi, \psi \in C_c(X)$,

$$\begin{aligned} |\langle P_0 \varphi, \psi \rangle| &= \left| \sum_{x, y \in X} p(x, y) \varphi(y) \psi(x) n(x) \right| \\ &\leq \left(\sum_{x \in X} \psi^2(x) n(x) \sum_{y \in X} p(x, y) \right)^{1/2} \left(\sum_{y \in X} \varphi^2(y) n(y) \sum_{x \in X} p(y, x) \right)^{1/2} \\ &\leq \|\psi\| \|\varphi\|, \end{aligned}$$

where we used that $p(x, y) = p(y, x) n(y)/n(x)$ in the second line. Hence, P_0 is bounded on $C_c(X)$ by 1 in $\ell^2(X, n)$ and can, therefore, be extended to a bounded operator P on $\ell^2(X, n)$. $\square$

By virtue of the lemma above, we can define powers of P and let

$$p_k(x, y) = P^k 1_y(x)$$

for $x, y \in X$ and $k \in \mathbb{N}_0$. Notice that $P^k 1_y(x)$ is the probability that the random walker of the Markov chain (Y_k) starting at x is at y after k jumps, i.e.,

$$\mathbb{P}_x(Y_k = y) = P^k 1_y(x),$$

where for an event A we define

$$\mathbb{P}_x(A) = \mathbb{P}(A \mid Y_0 = x).$$

Furthermore, we denote the conditioned expectation by

$$\mathbb{E}_x(A) = \mathbb{E}(A \mid Y_0 = x).$$

We now present the connection between the Markov chain and the notion of recurrence as presented in the preceding sections. Note, in particular, that recurrence is equivalent to the Markov chain visiting every vertex infinitely often, regardless of the starting vertex.

THEOREM 6.35 (Random walk perspective on recurrence). *Let (b, c) be a connected graph over (X, n). Then, for all $x, y \in X$,*

$$G_n(x, y) = \int_0^\infty e^{-tL_n} 1_y(x) dt = \sum_{k=0}^\infty p_k(x, y).$$

Moreover, for $c = 0$, the following statements are equivalent:

(xi) *For some (all) $x, y \in X$ and some (all) measure(s) m,*

$$G_m(x, y) = \infty.$$

(xi.c) *For some (all) $x, y \in X$,*

$$\mathbb{E}_x(\#\{k \in \mathbb{N}_0 \mid Y_k = y\}) = \infty.$$

(xi.d) *For some (all) $x, y \in X$,*

$$\mathbb{P}_x(Y_k = y \text{ for some } k \in \mathbb{N}) = 1.$$

(xi.e) *For some (all) $x, y \in X$,*

$$\mathbb{P}_x\left(Y_k = y \text{ for infinitely many } k \in \mathbb{N}_0\right) = 1.$$

REMARK. For a graph (b, c) over (X, m) with the counting measure $m = 1$ and associated self-adjoint operator L we have the following corresponding formula

$$\int_0^\infty e^{-tL} 1_y(x) dt = \frac{1}{\deg(x)} \sum_{k=0}^\infty p_k(x, y)$$

for $x, y \in X$ which relates the Green's function to the transition matrix (Exercise 6.23).

We start by proving the equality for the Green's function in the theorem above.

LEMMA 6.36. *Let (b, c) be a connected graph over (X, n). For all $x, y \in X$,*

$$G_n(x, y) = \int_0^\infty e^{-tL_n} 1_y(x) dt = \sum_{k=0}^\infty p_k(x, y).$$

PROOF. Let (K_j) be an increasing sequence of sets such that $X = \bigcup_j K_j$. Let P_{K_j} be the restriction of P to $C_c(K_j)$ which extends to an operator on $\ell^2(X, n)$ by projecting onto $C_c(K_j)$ first and extending by zero after applying P_{K_j}, $j \geq 1$. Since $p(x, y) \geq 0$ for all matrix elements of P, we have

$$P_{K_j}^k 1_x \leq P_{K_{j+1}}^k 1_x \leq P^k 1_x$$

for all $x \in X$, $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$.

On the other hand, for fixed $x, y \in X$ and $k \in \mathbb{N}_0$, there exists a j_0 such that for all K_j with $j \geq j_0$,

$$P_{K_j}^k 1_x(y) = P^k 1_x(y).$$

Namely, one chooses N such that K_N includes every combinatorial path of length k starting in x. Hence,

$$\lim_{j\to\infty}\sum_{k=0}^{\infty} P_{K_j}^k\varphi(x) = \sum_{k=0}^{\infty} P^k\varphi(x)$$

for all $\varphi \in C_c(X)$ and $x \in X$ by monotone convergence and decomposition of φ into positive and negative parts.

Now, the operator L_n on $\ell^2(X, n)$ can be written as $L_n = I - P$, where I is the identity operator and the same is true for any restriction of L_n. In particular, denoting the Dirichlet restriction of L_n to K_j by L_{K_j}, a direct algebraic computation now shows that

$$L_{K_j}^{-1} = (I_{K_j} - P_{K_j})^{-1} = \sum_{k=0}^{\infty} P_{K_j}^k.$$

Applying this to a function $\varphi \in C_c(X)$ and evaluating it at some $x \in X$, the left-hand side converges to $\int_0^\infty e^{-tL}\varphi(x)dt$ by Theorem 6.26. Furthermore, the right-hand side converges to $\sum_{k=0}^\infty P^k\varphi(x)$ by the considerations above. This finishes the proof. $\square$

For the proof of the equivalences in Theorem 6.35 we need some notation. First we introduce the following stopping times

$$s_x(y) = \min\{k \in \mathbb{N}_0 \mid Y_k = y, Y_0 = x\}$$
$$t_x(y) = \min\{k \in \mathbb{N} \mid Y_k = y, Y_0 = x\},$$

where $\min\emptyset = \infty$. Moreover, we define the functions

$$g_y(x) = \sum_{k=0}^{\infty} \mathbb{P}(s_x(y) = k) = \mathbb{P}_x\left(Y_k = y \text{ for some } k \in \mathbb{N}_0\right)$$
$$u_y(x) = \sum_{k=0}^{\infty} \mathbb{P}(t_x(y) = k) = \mathbb{P}_x(Y_k = y \text{ for some } k \in \mathbb{N}),$$

which we relate to the Green's function and show that g_y is superharmonic.

LEMMA 6.37. *Let (b, c) be a connected graph over (X, n). For all $x, y \in X$,*

$$G_n(y, y)g_y(x) = G_n(x, y).$$

Furthermore, for all $y \in X$,

$$\mathcal{L}_n g_y = 1_y(g_y(y) - u_y(y)) \geq 0.$$

PROOF. We calculate

$$p_k(x, y) = \mathbb{P}_x(Y_k = y) = \sum_{l=0}^{k} \mathbb{P}(s_x(y) = l)\mathbb{P}_y(Y_{k-l} = y).$$

Then, we get by the Cauchy product formula

$$\begin{aligned}
G_n(x,y) &= \sum_{k=0}^{\infty} p_k(x,y) \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \mathbb{P}(s_x(y)=l)\mathbb{P}_y(Y_{k-l}=y) \\
&= \left(\sum_{k=0}^{\infty} \mathbb{P}_y(Y_k=y)\right)\left(\sum_{l=0}^{\infty} \mathbb{P}(s_x(y)=l)\right) \\
&= G_n(y,y)g_y(x).
\end{aligned}$$

Furthermore, for $x \neq y$,

$$\begin{aligned}
Pg_y(x) &= \sum_{z\in X} p(x,z)g_y(z) \\
&= \sum_{z\in X} \mathbb{P}_x(Y_1=z) \sum_{k=0}^{\infty} \mathbb{P}(s_z(y)=k) \\
&= \sum_{k=0}^{\infty} \sum_{z\in X} \mathbb{P}_x(Y_1=z)\mathbb{P}(s_z(y)=k) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(s_x(y)=k+1) \\
&= g_y(x),
\end{aligned}$$

where we use $\mathbb{P}(s_x(y)=0)=0$ for $x \neq y$ in the last step. Analogously, we get for $x=y$,

$$Pg_y(y) = u_y(y).$$

Hence,

$$\mathcal{L}_n g_y = 1_y(g_y(y) - u_y(y)).$$

This completes the proof. $\square$

We introduce another function related to the return probability of the random walk. Let

$$v_y(x) = \mathbb{P}_x(Y_k = y \text{ for infinitely many } k \in \mathbb{N}_0).$$

We relate this function to the function g_y introduced above as follows.

LEMMA 6.38. *Let (b,c) be a graph over (X,n). For all $x,y \in X$,*

$$v_y(x) = v_y(y)g_y(x).$$

In particular, v_y is superharmonic for all y.

PROOF. We calculate

$$
\begin{aligned}
v_y(x) &= \mathbb{P}_x(Y_l = y \text{ for infinitely many } l \in \mathbb{N}_0) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(s_x(y) = k)\mathbb{P}_x(Y_{l+k} = y \text{ for infinitely many } l \in \mathbb{N}_0 \mid Y_k = y) \\
&= \mathbb{P}_y(Y_l = y \text{ for infinitely many } l \in \mathbb{N}_0) \sum_{k=0}^{\infty} \mathbb{P}(s_x(y) = k) \\
&= v_y(y)g_y(x).
\end{aligned}
$$

The "in particular" now follows from Lemma 6.37 above. □

PROOF OF THEOREM 6.35. The first equality follows from Lemma 6.36. We now prove the remaining equivalences.

(xi) $\iff$ (xi.c): By virtue of Lemma 6.36, we have

$$G_n(x, y) = \sum_{k=0}^{\infty} p_k(x, y) = \sum_{k=0}^{\infty} \mathbb{P}_x(Y_k = y).$$

On the other hand,

$$\mathbb{E}_x(\#\{k \in \mathbb{N}_0 \mid Y_k = y\}) = \sum_{k=0}^{\infty} \mathbb{P}_x(Y_k = y).$$

(xi) $\implies$ (xi.d)/(xi.e): Assume $G_n(y, y) = \infty$ for some (all) $y \in X$. Then, all positive superharmonic functions are constant by Theorem 6.1. Specifically, g_y and v_y are constant. We will show that $g_y = 1$, which is equivalent to (xi.d), and $v_y = 1$, which is equivalent to (xi.e) as y is chosen arbitrarily.

Obviously,

$$g_y(x) = g_y(y) = \mathbb{P}_y(Y_k = y \text{ for some } k \in \mathbb{N}_0) = 1$$

for all $x \in X$, which yields (xi.d).

Moreover, let

$$v_y^{(l)}(x) = \mathbb{P}_x(Y_k = y \text{ for at least } l \text{ numbers } k \in \mathbb{N}_0).$$

Then, by definition of u_y,

$$
\begin{aligned}
v_y^{(l)}(y) = &\mathbb{P}_y(Y_k = y \text{ for some } k \in \mathbb{N}) \\
&\cdot \mathbb{P}_y(Y_k = y \text{ for at least } l-1 \text{ numbers } k \in \mathbb{N}_0) \\
= &u_y(y)v_y^{(l-1)}(y) = \ldots = u_y(y)^l v_y^{(0)}(y) \\
= &u_y(y)^l
\end{aligned}
$$

since $v_y^{(0)}(y) = 1$. Hence,

$$v_y(y) = \lim_{l \to \infty} v_y^{(l)}(y) = \lim_{l \to \infty} u_y(y)^l,$$

which is either 0 or 1. Since v_y is constant it is either 0 or 1. Clearly,

$$u_y(y) = \mathbb{P}_y(Y_k = y \text{ for some } k \in \mathbb{N}) = 1,$$

which implies $v_y = 1$, which is (xi.e).

(xi.e) $\Longleftrightarrow$ (xi.d): This is clear.

(xi.d) $\Longrightarrow$ (xi): Assume $g_y = 1$ and $G_n(y, y) < \infty$. By Lemma 6.37, this implies that $G_n(\cdot, y)$ is constant. However, this is impossible by Theorem 6.26 (e) as $c = 0$. $\square$

Exercises

Excavation exercises.

Exercise 6.1 (Weakly convergent subsequence). Let H be a Hilbert space. Show that any bounded sequence (f_n) in H has a weakly convergent subsequence.

Exercise 6.2 (Minimizers on closed convex sets). Let H be a Hilbert space with norm $\|\cdot\|$ and let U be a closed convex subset. Show the map $U \longrightarrow [0,\infty)$ given by $f \mapsto \|f\|$ admits a unique minimizer.

Exercise 6.3 (Invertibility and positivity of the Dirichlet Laplacian). Let (b,c) be a connected graph over (X,m). Let $K \subseteq X$ be a finite subset of X and let $L_K^{(D)}$ be the associated Dirichlet Laplacian. Show that $L_K^{(D)}$ is an invertible operator. Furthermore, show that if $o \in K$, then $(L_K^{(D)})^{-1}1_o > 0$ on K.

Exercise 6.4 (Staying within the set of superharmonic functions). Let b be a graph over X. Show the following statements:

(a) The pointwise infimum of a set of superharmonic functions is superharmonic whenever it is a finite function.
(b) The sum of two superharmonic functions is superharmonic. More generally, the limit of any monotonically increasing sequence of superharmonic functions is superharmonic whenever the limit is pointwise finite.
(c) The composition $\varphi \circ u$ of a monotonically increasing concave function
$\varphi\colon [0,\infty) \longrightarrow [0,\infty]$ with a positive superharmonic function u is a superharmonic function which is non-harmonic whenever φ is strictly concave.

Example exercises.

Exercise 6.5 (Bounded degree and finite measure implies recurrence). Let b be a graph over (X,m) such that $m(X) < \infty$ and $\mathrm{Deg}(x) = (1/m(x))\sum_{y\in X} b(x,y)$ is bounded. Show that the graph is recurrent.

(Hint: Look for a sequence of functions e_n as needed to show recurrence.)

EXERCISE 6.6 (Hardy's inequality on the natural numbers*). Let $X = \mathbb{N}$ and let b be a graph over $\mathbb{N}$ with $b(x, y) = 1$ if and only if $|x - y| = 1$ and 0 otherwise. Prove that

$$\sum_{n=0}^{\infty}(\varphi(n) - \varphi(n + 1))^2 \geq \sum_{n=1}^{\infty} \frac{4}{n^2}\varphi^2(n)$$

for all $\varphi\colon \mathbb{N}_0 \longrightarrow \mathbb{R}$ with $\varphi(0) = 0$.

(Hint: Show the statement first for corresponding functions in $C_c(X)$ with the help of Exercises 6.18 and 6.4 (c). Then, extend to $\mathcal{D}$ via Exercise 6.19.)

EXERCISE 6.7 (Hardy's inequality for trees). Let b be a k-regular tree with standard weights over X (recall that a tree is a cycle-free graph which is called k-regular if all vertices have exactly $k + 1$ neighbors). Show that there exist a non-constant function $w \geq (k+1)-2\sqrt{k}$ such that $\mathcal{Q} \geq w$ on $C_c(X)$.

EXERCISE 6.8 (Pólya's theorem*). Let $d \in \mathbb{N}$ and consider the graph with standard weights over $\mathbb{Z}^d$. That is, the vertex set is given by $\{x = (x_1, \ldots , x_d) \mid x_j \in \mathbb{Z}, j = 1, \ldots , d\}$ and

$$b(x, y) = \left(2 - \sum_{j=1}^{d} |x_j - y_j|\right)_+$$

for $x \neq y$ and $b(x, x) = 0$. Show that this graph is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

(Hint: This is hard.)

Extension exercises.

EXERCISE 6.9 (Resistance metrics). Let b be a graph over X with associated form $\mathcal{Q}$.

(a) Show that

$$r(x, y) = \sup\{|f(x) - f(y)| \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

and

$$r_0(x, y) = \sup\{|f(x) - f(y)| \mid f \in \mathcal{D}_0, \mathcal{Q}(f) \leq 1\}$$

are metrics on X.

(b) Show that r^2 and r_0^2 are also metrics on X.

EXERCISE 6.10 (Decomposing $\mathcal{D}$). Consider a connected graph b over X. Let $o \in X$. Define $\mathcal{D}_o = \{f \in \mathcal{D} \mid f(o) = 0\}$. Show that the following statements hold:

(a) The form $\mathcal{Q}$ provides an inner product on $\mathcal{D}_o$ and $(\mathcal{D}_o, \mathcal{Q})$ is a Hilbert space with
$$\mathcal{D} = \mathcal{D}_o \oplus \mathrm{Lin}\{1\}.$$

(b) The map
$$\mathcal{D}_o \longrightarrow \mathcal{D}/\mathrm{Lin}\{1\}, f \mapsto [f],$$
is bijective and isometric where $\mathcal{D}_o$ is equipped with $\mathcal{Q}$ and $\mathcal{D}/\mathrm{Lin}\{1\}$ is equipped with $\mathcal{Q}([f]) = \mathcal{Q}(f)$.

Exercise 6.11 (The capacity dichotomy). Let b be a connected graph over X and let $x \in X$. Consider a sequence $\varphi_n \in C_c(X)$ such that $\varphi_n(x) = 1$ and $\mathcal{Q}(\varphi_n) \to 0$ as $n \to \infty$. Use Lemma 6.3 (a) to show that $\varphi_n(y) \to 1$ for all $y \in X$. Use this to conclude that the capacities of points in X either all vanish or are all positive.

Exercise 6.12 (Recurrence for finite measures). Let b be a connected graph over X. Show that the graph is recurrent if and only if there exists a finite measure m such that $D(Q_m^{(D)}) = D(Q_m^{(N)})$.

Exercise 6.13 (Recurrence and uniformly unbounded functions of finite energy). Let b be a connected recurrent graph over X. Show that there exists a function $f \in \mathcal{D}$ such that $f(x) \to \infty$ as $x \to \infty$ where $X \cup \{\infty\}$ is the one point compactification of X.

(Hint: Consider $f = \sum_{n=1}^{\infty}(1-e_n)$ where (e_n) is a suitable sequence of finitely supported functions approximating 1.)

Exercise 6.14 (Recurrence in terms of sequences*). Let b be a connected graph over (X, m). Show that b is recurrent if and only if there exists a sequence (e_n) in $C_c(X)$ such that $0 \leq e_n \leq 1$, $e_n(x) \to 1$ as $n \to \infty$ for every $x \in X$ and
$$\lim_{n\to\infty} \mathcal{Q}(e_n, f) = 0$$
for every $f \in \mathcal{D}_0$. Show, furthermore, that it suffices to consider $f \in \mathcal{D}_0$ such that there exists a $g > 0$ in $\ell^1(X, m)$ such that
$$f(x) = \sum_{y\in X} G_m(x, y)g(y)m(y)$$
for $x \in X$ (this is actually a hint).

Exercise 6.15 (A transient subgraph implies transience). Show that a connected graph is transient if and only if it contains a transient subgraph. In this context, a subgraph of a graph b over X is a restriction $b|_{Y\times Y}$ of b to some subset $Y \subseteq X$.

Exercise 6.16 (0 is an eigenvalue of $L^{(D)}$). Let (b, c) be a connected graph over (X, m). Show that the following are equivalent:

(i) 0 is an eigenvalue for $L^{(D)}$.
(ii) The graph b is recurrent, $c = 0$ and $m(X) < \infty$.

Exercise 6.17 (Existence of monopoles). Show that $\operatorname{cap}(x) > 0$ implies the existence of a monopole at x in the following way: If $\operatorname{cap}(x) > 0$, then $(\mathcal{D}_0, \mathcal{Q})$ is a Hilbert space and point evaluation is continuous. Thus, by the Riesz representation theorem, there exists an element $g_x \in \mathcal{D}_0$ with

$$\mathcal{Q}(g_x, f) = f(x)$$

for all $f \in \mathcal{D}_0$. Now, Green's formula easily shows that g_x must solve $\mathcal{L}g_x = 1_x$.

Exercise 6.18 (Hardy inequalities for general c). Let (b, c) be a graph over X. Let u be a non-trivial positive superharmonic function. Show that there exists a function $w \geq 0$ with $\mathcal{Q} \geq w$ on $C_c(X)$ such that w is non-trivial whenever u is non-constant.

(Hint: Use the ground state transform.)

Exercise 6.19 (Hardy inequalities on $\mathcal{D}$). Let (b, c) be a graph over X such that there exists a $w \geq 0$ with $\mathcal{Q} \geq w$ on $C_c(X)$. Assume that b over X is recurrent. Show that $\mathcal{Q} \geq w$ on $\mathcal{D}$.

Exercise 6.20 (Constant Green's function). Let $X = \mathbb{N}_0$ with $b(x, y) = 1$ if and only if $|x - y| = 1$ and 0 otherwise. Let $c = 1_0$ and $m = 1$. Show that the function $G_m(\cdot, 0)$ is constant, where G_m denotes the associated Green's function.

(Hint: Show that the function 1 is the unique function u in $\mathcal{D}$ which satisfies $\mathcal{L}_m u = 1_0$.)

Exercise 6.21 (Green's function for $\mathcal{L} + \lambda w$). Let (b, c) be a connected graph over (X, m) and let $w \geq 0$ be non-trivial. Show that for all non-trivial $\varphi \in C_c(X)$, $\varphi \geq 0$ and $\lambda > 0$ there exists a solution $u_\lambda \colon X \longrightarrow (0, \infty)$ to

$$(\mathcal{L}_m + \lambda w)u_\lambda = \varphi$$

and that for the smallest of these solutions u_λ,

$$\lim_{\lambda \to 0} u_\lambda(x) = \sum_{y \in X} G_m(x, y)\varphi(y)$$

for $x \in X$, where the limit is infinite if the Green's function G_m is infinite.

Exercise 6.22 (Green's formula for general c). Let (b, c) be a connected graph over (X, m). Show that the form $\mathcal{Q}$ is degenerate on $\mathcal{D}_0$ if and only if whenever $u \in \mathcal{D}$ with $\mathcal{L}_m u \in \ell^1(X, m)$, then

$$\sum_{x \in X} \mathcal{L}_m u(x) m(x) = 0$$

for some (all) measure(s) m.

Exercise 6.23 (Green's function and the counting measure). Let (b, c) be a graph over (X, m) with $m = 1$ and associated operator L. Denote by $p_k(x, y)$ for $x, y \in X$ and $k \in \mathbb{N}$ the matrix elements of the k-th power of the transition matrix p whose elements are given by $p(x, y) = b(x, y)/\deg(x)$ where $\deg(x) = \sum_{z \in X} b(x, z) + c(x)$. Show that, for all $x, y \in X$,

$$\int_0^\infty e^{-tL} 1_y(x) dt = \frac{1}{\deg(x)} \sum_{k=0}^\infty p_k(x, y).$$

Notes

Recurrence is a classical and well-established topic going back at least to the work of Pólya [**Pól21**]. As such, we do not even attempt to give an exhaustive nor detailed historical overview of the development of the subject. We only mention some major summarizing works and refer the reader to them for further historical background.

For Riemannian manifolds, a list of equivalences for recurrence can be found in the works of Grigor′yan [**Gri99, Gri09**], which also give a rather extensive historical overview starting with the work of Ahlfors on Riemannian surfaces [**Ahl52**]. For random walks on graphs, we mention the books of Woess [**Woe00**] as well as Soardi [**Soa94**] and Doyle/Snell [**DS84**], which also provide a connection to electrical networks, see also [**LP16, JP**]. These works also have a strong probabilistic focus which we only touch upon in Section 6, which is basically taken from [**Woe00**]. Let us mention that the works above for graphs deal mostly with discrete time. The connection between recurrence for discrete and continuous time can be found, for instance, in [**Sch17b**].

Fundamental to our approach is the form perspective on recurrence based on the idea of the extended Dirichlet space which appears as $\mathcal{D}_0$ in our presentation. This approach goes back to work of Silverstein [**Sil74**] and was further developed in the work of Fukushima/Ōshima/Takeda [**FŌT11**] and Chen/Fukushima [**CF12**].

Let us also mention the approach via potential theory, for which a historical overview is provided in the book edited by Brelot [**Bre10**]. Potential theory focuses on the study of the Green's function in the context of superharmonic functions. For a classical overview of potential theory we refer to the book of Helms [**Hel14**], for the graph case see the book of Soardi [**Soa94**] and Anandam [**Ana11**].

There are a few non-classical characterization of recurrence in Theorem 6.1. The first involves the Green's formula (iii), which is developed in Section 5. This goes back to the work of Grigor′yan/Masamune [**GM13**] for manifolds and Schmidt [**Sch17b**] for graphs. In the form presented here, this characterization can be found in [**HKL$^+$17**], where it is shown for general Dirichlet forms. Moreover, the characterization in terms of Hardy's inequality found in (viii) does not seem to appear in the standard textbooks and originates in the work of Fitzsimmons [**Fit00**]. For some more recent characterizations of recurrence in the graph setting, see Theorem 11.6.15 in [**Sch20b**].

We also note that our equivalent notions of recurrence also hold more generally for Schrödinger operators. In this case, this is referred to as criticality theory, see [**Pin88**] for elliptic operators on domains in Euclidean space, [**Tak14**] for Schrödinger forms and [**KPP20**] for the graph setting.

CHAMBER 7

Stochastic Completeness

All the math got me open like fallopian tubes.
RZA.

In this chapter we study a phenomenon called stochastic completeness. At a basic level, this phenomenon concerns the heat equation on $\ell^\infty(X)$ and the preservation of heat for a graph b over (X, m). We will give a variety of perspectives on this property. In particular, we will show that a Green's formula, uniqueness of bounded solutions of the Poisson equation, triviality of α-harmonic bounded functions for $\alpha > 0$, a maximum principle and uniqueness of bounded solutions of the heat equation and stochastic completeness are all equivalent. We also introduce the more general notion of stochastic completeness at infinity, which allows us to carry out a similar analysis for the heat equation on $\ell^\infty(X)$ for graphs (b, c) with a general killing term over (X, m).

We start by recalling some basic definitions concerning the heat equation. Let (b, c) be a graph over a discrete measure space (X, m). A function $u\colon [0,\infty)\times X \longrightarrow \mathbb{R}$ is called a solution of the heat equation with initial condition $f \in C(X)$ if

- $t \mapsto u_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ for all $x \in X$
- u_t belongs to $\mathcal{F}$ for all $t > 0$

and u satisfies $u_0 = f$ as well as

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t > 0$.

A solution of the heat equation u is said to be *bounded* if u is a bounded function, i.e., $\sup_{t\in[0,\infty)} \sup_{x\in X} |u_t(x)| < \infty$. The function u_0 is called the initial condition for the solution u and $(\mathcal{L} + \partial_t)u = 0$ is called the heat equation. If u instead satisfies

$$(\mathcal{L} + \partial_t)u \geq 0,$$

then u is called a supersolution of the heat equation.

We now recall the basics of the $\ell^2(X, m)$ theory for the heat equation in order to motivate our discussion of the heat equation for bounded functions. Let $L = L^{(D)}_{b,c,m}$ be the Laplacian of the regular Dirichlet form $Q = Q^{(D)}_{b,c,m}$. As can be seen from the spectral theorem, for $f \in \ell^2(X, m)$, the heat equation admits a solution $u_t = e^{-tL}f$ which is in $\ell^2(X,m)$ for all $t \geq 0$ and has initial condition $u_0 = f$. Furthermore, it can be shown that this is the unique solution in $D(L)$ which has f as the initial condition. This case is discussed in detail in Appendices A and D, see, in particular, Theorems A.33 and D.6.

As we are interested in bounded solutions of the heat equation for bounded initial conditions, we have to extend the above discussion to the space of bounded functions. Thus, we now recall the necessary basics of the extension theory of the semigroups and resolvents on $\ell^2(X,m)$ to all $\ell^p(X,m)$ spaces for $p \in [1,\infty]$, see Section 1 for further details. We recall that

$$\ell^p(X,m) = \{f \in C(X) \mid \sum_{x\in X} |f(x)|^p m(x) < \infty\}$$

for $p \in [1,\infty)$ and

$$\ell^\infty(X) = \{f \in C(X) \mid \sup_{x\in X} |f(x)| < \infty\}.$$

As shown in Theorems 2.9 and 2.11, the semigroup e^{-tL} for $t > 0$ and the resolvent $\alpha(L+\alpha)^{-1}$ for $\alpha > 0$ on $\ell^2(X,m)$ extend to bounded operators on $\ell^p(X,m)$ for all $p \in [1,\infty]$. The extensions to different $\ell^p(X,m)$ spaces agree on the intersection of the $\ell^p(X,m)$ spaces in question. Therefore, with a slight abuse of notation, we do not distinguish in notation between the semigroup (or resolvent) on $\ell^p(X,m)$ and the semigroup (or resolvent) on $\ell^2(X,m)$ in this chapter.

These extended semigroups and resolvents are Markov, i.e., they map positive functions to positive functions and functions bounded above by 1 to functions bounded above by 1. Furthermore, the semigroups and resolvents are also contractions in the sense that

$$\|e^{-tL}\| \leq 1 \qquad \text{and} \qquad \|\alpha(L+\alpha)^{-1}\| \leq 1$$

for all $t \geq 0$ and $\alpha > 0$.

We will use throughout that the semigroup on $\ell^p(X,m)$ and the semigroup on $\ell^q(X,m)$ for $1/p + 1/q = 1$ are dual to each other. Specifically, if $f \in \ell^p(X,m)$ and $g \in \ell^q(X,m)$ with $1/p + 1/q = 1$ and $(f,g) = \sum_{x\in X} f(x)g(x)m(x)$ denotes the dual pairing between these spaces, then

$$(e^{-tL}f, g) = (f, e^{-tL}g)$$

for all $t \geq 0$. As they are Markov, the semigroup and resolvent admit a positive kernel. As a consequence, for a positive function $f \in \ell^p(X,m)$ for $p \in [1,\infty]$ the semigroup $e^{-tL}f$ and the resolvent $(L+\alpha)^{-1}f$ can be obtained via monotone limits $e^{-tL}\varphi_n$ and $(L+\alpha)^{-1}\varphi_n$, where φ_n are positive functions in $C_c(X)$ such that $\varphi_n \nearrow f$ pointwise.

We now focus on the heat equation and introduce the property of interest in this chapter. As is the case for $\ell^2(X,m)$, we will show that if $f \in \ell^\infty(X)$, then $u_t = e^{-tL}f$ is a bounded solution of the heat equation with initial condition f. This applies, in particular, to the constant function 1 which is equal to 1 everywhere on X. Since the semigroup and the resolvent are Markov on $\ell^\infty(X)$, we have

$$0 \leq e^{-tL}1 \leq 1 \qquad \text{and} \qquad 0 \leq \alpha(L+\alpha)^{-1} \leq 1$$

for $t \geq 0$ and $\alpha > 0$.

DEFINITION 7.1 (Stochastic completeness). A graph (b, c) over (X, m) which satisfies one (equivalently, both) of the following equalities

$$e^{-tL}1 = 1 \qquad \text{and} \qquad \alpha(L+\alpha)^{-1}1 = 1$$

is called *stochastically complete* or *conservative* for all $t \geq 0$ and $\alpha > 0$. Otherwise, (b, c) over (X, m) is called *stochastically incomplete*.

It is not hard to show that whenever $c \neq 0$ neither of the equalities hold, see the remark after the proof of Theorem 7.2 below. So, Theorem 7.2 characterizes the validity of the equalities in the case $c = 0$, that is, in the case of no killing term. Later in this chapter, we will also address the general case of $c \geq 0$ under the name of stochastic completeness at infinity. If a graph is stochastically complete, then $c = 0$ and stochastic completeness and stochastic completeness at infinity are the same. However, when $c \neq 0$, it is possible for a graph to be stochastically incomplete while being stochastically complete at infinity. The basic idea for stochastic completeness at infinity is that we store the heat removed by c and add it to $e^{-tL}1$. In order to make this idea precise, we have to extend the semigroup to general positive functions.

Let us next give an interpretation of the equation $e^{-tL}1 = 1$ in terms of the preservation of heat. For this discussion, we will assume $c = 0$ as otherwise $e^{-tL}1 = 1$ does not hold. Let $f \in \ell^1(X, m)$ with $f \geq 0$ represent a distribution of heat on X at time $t = 0$. In other words, $f \in \ell^1(X, m)$ is an initial condition for the heat equation. Then, the amount of heat at time $t \geq 0$ at a vertex x is given by $e^{-tL}f(x)$. Using the dual pairing $(\cdot, \cdot)$ between $\ell^\infty(X)$ and $\ell^1(X, m)$, the fact that the heat semigroups on these spaces are dual to one another and $e^{-tL}1 \leq 1$, we calculate

$$\sum_{x\in X} e^{-tL}f(x)m(x) = (1, e^{-tL}f) = (e^{-tL}1, f) \leq (1, f) = \sum_{x\in X} f(x)m(x).$$

The left-hand side of the equation is the amount of heat in X at time $t \geq 0$ and the right-hand side of the equation is the amount of heat in X at time $t = 0$. Hence, whenever the inequality is strict, there is less heat in the graph at time $t > 0$ than at the beginning, in other words, the system has lost heat.

From another viewpoint, the equality $e^{-tL}1 = 1$ can be understood as a uniqueness condition on bounded solutions of the heat equation.

Specifically, both $e^{-tL}1$ and 1 are a bounded solutions of the heat equation with initial condition 1. Hence, if bounded solutions of the heat equation are uniquely determined by initial conditions, then $e^{-tL}1 = 1$ and the graph is stochastically complete. It turns out that the opposite implication also holds, that is, stochastic completeness implies that bounded solutions of the heat equation are uniquely determined by initial conditions.

Informally, the reason for the possible loss of heat is strong growth of the geometry of (b, c) which pushes heat to infinity. In fact, if the graph is connected and if heat is lost at some time, then it is lost at any other time. Hence, stochastic incompleteness is an instantaneous phenomenon. From a probabilistic perspective, this means that the process has a finite lifetime.

Finally, we recall that $u \in \mathcal{F}$ is called α-harmonic for $\alpha \in \mathbb{R}$ if $(\mathcal{L} + \alpha)u = 0$ and α-subharmonic if $(\mathcal{L} + \alpha)u \leq 0$. We will also see that stochastic completeness is equivalent to the triviality of bounded positive α-(sub)harmonic functions for $\alpha > 0$.

After these preparations, we now state the various characterizations of stochastic completeness.

THEOREM 7.2 (Characterization of stochastic completeness). *Let b be a connected graph over (X, m). Then, the following statements are equivalent:*

(i) *For some (all) $t > 0$ and some (all) $x \in X$,*
$$e^{-tL}1(x) = 1.$$

(i.a) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*
$$\alpha(L + \alpha)^{-1}1(x) = 1$$

(ii) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, v) = 0$$
for all $v \in D(Q) \cap \ell^1(X, m)$.

(ii.a) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, (L + \alpha)^{-1}v) = 0$$
for one $v \in \ell^2(X, m) \cap \ell^1(X, m)$ with $v > 0$ and some (all) $\alpha > 0$.

(iii) *If $v \in \mathcal{D} \cap \ell^1(X, m) \cap \ell^2(X, m)$ satisfies $\mathcal{L}v \in \ell^1(X, m)$, then*
$$\sum_{x\in X} \mathcal{L}v(x)m(x) = 0. \qquad (\text{“Green's formula”})$$

(iii.a) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m) \cap \ell^2(X,m)$, *then*

$$\sum_{x\in X} \mathcal{L}v(x)m(x) = 0.$$

(iv) *If* $u \in \mathcal{F}$ *satisfies* $\sup u \in (0,\infty)$ *and* $\beta \in (0, \sup u)$, *then*

$$\sup_{X_\beta} \mathcal{L}u \geq 0,$$

where $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$.

(*"Omori–Yau maximum principle"*)

(v) *For some (all)* $\alpha > 0$ *and every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the Poisson equation*

$$(\mathcal{L} + \alpha)u = f. \qquad \text{(“Poisson equation”)}$$

(v.a) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u \leq 0$ *is trivial.*

(v.b) *For some (all)* $\alpha > 0$ *every* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(v.c) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L} + \alpha)u = 0$ *is trivial.*

(vi) *For every* $f \in \ell^\infty(X)$ *there exists a unique bounded solution* u *of the heat equation*

$$(\mathcal{L} + \partial_t)u = 0 \quad \text{with} \quad u_0 = f. \quad \text{(“Heat equation”)}$$

(vi.a) *Every bounded solution* u *of the heat equation* $(\mathcal{L} + \partial_t)u = 0$ *with* $u_0 = 0$ *is trivial.*

Remark. The characterizations given in (v) and (vi) above should be understood as uniqueness statements since the existence of bounded solutions is always guaranteed by means of the extended semigroups and resolvents.

The proof of Theorem 7.2 is divided into several theorems which we prove in the upcoming sections. Indeed, we prove the equivalence of these statements for the more general notion of stochastic completeness at infinity, which allows for a non-vanishing killing term c. Below we give a summary of how the results proven in the upcoming sections come together.

Proof of Theorem 7.2. The equivalences (i) $\Longleftrightarrow$ (i.a) $\Longleftrightarrow$ (vi) $\Longleftrightarrow$ (vi.a) which connect stochastic completeness and uniqueness of bounded solutions of the heat equation are proven in Theorem 7.16 in Section 3.

The equivalences (i) $\Longleftrightarrow$ (v) $\Longleftrightarrow$ (v.a) $\Longleftrightarrow$ (v.b) $\Longleftrightarrow$ (v.c) which relate stochastic completeness to the Poisson equation as well as α-(sub)harmonic functions are proven in Theorem 7.18 in Section 4.

The equivalences (i.a) $\iff$ (ii) $\iff$ (ii.a) which characterize stochastic completeness in terms of the constant function 1 being suitably approximated are proven in Theorem 7.23 in Section 5.

The equivalence (ii) $\iff$ (ii.a) $\iff$ (iii) $\iff$ (iii.a) connecting Green's formula and the ability to approximate 1 is proven in Theorem 7.26 in Section 6. Combined with the results above, this gives the Green's formula perspective on stochastic completeness.

The equivalence (iv) $\iff$ (v.a)/(v.c) connecting the Omori–Yau maximum principle and triviality of bounded positive α-(sub)harmonic functions for $\alpha > 0$ is proven in Theorem 7.28 in Section 7. Combined with the results above, this gives the Omori–Yau maximum perspective on stochastic completeness. $\square$

The proofs of the theorems which combine to prove Theorem 7.2 are given in the subsequent sections. More specifically, we start by discussing properties of bounded solutions to the heat equation in Section 1. In Section 2 we introduce the concept of stochastic completeness at infinity. This requires a bit of work as we have to extend the semigroup and resolvent on bounded functions to general positive functions by monotone approximation and the use of nets. The reader who is only interested in stochastic completeness may skip Section 2 as we point out how to substitute the results needed from this section in subsequent proofs. Sections 3, 4, 5, 6 and 7 are dedicated to the proof of the results mentioned above. Section 8 gives an additional criterion for stochastic completeness at infinity which is useful in certain situations such as comparison results. Finally, in Section 9 we discuss the probabilistic point of view on stochastic completeness and stochastic completeness at infinity.

Remark (Stochastic completeness implies $c = 0$). We have already discussed stochastic completeness for finite graphs in Section 8. In the case of finite graphs, a graph is stochastically complete if and only if $c = 0$, see Theorem 0.65. In particular, whenever we take the Dirichlet restriction of the form $Q = Q^{(D)}_{b,c,m}$ to a finite subset of an infinite connected graph, the resulting graph is stochastically incomplete as $c \neq 0$ in this case. However, for infinite graphs, we will see that a graph can be stochastically incomplete even if $c = 0$. On the other hand, stochastic completeness always implies $c = 0$ (Exercise 7.5).

1. The heat equation on ℓ^∞

In this section we study the heat equation on the space of bounded functions. In particular, we show that there always exists a bounded solution of the heat equation for a given bounded initial condition. We also show how bounded solutions of the heat equation with zero initial conditions generate bounded α-harmonic functions for $\alpha > 0$.

Excavation Exercise 7.1 recalls Dini's theorem while Excavation Exercise 7.2 discusses the convergence of continuously differentiable functions. These are used in the proof of Theorem 7.3.

To show the existence of bounded solutions of the heat equation for any bounded initial condition, we apply the semigroup e^{-tL} originally defined on $\ell^2(X,m)$ and then extended to $\ell^\infty(X)$ to a given bounded function. We will show that such a solution is positive whenever the initial condition is positive and that the heat semigroup generates the minimal solution of the heat equation whenever the initial condition is positive. This extends the corresponding statement for $\ell^2(X,m)$ in Lemma 1.24 and for $\ell^p(X,m)$ for $p \in [1,\infty)$ in Theorem 2.14.

We note that for the existence part of the proof in the $\ell^2(X,m)$ case we used the spectral theorem found in Appendix A and in the $\ell^p(X,m)$ case for $p \in [1,\infty)$ and initial conditions in $D(L^{(p)})$ we used the general theory of strongly continuous semigroups found in Appendix D. However, the semigroup on $\ell^\infty(X)$ is only weak* continuous and as such, we cannot use the general theory. Furthermore, we are interested in solutions to general bounded initial conditions. Thus, we give full details for the existence proof in this case below.

Theorem 7.3 (Existence of bounded solutions of the heat equation). *Let (b,c) be a graph over (X,m) and let $f \in \ell^\infty(X)$. If*

$$u_t(x) = e^{-tL} f(x)$$

for $t \geq 0$ and $x \in X$, then u is a bounded solution of the heat equation with initial condition f.

Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

Proof. We start by showing the continuity and boundedness of u. We denote the dual pairing between $\ell^1(X,m)$ and $\ell^\infty(X)$ by $(\cdot,\cdot)$ and let $\eta_x \in \ell^1(X,m)$ for $x \in X$ be given by $\eta_x = 1_x/m(x)$. Since

$$u_t(x) = (\eta_x, e^{-tL} f)$$

for $x \in X$ and $t \geq 0$, continuity of the function $t \mapsto u_t(x)$ for $t \geq 0$ and $x \in X$ follows from the weak* continuity of the semigroup on $\ell^\infty(X)$ established in Theorem 2.9. Furthermore, as the semigroup on $\ell^1(X,m)$ is strongly continuous, we have $u_0 = f$. Finally, as e^{-tL} is a contraction semigroup on $\ell^\infty(X)$ by Theorem 2.9, it follows that u_t is bounded by $\|f\|_\infty$ for every $t \geq 0$. In particular, $u_t \in \mathcal{F}$ for every $t \geq 0$.

As an intermediate step, we next show the continuity of

$$t \mapsto \mathcal{L}u_t(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x,y)(u_t(x) - u_t(y)) + c(x) u_t(x) \right)$$

on $[0,\infty)$ for every $x \in X$. Indeed, this is immediate from the continuity of $t \mapsto u_t(y)$ for each $y \in X$, the uniform boundedness of u in both variables, and the summability of $b(x,\cdot)$ for every $x \in X$.

We will now show differentiability and the fact that u satisfies the heat equation. We will do so by approximating f by functions with finite support and using that the heat equation holds for functions in $\ell^2(X,m)$ and, hence, for functions with finite support.

We note that the preceding considerations hold for any bounded function f and, in particular, for the elements of the approximating sequence f_n and $u_t^{(n)} = e^{-tL} f_n$ which we introduce below. Hence, the functions $t \mapsto u_t(x), t \mapsto u_t^{(n)}(x)$, $t \mapsto \mathcal{L}u_t(x)$ and $t \mapsto \mathcal{L}u_t^{(n)}(x)$ will be continuous for all $x \in X$ and all $n \in \mathbb{N}$.

Let $t > 0$. By decomposing f into positive and negative parts, we can assume without loss of generality that f is positive. Let (K_n) be a sequence of finite increasing subsets of X such that $X = \bigcup_n K_n$. Furthermore, let $f_n = f 1_{K_n}$ and let

$$u_t^{(n)} = e^{-tL} f_n.$$

Since e^{-tL} on $\ell^\infty(X)$ is a bounded Markov semigroup by Theorem 2.9, e^{-tL} admits a positive kernel p. That is,

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x,y) f(y) m(y),$$

where $p_t(x,y) \geq 0$ for all $x, y \in X$ and $t > 0$. Thus, $u_t^{(n)}(x) \nearrow u_t(x)$ as $n \to \infty$ for all $x \in X$ and $t > 0$. Moreover, the convergence is uniform on compact subintervals of $(0,\infty)$ by Dini's theorem as both $u_t^{(n)}(x)$ and $u_t(x)$ are continuous functions.

Since $f_n \in C_c(X) \subseteq \ell^2(X,m) \cap \ell^\infty(X)$ and the semigroup on $\ell^\infty(X)$ agrees with the semigroup on $\ell^2(X,m)$ for all functions in $\ell^2(X,m) \cap \ell^\infty(X)$ by Theorem 2.9, it follows that $u_t^{(n)} \in \ell^2(X,m)$. Therefore, for all $x \in X$, $t \geq 0$ and $n \in \mathbb{N}$, we infer by Lemma 1.24 that

$$\begin{aligned}
\partial_t u_t^{(n)}(x) &= -L u_t^{(n)}(x) \\
&= -\mathcal{L} u_t^{(n)}(x) \\
&= -\frac{1}{m(x)} \left(\sum_{y \in X} b(x,y)(u_t^{(n)}(x) - u_t^{(n)}(y)) + c(x) u_t^{(n)}(x) \right).
\end{aligned}$$

Monotone convergence of $u_t^{(n)}(y)$ to $u_t(y)$ for all $y \in X$, and the fact that $u_t \in \ell^\infty(X) \subseteq \mathcal{F}$, yields the convergence of the right-hand side to $-\mathcal{L}u_t(x)$ as $n \to \infty$ for each $x \in X$ and $t \geq 0$. Therefore, we obtain the convergence of $\partial_t u_t^{(n)}(x)$ to $\mathcal{L}u_t(x)$ as $n \to \infty$ for each $x \in X$ and $t > 0$. In fact, this convergence is uniform in t on compact subintervals of $(0,\infty)$ as the convergence of the $u_t^{(n)}(y)$ to $u_t(y)$ is uniform on compact subintervals of $(0,\infty)$ for each $y \in X$ and $b(x,\cdot)$ is summable for each $x \in X$.

Altogether we have established that the $u_t^{(n)}(x)$ converge uniformly on compact subintervals of $(0,\infty)$ to $u_t(x)$ and the $\partial_t u_t^{(n)}(x)$ converge uniformly on compact subintervals of $(0,\infty)$ to $-\mathcal{L}u_t(x)$ as $n \to \infty$ for each $x \in X$. As discussed above, all involved functions are continuous. Thus, this gives that $t \mapsto u_t(x)$ is differentiable with the desired derivative.

It remains to show the last statement of the theorem. That u is positive whenever f is positive follows immediately from the fact that the semigroup on $\ell^\infty(X)$ is Markov and, in particular, positivity preserving by Theorem 2.9. We now show the minimality statement. Let w be a supersolution of the heat equation with initial condition greater than or equal to f. From what we have shown above, $u^{(n)}$ satisfies

$$(\mathcal{L} + \partial_t)u_t^{(n)} = 0$$

for $t > 0$ and $u_0^{(n)} = f_n$ for $n \in \mathbb{N}$. Furthermore, the $u^{(n)}$ agree with the solution generated by the semigroup of $\ell^2(X,m)$ as the semigroups agree on their common domain by Theorem 2.9. As w is a positive supersolution with initial condition greater than or equal to f_n we obtain $u^{(n)} \leq w$ for all n by Lemma 1.24. Letting $n \to \infty$ gives $u \leq w$ which completes the proof. $\square$

The argument in the preceding proof deals with compact subintervals of $(0,\infty)$. We could equally well work with compact subintervals of $[0,\infty)$, as can be seen by carefully going through the argument. This would yield that the function u_t satisfies the heat equation at $t = 0$ as well, i.e., on the entire interval $[0,\infty)$. In the next proposition we provide a more general argument showing this for all bounded solutions of the heat equation irrespective of whether they have the form $e^{-tL}f$ or not.

Proposition 7.4 (Heat equation at $t = 0$). *Let (b,c) be a graph over (X,m) and let u be a bounded solution of the heat equation. Then, the function $[0,\infty) \longrightarrow \mathbb{R}$,*

$$t \mapsto \mathcal{L}u_t(x),$$

is continuous for all $x \in X$, the limit

$$\partial_t u_t(x)|_{t=0} = \lim_{h \to 0^+} \frac{u_h(x) - u_0(x)}{h}$$

exists for all $x \in X$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t \geq 0$.

PROOF. We first show the continuity of $t \mapsto \mathcal{L}u_t(x)$ on $[0,\infty)$: As u is a solution of the heat equation the map $t \mapsto u_t(y)$ is continuous for each $y \in X$. As $b(x,\cdot)$ is summable for each $x \in X$ and u is bounded, we then obtain that

$$\mathcal{L}u_t(x) = \frac{1}{m(x)}\left(\sum_{y\in X} b(x,y)(u_t(x) - u_t(y)) + c(x)u_t(x)\right)$$

is continuous at every $t \in [0,\infty)$. Moreover, as u is a solution of the heat equation, we have

$$\partial_t u_t(x) = -\mathcal{L}u_t(x)$$

for all $t > 0$.

Altogether, for each $x \in X$, the function $t \mapsto u_t(x)$ is a continuous function on $[0,\infty)$ which is differentiable on $(0,\infty)$ and $t \mapsto -\mathcal{L}u_t(x)$ is a continuous function on $[0,\infty)$ which agrees with the derivative of $t \mapsto u_t(x)$ on $(0,\infty)$. This implies that $u_t(x)$ is differentiable at $t = 0$ with derivative given by $-\mathcal{L}u_0(x)$: Indeed, by the mean value theorem, for every $x \in X$ and $h > 0$, there exists a $\zeta(h) \in (0,h)$ such that

$$\frac{u_h(x) - u_0(x)}{h} = \partial_t u_t|_{t=\zeta(h)} = -\mathcal{L}u_{\zeta(h)}(x).$$

This implies

$$\lim_{h\to 0^+} \frac{u_h(x) - u_0(x)}{h} = -\mathcal{L}u_0(x)$$

by continuity. □

REMARK. We observe that the proofs of Theorem 7.3 and Proposition 7.4 given above extend to initial conditions $f \in \ell^p(X,m)$ for $p \in [1,\infty]$ whenever the graph is locally finite. Indeed, the actual argument works as soon as continuity of the function $t \mapsto u_t(y)$ for all $y \in X$ implies continuity of $t \mapsto \mathcal{L}u_t(x)$ for all $x \in X$.

REMARK. In the case of a uniformly positive measure, which is treated in the next chapter, we have $\ell^p(X,m) \subseteq \ell^\infty(X)$ for all $p \in [1,\infty]$ with continuous inclusions. In this case, the theorem above implies the existence of solutions to the heat equation for initial conditions $f \in \ell^p(X,m)$ for $p \in [1,\infty]$ as well.

The preceding considerations apply to the heat equation with initial condition 1. In this case, $u_t = e^{-tL}1$ is the minimal positive solution of the heat equation with $u_0 = 1$. The constant function 1 is also such a solution and stochastic completeness is defined as the equivalence of these two solutions, that is, $e^{-tL}1 = 1$. From this discussion, it is clear that if bounded solutions of the heat equation are uniquely determined by initial conditions, then a graph is stochastically complete. We will see later that the converse is true as well.

We now present an easy consequence of the semigroup and Markov properties of the heat semigroup on $\ell^\infty(X)$. This result will be useful in showing that if the total amount of heat in the graph drops below 1 at some time, then it drops below 1 for all times.

LEMMA 7.5. *Let (b,c) be a graph over (X,m). If $s \geq t \geq 0$, then*
$$e^{-sL}1 \leq e^{-tL}1.$$

PROOF. From Theorem 2.9, we get that the heat semigroup is both positivity preserving and contracting. Let $s = t + h$ with $t, h \geq 0$. As the semigroup is contracting we have
$$e^{-hL}1 \leq 1.$$
As the semigroup is positivity preserving, this gives, after we apply e^{-tL} to both sides,
$$e^{-sL}1 = e^{-tL}e^{-hL}1 \leq e^{-tL}1.$$
This is the desired statement. □

The next proposition connects bounded solutions of an inhomogeneous heat equation with solutions of the Poisson equation.

PROPOSITION 7.6 (Solutions of heat and Poisson equations). *Let (b,c) be a graph over (X,m). Let $f, g \in \ell^\infty(X)$ and let $u\colon [0,\infty) \times X \longrightarrow \mathbb{R}$ be a bounded solution of*
$$(\mathcal{L} + \partial_t)\, u_t = f$$
with initial condition $u_0 = g$. Then, for $\alpha > 0$ the function
$$v = \int_0^\infty \alpha e^{-t\alpha} u_t dt$$
is bounded and satisfies
$$(\mathcal{L} + \alpha)v = f + \alpha g.$$
Moreover, if additionally $f, g \geq 0$, then
$$w = \int_0^\infty e^{-t\alpha} e^{-tL}(f + \alpha g) dt$$
is the smallest positive function $w \in \mathcal{F}$ with $(\mathcal{L} + \alpha)w \geq f + \alpha g$. In particular,
$$\int_0^\infty e^{-t\alpha} e^{-tL}(f + \alpha g) dt \leq \int_0^\infty \alpha e^{-t\alpha} u_t dt.$$

PROOF. The boundedness of v follows since we assume that u is bounded and since $\alpha e^{-t\alpha} dt$ has total measure 1 on $[0,\infty)$.

Furthermore, by the boundedness of u and Fubini's theorem, we have for all $x \in X$
$$\mathcal{L}v(x) = \int_0^\infty \alpha e^{-t\alpha} \mathcal{L}u_t(x) dt = \lim_{T\to\infty} \int_0^T \alpha e^{-t\alpha} \mathcal{L}u_t(x) dt.$$

Since u satisfies $\mathcal{L}u_t = -\partial_t u_t + f$, we infer

$$\begin{aligned}\ldots &= \lim_{T\to\infty}\int_0^T \alpha e^{-t\alpha}(-\partial_t u_t(x))dt + \int_0^\infty \alpha e^{-t\alpha} f(x)dt\\ &= \lim_{T\to\infty}\left(-\alpha e^{-t\alpha}u_t(x)|_0^T - \int_0^T \alpha^2 e^{-t\alpha}u_t(x)dt\right) + f(x),\end{aligned}$$

where we used integration by parts and the fact that $\alpha e^{-t\alpha}dt$ has total measure 1 on $[0,\infty)$. Next, we conclude from the boundedness of u and $u_0 = g$ that the first term tends to αg and, therefore,

$$\begin{aligned}\ldots &= \alpha g(x) - \alpha\int_0^\infty \alpha e^{-t\alpha}u_t(x)dt + f(x)\\ &= \alpha g(x) - \alpha v(x) + f(x)\end{aligned}$$

by the definition of v. Therefore, v is bounded and satisfies $(\mathcal{L}+\alpha)v = f + \alpha g$.

If $f, g \in \ell^\infty(X)$, then

$$\int_0^\infty e^{-t\alpha}e^{-tL}(f+\alpha g)dt = (L+\alpha)^{-1}(f+\alpha g)$$

by the Laplace transform formula, see Theorem 2.11. Furthermore, if f and g are positive, Theorem 2.12 gives that $(L+\alpha)^{-1}(f+\alpha g)$ is the minimal positive function $w \in \mathcal{F}$ with $(\mathcal{L}+\alpha)w \geq f + \alpha g$. As v is such a function by what we have shown above, $(L+\alpha)^{-1}(f+\alpha g) \leq v$ follows. □

As a particular consequence of the proposition above, we get that bounded solutions of the heat equation with vanishing initial conditions give rise to α-harmonic functions, i.e., to solutions of the equation $(\mathcal{L}+\alpha)v = 0$. In particular, if a bounded solution of the heat equation is positive and non-trivial, then the arising α-harmonic function is positive and non-trivial. If a graph is stochastically incomplete, i.e., $e^{-tL}1 < 1$, we get that $u_t = 1 - e^{-tL}1$ is a positive non-trivial bounded solution of the heat equation with trivial initial condition. Therefore, there exists a positive bounded non-trivial α-harmonic function for any $\alpha > 0$ in this case. This will be used later in the proof of our characterization of stochastic completeness.

Corollary 7.7 (Solutions of the heat equation and α-harmonic functions). *Let (b,c) be a graph over (X,m) and let u be a bounded solution of the heat equation with $u_0 = 0$. Then, for $\alpha > 0$ the function*

$$v = \int_0^\infty e^{-t\alpha}u_t dt$$

is bounded and satisfies

$$(\mathcal{L}+\alpha)v = 0.$$

In particular, if there exists a positive non-trivial bounded solution of the heat equation with trivial initial conditions, then there exists a positive non-trivial bounded α-harmonic function for any $\alpha > 0$.

PROOF. This follows immediately from Proposition 7.6 by letting f and g be 0. $\square$

2. Stochastic completeness at infinity

In this section we introduce the concept of stochastic completeness at infinity. This will allow us to discuss the notion of conservation of heat in the case of a non-vanishing killing term. In order to do so, we extend the semigroup and the resolvent of a graph to arbitrary positive functions by means of monotone convergence.

We have already introduced the notion of stochastic completeness via the equality $e^{-tL}1 = 1$ for all $t > 0$. It was discussed in the remark following Theorem 7.2 that a non-vanishing killing term instantly removes heat from a space so that $e^{-tL}1 < 1$ for all $t > 0$ whenever the graph is connected and $c \neq 0$. In particular, all connected graphs with a non-vanishing killing term are stochastically incomplete.

Therefore, in order to deal with the case of a general killing term, we need to introduce a new concept. We will use monotone limits to apply the heat semigroup e^{-tL}, originally defined on $\ell^2(X,m)$ and then extended to $\ell^\infty(X)$, to arbitrary positive functions. In particular, we apply the heat semigroup to the function c/m and define the quantity

$$M_t(x) = e^{-tL}1(x) + \int_0^t \left(e^{-sL}\frac{c}{m}\right)(x)ds$$

for $x \in X$ and $t \geq 0$. The function M_t serves as a replacement of $e^{-tL}1$ in the case of a non-vanishing c. We note that some care has to be taken since even the finiteness of M is not immediately clear. However, we will show that $0 \leq M_t \leq 1$ for all $t \geq 0$ and that the function $t \mapsto M_t(x)$ is continuous and differentiable for all $x \in X$. Furthermore, we will see that M satisfies a modified heat equation.

Clearly, $M_t = e^{-tL}1$ if $c = 0$. On the other hand, we will see that $M_t > e^{-tL}1$ on any connected component on which c does not vanish identically since the extended semigroup is positivity improving. The term $e^{-tL}1$ can be interpreted as the amount of heat in the graph at time t given a constant initial distribution of heat. The integral term in M_t can be interpreted as the amount of heat killed by c in the graph up to the time t. Hence, adding these two terms gives the total amount of heat that is either in the graph at time t or has been removed from the graph by c up to time t.

While c directly removes heat from the graph, we will see that heat can also disappear from the graph by being transported to "infinity" via the geometry even when no killing term is present. In the case of non-vanishing c, the function M_t can be interpreted as the amount of heat not transported to "infinity" via the geometry so that $1 - M_t$ is the amount of heat transported to "infinity" via geometry at time t. In this section we study the question if

$$M_t = 1$$

for all $t > 0$, that is, if none of the heat has been transported to "infinity" via the geometry.

After this discussion, we now work towards proving the statements above. In order to define M, we have to extend the semigroup to general positive functions to make sense of the term $e^{-tL}(c/m)$. To this end, we note that for a positive function $v \in C(X)$ the functions $\varphi \in C_c(X)$ with $0 \leq \varphi \leq v$ form a net with respect to the partial ordering $g \prec h$ whenever $g \leq h$ for $g, h \in C(X)$. More specifically, the set $C_v(X) = \{\varphi \in C_c(X) \mid 0 \leq \varphi \leq v\}$ is both a directed set with the partial ordering $\prec$ as well as a topological space with respect to pointwise convergence. We denote limits along this net by $\lim_{\varphi \prec v}$.

By Theorems 2.9 and 2.11, the semigroup and the resolvent are positivity preserving on $C_c(X)$. Therefore, for $f \in C(X)$ with $f \geq 0$, we can define the functions $e^{-tL}f \colon X \longrightarrow [0,\infty]$ for $t \geq 0$ and $(L+\alpha)^{-1}f \colon X \longrightarrow [0,\infty]$ for $\alpha > 0$ via

$$\begin{aligned} e^{-tL}f(x) &= \lim_{\varphi \prec f} e^{-tL}\varphi(x) \\ (L+\alpha)^{-1}f(x) &= \lim_{\varphi \prec f}(L+\alpha)^{-1}\varphi(x). \end{aligned}$$

We refer to $e^{-tL}f$ and $(L+\alpha)^{-1}f$ for $f \in C(X)$ with $f \geq 0$ as the *extended semigroup* and *extended resolvent*, respectively. We note that they may both take the value infinity. We will give some abstract criteria for when they are finite below. Furthermore, since the semigroup and resolvent are bounded Markov operators on $\ell^p(X,m)$, they admit positive kernels. Therefore, the definitions above agree with the semigroup and the resolvent for functions in $\ell^p(X,m)$ for $p \in [1,\infty]$. For this reason we do not distinguish between them in notation.

We next collect some basic properties of the extended semigroup and resolvent. As we define the extended semigroup and resolvent as limits over finitely supported functions, where all semigroups and resolvent and their generators agree, we note that we can use properties of either $L = L^{(D)}$ or $L = L^{(p)}$ before passing to the limit.

We first note that the extended semigroup e^{-tL} satisfies the semigroup property, that is,

$$e^{-(s+t)L} = e^{-sL}e^{-tL}$$

for $s, t \geq 0$. This can be seen directly by taking monotone limits.

Next, we show that the extended semigroup and resolvent can be approximated by the restrictions of the Dirichlet Laplacian L to an exhaustion. To this end, we recall that for any finite $K \subseteq X$ the operator $L_K^{(D)}$ is the restriction of $\mathcal{L}$ to functions in $\ell^2(K, m_K)$ which are considered as functions on X by extending by 0. Thus,

$$L_K^{(D)} f(x) = \mathcal{L}\widetilde{f}(x)$$

for all $x \in K$ and $f\colon K \longrightarrow \mathbb{R}$, where $\widetilde{f}\colon K \longrightarrow \mathbb{R}$ satisfies $\widetilde{f} = f$ on K and $\widetilde{f} = 0$, otherwise.

In Section 3, we have shown that the resolvent and semigroup of the Dirichlet Laplacian of an exhaustion converge in a pointwise monotonically increasing manner to the resolvent and semigroup of the Laplacian on the entire space, see Lemma 1.21. We now show that the same is true for the extended resolvent and semigroup.

Lemma 7.8 (Convergence of finite approximations). *Let (b, c) be a graph over (X, m). Let $f \in C(X)$ with $f \geq 0$, let (K_n) be an increasing sequence of finite subsets of X such that $X = \bigcup_n K_n$ and let $f_n = 1_{K_n} f$. Then, for all $x \in X$, $t \geq 0$ and $\alpha > 0$*

$$(L_{K_n}^{(D)} + \alpha)^{-1} f_n(x) \nearrow (L + \alpha)^{-1} f(x)$$
$$e^{-tL_{K_n}^{(D)}} f_n(x) \nearrow e^{-tL} f(x)$$

as $n \to \infty$, where the right-hand sides are allowed to take the value ∞.

Consequently, both $(L + \alpha)^{-1} f$ and $e^{-tL} f$ are strictly positive on any connected component of the graph on which f does not vanish. In particular, if the graph is connected, then the extended resolvent and semigroup are positivity improving.

Proof. We only show the statement for the resolvent as the statement for the semigroup is proven analogously. Let $L_{K_n}^{(D)}$ be the restriction of L to K_n for $n \in \mathbb{N}$. As

$$(L_{K_n}^{(D)} + \alpha)^{-1} f_n \leq (L_{K_{n+1}}^{(D)} + \alpha)^{-1} f_n$$

by domain monotonicity, Proposition 1.20 (c), and $(L_{K_k}^{(D)} + \alpha)^{-1} f_n$ converge to $(L + \alpha)^{-1} f_n$ as $k \to \infty$ by Lemma 1.21, we obtain

$$(L_{K_n}^{(D)} + \alpha)^{-1} f_n \leq (L + \alpha)^{-1} f_n.$$

Combining this with the fact that $f_n \leq f$ we have

$$(L_{K_n}^{(D)} + \alpha)^{-1} f_n \leq (L + \alpha)^{-1} f_n \leq (L + \alpha)^{-1} f$$

for all $n \in \mathbb{N}$.

It remains to show the "reverse" inequality. We consider two cases.

Case 1. $(L + \alpha)^{-1} f(x) < \infty$: Let $\varepsilon > 0$. Then, by the definition of the extended resolvent, there exists a $\varphi \in C_c(X)$ such that $0 \leq \varphi \leq f$ and

$$(L + \alpha)^{-1} f(x) - \varepsilon \leq (L + \alpha)^{-1} \varphi(x).$$

Since $\varphi \in C_c(X)$, the support of φ is included in K_n for all n large. Furthermore, as the resolvents $(L_{K_n} + \alpha)^{-1}$ converge to $(L+\alpha)^{-1}$ on $\ell^2(X,m)$ by Lemma 1.21, we get

$$(L+\alpha)^{-1}\varphi(x) - \varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}\varphi(x)$$

for all n sufficiently large. Together, these two inequalities give for all n large enough that

$$(L+\alpha)^{-1}f(x) - 2\varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}\varphi(x).$$

Since the support of φ is included in K_n, we have $0 \leq \varphi \leq f_n$ as $f = f_n$ on K_n. Thus, as $(L_{K_n}^{(D)} + \alpha)^{-1}$ is positivity preserving by Proposition 1.20 (b),

$$(L+\alpha)^{-1}f(x) - 2\varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x),$$

which finishes the proof in this case as $\varepsilon > 0$ can be chosen arbitrarily small.

Case 2. $(L+\alpha)^{-1}f(x) = \infty$: Let $C > 0$. By the definition of the extended resolvent, there exists a $\varphi \in C_c(X)$ such that $0 \leq \varphi \leq f$ and

$$C \leq (L+\alpha)^{-1}\varphi(x).$$

By similar considerations as in Case 1, we obtain for all $\varepsilon > 0$ and all sufficiently large n

$$C - \varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x),$$

which finishes the proof of the convergence statement.

The fact that $(L+\alpha)^{-1}f(x) > 0$ for all x in a connected component of (b,c) on which f does not vanish follows from the positivity improving property of the resolvent on connected graphs shown in Theorem 1.26. $\square$

In the following lemma we show how the finiteness of the extended resolvents can be characterized via the existence of minimal supersolutions. We note that, in the case of bounded positive functions, the resolvent always gives the minimal solution by Theorem 2.12. The following result shows that the same is true for general positive functions whenever the extended resolvent is finite.

LEMMA 7.9 (Characterizing resolvents as supersolutions). *Let (b,c) be a graph over (X,m). Let $\alpha > 0$ and let $f \in C(X)$ with $f \geq 0$. Then, the following statements are equivalent:*

(i) $(L+\alpha)^{-1}f(x) < \infty$ *for all* $x \in X$.

(ii) *There exists a $v \in \mathcal{F}$ with $v \geq 0$ such that $(\mathcal{L}+\alpha)v \geq f$.*

In this case, $u = (L+\alpha)^{-1}f$ satisfies $(\mathcal{L}+\alpha)u = f$ and is the smallest function $v \in \mathcal{F}$ with $v \geq 0$ and $(\mathcal{L}+\alpha)v \geq f$.

PROOF. (i) $\Longrightarrow$ (ii): Let $\varphi \in C_c(X)$ satisfy $0 \le \varphi \le f$. Since L is a restriction of $\mathcal{L}$ on $C_c(X)$ by Theorem 1.6 we have

$$(\mathcal{L}+\alpha)(L+\alpha)^{-1}\varphi = \varphi.$$

Taking monotone limits on both sides of the equation we obtain that $(L+\alpha)^{-1}f \in \mathcal{F}$ and

$$(\mathcal{L}+\alpha)(L+\alpha)^{-1}f = f$$

by Lemma 1.8. Since $\varphi \ge 0$ it follows that $(L+\alpha)^{-1}\varphi \ge 0$ as the resolvent is positivity preserving by Corollary 1.22. Therefore, we have $(L+\alpha)^{-1}f \ge 0$. This shows (ii) for $v = (L+\alpha)^{-1}f$. Furthermore, this also shows that $u = (L+\alpha)^{-1}f$ solves $(\mathcal{L}+\alpha)u = f$.

(ii) $\Longrightarrow$ (i): Let $v \in \mathcal{F}$ with $v \ge 0$ satisfy $(\mathcal{L}+\alpha)v \ge f$. Let (K_n) be an increasing sequence of finite sets such that $X = \bigcup_n K_n$ and let $f_n = f1_{K_n}$ for $n \in \mathbb{N}$. Let $u_n = (L_{K_n}^{(D)} + \alpha)^{-1}f_n$ and extend u_n by zero outside of K_n. Then, letting $w_n = v - u_n$, we get that w_n satisfies:

- $(\mathcal{L}+\alpha)w_n = (\mathcal{L}+\alpha)v - (\mathcal{L}+\alpha)u_n \ge f - f_n = 0$ on K_n
- $w_n \wedge 0 = \min\{w_n, 0\}$ attains a minimum on the finite set K_n
- $w_n = v \ge 0$ on $X \setminus K_n$.

Therefore, by the minimum principle, Theorem 1.7, we infer

$$w_n = v - u_n \ge 0.$$

Since $u_n(x)$ converges to $(L+\alpha)^{-1}f(x)$ as $n \to \infty$ for every $x \in X$ by Lemma 7.8, we infer

$$v \ge (L+\alpha)^{-1}f.$$

Therefore, $(L+\alpha)^{-1}f(x) < \infty$ for all $x \in X$. This shows (i).

Furthermore, letting $u = (L+\alpha)^{-1}f$, we have $0 \le u \le v$ and thus $u \in \mathcal{F}$. Since v was an arbitrary solution of $(\mathcal{L}+\alpha)v \ge f$, we infer the minimality of u. Finally, the fact that u solves $(\mathcal{L}+\alpha)u = f$ follows from (i), as discussed at the end of the proof of (i) $\Longrightarrow$ (ii). $\square$

We now apply the considerations above to the strictly positive function $f = \alpha 1 + c/m$ for $\alpha > 0$. In particular, we show that the resolvent applied to f is bounded between 0 and 1. It will turn out that this resolvent being equal to 1 is equivalent to stochastic completeness at infinity.

LEMMA 7.10. *Let (b,c) be a graph over (X,m) and let $\alpha > 0$. Then,*

$$0 \le (L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right) \le 1.$$

PROOF. Let $f = \alpha 1 + c/m$. Then, the constant function 1 solves

$$(\mathcal{L}+\alpha)1 = \left(\frac{c}{m} + \alpha 1\right) = f.$$

Hence, Lemma 7.9 yields that $(L+\alpha)^{-1}f$ is the smallest solution v to $(\mathcal{L}+\alpha)v \ge f$. Therefore, $(L+\alpha)^{-1}f \le 1$. Since $f > 0$ and the resolvent is positivity preserving, we also get $(L+\alpha)^{-1}f \geq 0$. This completes the proof. $\square$

In the lemma below we show that the connections between the resolvent and the semigroup also hold for the extended resolvent and semigroup. In particular, we extend the Laplace transform formula to all positive functions.

LEMMA 7.11. *Let (b,c) be a graph over (X,m) and let $f \in C(X)$ with $f \geq 0$.*
(a) *For every $\alpha > 0$,*

$$(L+\alpha)^{-1}f = \int_0^\infty e^{-t\alpha}e^{-tL}f dt.$$

(*"Laplace transform"*)

(b) *For every $t > 0$,*

$$e^{-tL}f = \lim_{n\to\infty}\left(\frac{n}{t}\left(L+\frac{n}{t}\right)^{-1}\right)^n f.$$

PROOF. From the spectral theorem we have the statement for all $\varphi \in C_c(X)$, see Theorem A.35. Thus, the statements follow by taking monotone limits. $\square$

We recall that the goal of this section is to investigate the properties of $M_t = e^{-tL}1 + \int_0^t e^{-sL}(c/m)ds$, where the integrand has now been defined via monotone limits. In the next lemma we show that the integral part of M is finite and even bounded by 1.

LEMMA 7.12. *Let (b,c) be a graph over (X,m). Then, the function $u\colon X \longrightarrow [0,\infty]$ defined by*

$$u(x) = \int_0^\infty \left(e^{-tL}\frac{c}{m}\right)(x)dt$$

satisfies

$$0 \leq u \leq 1 \qquad \textit{and} \qquad \mathcal{L}u = \frac{c}{m}.$$

PROOF. Let $\varphi \in C_c(X)$ be such that $0 \leq \varphi \leq c/m$. For $\alpha > 0$, we define

$$u_{\varphi,\alpha} = \int_0^\infty e^{-t\alpha}e^{-tL}\varphi\, dt \geq 0$$

and

$$u_\varphi = \lim_{\alpha\to 0^+} u_{\varphi,\alpha},$$

where the limit exists since $u_{\varphi,\alpha}$ is monotonically increasing as $\alpha \to 0^+$ and $u_{\varphi,\alpha} \leq 1$. Then, by the Laplace transform, we have

$$u_{\varphi,\alpha} = (L+\alpha)^{-1}\varphi$$

so that $(\mathcal{L}+\alpha)u_{\varphi,\alpha} = \varphi$ as L is a restriction of $\mathcal{L}$. Since $\varphi \leq \alpha 1 + c/m$ for all $\alpha > 0$ and since the resolvent is positivity preserving, we conclude

$$u_{\varphi,\alpha} = (L+\alpha)^{-1}\varphi \leq (L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right) \leq 1,$$

where the second inequality follows from Lemma 7.10. Therefore, $0 \leq \lim_{\alpha\to 0^+} u_{\varphi,\alpha} \leq 1$ and thus $0 \leq u_\varphi \leq 1$.

Using the uniform bound on $u_{\varphi,\alpha}$ and taking the limit $\alpha \to 0^+$ in the equation

$$(\mathcal{L}+\alpha)u_{\varphi,\alpha} = \varphi$$

yields

$$\mathcal{L}u_\varphi = \varphi \geq 0.$$

As $u = \lim_{\varphi \prec c/m} u_\varphi$ and $u_\varphi \leq 1$, we obtain the statement by taking monotone limits and using Lemma 1.8. □

The next lemma shows that the extended semigroup and resolvent contract any positive superharmonic function.

LEMMA 7.13. *Let (b,c) be a graph over (X,m) and let $f \in C(X)$ with $f \geq 0$. Then, the following statements are equivalent:*
(i) *$e^{-tL}f \leq f$ for all $t \geq 0$.*
(ii) *$\alpha(L+\alpha)^{-1}f \leq f$ for all $\alpha > 0$.*
If, additionally, $f \in \mathcal{F}$ and $\mathcal{L}f \geq 0$, then f satisfies the above conditions.

PROOF. The implications (i) $\Longleftrightarrow$ (ii) follow by Lemma 7.11.

Now, any $f \in \mathcal{F}$ with $f \geq 0$ and $\mathcal{L}f \geq 0$ satisfies

$$(\mathcal{L}+\alpha)\, f \geq \alpha f$$

for $\alpha > 0$. By Lemma 7.9 we infer $\alpha(L+\alpha)^{-1}f \leq f$ for $\alpha > 0$ since $\alpha(L+\alpha)^{-1}f$ is the smallest supersolution. □

The next theorem uses the properties established above to prove all of the facts we announced in the beginning of the section about the function M. Analogous properties have already been established for the heat semigroup acting on bounded functions. In particular, statement (d) below gives a minimality statement for M which gives an analogue to Proposition 7.6 concerning the heat semigroup.

THEOREM 7.14 (Properties of M). *Let (b,c) be a graph over (X,m). Then, the function $M\colon [0,\infty)\times X \longrightarrow \mathbb{R}$ given by*

$$M_t(x) = e^{-tL}1(x) + \int_0^t \left(e^{-sL}\frac{c}{m}\right)(x)ds$$

has the following properties:
(a) *$0 \leq M_s \leq M_t \leq 1$ for all $s \geq t \geq 0$.*
(b) *The function $t \mapsto M_t(x)$ is differentiable for all $x \in X$.*

(c) *For all $x \in X$ and $t > 0$*

$$(\mathcal{L} + \partial_t)\, M_t(x) = \frac{c}{m}(x).$$

Furthermore, M is the smallest positive solution of $(\mathcal{L} + \partial_t)\, u = c/m$ with $u_0 = 1$.

(d) *For all $\alpha > 0$ and $x \in X$*

$$(L + \alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = \int_0^\infty \alpha e^{-t\alpha} M_t(x) dt.$$

In particular,

$$w = \int_0^\infty \alpha e^{-t\alpha} M_t dt$$

satisfies $(\mathcal{L} + \alpha) w = \alpha 1 + c/m$ and is the smallest function $v \in \mathcal{F}$ with $v \geq 0$ and

$$(\mathcal{L} + \alpha) v \geq \alpha 1 + \frac{c}{m}.$$

Proof. We recall from Lemma 7.12 that the function

$$u(x) = \int_0^\infty \left(e^{-sL}\frac{c}{m}\right)(x) ds$$

satisfies $0 \leq u \leq 1$ and $\mathcal{L}u = c/m$. The key observation for the proof of the first few properties is shown in the proof of (a) below and states that

$$M_t = u + e^{-tL}(1 - u).$$

(a) Positivity of M follows since the extended semigroup is positivity preserving by definition.

Next, we show $M_t \leq 1$ for $t \geq 0$. By Lemma 7.12 the function $u = \int_0^\infty \left(e^{-sL}(c/m)\right) ds$ is bounded and, therefore, we can calculate

$$\begin{aligned}
\int_0^t \left(e^{-sL}\frac{c}{m}\right)(x) ds &= u(x) - \int_t^\infty \left(e^{-sL}\frac{c}{m}\right)(x) ds \\
&= u(x) - \int_0^\infty \left(e^{-(s+t)L}\frac{c}{m}\right)(x) ds \\
&= u(x) - e^{-tL} u(x),
\end{aligned}$$

where the last equality follows by the semigroup property for the extended semigroup and the fact that u is bounded. Thus,

$$M_t = e^{-tL}1 + u - e^{-tL}u = u + e^{-tL}(1 - u).$$

By Lemma 7.12, we have $1 - u \geq 0$ and $\mathcal{L}u = c/m$ so that

$$\mathcal{L}(1 - u) = \mathcal{L}1 - \mathcal{L}u = \frac{c}{m} - \frac{c}{m} = 0.$$

By Lemma 7.13, we infer for all $t > 0$,

$$e^{-tL}(1 - u) \leq 1 - u.$$

Therefore,
$$M_t = u + e^{-tL}(1-u) \leq u + 1 - u = 1.$$
This establishes the desired inequality.

Let $s \geq t \geq 0$. Then, $e^{-(s-t)L}(1-u) \leq (1-u)$ and, since the semigroup is positivity preserving, we get by the semigroup property that
$$e^{-sL}(1-u) = e^{(-t-(s-t))L}(1-u) = e^{-tL}e^{-(s-t)L}(1-u) \leq e^{-tL}(1-u).$$
Hence, as $M_t = u + e^{-tL}(1-u)$, it follows that $M_s \leq M_t$. Putting all of these properties together, we get
$$0 \leq M_s \leq M_t \leq 1$$
whenever $s \geq t \geq 0$.

(b) As the constant function 1 and u are both bounded, we can apply Theorem 7.3 with $f = 1-u$ to conclude that
$$t \mapsto M_t(x) = e^{-tL}(1-u)(x) + u(x)$$
is differentiable for $t > 0$ as $e^{-tL}(1-u)$ is a bounded solution of the heat equation. Furthermore, $M_t(x)$ is differentiable for $t \geq 0$ by Proposition 7.4.

(c) Using $M_t = u + e^{-tL}(1-u)$, the fact that $e^{-tL}(1-u)$ is a bounded solution of the heat equation and $\mathcal{L}u = c/m$ allows us to calculate, for all $x \in X$ and $t > 0$,
$$\begin{aligned}\partial_t M_t(x) &= \partial_t e^{-tL}(1-u)(x)\\ &= -\mathcal{L}e^{-tL}(1-u)(x)\\ &= -\mathcal{L}M_t(x) + \mathcal{L}u(x)\\ &= -\mathcal{L}M_t(x) + \frac{c(x)}{m(x)}.\end{aligned}$$
This proves the first statement.

The minimality statement follows by approximating M and using the minimum principle for the heat equation, Theorem 1.10, as follows: Let (K_n) be an increasing sequence of finite sets whose union equals X. For the Dirichlet Laplacian $L_n = L^{(D)}_{K_n}$ with respect to the finite set K_n, we let
$$M^{(n)}_t = e^{-tL_n}1_{K_n} + \int_0^t e^{-sL_n}\left(\frac{c}{m}1_{K_n}\right) ds$$
on K_n for $n \geq 0$ and extend it by 0 to X. Then, $M^{(n)}$ satisfies
$$\begin{aligned}(\mathcal{L} + \partial_t)M^{(n)} &= \frac{c}{m} && \text{on } [0,\infty) \times K_n\\ M^{(n)}_0 &= 1_{K_n} && \text{on } X\\ M^{(n)} &= 0 && \text{on } [0,\infty) \times X \setminus K_n.\end{aligned}$$

For any other positive solution w of $(\mathcal{L}+\partial_t)w = c/m$ with $w_0 = 1$, let

$$v^{(n)} = w - M^{(n)}.$$

Then, for any $T \geq 0$,

- $(\mathcal{L}+\partial_t)v^{(n)} = 0$ on $(0,T) \times K_n$
- $v^{(n)} \wedge 0$ attains a minimum on the compact set $[0,T] \times K_n$ since $v^{(n)}$ is continuous
- $v^{(n)} \geq 0$ on $([0,T) \times (X \setminus K_n)) \cup (\{0\} \times K_n)$.

Thus, $w - M^{(n)} = v^{(n)} \geq 0$ by the minimum principle for the heat equation, Theorem 1.10. By monotone convergence, Lemma 7.8, we get $w \geq M$. This proves the minimality.

(d) By the Laplace transform, Lemma 7.11, applied to $f = \alpha 1 + c/m$ for $\alpha > 0$,

$$(L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = \int_0^\infty e^{-t\alpha} e^{-tL}\left(\alpha 1 + \frac{c}{m}\right)(x)dt.$$

The function $t \mapsto \int_0^t e^{-sL}(c/m)ds = M_t - e^{-tL}1$ is continuously differentiable, so by integration by parts, we have

$$\begin{aligned}
&\int_0^\infty e^{-t\alpha}\left(e^{-tL}\frac{c}{m}\right)(x)dt \\
&\qquad = e^{-t\alpha}\int_0^t \left(e^{-sL}\frac{c}{m}\right)(x)ds\bigg|_0^\infty + \int_0^\infty \alpha e^{-t\alpha}\left(\int_0^t e^{-sL}\frac{c}{m}(x)ds\right)dt \\
&\qquad = \int_0^\infty \alpha e^{-t\alpha}\left(M_t(x) - e^{-tL}1(x)\right)dt,
\end{aligned}$$

where the first term vanishes due to the boundedness of u. Putting these two calculations together yields

$$(L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = \int_0^\infty \alpha e^{-t\alpha} M_t(x)dt$$

for all $x \in X$, which gives the first statement.

The fact that

$$w = \int_0^\infty \alpha e^{-t\alpha} M_t dt$$

satisfies $(\mathcal{L}+\alpha)w = \alpha 1 + c/m$ and is the smallest positive $v \in \mathcal{F}$ with $(\mathcal{L}+\alpha)v \geq \alpha 1 + c/m$ follows immediately from Lemma 7.9 since $w = (L+\alpha)^{-1}(\alpha 1 + c/m)$ and since $0 \leq w \leq 1$ as $0 \leq M_t \leq 1$ by part (a). □

DEFINITION 7.15 (Stochastic completeness at infinity). A graph (b,c) over (X,m) is called *stochastically complete at infinity* if

$$M_t(x) = 1$$

for all $x \in X$ and all $t > 0$. Otherwise, (b,c) over (X,m) is called *stochastically incomplete at infinity*.

REMARK (A word of caution). We note that by Theorem 7.14 (d) we have by integrating M_t with respect to the probability measure $\alpha e^{-t\alpha}dt$ that stochastic completeness at infinity is equivalent to

$$(L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right) = 1.$$

Let us now stress that the care taken with the monotone convergence arguments in the proof of the theorem above is quite necessary. For example, one might think that since $(\mathcal{L}+\alpha)(L+\alpha)^{-1}1 = 1$, one also has $(L+\alpha)^{-1}(\mathcal{L}+\alpha)1 = 1$. However, by a direct calculation, this would yield

$$(L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right) = (L+\alpha)^{-1}(\mathcal{L}+\alpha)1 = 1,$$

which would imply that all graphs are stochastically complete at infinity. We will have ample opportunity to see that this is not the case.

Furthermore, in Section 6 we show that stochastic completeness at infinity is equivalent to the function 1 being in the domain of the adjoint of $\mathcal{L}$ on $D(Q^{(N)}) \cap \ell^1(X,m)$. It turns out that this is equivalent to 1 being in the domain of the generator L_∞ of the semigroup on $\ell^\infty(X)$, in which case

$$(L+\alpha)^{-1}(\mathcal{L}+\alpha)1 = (L+\alpha)^{-1}(L_\infty+\alpha)1 = 1$$

holds (Exercise 7.6).

3. The heat equation perspective

In this section we present the heat equation viewpoint on stochastic completeness at infinity. This states that stochastic completeness at infinity is equivalent to the uniqueness of bounded solutions of the heat equation.

In the previous section we introduced stochastic completeness at infinity of a graph (b,c) over (X,m). Specifically, we first defined the function

$$M_t = e^{-tL}1 + \int_0^t \left(e^{-sL}\frac{c}{m}\right) ds$$

by using monotone limits to apply the heat semigroup to general positive functions in order to define the integrand. The function M_t is a replacement for $e^{-tL}1$ in the case of a general killing term. The question of stochastic completeness at infinity is the question if $M_t = 1$ for all $t \geq 0$. By contrast, the question of stochastic completeness is if $e^{-tL}1 = 1$ for all $t \geq 0$. As the extended semigroup is positivity preserving, it is clear that stochastic completeness always implies stochastic completeness at infinity but the converse is not always true. In fact, stochastic completeness implies $c = 0$ while graphs with $c \neq 0$ may or may not be stochastically complete at infinity.

In this section, we start to prove our main characterization of stochastic completeness and stochastic completeness at infinity by connecting these notions to the uniqueness of bounded solutions of the heat equation. That is, given a bounded function f we consider the heat equation with initial condition f, i.e.,

$$(\mathcal{L} + \partial_t)\, u = 0$$

with $u_0 = f$. As we have seen, the semigroup e^{-tL} on $\ell^\infty(X)$ applied to f generates a bounded solution of the heat equation. It is clear in the case of $c = 0$ that if bounded solutions of the heat equation are uniquely determined by initial data, then $e^{-tL}1 = 1$ since both $e^{-tL}1$ and 1 satisfy the heat equation with initial condition 1. A similar reasoning shows that $M_t = 1$ if bounded solutions are uniquely determined. In this section, we will show that the converse is also true. That is, we show that stochastic completeness at infinity is equivalent to the uniqueness of this solution in the class of bounded solutions.

We note that stochastic completeness at infinity reduces to stochastic completeness whenever $c = 0$. Hence, the reader who is only interested in stochastic completeness can substitute $e^{-tL}1$ for M_t and let $c = 0$ in the statements and proofs below. We give a more detailed discussion of how the proof of stochastic completeness can be simplified in a remark before we commence the proof below.

THEOREM 7.16 (Stochastic completeness at infinity and the heat equation). *Let (b, c) be a connected graph over (X, m). Then, the following statements are equivalent:*

(i′) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$M_t(x) = 1.$$

(*"Stochastic completeness at infinity"*)

(i.a′) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*

$$(L + \alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = 1.$$

(vi′) *For every $f \in \ell^\infty(X)$ there exists a unique bounded solution u of the heat equation*

$$(\mathcal{L} + \partial_t)u = 0 \quad \textit{with} \quad u_0 = f. \quad (\textit{"Heat equation"})$$

(vi.a′) *Every bounded solution u of the heat equation $(\mathcal{L}+\partial_t)u = 0$ with $u_0 = 0$ is trivial.*

REMARK (What is needed for stochastic completeness). Before starting the proof of Theorem 7.16 we give a roadmap on what is required for the reader who is only interested in stochastic completeness, i.e., in the case $c = 0$, in the proofs below.

- The formula for M_{s+t} found in the proof of Lemma 7.17 can be replaced by the semigroup property $e^{-(s+t)L}1 = e^{-sL}e^{-tL}1$ of the semigroup on $\ell^\infty(X)$.
- The positivity improving property of the semigroup on $\ell^\infty(X)$ for connected graphs is used in the proof of Lemma 7.17. This follows by extending the corresponding property for the semigroup on $\ell^2(X,m)$ from Theorem 1.26.
- The fact that $M_t \leq 1$, used in the proof of Lemma 7.17, can be replaced by $e^{-tL}1 \leq 1$, which follows from the Markov property of the semigroup on $\ell^\infty(X)$ found in Theorem 2.9.
- The fact that $M_s \leq M_t$ for all $s \geq t$, which is used in the proof of Lemma 7.17, can be replaced by $e^{-sL}1 \leq e^{-tL}1$, which is shown in Lemma 7.5.
- The proof of the equivalence of (i′) and (i.a′) relies on the Laplace transform formula, i.e., $(L+\alpha)^{-1}f = \int_0^\infty e^{-t\alpha}e^{-tL}f dt$ for all $\alpha > 0$ and $f \in \ell^\infty(X)$. This is shown in Theorem 2.11.
- The fact that $e^{-tL}1$ is a bounded solution of the heat equation with initial condition 1 was shown in Theorem 7.3. This is used in the proof of (vi.a′) $\Longrightarrow$ (i′).
- The proof of (i′) $\Longrightarrow$ (vi.a′) uses the fact that M is the smallest positive solution to $(\mathcal{L} + \partial_t)v = c/m$ with $v_0 = 1$, which is shown in Theorem 7.14 (c). The analogous fact that $u_t = e^{-tL}1$ is the smallest positive solution of $(\mathcal{L}+\partial_t)v = 0$ with $v_0 = 1$ is found in Theorem 7.3.

We start the proof of Theorem 7.16 by showing the equivalence of the "for some" and "for all" statements in (i′). For this, the connectedness of the graph is essential.

LEMMA 7.17. *Let (b,c) be a connected graph over (X,m). If $M_t(x) < 1$ for some $t > 0$ and some $x \in X$, then $M_t(x) < 1$ for all $t > 0$ and all $x \in X$.*

PROOF. Recall that the extended semigroup satisfies the semigroup property, i.e., $e^{-(s+t)L}f = e^{-sL}e^{-tL}f$ for all $s,t \geq 0$ and all $f \in C(X)$ with $f \geq 0$. While M does not satisfy such a property, it satisfies a similar property which will imply the statement of the lemma. More specifically, a direct calculation using the semigroup property for the extended semigroup yields, for $s,t \geq 0$,

$$\begin{aligned}
M_{s+t} &= e^{-(s+t)L}1 + \int_s^{s+t} e^{-rL}\frac{c}{m}dr + \int_0^s e^{-rL}\frac{c}{m}dr \\
&= e^{-sL}e^{-tL}1 + \int_0^t e^{-sL}e^{-rL}\frac{c}{m}dr + \int_0^s e^{-rL}\frac{c}{m}dr \\
&= e^{-sL}\left(e^{-tL}1 + \int_0^t e^{-rL}\frac{c}{m}dr\right) + \int_0^s e^{-rL}\frac{c}{m}dr \\
&= e^{-sL}M_t + \int_0^s e^{-rL}\frac{c}{m}dr,
\end{aligned}$$

where interchanging the integration $\int_0^t \dots dr$ and e^{-sL} is justified by Fubini's theorem.

We obtain the following two facts from this equality:

Fact 1. If $M_t = 1$ for some $t > 0$, then $M_{nt} = M_t = 1$ for all $n \in \mathbb{N}$.
Proof. This follows easily by induction.

Fact 2. If $M_t(x) < 1$ for some $t > 0$ and some $x \in X$, then $M_{t+s} < 1$ for all $s > 0$.

Proof. Since $M_t \le 1$ by Theorem 7.14 (a), we get that $1 - M_t \ge 0$. This function is non-trivial by assumption. As the graph is connected, the extended semigroup is positivity improving by Lemma 7.8 and, therefore, $e^{-sL}(1 - M_t) > 0$, i.e.

$$e^{-sL} M_t < e^{-sL} 1.$$

Combined with the equality above we infer

$$M_{s+t} < e^{-sL} 1 + \int_0^s e^{-rL} \frac{c}{m} dr = M_s \le 1,$$

where the last inequality follows from Theorem 7.14 (a).

We now complete the proof. Assume that $M_t(x) < 1$ for some $t > 0$ and some $x \in X$. We note that it follows from Fact 2 that $M_s < 1$ for all $s > t$. Therefore, let $s \le t$. We aim to show that $M_s < 1$. Suppose not. Then, there exists a $y \in X$ such that $M_s(y) = 1$. By what we have already shown, it follows that $M_r = 1$ for all r with $0 \le r < s$. Fix any such $r > 0$. Let $n \in \mathbb{N}$ be such that $nr > t$. By Fact 1, we get $M_{nr} = 1$. Thus, $M_t = 1$ as $nr > t$, which yields a contradiction to the assumption that $M_t(x) < 1$. This completes the proof. $\square$

Proof of Theorem 7.16. The equivalence of the "for some" and "for all" statements found in (i$'$) has already been shown in Lemma 7.17.

(i$'$) $\iff$ (i.a$'$): This follows readily from Theorem 7.14 (d) and Lemma 7.17. This also shows the equivalence of the "for some" and "for all" statements found in (i.a$'$).

(vi$'$) $\Longrightarrow$ (vi.a$'$): This is clear since if we let $f = 0$, then $u = 0$ is the unique bounded solution of the heat equation with initial condition 0.

(vi.a$'$) $\Longrightarrow$ (vi$'$): Given $f \in \ell^\infty(X)$, the existence of a bounded solution of the heat equation with initial condition f has been shown in Theorem 7.3. Hence, we only need to establish uniqueness. Therefore, let u and v be two bounded solutions of the heat equation with initial condition f. Then, $w = u - v$ is a bounded solution of the heat equation with initial condition $w_0 = 0$. By (vi.a$'$) we get that $w = 0$, so that $u = v$.

(vi.a$'$) $\Longrightarrow$ (i$'$): We show this by contraposition. If $M_t(x) < 1$ for some $x \in X$ and some $t > 0$, then $M_t < 1$ for all $t > 0$ by Lemma 7.17.

Hence, suppose that $M_t < 1$. By Theorem 7.14 (a) M is bounded, $M_0 = 1$ by the definition of M and M_t satisfies

$$(\mathcal{L} + \partial_t)M_t = \frac{c}{m}$$

by Theorem 7.14 (c). Furthermore, it is clear that $(\mathcal{L} + \partial_t)1 = c/m$. Therefore, it follows that

$$u_t = 1 - M_t$$

is a bounded solution of the heat equation with initial condition 0, which is non-trivial since $M_t < 1$.

(i′) $\Longrightarrow$ (vi.a′): We show this by contraposition as well. Let u be a non-zero bounded solution of the heat equation with $u_0 = 0$. Without loss of generality, we may assume that $u_{t_0}(x_0) > 0$ for some $t_0 > 0$ and some $x_0 \in X$ as otherwise we work with $-u$. Furthermore, by rescaling, we may assume $|u| \leq 1$.

Let $w = 1 - u$. Then, w is positive bounded and $w_{t_0}(x_0) < 1$. Furthermore, since $u_0 = 0$, we get $w_0 = 1$ and since u solves the heat equation, we get

$$(\mathcal{L} + \partial_t)\, w = \frac{c}{m}$$

for all $t > 0$. Since M is the smallest positive function with these properties by Theorem 7.14 (c), we infer $M_{t_0}(x_0) \leq w_{t_0}(x_0) < 1$. Therefore, $M_t(x) < 1$ for all $t > 0$ and all $x \in X$ by Lemma 7.17. This completes the proof. □

4. The Poisson equation perspective

In this section we connect the notion of stochastic completeness at infinity to the uniqueness of bounded solutions of the Poisson equation. As a consequence we show that stochastic completeness at infinity implies uniqueness of associated forms.

In this section we investigate the Poisson equation perspective on stochastic completeness at infinity. In particular, we characterize stochastic completeness at infinity via the uniqueness of bounded solutions of the Poisson equation

$$(\mathcal{L} + \alpha)u = f$$

for $f \in \ell^\infty(X)$ and $\alpha > 0$. We recall that applying the resolvent $(L + \alpha)^{-1}$ on $\ell^\infty(X)$ to f gives a bounded solution of the Poisson equation. Hence, the issue here is the uniqueness of this solution.

We will also connect stochastic completeness at infinity to the non-existence of non-trivial bounded α-harmonic and non-trivial positive bounded α-subharmonic functions, i.e., functions $u \in \ell^\infty(X)$ satisfying $(\mathcal{L} + \alpha)u = 0$ and $(\mathcal{L} + \alpha)u \leq 0$ for $\alpha > 0$. These criteria for stochastic completeness at infinity are quite useful in practice.

As previously noted, the reader only interested in stochastic completeness can substitute $e^{-tL}1$ for M_t and let $c = 0$ in the statements and proofs found below. The additional ingredients which are needed to carry out the proof are listed in a remark before the proof of Lemma 7.19.

THEOREM 7.18 (Stochastic completeness at infinity and the Poisson equation). *Let (b, c) be a connected graph over (X, m). Then, the following statements are equivalent:*

(i′) *For some (all) $t > 0$ and some (all) $x \in X$,*
$$M_t(x) = 1.$$
(*"Stochastic completeness at infinity"*)

(v′) *For some (all) $\alpha > 0$ and every $f \in \ell^\infty(X)$ there exists a unique $u \in \ell^\infty(X)$ satisfying*
$$(\mathcal{L} + \alpha)u = f. \qquad \text{("Poisson equation")}$$

- (v.a′) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u \leq 0$ is trivial.*
- (v.b′) *For some (all) $\alpha > 0$ every $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*
- (v.c′) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u = 0$ is trivial.*

REMARK. We first discuss some of the informal intuition behind the equivalences. As we have seen, stochastic completeness and stochastic completeness at infinity concern the preservation of heat. Now, a non-vanishing killing term instantly removes heat from the graph. This is the reason why we add the integral terms to $e^{-tL}1$ in the definition of M_t. However, even with the addition of this term, heat can still vanish for geometric reasons. Geometrically, the way that heat can escape is due to an intense growth of the geometry which forces heat to infinity in a finite time. The notion of the growth of geometry will be made precise in Chapter 14.

A positive α-harmonic function for $\alpha > 0$ whose value is strictly positive at some vertex has to have a strictly bigger value at some neighbor of that vertex, as can be seen from the equation $(\mathcal{L}+\alpha)u = 0$. Therefore, such functions must increase and the only way that they can remain bounded, if the graph does not grow strongly, is that they are trivial. This gives an informal intuition for the equivalence of stochastic completeness at infinity and the non-existence of non-trivial positive α-harmonic functions for $\alpha > 0$.

REMARK (What is needed for stochastic completeness). Before starting the proof of Theorem 7.18, we give a roadmap of what is required for the reader who is only interested in stochastic completeness, i.e., in the case $c = 0$ in the proof of the lemmas below.

- The fact that $(L+\alpha)^{-1}(\alpha 1 + c/m) = \int_0^\infty \alpha e^{-t\alpha} M_t dt$ is the minimal positive solution of $(\mathcal{L}+\alpha)v \geq \alpha 1 + c/m$, proven in Theorem 7.14 (d), can be replaced as follows: The Laplace transform formula proven in Theorem 2.11 gives $\alpha(L+\alpha)^{-1}1 = \int_0^\infty \alpha e^{-t\alpha} e^{-tL} 1 dt$. Furthermore, that $\alpha(L+\alpha)^{-1}1$ generates the minimal positive function $v \in \mathcal{F}$ with $(\mathcal{L}+\alpha)v \geq \alpha 1$ was shown in Theorem 2.12. This is used in the proof of Lemma 7.19.
- The fact that $0 \leq M_t \leq 1$ shown in Theorem 7.14 (a) can be replaced by $0 \leq e^{-tL}1 \leq 1$, which is the Markov property of the semigroup on $\ell^\infty(X)$ found in Theorem 2.9. This fact is also used in the proof of Lemma 7.19.
- The fact that $M_t < 1$ for some t implies that $M_s < 1$ for all $s \geq t$ can be replaced by the fact that if $e^{-tL}1 < 1$ for some t, then $e^{-sL}1 \leq e^{-tL}1 < 1$ for all $s \geq t$, which is shown in Lemma 7.5. This is used in the proof of Lemma 7.20.

The following lemma is the key to proving Theorem 7.18. It connects M_t with bounded α-harmonic functions for $\alpha > 0$.

Lemma 7.19 (Largest α-subharmonic function). *Let (b,c) be a graph over (X,m). For $\alpha > 0$, the function*

$$w_\alpha = \int_0^\infty \alpha e^{-t\alpha}(1-M_t)dt = 1 - (L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)$$

satisfies $0 \leq w_\alpha \leq 1$, solves $(\mathcal{L}+\alpha)w_\alpha = 0$ and is the largest function $u \in \mathcal{F}$ with $0 \leq u \leq 1$ such that $(\mathcal{L}+\alpha)u \leq 0$.

Proof. We note that by Theorem 7.14 (d), for every $\alpha > 0$ the function

$$v_\alpha = \int_0^\infty \alpha e^{-t\alpha} M_t dt = (L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)$$

satisfies $(\mathcal{L}+\alpha)v_\alpha = \alpha 1 + c/m$ and is the minimal positive $v \in \mathcal{F}$ such that $(\mathcal{L}+\alpha)v \geq \alpha 1 + c/m$. Furthermore, as $0 \leq M_t \leq 1$ by Theorem 7.14 (a), we get $0 \leq v_\alpha \leq 1$. Therefore,

$$\begin{aligned} w_\alpha = 1 - v_\alpha &= 1 - (L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right) \\ &= 1 - \int_0^\infty \alpha e^{-t\alpha} M_t dt \\ &= \int_0^\infty \alpha e^{-t\alpha} dt - \int_0^\infty \alpha e^{-t\alpha} M_t dt \\ &= \int_0^\infty \alpha e^{-t\alpha}(1-M_t)dt, \end{aligned}$$

and $0 \leq w_\alpha \leq 1$. Furthermore, as v_α satisfies $(\mathcal{L}+\alpha)v_\alpha = \alpha 1 + c/m$ and since $(\mathcal{L}+\alpha)1 = \alpha 1 + c/m$ by a direct calculation, we get

$$(\mathcal{L}+\alpha)\, w_\alpha = 0.$$

We now show the maximality of w_α. Hence, let u satisfy $(\mathcal{L}+\alpha)u \leq 0$ with $0 \leq u \leq 1$. Then, $1-u \geq 0$ satisfies $(\mathcal{L}+\alpha)(1-u) \geq \alpha 1 + c/m$. As v_α is the minimal such positive function by Theorem 7.14 (d), we get $v_\alpha \leq 1-u$. As $w_\alpha = 1 - v_\alpha$, $w_\alpha \geq u$ follows. This completes the proof. □

The equivalence of the "for some" and "for all" statements in (iv.a′), (iv.b′) and (iv.c′) is shown in the next lemma.

Lemma 7.20. *Let (b,c) be a graph over (X,m). If there exists a bounded non-trivial $v \geq 0$ such that $(\mathcal{L}+\alpha)v \leq 0$ for some $\alpha > 0$, then for every $\alpha > 0$ there exists a bounded non-trivial $v \geq 0$ such that $(\mathcal{L}+\alpha)v = 0$.*

Proof. Let $\alpha > 0$ and let v be a bounded non-trivial positive function on X satisfying $(\mathcal{L}+\alpha)v \leq 0$. By rescaling, we may assume that $0 \leq v \leq 1$. By Lemma 7.19, $w_\alpha = \int_0^\infty \alpha e^{-t\alpha}(1-M_t)dt$ is the maximal function $u \in \mathcal{F}$ with $0 \leq u \leq 1$ such that $(\mathcal{L}+\alpha)u \leq 0$. Therefore, $v \leq w_\alpha$. As v is non-trivial, w_α is non-trivial and we conclude that $M_t < 1$ for some t. Therefore, $M_t < 1$ for all $t > 0$ by Theorem 7.17. Hence, for all $\beta > 0$, the function $w_\beta = \int_0^\infty \beta e^{-t\beta}(1-M_t)dt$ is non-trivial. Furthermore, by Lemma 7.19 we have $0 \leq w_\beta \leq 1$ and $(\mathcal{L}+\beta)w_\beta = 0$ for $\beta > 0$. This completes the proof. □

Proof of Theorem 7.18. The equivalence of the "for some" and "for all" statements in (v.a′), (v.b′) and (v.c′) follows from Lemma 7.20. The equivalence of the "for some" and "for all" statements in (v′) will follow from the arguments given below.

For the rest of the proof recall that $w_\alpha = \int_0^\infty \alpha e^{-t\alpha}(1-M_t)dt$ solves $(\mathcal{L}+\alpha)w_\alpha = 0$ and is the largest function u with $0 \leq u \leq 1$ and $(\mathcal{L}+\alpha)u \leq 0$ by Lemma 7.19. Obviously, $w_\alpha = 0$ for some (all) $\alpha > 0$ if and only if $M_t = 1$ for some (all) $t > 0$, i.e., if and only if the graph is stochastically complete at infinity.

We first show (i′) $\Longrightarrow$ (v.a′) $\Longrightarrow$ (v.b′) $\Longrightarrow$ (v.c′) $\Longrightarrow$ (i′). To this end let $\alpha > 0$ be fixed.

(i′) $\Longrightarrow$ (v.a′): Let $u \geq 0$ be a bounded solution of $(\mathcal{L}+\alpha)u \leq 0$. By rescaling, we may assume that $u \leq 1$. Then, $0 \leq u \leq w_\alpha$ since w_α is the largest such solution. If $M_t = 1$, then $w_\alpha = 0$ and, therefore, $u = 0$.

(v.a′) $\Longrightarrow$ (v.b′): This follows immediately from Lemma 1.9, which states that if $u \in \mathcal{F}$ is α-harmonic, then $|u|$ is α-subharmonic.

(v.b′) $\Longrightarrow$ (v.c′): This is clear.

(v.c′) $\Longrightarrow$ (i′): If there do not exist non-trivial positive functions $u \leq 1$ such that $(\mathcal{L}+\alpha)u = 0$, then the largest such function w_α satisfies $w_\alpha = 0$. Therefore, $M_t = 1$ for all $t > 0$.

Next, we show the implications (v′) $\implies$ (i′) $\implies$ (v.b′) $\implies$ (v′) which will complete the proof. In particular, we show the equivalence of the "for some" and "for all" statements in (v′).

(v′) $\implies$ (i′): We show this by contraposition. So, suppose that $M_t < 1$. Let $\alpha > 0$ and let $f \in \ell^\infty(X)$. Then, $u = (L+\alpha)^{-1}f \in \ell^\infty(X)$ solves $(\mathcal{L} + \alpha)u = f$ by Theorem 2.12. As we assume that $M_t < 1$, $w_\alpha > 0$ and, therefore, $v = u + w_\alpha > u$ also solves $(L + \alpha)v = f$ since $(\mathcal{L} + \alpha)w_\alpha = 0$. Therefore, there is no uniqueness of solutions to the Poisson equation for any $\alpha > 0$.

(i′) $\implies$ (v.b′): We have already shown this in the first round of equivalences above.

(v.b′) $\implies$ (v′): Let $f \in \ell^\infty(X)$ and let $\alpha > 0$. The existence of solutions to the Poisson equation for $\alpha > 0$ is given by $u = (L+\alpha)^{-1}f$. So, we have to show uniqueness. Therefore, assume that there exists $f \in \ell^\infty(X)$ and two bounded solutions u_1, u_2 such that $(\mathcal{L} + \alpha)u_1 = f = (\mathcal{L}+\alpha)u_2$. Then, $u = u_1 - u_2$ is bounded and satisfies $(\mathcal{L}+\alpha)u = 0$. From (v.b′) we infer $u = 0$ and, therefore, $u_1 = u_2$. □

We end this section with a corollary which connects stochastic completeness at infinity and the property of form uniqueness found in Chapter 3. In particular, if $Q^{(D)} = Q^{(N)}$, then there is only one form associated to a graph (b, c) over (X, m). We next show that this is always the case when a graph is stochastically complete at infinity.

Corollary 7.21 (Stochastic completeness implies $Q^{(D)} = Q^{(N)}$). *If (b, c) is a connected graph over (X, m) which is stochastically complete at infinity, then $Q^{(D)} = Q^{(N)}$.*

Proof. If $Q^{(D)} \neq Q^{(N)}$, then there is a non-trivial bounded solution to $(\mathcal{L} + \alpha)u = 0$ for $\alpha > 0$ by Theorem 3.2. By Theorem 7.18, this implies the graph is stochastically incomplete at infinity. □

5. The form perspective

In this section we show that a graph is stochastically complete at infinity if and only if the constant function 1 can be approximated by functions in the form domain or, equivalently, by compactly supported functions, in a weak sense. As a consequence we show that recurrence implies stochastic completeness.

For a connected graph b over (X, m) in Chapter 6 we proved that recurrence is equivalent to the fact that the constant function 1 can be approximated by compactly supported functions with respect to pointwise convergence and convergence in the form sense, see Theorem 6.1 (i.d). Here, we give an analogous criterion for stochastic completeness. More specifically, we show that stochastic completeness at infinity is equivalent to the ability to approximate 1 in a weak sense.

As a consequence we get that recurrence always implies stochastic completeness.

As usual, the reader who is only interested in stochastic completeness and not stochastic completeness at infinity can let $c = 0$ in all of the statements below. In particular, Lemma 7.22 will not be needed for the discussion of stochastic completeness.

We denote by $\ell^1(X,c)$ the vector space of all functions $f \in C(X)$ such that $\sum_{x\in X} c(x)|f(x)| < \infty$. We note that this is not necessarily a normed space since c is not assumed to be strictly positive. In the following lemma we show that $\sum_{x\in X} c(x)(L+\alpha)^{-1}f(x)$ converges absolutely for all $f \in \ell^1(X,m)$ and $\alpha > 0$.

LEMMA 7.22. *Let (b,c) be a graph over (X,m) and let $\alpha > 0$. Then,*

$$(L+\alpha)^{-1}\ell^1(X,m) \subseteq \ell^1(X,c)$$

and for all $v \in \ell^1(X,m)$,

$$\sum_{x\in X} \left((L+\alpha)^{-1}\frac{c}{m}\right)(x)v(x)m(x) = \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x),$$

where all sums converge absolutely.

PROOF. By Lemma 7.10 we have $0 \le (L+\alpha)^{-1}(\alpha 1 + c/m) \le 1$. In particular, $(L+\alpha)^{-1}(c/m) \in \ell^\infty(X)$. Thus, the sum on the left-hand side of the asserted equality is equal to $((L+\alpha)^{-1}(c/m), v)$ for $v \in \ell^1(X,m)$, where $(\cdot,\cdot)$ denotes the dual pairing between $\ell^\infty(X)$ and $\ell^1(X,m)$. Therefore, the sum on the left-hand side converges absolutely.

Let (c_n) be a sequence of finitely supported functions such that $c_n \nearrow c$ pointwise as $n \to \infty$. Then, by the definition of the extended resolvent, $(L+\alpha)^{-1}(c_n/m) \nearrow (L+\alpha)^{-1}(c/m)$ as $n \to \infty$. Let $\varphi_n = c_n/m$. By using the symmetry of the resolvents shown in Theorem 2.11 and by decomposing v into positive and negative parts we conclude by the monotone convergence theorem

$$\begin{aligned}\sum_{x\in X} \left((L+\alpha)^{-1}\frac{c}{m}\right)(x)v(x)m(x) &= \left((L+\alpha)^{-1}\frac{c}{m}, v\right)\\ &= \lim_{n\to\infty} \left((L+\alpha)^{-1}\varphi_n, v\right)\\ &= \lim_{n\to\infty} \left(\varphi_n, (L+\alpha)^{-1}v\right)\\ &= \lim_{n\to\infty} \sum_{x\in X} \varphi_n(x)(L+\alpha)^{-1}v(x)m(x)\\ &= \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x).\end{aligned}$$

In particular, this shows $(L+\alpha)^{-1}v \in \ell^1(X,c)$ for all $v \in \ell^1(X,m)$, which shows the inclusion asserted in the statement of the lemma. This completes the proof. $\square$

We now state the characterization of stochastic completeness at infinity via two approximation schemes for the constant function 1. We note that this proves the equivalence of (i.a), (ii) and (ii.a) in Theorem 7.2 for the characterization of stochastic completeness.

THEOREM 7.23 (Stochastic completeness at infinity and approximating 1). *Let (b, c) be a connected graph over (X, m). The following statements are equivalent:*

(i.a′) *For some (all) $\alpha > 0$ and some (all) $x \in X$,*

$$(L+\alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = 1.$$

(ii′) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \le e_n \le 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*

$$\lim_{n\to\infty} Q(e_n, v) = \sum_{x\in X} c(x)v(x)$$

for all $v \in D(Q) \cap \ell^1(X, m) \cap \ell^1(X, c)$.

(ii.a′) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \le e_n \le 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*

$$\lim_{n\to\infty} Q(e_n, (L+\alpha)^{-1}v) = \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x)$$

for one $v \in \ell^2(X, m)\cap\ell^1(X, m)$ with $v > 0$ and some (all) $\alpha > 0$.

We start by showing that we can always pass from a sequence in $D(Q)$ to a sequence in $C_c(X)$ in the approximation schemes in (ii′) and (ii.a′) above. For this, recall that for a vertex $o \in X$ and $f \in \mathcal{D}$ we define the norm $\|f\|_o = (\mathcal{Q}(f) + f^2(o))^{1/2}$. If a graph is connected, then the norms for different $o \in X$ are equivalent and convergence with respect to any of these norms is equivalent to convergence pointwise and in $\mathcal{Q}$. See Lemma 6.3 for further basic facts about this norm.

LEMMA 7.24. *Let (b, c) be a connected graph over (X, m). Let $e_n \in D(Q)$ be a sequence with $0 \le e_n \le 1$, $e_n \to 1$ pointwise and $Q(e_n, v) \to C$ as $n \to \infty$ for some $v \in \mathcal{D}$ and some constant C. Then, there exist $\varphi_n \in C_c(X)$ with $0 \le \varphi_n \le 1$, $\varphi_n \to 1$ pointwise and*

$$\lim_{n\to\infty} Q(\varphi_n, v) = C.$$

PROOF. As $D(Q) = \overline{C_c(X)}^{\|\cdot\|_Q}$, where $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$, and convergence in $\ell^2(X, m)$ implies pointwise convergence, it follows that for every e_n, there exists a sequence $(\varphi_k^{(n)})$ in $C_c(X)$ such that

$$\varphi_k^{(n)} \to e_n$$

pointwise and with respect to Q as $k \to \infty$. In particular, $\|e_n - \varphi_k^{(n)}\|_o \to 0$ as $k \to \infty$ for all $o \in X$. We note that since $0 \leq e_n \leq 1$, the negative part of e_n is 0 while the positive part of e_n is e_n. Hence, by Lemma 6.6, for every $n \in \mathbb{N}$, the sequence

$$\psi_k^{(n)} = 0 \vee \varphi_k^{(n)} \wedge e_n$$

also converges to e_n in $\|\cdot\|_o$ as $k \to \infty$. In particular, $\psi_k^{(n)} \in C_c(X)$ with $0 \leq \psi_k^{(n)} \leq 1$, $\psi_k^{(n)}(x) \to e_n(x)$ for all $x \in X$ as $k \to \infty$ and

$$\lim_{k\to\infty} Q(e_n - \psi_k^{(n)}) = 0.$$

By applying a diagonalization procedure to the family of functions $\{\psi_k^{(n)}\}_{k,n=1}^{\infty}$ we obtain a subsequence (φ_n) in $C_c(X)$ with $0 \leq \varphi_n \leq 1$ and which converges pointwise at all $x \in X$. In particular, from the properties above, $\varphi_n \to 1$ pointwise and $\lim_{n\to\infty} Q(\varphi_n, v) = C$. This completes the proof. $\square$

Proof of Theorem 7.23. In the proof we will make repeated use of the formula

$$Q((L+\alpha)^{-1}f, g) = \langle f - \alpha(L+\alpha)^{-1}f, g\rangle$$

for $f, g \in D(Q)$ and $\alpha > 0$, which follows by a simple calculation.

(i.a$'$) $\Longrightarrow$ (ii$'$): Let (φ_n) be a sequence in $C_c(X)$ such that $0 \leq \varphi_n \leq 1$ and $\varphi_n \nearrow 1$ pointwise as $n \to \infty$. Let $\psi_n = \varphi_n(1 + c/m)$ so that $\psi_n \nearrow 1 + c/m$ pointwise and define

$$e_n = (L+1)^{-1}\psi_n$$

for $n \in \mathbb{N}$. Then, as $\psi_n \geq 0$ and $(L+1)^{-1}$ is positivity preserving, we infer $e_n \geq 0$. Furthermore, since $0 \leq \varphi_n \leq 1$ we get $\psi_n = \varphi_n(1+c/m) \leq 1 + c/m$. As the resolvent is positivity preserving we get

$$e_n \leq (L+1)^{-1}\left(1 + \frac{c}{m}\right) \leq 1,$$

where the last inequality follows from Lemma 7.10. Thus, $0 \leq e_n \leq 1$.

Furthermore, $e_n \in D(L) \subseteq D(Q)$ for all $n \in \mathbb{N}$ and

$$e_n \nearrow (L+1)^{-1}\left(1 + \frac{c}{m}\right) = 1$$

as $n \to \infty$ where the equality follows by (i.a$'$). By using the formula at the start of the proof and the fact that $\psi_n = \varphi_n + \varphi_n c/m$, we infer for $v \in D(Q) \cap \ell^1(X, m) \cap \ell^1(X, c)$ with $v \geq 0$

$$\begin{aligned}
Q(e_n, v) &= Q\left((L+1)^{-1}\psi_n, v\right) \\
&= \langle \psi_n - (L+1)^{-1}\psi_n, v\rangle \\
&= \langle \varphi_n, v\rangle - \langle (L+1)^{-1}\psi_n, v\rangle + \langle \varphi_n \frac{c}{m}, v\rangle \\
&= \langle \varphi_n, v\rangle - \langle e_n, v\rangle + \langle \varphi_n \frac{c}{m}, v\rangle
\end{aligned}$$

$$\to \sum_{x\in X} c(x)v(x)$$

as $n \to \infty$, where the convergence follows by the monotone convergence theorem. We note that the first two terms converge to $\|v\|_1$ as we assume that $v \in \ell^1(X,m)$ and the third term converges to the sum as we assume that $v \in \ell^1(X,c)$.

For a general $v \in D(Q) \cap \ell^1(X,m) \cap \ell^1(X,c)$, we obtain the convergence by decomposing v into positive and negative parts.

(ii′) $\Longrightarrow$ (ii.a′): This is clear as $(L+\alpha)^{-1}v \in D(L) \cap \ell^1(X,m) \cap \ell^1(X,c)$ for $v \in \ell^2(X,m) \cap \ell^1(X,m)$ by Lemma 7.22.

(ii.a′) $\Longrightarrow$ (i.a′): Let (e_n) and v be as assumed in (ii.a′) and let $\alpha > 0$. By Lemma 7.22 and the formula at the start of the proof,

$$\begin{aligned}\sum_{x\in X}\left((L+\alpha)^{-1}\frac{c}{m}\right)(x)v(x)m(x) &= \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x)\\ &= \lim_{n\to\infty} Q(e_n,(L+\alpha)^{-1}v)\\ &= \lim_{n\to\infty}\langle e_n, v-\alpha(L+\alpha)^{-1}v\rangle\\ &= \lim_{n\to\infty}\langle e_n-\alpha(L+\alpha)^{-1}e_n, v\rangle\\ &= \sum_{x\in X}\left(1-\alpha(L+\alpha)^{-1}1\right)(x)v(x)m(x),\end{aligned}$$

where the convergence follows by the Lebesgue dominated convergence theorem as we assume that $v \in \ell^1(X,m)$. Hence,

$$\sum_{x\in X}\left(1-\alpha(L+\alpha)^{-1}1-(L+\alpha)^{-1}\frac{c}{m}\right)(x)v(x)m(x)=0.$$

Since $v > 0$, we infer

$$(L+\alpha)^{-1}\left(\alpha 1+\frac{c}{m}\right)=1,$$

which is (i.a′). This finishes the proof. $\square$

We recall that a connected graph b over (X,m) is called recurrent if $1 \in \mathcal{D}_0$, where $\mathcal{D}_0 = \overline{C_c(X)}^{\|\cdot\|_o}$ and $\|\varphi\|_o = (\mathcal{Q}(\varphi)+\varphi^2(o))^{1/2}$ for $\varphi \in C_c(X)$ and an arbitrary vertex $o \in X$. This means that 1 can be approximated by finitely supported functions via pointwise convergence and convergence in the form sense, see Theorem 6.1. Comparing with the result above, we obtain the following immediate corollary.

Corollary 7.25 (Recurrence implies stochastic completeness). *Let b be a connected graph over (X,m). If b is recurrent, then b is stochastically complete.*

Proof. By Theorem 6.1 (i.d), if b is recurrent, then there exists a sequence of functions $e_n \in C_c(X)$ with $0 \le e_n \le 1$, $e_n \to 1$ pointwise and $Q(e_n) \to 0$ as $n \to \infty$. Hence, for all $v \in D(Q) \cap \ell^1(X,m)$ we get

$$Q(e_n, v) \leq Q^{1/2}(e_n)Q^{1/2}(v) \to 0$$

as $n \to \infty$. Therefore, the graph is stochastically complete by Theorem 7.23 as $c = 0$. $\square$

Remark. We have seen in Corollaries 7.21 and 7.25 that recurrence implies stochastic completeness, which implies form uniqueness. As we will see via examples, the reverse implications do not hold. On the other hand, if the measure satisfies $m(X) < \infty$, then all three properties are equivalent (Exercise 7.7).

6. The Green's formula perspective

In this section we discuss stochastic completeness from the perspective of Green's formula. This formula allows us to move the Laplacian between functions when summing over the set of vertices. We will show that stochastic completeness at infinity is equivalent to the validity of such a formula for a class of functions satisfying several summability conditions.

As we have seen, Green's formulas assert the validity of summation formulas such as

$$\sum_{x \in X} \mathcal{L}f(x)g(x)m(x) = \sum_{x \in X} f(x)\mathcal{L}g(x)m(x)$$

for functions f and g. For example, if $f \in \mathcal{F} \cap \ell^2(X,m)$ and $\varphi \in C_c(X)$ with $\mathcal{L}f, \mathcal{L}\varphi \in \ell^2(X,m)$, then this can be written more simply via the inner product as

$$\langle \mathcal{L}f, \varphi \rangle = \langle f, \mathcal{L}\varphi \rangle,$$

see Proposition 1.5. In this section we characterize stochastic completeness at infinity via the validity of the formula

$$\sum_{x \in X} \mathcal{L}v(x)1(x)m(x) = \sum_{x \in X} v(x)\mathcal{L}1(x)m(x)$$

for suitable functions v and the constant function 1. We note that $\mathcal{L}1 = c/m$ so that the right-hand side in the equality above vanishes in the case $c = 0$. In particular, this is the case when studying stochastic completeness as opposed to stochastic completeness at infinity.

We will connect a Green's formula with the possibility to approximate the constant function 1 as discussed in the previous section. As a consequence, we get that the validity of a Green's formula is equivalent to stochastic completeness at infinity. We recall that

$$\ell^1(X,c) = \{f \in C(X) \mid \sum_{x \in X} c(x)|f(x)| < \infty\}$$

and that $(L + \alpha)^{-1}v \in \ell^1(X,c)$ for all $v \in \ell^1(X,m)$ and all $\alpha > 0$ by Lemma 7.22.

THEOREM 7.26 (Approximating 1 and Green's formula). *Let (b, c) be a connected graph over (X, m). Then, the following statements are equivalent:*

(ii′) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, v) = \sum_{x\in X} c(x)v(x)$$
for all $v \in D(Q) \cap \ell^1(X, m) \cap \ell^1(X, c)$.

(ii.a′) *There exists a sequence of functions $e_n \in D(Q)$ (equivalently, $e_n \in C_c(X)$) with $0 \leq e_n \leq 1$ for all $n \in \mathbb{N}$ such that $e_n \to 1$ pointwise and*
$$\lim_{n\to\infty} Q(e_n, (L+\alpha)^{-1}v) = \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x)$$
for one $v \in \ell^2(X, m) \cap \ell^1(X, m)$ with $v > 0$ and some (all) $\alpha > 0$.

(iii′) *If $v \in \mathcal{D} \cap \ell^1(X, m) \cap \ell^2(X, m) \cap \ell^1(X, c)$ satisfies $\mathcal{L}v \in \ell^1(X, m)$, then*
$$\sum_{x\in X} \mathcal{L}v(x)m(x) = \sum_{x\in X} c(x)v(x).$$
(*"Green's formula"*)

(iii.a′) *If $v \in \mathcal{D} \cap \ell^1(X, m) \cap \ell^2(X, m) \cap \ell^1(X, c)$ satisfies $\mathcal{L}v \in \ell^1(X, m) \cap \ell^2(X, m)$, then*
$$\sum_{x\in X} \mathcal{L}v(x)m(x) = \sum_{x\in X} c(x)v(x).$$

PROOF. (ii′) $\Longleftrightarrow$ (ii.a′): This was already shown in Theorem 7.23.

(ii′) $\Longrightarrow$ (iii′): At first let $v \in D(Q) \cap \ell^1(X, m) \cap \ell^1(X, c)$ be such that $\mathcal{L}v \in \ell^1(X, m)$. As given in (ii′), there exist $e_n \in C_c(X)$ such that $0 \leq e_n \leq 1$, $e_n \to 1$ pointwise and
$$\lim_{n\to\infty} Q(e_n, v) = \sum_{x\in X} c(x)v(x).$$
As $v \in D(Q) \subseteq \mathcal{D} \subseteq \mathcal{F}$ by Green's formula, Proposition 1.5, we compute that
$$Q(e_n, v) = \sum_{x\in X} e_n(x)\mathcal{L}v(x)m(x) \to \sum_{x\in X} \mathcal{L}v(x)m(x)$$
as $n \to \infty$, where the convergence follows from Lebesgue's's dominated convergence theorem using the assumptions that $e_n \to 1$ pointwise and $\mathcal{L}v \in \ell^1(X, m)$. Hence,
$$\sum_{x\in X} \mathcal{L}v(x)m(x) = \sum_{x\in X} c(x)v(x)$$
for all $v \in D(Q) \cap \ell^1(X, m) \cap \ell^1(X, c)$.

It remains to extend the equality to all $v \in \mathcal{D}\cap\ell^1(X,m)\cap\ell^2(X,m)\cap\ell^1(X,c)$ with $\mathcal{L}v \in \ell^1(X,m)$. First, by definition, $\mathcal{D} \cap \ell^2(X,m) = D(Q^{(N)})$. Moreover, by Corollary 7.21 stochastic completeness at infinity implies $D(Q^{(D)}) = D(Q^{(N)})$, where $D(Q) = D(Q^{(D)})$. As we have shown that stochastic completeness at infinity is equivalent to (ii′) in Theorem 7.23 it follows that

$$D(Q) = D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m).$$

This finishes the proof.

(iii′) $\Longrightarrow$ (iii.a′): This is obvious.

(iii.a′) $\Longrightarrow$ (ii.a′): Let $v \in \ell^1(X,m)\cap\ell^2(X,m)$ satisfy $v > 0$ and let $w = (L+\alpha)^{-1}v$. Then, as $v \in \ell^1(X,m)$, $w \in \ell^1(X,c)$ by Lemma 7.22. Furthermore, as resolvents preserve ℓ^p spaces and map into the operator domain, which is included in $\mathcal{D}$, we get that $w \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m) \cap \ell^1(X,c)$. As L is a restriction of $\mathcal{L}$ by Theorem 1.6 we get

$$\mathcal{L}w = \mathcal{L}(L+\alpha)^{-1}v = v - \alpha(L+\alpha)^{-1}v \in \ell^1(X,m)\cap\ell^2(X,m).$$

Hence, by (iii.a′) applied to w, we obtain

$$\sum_{x\in X} \mathcal{L}w(x)m(x) = \sum_{x\in X} c(x)w(x).$$

Now, let $e_n \in C_c(X)$ satisfy $0 \le e_n \le 1$ and $e_n \to 1$ pointwise. Then, by Green's formula, Proposition 1.5, we get

$$\begin{aligned} Q(e_n,(L+\alpha)^{-1}v) &= Q(e_n,w) \\ &= \sum_{x\in X} e_n(x)\mathcal{L}w(x)m(x) \\ &\to \sum_{x\in X} \mathcal{L}w(x)m(x) \\ &= \sum_{x\in X} c(x)w(x) \\ &= \sum_{x\in X} c(x)(L+\alpha)^{-1}v(x) \end{aligned}$$

as $n \to \infty$, where the convergence follows by Lebesgue's dominated convergence theorem since $\mathcal{L}w \in \ell^1(X,m)$. This shows (ii.a′) and completes the proof. □

As we have shown previously, the possibility to approximate 1 as in (ii′) is equivalent to stochastic completeness at infinity. Hence, we get an immediate characterization of stochastic completeness at infinity via the Green's formula.

Corollary 7.27 (Stochastic completeness at infinity and Green's formula). *Let (b,c) be a connected graph over (X,m). Then, the following statements are equivalent:*

(i′) *For some (all)* $t > 0$ *and some (all)* $x \in X$,

$$M_t(x) = 1.$$

(*"Stochastic completeness at infinity"*)

(iii′) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m) \cap \ell^1(X,c)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m)$, *then*

$$\sum_{x \in X} \mathcal{L}v(x)m(x) = \sum_{x \in X} c(x)v(x).$$

(*"Green's formula"*)

(iii.a′) *If* $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m) \cap \ell^1(X,c)$ *satisfies* $\mathcal{L}v \in \ell^1(X,m) \cap \ell^2(X,m)$, *then*

$$\sum_{x \in X} \mathcal{L}v(x)m(x) = \sum_{x \in X} c(x)v(x).$$

Proof. That (i′) is equivalent to $(L+\alpha)^{-1}(\alpha 1 + c/m) = 1$ for some (all) $\alpha > 0$ was shown in Theorem 7.16. Hence, the conclusion follows by combining Theorems 7.23 and 7.26. □

Remark (Relation to recurrence). By Theorem 6.1 (iii.a), for a graph b over (X,m) recurrence is equivalent to the fact that for every $v \in \mathcal{D}$ with $\mathcal{L}v \in \ell^1(X,m)$, we have $\sum_{x\in X} \mathcal{L}v(x)m(x) = 0$. Comparing with (iii′) above, this gives another proof that recurrence implies stochastic completeness when $c = 0$.

Remark (Abstract version of Green's formula). The condition (iii′) can be understood on a more abstract level. For the sake of clarity, we discuss this here for $c = 0$. When $c = 0$, the corresponding statement in Corollary 7.27 is

(iii) If $v \in \mathcal{D} \cap \ell^1(X,m) \cap \ell^2(X,m)$ satisfies $\mathcal{L}v \in \ell^1(X,m)$, then

$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

To discuss an abstract version of this, recall that the generator of the semigroup on $\ell^1(X,m)$ is denoted by $L^{(1)}$ and that the dual of $L^{(1)}$ on $\ell^\infty(X)$ is denoted by $L^{(\infty)}$. In Theorem 2.13 from Section 1 we have shown that these generators are restrictions of $\mathcal{L}$. Then, (iii) can be shown to be equivalent to either of the following statements:

(iii.a) For all $u \in D(L^{(1)})$

$$\sum_{x \in X} L^{(1)}u(x)m(x) = 0.$$

(iii.b) The constant function 1 is in $D(L^{(\infty)})$ and $L^{(\infty)}1 = 0$.

We leave the proof of the equivalence as an exercise (Exercise 7.6).

7. The Omori–Yau maximum principle

In this section we prove the Omori–Yau maximum principle characterization of stochastic completeness at infinity. This maximum principle that every function bounded above must be close to superharmonic on the set where the function takes values near its supremum. For some considerations, this is a more flexible criterion than analyzing the behavior of α-subharmonic functions on the entire vertex set.

In Theorem 7.18 we have already shown that stochastic incompleteness at infinity is equivalent to the existence of a non-trivial positive bounded α-(sub)harmonic function defined on the entire vertex set. Specifically, there exists a non-trivial $u \in \ell^\infty(X)$ with $u \geq 0$ and

$$(\mathcal{L} + \alpha)u \leq 0$$

for $\alpha > 0$ if and only if the graph is stochastically incomplete at infinity.

In this section we prove the Omori–Yau maximum principle criterion for stochastic incompleteness at infinity. In particular, the existence of a bounded function u whose supremum is strictly positive and which satisfies

$$\mathcal{L}u \leq -C$$

for some constant $C > 0$ and some set of vertices where u is near its supremum is equivalent to stochastic incompleteness at infinity. A basic intuition behind the Omori–Yau maximum principle is that the equation $\mathcal{L}u \leq -C$ implies that u is strictly increasing in some direction. Hence, if u still has some direction to increase even as we get near the supremum of u and the graph can accommodate this increase to allow u to be bounded, then the graph must have large growth and, hence, be stochastically incomplete at infinity.

We first connect the Omori–Yau maximum principle with the nonexistence of α-(sub)harmonic functions. As usual, the reader who is only interested in stochastic completeness and not stochastic completeness at infinity can let $c = 0$ and substitute $e^{-tL}1$ for M_t in the statements below.

Theorem 7.28 (Omori–Yau maximum principle and α-harmonic functions). *Let (b, c) be a connected graph over (X, m). Then, the following statements are equivalent:*

(iv′) *If $u \in \mathcal{F}$ satisfies $\sup u \in (0, \infty)$ and $\beta \in (0, \sup u)$, then*

$$\sup_{X_\beta} \mathcal{L}u \geq 0,$$

where $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$.

(*"Omori–Yau maximum principle"*)

(v.a′) *For some (all) $\alpha > 0$ every positive $u \in \ell^\infty(X)$ which satisfies $(\mathcal{L} + \alpha)u \leq 0$ is trivial.*

(v.c′) *For some (all)* $\alpha > 0$ *every positive* $u \in \ell^\infty(X)$ *which satisfies* $(\mathcal{L}+\alpha)u = 0$ *is trivial.*

PROOF. (v.a′) $\Longleftrightarrow$ (v.c′): This was shown in Theorem 7.18.

(iv′) $\Longrightarrow$ (v.c′): We show this by contraposition. Let u be a non-trivial positive bounded function which satisfies $(\mathcal{L}+\alpha)u = 0$ for $\alpha > 0$. Let $\beta = \sup u/2 > 0$, which is strictly positive since u is non-trivial. Then, for all $x \in X_\beta = \{y \in X \mid u(y) > \sup u/2\}$ we get

$$\mathcal{L}u(x) = -\alpha u(x) < -\alpha \frac{\sup u}{2} < 0.$$

Hence, (iv′) fails for this u and for this choice of β.

(v.a′) $\Longrightarrow$ (iv′): We also show this by contraposition. Assume that (iv′) fails. Then, there exists a constant $C > 0$, a function $u \in \mathcal{F}$ with $\sup u \in (0,\infty)$ and a $\beta \in (0, \sup u)$ such that

$$\mathcal{L}u \leq -C$$

on $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$. Define

$$u_\beta = (u + \beta - \sup u)_+,$$

where $f_+ = f \vee 0$ is the positive part of a function f. We will show that u_β is positive bounded non-trivial and α-subharmonic for $\alpha > 0$, which will show that (v.a′) fails.

Since u is bounded, it is clear from the definition that u_β is positive and bounded. If u_β were trivial, then $u(x) \leq \sup u - \beta$ for all $x \in X$, where $\beta > 0$, which contradicts the definition of the supremum. Hence, u_β is non-trivial.

Let $\alpha = C/\beta > 0$, where $\mathcal{L}u \leq -C$ on X_β. We will now show that u_β is α-subharmonic, that is,

$$(\mathcal{L}+\alpha)u_\beta \leq 0.$$

If $x \notin X_\beta$, then $u_\beta(x) = 0$. Therefore,

$$(\mathcal{L}+\alpha)u_\beta(x) = -\frac{1}{m(x)} \sum_{y \in X} b(x,y) u_\beta(y) \leq 0.$$

For $x \in X_\beta$, we first note that

$$\alpha u_\beta(x) = \alpha(u(x) + \beta - \sup u)_+ \leq \alpha\beta = C$$

by the choice of α. Moreover, since $\beta < \sup u$ by assumption, we have

$$u_\beta(x) = u(x) + \beta - \sup u < u(x).$$

For $y \in X$ we obtain

$$\begin{aligned} u_\beta(x) - u_\beta(y) &= u(x) + \beta - \sup u - u_\beta(y) \\ &= \begin{cases} u(x) - u(y) & : y \in X_\beta \\ u(x) + \beta - \sup u - 0 & : y \notin X_\beta \end{cases} \\ &\leq u(x) - u(y) \end{aligned}$$

as $u(y) \leq \sup u - \beta$ for $y \notin X_\beta$. Thus, combining the three inequalities above with $\mathcal{L}u \leq -C$ on X_β yields that, for $x \in X_\beta$,

$$\begin{aligned}(\mathcal{L}+\alpha)u_\beta(x) &= \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u_\beta(x)-u_\beta(y)) + \left(\frac{c(x)}{m(x)}+\alpha\right)u_\beta(x)\\ &\leq \frac{1}{m(x)}\sum_{y\in X} b(x,y)(u(x)-u(y)) + \frac{c(x)}{m(x)}u(x) + C\\ &= \mathcal{L}u + C\\ &\leq 0.\end{aligned}$$

Therefore, u_β is a positive bounded non-trivial function satisfying $(\mathcal{L}+\alpha)u_\beta \leq 0$ for $\alpha > 0$, which shows that (v.a′) does not hold. This completes the proof. □

As we have already shown that the statements above concerning α-(sub)harmonic functions are equivalent to stochastic completeness at infinity, we get the following immediate corollary which links the Omori–Yau maximum principle with stochastic completeness at infinity.

Corollary 7.29 (Stochastic completeness at infinity and Omori–Yau maximum principle). *Let (b,c) be a connected graph over (X,m). Then, the following statements are equivalent:*

(i′) *For some (all) $t > 0$ and some (all) $x \in X$,*

$$M_t(x) = 1.$$

(*"Stochastic completeness at infinity"*)

(iv′) *If $u \in \mathcal{F}$ satisfies $\sup u \in (0,\infty)$ and $\beta \in (0, \sup u)$, then*

$$\sup_{X_\beta} \mathcal{L}u \geq 0,$$

where $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$.

(*"Omori–Yau maximum principle"*)

Proof. The result follows immediately by combining the previously proven Theorems 7.18 and 7.28. □

8. A stability criterion and Khasminskii's criterion

In this section we use the Omori–Yau maximum principle to prove a stability result for stochastic completeness at infinity. This result gives that stochastic incompleteness at infinity of a subgraph implies stochastic incompleteness at infinity of the entire graph under some additional conditions. Furthermore, we state and prove a Khasminskii criterion which states that the existence of a positive α-superharmonic function which goes to infinity implies stochastic completeness at infinity.

We start with some general remarks regarding subgraphs and the stability question. Whenever Y is a subset of X, the graph (b_Y, c_Y) over (Y, m_Y) with $b_Y\colon Y \times Y \longrightarrow [0,\infty)$ satisfying $b_Y(x,y) = b(x,y)$, $c_Y\colon Y \longrightarrow [0,\infty)$ satisfying $c_Y(x) = c(x)$ and $m_Y\colon Y \longrightarrow (0,\infty)$ satisfying $m_Y(x) = m(x)$ is called a subgraph of (b,c). In other words, we restrict the functions b, c and m to the subset Y of X.

We note that any graph that has a transient subgraph is also transient, cf. Exercise 6.15 in the previous chapter. In contrast, a graph which has a stochastically incomplete subgraph is not, in general, stochastically incomplete. In fact, every stochastically incomplete graph is a subgraph of a graph which is stochastically complete (Exercise 7.8). Intuitively, this can be achieved by diverting heat from regions of large growth by creating regions with slow growth via additional edges. Alternatively, by adding a killing term c, any stochastically incomplete graph is a subgraph of a graph which is stochastically complete at infinity (Exercise 7.9). The reason why this works is that heat is removed via c so that it is not lost at infinity.

However, there are several conditions yielding that stochastic incompleteness at infinity of a subgraph does imply stochastic incompleteness at infinity of the entire graph. We now present one such criterion and leave other criteria as exercises. Specifically, we will use the Omori–Yau maximum principle to show that if the stochastically incomplete subgraph is not too connected to vertices outside of the subgraph, then the entire graph is stochastically incomplete.

THEOREM 7.30 (Stability of stochastic incompleteness). *Let (b,c) be a connected graph over (X,m). Let $Y \subseteq X$ and let (b_Y, c_Y) over (Y, m_Y) be a connected subgraph of (b,c). If (b_Y, c_Y) is stochastically incomplete at infinity and there exists a constant C such that*

$$\frac{1}{m(x)} \sum_{y \in X\setminus Y} b(x,y) \leq C$$

for all $x \in Y$, then (b,c) is stochastically incomplete at infinity.

PROOF. As (b_Y, c_Y) is stochastically incomplete at infinity, it follows from Corollary 7.29 that there exists a function $u \in C(Y)$ with $\sup_Y u \in (0,\infty)$ and $\beta_1 \in (0, \sup_Y u)$ such that

$$\mathcal{L}_Y u \leq -C_1$$

on $Y_{\beta_1} = \{x \in Y \mid u(x) > \sup_Y u - \beta_1\}$ for some constant $C_1 > 0$. Here,

$$\mathcal{L}_Y u(x) = \frac{1}{m(x)} \sum_{y \in Y} b(x,y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x),$$

for $x \in Y$, denotes the Laplacian on Y.

We first note that by letting $\beta = \beta_1 \wedge C_1$, we get $Y_\beta \subseteq Y_{\beta_1}$ and so

$$\sup_{Y_\beta} \mathcal{L}_Y u \leq \sup_{Y_{\beta_1}} \mathcal{L}_Y u \leq -C_1 \leq -\beta,$$

yielding that $\sup_{Y_\beta} \mathcal{L}_Y u \leq -\beta$. We will work with this β in what follows.

We wish to extend u to the entire space X in such a way that the resulting function violates the Omori–Yau maximum principle. We let

$$v(x) = \begin{cases} \left(u(x) - \sup_Y u + \frac{\beta}{2C}\right)_+ & \text{if } x \in Y \\ 0 & \text{otherwise,} \end{cases}$$

where we choose $C \geq 1$ such that $(1/m(x)) \sum_{y \in X \setminus Y} b(x,y) \leq C$ as given by assumption. We note that

$$\sup v = \frac{\beta}{2C},$$

so that

$$\begin{aligned} X_{\beta/2C} &= \{x \in X \mid v(x) > \sup v - \beta/2C\} \\ &= \{x \in X \mid v(x) > 0\} \\ &= \{x \in Y \mid u(x) > \sup u - \beta/2C\}. \end{aligned}$$

In particular, as $C \geq 1$ so that $\beta/2C < \beta$, we get $X_{\beta/2C} \subseteq Y_\beta$, where $Y_\beta = \{x \in Y \mid u(x) > \sup_Y u - \beta\}$ is the set where $\mathcal{L}_Y u \leq -\beta$.

Let $x \in X_{\beta/2C}$. Then,

$$v(x) = u(x) - \sup_Y u + \frac{\beta}{2C} < u(x)$$

since $\beta \in (0, \sup_Y u)$ and $C \geq 1$. If $y \in X_{\beta/2C}$, then

$$v(x) - v(y) = u(x) - u(y).$$

If $y \notin X_{\beta/2C}$ and $y \in Y$, then

$$\begin{aligned} v(x) - v(y) &= u(x) - \sup_Y u + \frac{\beta}{2C} \\ &\leq u(x) - u(y) \end{aligned}$$

since $u(y) \leq \sup_Y u - \beta/2C$ in this case. Finally, if $y \notin X_{\beta/2C}$ and $y \notin Y$, then

$$\begin{aligned} v(x) - v(y) &= v(x) \\ &= u(x) - \sup_Y u + \frac{\beta}{2C} \\ &\leq \frac{\beta}{2C}. \end{aligned}$$

Putting these inequalities together and using the fact that $X_{\beta/2C} \subseteq Y_\beta$ mentioned above, we get for $x \in X_{\beta/2C}$,

$$\begin{aligned}\mathcal{L}v(x) &= \frac{1}{m(x)} \sum_{y \in Y} b(x,y)(v(x)-v(y)) + \frac{c(x)}{m(x)} v(x) \\ &\quad + \frac{1}{m(x)} \sum_{y \notin Y} b(x,y)(v(x)-v(y)) \\ &\leq \mathcal{L}_Y u(x) + \frac{1}{m(x)} \sum_{y \notin Y} b(x,y) \frac{\beta}{2C} \\ &\leq -\beta + \frac{\beta}{2} = -\frac{\beta}{2} < 0.\end{aligned}$$

Therefore,

$$\sup_{X_{\beta/2C}} \mathcal{L}v \leq -\frac{\beta}{2},$$

so that the graph is stochastically incomplete at infinity by Corollary 7.29. □

Remark. A similar argument gives that if the degree within Y is bounded on the set of vertices in Y which have a neighbor outside of Y, then (b,c) over (X,m) is stochastically incomplete at infinity (Exercise 7.10).

Remark. If the graph associated to a Dirichlet restriction $Q_Y^{(D)}$ for some subset $Y \subset X$ is stochastically incomplete at infinity, then the entire graph (b,c) over (X,m) is stochastically incomplete at infinity (Exercise 7.11).

We next present the Khasminskii criterion, which is a useful test for stochastic completeness. It states that the existence of a positive α-superharmonic function for $\alpha > 0$ which grows at infinity implies that the graph is stochastically complete at infinity. The idea of the proof is that a growing α-superharmonic function will dominate any positive bounded α-harmonic function up to an arbitrary scale, which forces the bounded α-harmonic function to be zero. This result will be used in our comparison results for weakly spherically symmetric graphs.

We start by making precise the notion of growing at infinity. We say that $f \in C(X)$ satisfies $f(x) \to \infty$ as $x \to \infty$ if for every $C \in \mathbb{R}$, there exists a finite set $K \subseteq X$ such that $f|_{X \setminus K} \geq C$. Similarly, we define $f(x) \to -\infty$ by replacing $\geq$ with $\leq$. In particular, for a function $f \in C(X)$ and $a \in \mathbb{R} \cup \{\pm\infty\}$ we have

$$f(x) \to a \text{ as } x \to \infty$$

if for every sequence of (x_n) of distinct elements in X we have

$$f(x_n) \to a \quad \text{as} \quad n \to \infty.$$

We note that this can be understood in terms of the one-point compactification $\widehat{X} = X \cup \{\infty\}$ of X, where ∞ is an additional point.

Theorem 7.31 (Khasminskii criterion for stochastic completeness). *Let (b, c) be a connected graph over (X, m). If there exists a positive function $v \in \mathcal{F}$ such that $v(x) \to \infty$ as $x \to \infty$ and*

$$(\mathcal{L} + \alpha)v \geq 0$$

for $\alpha > 0$, then (b, c) is stochastically complete at infinity.

Proof. Let $\alpha > 0$ and let v be a function as assumed in the statement of the theorem. Let $u \in \ell^\infty(X)$ be a positive solution of $(\mathcal{L}+\alpha)u = 0$. We can, by rescaling, assume that $u \leq 1$. By Theorem 7.18 in order to prove stochastic completeness at infinity, it suffices to show that $u = 0$.

For a given $C > 0$ let $K \subseteq X$ be a finite subset such that $v|_{X \setminus K} \geq C$. The function $w = v - Cu$ then satisfies

- $(\mathcal{L} + \alpha)w \geq 0$ on K
- $w \wedge 0$ assumes a minimum on K as K is finite
- $w \geq 0$ on $X \setminus K$.

By the minimum principle, Theorem 1.7, we infer $w \geq 0$. Therefore, $v \geq Cu$. As $C > 0$ was chosen arbitrarily, we infer $u = 0$. Hence, every bounded positive solution u of $(\mathcal{L}+\alpha)u = 0$ is trivial, which completes the proof. □

9. A probabilistic interpretation*

In this section we give a probabilistic interpretation of stochastic completeness at infinity. This relies heavily on the Feynman–Kac formula. As such, this section is marked as optional.

Let (b, c) be a graph over (X, m). Let $(\mathbb{X}_t) = (\mathbb{X}_t^b)$ denote the Markov process with respect to the graph b over (X, m) which was introduced in Section 5. In this context, ζ denotes the lifetime of the process $(\mathbb{X}_t)$ and $\mathbb{P}_x$ denotes the probability measure conditioned on $\mathbb{X}_0 = x$ for $x \in X$.

Let $f \in \ell^2(X, m)$, $x \in X$ and $t \geq 0$. The Feynman–Kac formula, Theorem 2.31, gives

$$e^{-tL} f(x) = \mathbb{E}_x \left(1_{\{t<\zeta\}} e^{-\int_0^t (c/m)(\mathbb{X}_s) ds} f(\mathbb{X}_t) \right),$$

where $\mathbb{E}_x$ is the expected value of the process conditioned on $\mathbb{X}_0 = x$. This formula obviously extends to all positive functions $f \geq 0$ by monotone convergence. We make this explicit for the constant function 1 below.

We start with a discussion concerning stochastic completeness, i.e., the equality $e^{-tL}1 = 1$. The result below shows that stochastic completeness is equivalent to the process having infinite lifetime and the fact that the killing term vanishes.

THEOREM 7.32 (Probabilistic characterization of stochastic completeness). *Let (b, c) be a graph over (X, m). Let $(\mathbb{X}_t)$ be the process associated to b with lifetime ζ. Then,*

$$e^{-tL}1(x) = \mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right).$$

Furthermore, the following statements are equivalent:

(i) $e^{-tL}1 = 1$ *for all* $t > 0$.
(ii) $\mathbb{P}_x(\zeta = \infty) = 1$ *for* $x \in X$ *and* $c = 0$.

PROOF. We first show the stated equality. Let $0 \leq \eta_k \leq 1$ for $k \in \mathbb{N}$ be a sequence of compactly supported functions such that $\eta_k \nearrow 1$ pointwise. Then, the Feynman–Kac formula, Theorem 2.31, combined with monotone convergence yields

$$\begin{aligned} e^{-tL}1(x) &= \lim_{k\to\infty} e^{-tL}\eta_k(x) \\ &= \lim_{k\to\infty} \mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\eta_k(\mathbb{X}_t)\right) \\ &= \mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right). \end{aligned}$$

This proves the stated equality. We now show the equivalence.

(i) $\Longrightarrow$ (ii): We show this by contraposition. First, by the formula above and the Cauchy–Schwarz inequality we get

$$\begin{aligned} e^{-tL}1(x) &= \mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right) \\ &\leq \mathbb{E}_x\left(1_{\{t<\zeta\}}\right)^{1/2}\mathbb{E}_x\left(e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right)^{1/2} \\ &= \mathbb{P}_x(t<\zeta)^{1/2}\mathbb{E}_x\left(e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right)^{1/2}. \end{aligned}$$

If $\mathbb{P}_x(\zeta = \infty) < 1$, then $\mathbb{P}_x(t < \zeta) < 1$ for t large enough. If $c(x) > 0$ for some $x \in X$ and we let J_1 be the first jumping time of the process which is strictly positive almost surely, then

$$\mathbb{E}_x\left(e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right) \leq \mathbb{E}_x\left(e^{-J_1(c/m)(x)}\right) < 1$$

by the Taylor expansion of the exponential function. Therefore, if either $\mathbb{P}_x(\zeta = \infty) < 1$ or $c(x) > 0$ for some $x \in X$, then $e^{-tL}1(x) < 1$ and, consequently, $e^{-tL}1 < 1$ for all $t > 0$ by Lemma 7.17.

(ii) $\Longrightarrow$ (i): This is immediate from the equality as $e^{-tL}1(x) = \mathbb{E}_x(1) = 1$. $\square$

Next, we turn to a probabilistic characterization of stochastic completeness at infinity, i.e., the equality $M_t = 1$. For two events A and B, we denote the probability of A conditioned on B by $\mathbb{P}(A \mid B)$.

THEOREM 7.33 (Probabilistic characterization of stochastic completeness at infinity). *Let (b,c) be a graph over (X,m). Let $(\mathbb{X}_t)$ be the process associated to b with lifetime ζ. Then,*

$$M_t(x) = 1 - \mathbb{E}_x\left(1_{\{t\geq\zeta\}} e^{-\int_0^\zeta (c/m)(\mathbb{X}_s)ds}\right).$$

Furthermore, the following statements are equivalent:

(i) $M_t = 1$ *for all* $t > 0$.

(ii) $\mathbb{P}_x(\zeta = \infty) = 1$ *or* $\mathbb{P}_x\left(\int_0^\zeta \frac{c}{m}(\mathbb{X}_t)dt = \infty \mid \zeta < \infty\right) = 1$ *for all* $x \in X$.

PROOF. For ease of notation, we denote c/m by q. By the Feynman–Kac formula, Theorem 2.31, which extends to arbitrary positive functions by monotone convergence, we compute $M_t = e^{-tL}1 + \int_0^t e^{-sL}q ds$ in probabilistic terms. We start with the second term using Fubini's theorem in the second step

$$\begin{aligned}
\int_0^t e^{-sL}q(x)ds &= \int_0^t \mathbb{E}_x\left(1_{\{s<\zeta\}} e^{-\int_0^s q(\mathbb{X}_r)dr} q(\mathbb{X}_s)\right) ds \\
&= \mathbb{E}_x\left(\int_0^t 1_{\{s<\zeta\}} e^{-\int_0^s q(\mathbb{X}_r)dr} q(\mathbb{X}_s)ds\right) \\
&= \mathbb{E}_x\left(\int_0^{t\wedge\zeta} e^{-\int_0^s q(\mathbb{X}_r)dr} q(\mathbb{X}_s)ds\right).
\end{aligned}$$

We can now apply the fundamental theorem of calculus to the function $F(s) = -e^{-\int_0^s q(\mathbb{X}_r)dr}$, whose derivative is

$$F'(s) = e^{-\int_0^s q(\mathbb{X}_r)dr} q(\mathbb{X}_s).$$

Note that for $t < \zeta$ the fundamental theorem of calculus applies as the function to be integrated has only finitely many points of discontinuity. By a limiting procedure the fundamental theorem of calculus then applies for $t = \zeta$ as well. Thus, we obtain

$$\begin{aligned}
\ldots &= 1 - \mathbb{E}_x\left(e^{-\int_0^{t\wedge\zeta} q(\mathbb{X}_r)dr}\right) \\
&= 1 - \mathbb{E}_x\left(1_{\{t\geq\zeta\}} e^{-\int_0^\zeta q(\mathbb{X}_r)dr}\right) - \mathbb{E}_x\left(1_{\{t<\zeta\}} e^{-\int_0^t q(\mathbb{X}_r)dr}\right) \\
&= 1 - \mathbb{E}_x\left(1_{\{t\geq\zeta\}} e^{-\int_0^\zeta q(\mathbb{X}_r)dr}\right) - e^{-tL}1(x),
\end{aligned}$$

where the last equality follows by Theorem 7.32. This proves the asserted formula. The equivalence (i) $\iff$ (ii) is immediate from this formula. $\square$

Remark. We now present an interpretation of the formula and equivalence above. Theorem 7.32 gives

$$e^{-tL}1(x) = \mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}\right),$$

which can be interpreted as the heat present in the graph at time t. We can interpret $e^{-\int_0^t (c/m)(\mathbb{X}_s)ds}$ as the heat staying in the graph along the path taken by $\mathbb{X}_s$ up to time t if $t < \zeta$, i.e., heat which is not transferred to the cemetery via c. Earlier we interpreted $1 - M_t$ as the heat transferred to infinity via the geometry. So, by Theorem 7.33,

$$1 - M_t(x) = \mathbb{E}_x\left(1_{\{t\geq\zeta\}}e^{-\int_0^\zeta (c/m)(\mathbb{X}_s)ds}\right),$$

which gives a probabilistic version of this interpretation. By the discussion above, we think of the term $e^{-\int_0^\zeta (c/m)(\mathbb{X}_s)ds}$ as the heat not transferred to the cemetery via c, i.e., the heat remaining in the graph along the path taken by $\mathbb{X}_s$ and the function $1_{\{t\geq\zeta\}}$ indicates that this heat is lost via the geometry.

Moreover, the proof above shows that

$$\int_0^t e^{-sL}\frac{c}{m}(x)ds = \mathbb{E}_x\left(1 - e^{-\int_0^{t\wedge\zeta} (c/m)(\mathbb{X}_r)dr}\right).$$

Hence, this term can be understood as the expected heat transferred to the cemetery via c up to time t and one can think of $1 - e^{-\int_0^{t\wedge\zeta} (c/m)(\mathbb{X}_r)dr}$ as the heat transferred to the cemetary along the path taken by $\mathbb{X}_r$ up to $t \wedge \zeta$.

We finish the section with another probabilistic formula, this time for the resolvent.

Theorem 7.34. *Let (b, c) be a graph over (X, m). Let $(\mathbb{X}_t)$ be the process associated to b with lifetime ζ. Then,*

$$(L + \alpha)^{-1}\left(\alpha 1 + \frac{c}{m}\right)(x) = 1 - \mathbb{E}_x\left(e^{-\alpha\zeta-\int_0^\zeta (c/m)(\mathbb{X}_s)ds}\right).$$

Proof. Let $q = c/m$. We recall the Laplace transform, Lemma 7.11, which states

$$(L + \alpha)^{-1}f = \int_0^\infty e^{-t\alpha}e^{-tL}f dt$$

for all $f \in C(X)$ with $f \geq 0$. We use this to apply the Feynman–Kac formula to the right-hand side for the functions 1 and q. To this end we approximate the constant function 1 and q by compactly supported functions from below. Let (η_n) in $C_c(X)$ such that $\eta_n \nearrow 1$ as $n \to \infty$. Then, we obtain by the Feynman–Kac formula, Theorem 2.31, and

Fubini's theorem

$$\begin{aligned}\int_0^\infty e^{-\alpha t}e^{-tL}\eta_n(x)dt &= \int_0^\infty e^{-\alpha t}\left(\mathbb{E}_x\left(1_{\{t<\zeta\}}e^{-\int_0^t q(\mathbb{X}_s)ds}\eta_n(\mathbb{X}_t)\right)\right)dt\\ &= \mathbb{E}_x\left(\int_0^\zeta e^{-\alpha t-\int_0^t q(\mathbb{X}_s)ds}\eta_n(\mathbb{X}_t)dt\right).\end{aligned}$$

Hence, by the Laplace transform and by taking monotone limits, we obtain

$$(L+\alpha)^{-1}1(x) = \mathbb{E}_x\left(\int_0^\zeta e^{-\alpha t-\int_0^t q(\mathbb{X}_s)ds}dt\right).$$

Similarly, by the Feynman–Kac formula, Theorem 2.31 and monotone convergence

$$\begin{aligned}(L+\alpha)^{-1}q(x) &= \mathbb{E}_x\left(\int_0^\zeta e^{-\alpha t}e^{-\int_0^t q(\mathbb{X}_s)ds}q(\mathbb{X}_t)dt\right)\\ &=1-\mathbb{E}_x\left(e^{-\alpha\zeta-\int_0^\zeta q(\mathbb{X}_s)ds}\right)-\alpha\mathbb{E}_x\left(\int_0^\zeta e^{-\alpha t}e^{-\int_0^t q(\mathbb{X}_s)ds}dt\right),\end{aligned}$$

where we applied partial integration. Together with the calculation above this yields

$$(L+\alpha)^{-1}(\alpha 1+q)(x) = 1-\mathbb{E}_x\left(e^{-\alpha\zeta-\int_0^\zeta q(\mathbb{X}_s)ds}\right).$$

This completes the proof. □

Remark (Boundedness and stochastic completeness). We finish this chapter by noting that stochastic incompleteness and stochastic incompleteness at infinity are only possible for unbounded operators. In particular, if the formal Laplacian gives a bounded operator on any $\ell^p(X,m)$ space, then the graph is stochastically complete at infinity. We challenge the reader to give as many proofs as possible for this result (Exercise 7.12).

Exercises

Excavation exercises.

Exercise 7.1 (Dini's theorem). Let (f_n) be a sequence of functions $f_n\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that f_n is continuous for every $n \in \mathbb{N}$ and such that $f_n \nearrow f$ pointwise to a continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. Show that (f_n) converges uniformly to f on every compact subset of $\mathbb{R}$.

Exercise 7.2 (Uniform convergence of continuously differentiable functions). Let (f_n) be a sequence of continuously differentiable functions $f_n\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that the sequence (f_n) as well as the sequence of derivatives (f_n') converge uniformly on compact subintervals to continuous functions $f, g\colon \mathbb{R} \longrightarrow \mathbb{R}$, respectively. Then, g is continuously differentiable with $g' = f$.

(Hint: Use the fundamental theorem of calculus.)

Example exercises.

Exercise 7.3 (Stochastically complete but not recurrent). Give an example of a graph b over (X, m) which is stochastically complete but not recurrent.

Exercise 7.4 (Form unique but not stochastically complete at infinity). Give an example of a graph (b, c) over (X, m) which satisfies $Q^{(D)} = Q^{(N)}$ but is not stochastically complete at infinity.

Extension exercises.

Exercise 7.5 (Stochastic completeness implies $c = 0$). Let (b, c) be a connected graph over (X, m). Show that if (b, c) is stochastically complete, then $c = 0$.

Exercise 7.6 (Stochastic completeness and abstract Green's formula). Let b be a graph over (X, m). Let $L^{(1)}$ be the generator of the semigroup e^{-tL} on $\ell^1(X, m)$ and denote the dual of $L^{(1)}$ on $\ell^\infty(X)$ by $L^{(\infty)}$. Show that the following statements are equivalent:

(iii) For every $v \in \mathcal{D} \cap \ell^1(X, m) \cap \ell^2(X, m)$ such that $\mathcal{L}v \in \ell^1(X, m) \cap \ell^2(X, m)$ we have
$$\sum_{x \in X} \mathcal{L}v(x)m(x) = 0.$$

(iii.a) For all $u \in D(L^{(1)})$,
$$\sum_{x \in X} (L^{(1)}u)(x)m(x) = 0.$$

(iii.b) The constant function 1 belongs to $D(L^{(\infty)})$ and $L^{(\infty)}1 = 0$.

Exercise 7.7 (Graphs of finite measure and stochastic completeness). Let b be a connected graph over (X, m). Suppose that m satisfies $m(X) < \infty$. Show that the following statements are equivalent:

(i) b is recurrent.
(ii) b is stochastically complete.
(iii) $Q^{(D)} = Q^{(N)}$.

Exercise 7.8 (Every stochastically incomplete graph is a subgraph of a stochastically complete graph). Let b be a stochastically incomplete graph over (X, m). Show that there exist X' with $X \subseteq X'$, b' and m' which extend b and m to X' such that b' over (X', m') is stochastically complete.

Exercise 7.9 (Every stochastically incomplete graph is a subgraph of a stochastically complete at infinity graph). Let b be a stochastically incomplete graph over (X, m). Show that there exists a $c\colon X \longrightarrow [0, \infty)$ such that (b, c) over (X, m) is stochastically complete at infinity.

Exercise 7.10 (Stochastic incompleteness of a subgraph and the weighted degree). Let (b, c) be a graph over (X, m). Let $Y \subseteq X$ and suppose that the associated subgraph (b_Y, c_Y) over (Y, m_Y) is stochastically incomplete at infinity. Let

$$\mathrm{Deg}_Y(x) = \frac{1}{m(x)} \sum_{y \in Y} b(x, y)$$

for $x \in Y$. Suppose that Deg_Y is bounded on the set

$$\{x \in Y \mid \text{there exists a } y \sim x, y \notin Y\}.$$

Show that (b, c) over (X, m) is stochastically incomplete at infinity.

Exercise 7.11 (Stochastic incompleteness of a Dirichlet restriction implies
stochastic incompleteness of the entire graph). Let (b, c) be a graph over (X, m) with associated form $Q = Q^{(D)}$ and let $Y \subset X$. Let $(b_Y^{(D)}, c_Y^{(D)})$ be the graph associated to the Dirichlet restriction of Q to Y, i.e., to $Q_Y^{(D)}$. Show that if $(b_Y^{(D)}, c_Y^{(D)})$ is stochastically incomplete at infinity, then (b, c) is stochastically incomplete at infinity.

Show that an analogous statement is true for $c = 0$ by replacing stochastically complete at infinity by stochastically complete.

Exercise 7.12 (Boundedness and stochastic completeness). Let b be a graph over (X, m) with associated regular Dirichlet form $Q = Q^{(D)}$ and Laplacian $L = L^{(D)}$. Assume that L is a bounded operator. How many proofs can you give that the graph is stochastically complete in this case?

If (b, c) is a graph over (X, m), how many of the proofs carry over to show that (b, c) is stochastically complete at infinity if L is bounded?

Notes

The study of stochastic completeness has a long history in both the continuous setting, see, e.g., the survey of Grigor′yan [**Gri99**], and the discrete setting, see, e.g., the early work of Feller [**Fel57**] and Reuter [**Reu57**], often going by the name of conservativeness. In the specific case of graphs with standard weights and counting measure, parts of Theorem 7.2 are worked out in [**Woj08**]. Stochastic completeness at infinity for arbitrary regular Dirichlet forms on discrete spaces is then introduced in [**KL12**]. The corresponding extensions of some of the characterizations found in Theorem 7.2 are then also established in [**KL12**]. The concept of stochastic completeness at infinity was extended to the weighted manifold case in [**MS20**].

We now mention some of the history of specific equivalences found in Theorem 7.2 in both the continuous and discrete settings. As mentioned above, the extension to stochastic completeness at infinity is, for the most part, carried out in [**KL12**] or presented here for the first time.

The equivalence between stochastic completeness, uniqueness of bounded solutions of the heat equation and the triviality of positive bounded α-harmonic functions goes back to [**Fel54**] in the case of one-dimensional diffusions, [**Fel57, Reu57**] for discrete Markov chains and to [**Has60**] in the case of Euclidean space. For graphs with standard weights and counting measure it can be found in [**Woj08**].

The equivalence between stochastic completeness and the ability to approximate the function 1 seems to be a part of the standard theory of stochastic completeness. It can be found, for example, in the textbook [**FŌT11**].

The Green's formula characterization of stochastic completeness at infinity was first shown for weighted manifolds in [**GM13**] and then extended to general Dirichlet forms in [**HKL$^+$17**]. In particular, the general Dirichlet form setting covers both Riemannian manifolds as well as graphs.

The Omori–Yau maximum principle for Riemannian manifolds was introduced in [**Omo67, Yau75**]. The equivalence of a weak form of the Omori–Yau principle and stochastic completeness was first shown in [**PRS03**] for manifolds and [**Hua11b**] for graphs. Furthermore, [**Hua11b**] contains several stability results as a consequence of the Omori–Yau maximum principle, see also [**Woj08, Woj09, Woj11, KL12**] for further investigations into this question.

The Khasminskii criterion for stochastic completeness can be found for the Euclidean case in [**Has60**]. For graphs, it can be found in [**Hua11b**] as well as [**KLW13**]. The idea that a stochastically incomplete graph can be a subgraph of a stochastically complete graph either via the geometry or via the killing term is found in [**KL12, Woj11**]. For manifolds, this can be found in [**MS20**].

The probabilistic viewpoint on stochastic completeness is classical. It can, for instance, be found in [**FŌT11**].

Part 2

Classes of Graphs

Synopsis

In this part we look at three classes of graphs by imposing additional restrictions on the graph structure. For the first class, found in Chapter 8, we impose lower bounds on the measure. For the second class, found in Chapter 9, we impose a very mild spherical symmetry assumption. For the third class, found in Chapter 10, we impose sparseness assumptions and relate them to isoperimetric inequalities. For all three classes, there is a surprising number of consequences of these assumptions for both the spectral theory and stochastic properties.

CHAMBER 8

Uniformly Positive Measure

Here I go, deep type flow Jacques Cousteau could never get this low.
ODB.

In this chapter we look at consequences of lower bounds on the measure m for a graph (b, c) over a discrete measure space (X, m). We formulate the lower bound assumptions in two ways. One way does not take the graph structure into account and one does. We will look at consequences of these lower bounds for the uniqueness of forms and operators as well as for spectral properties.

We first present a Liouville theorem in Section 1. This theorem states that any positive α-subharmonic function for $\alpha \geq 0$ which is additionally in $\ell^p(X, m)$ for $p \in [1, \infty)$ must be zero whenever all infinite paths have infinite measure. In particular, this shows that there are no non-trivial harmonic functions in $\ell^p(X, m)$ in this case.

This Liouville theorem has strong consequences for both uniqueness of the forms and of the operators and allows us to determine the domains of generators for semigroups and resolvents on $\ell^p(X, m)$ for $p \in [1, \infty)$. These consequences are discussed in Section 2. Furthermore, our assumptions also naturally lead to spectral inclusions between the generators on $\ell^p(X, m)$ and solvability of the heat equation for initial conditions in $\ell^p(X, m)$. This is discussed in Sections 3 and 4. Finally, we discuss applications to graphs with standard weights in Section 5.

As should be clear from the discussion above, ℓ^p spaces play a significant role in this chapter. These spaces were introduced in Section 1. However, let us mention that we mostly require only the basics of this theory in this chapter and will introduce these basics as we go along. The only real place where the material of Section 1 is required is when we discuss the domains of the generators on $\ell^p(X, m)$ and their spectral properties in Sections 2 and 3.

We first introduce the assumptions on the measure which will play a role throughout this chapter. The first type of assumption that we consider consists of a uniform lower bound on m and does not involve a graph structure. We denote this assumption by (M). Specifically, we call a measure m over X *uniformly positive* if

(M) $\qquad \inf_{x \in X} m(x) > 0.$

We now recall two prominent examples of measures found in the literature. These examples were first introduced in Section 1 and also played a prominent role in Section 6.

EXAMPLE 8.1 (Counting measure). The counting measure $m = 1$ clearly satisfies (M).

EXAMPLE 8.2 (Normalizing measure). Suppose that b is a graph over X. Then, the the normalizing measure n is given by

$$n(x) = \sum_{y \in X} b(x, y).$$

In particular, if every vertex has at least one neighbor and b is uniformly bounded below on neighbors, then n satisfies (M).

The second type of assumption that we consider requires a graph structure. Given a graph b over X, we recall that a path is a sequence of pairwise distinct vertices (x_n) such that $b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. We call a graph connected if for every pair of vertices there exists a path which includes both vertices. We call a path infinite if the path consists of infinitely many vertices. With these definitions, we now introduce condition (M*) as follows:

(M*) Every infinite path has infinite measure, i.e., for every infinite path (x_n) we have $\sum_n m(x_n) = \infty$.

Let us emphasize that, unlike (M), (M*) is a condition on both the graph b and the measure m. Clearly, (M) implies (M*) whenever we have a graph b over (X, m). In particular, Examples 8.1 and 8.2 discuss some measures satisfying (M*).

Although less frequently arising in the literature, we will see that (M*) is the condition that we need to prove our Liouville theorem in the next section. From this theorem, we can derive quite strong consequences for both uniqueness of associated forms and essential self-adjointness. Furthermore, we will be able to explicitly determine the domains of the generators on all $\ell^p(X, m)$ spaces for measures satisfying (M*). On the other hand, we will see that (M) plays a role in the spectral inclusion results.

1. A Liouville theorem

In this section we prove a Liouville theorem for functions in $\ell^p(X, m)$. It states that every positive α-subharmonic function in $\ell^p(X, m)$ for $p \in [1, \infty)$ and $\alpha \geq 0$ is identically zero whenever the measure of any infinite path is infinite.

Let (b, c) be a graph over (X, m) and let $\mathcal{L} = \mathcal{L}_{b,c,m}$ be the formal Laplacian with domain $\mathcal{F}$. We recall that $u \in \mathcal{F}$ is called α-subharmonic for $\alpha \in \mathbb{R}$ if

$$(\mathcal{L} + \alpha) u \leq 0$$

and is called α-harmonic if

$$(\mathcal{L}+\alpha)u = 0.$$

We also recall that Lemma 1.9 allows us to reduce the study of α-harmonic functions to that of positive α-subharmonic functions. More precisely, Lemma 1.9 shows that if u is α-harmonic, then u_+, u_- and $|u|$ are all α-subharmonic, where $u_+ = u \vee 0$ and $u_- = -u \vee 0$ denote the positive and negative parts of u. Consequently, if all positive α-subharmonic functions are trivial, then all α-harmonic functions are trivial as well. This will be used below.

We recall that, for $p \in [1,\infty)$,

$$\ell^p(X,m) = \{f \in C(X) \mid \sum_{x\in X} |f(x)|^p m(x) < \infty\}.$$

These are Banach spaces with norm

$$\|f\|_p^p = \sum_{x\in X} |f(x)|^p m(x).$$

In particular, when $p=2$, we get our usual Hilbert space $\ell^2(X,m)$.

We now show that positive α-subharmonic functions in $\ell^p(X,m)$ for any $p \in [1,\infty)$ are zero whenever $\alpha \geq 0$ and infinite paths have infinite measure. We refer to statements concerning the constancy or triviality of α-harmonic functions as Liouville theorems. The crucial observation for the proof is that a positive α-subharmonic function for $\alpha \geq 0$ must be strictly increasing in some direction if the function is not constant. Hence, as we assume that the measure is not summable along any infinite path, we obtain that such functions cannot be in $\ell^p(X,m)$ for any $p \in [1,\infty)$.

THEOREM 8.3 (l^p-Liouville theorem). *Let (b,c) be a connected graph over an infinite measure space (X,m) which satisfies* (M*)*. Let $\alpha \geq 0$ and $u \in \mathcal{F}$ with $u \geq 0$ satisfy*

$$(\mathcal{L}+\alpha)u \leq 0.$$

If $u \in \ell^p(X,m)$ for any $p \in [1,\infty)$, then $u = 0$.

In particular, if $u \in \ell^p(X,m)$ for any $p \in [1,\infty)$ is α-harmonic for $\alpha \geq 0$, then $u = 0$.

PROOF. If u is constant, then $u \in \ell^p(X,m)$ for any $p \in [1,\infty)$ if and only if $u = 0$ by (M*) and the assumption that the graph is infinite and connected. So, we assume that $u \geq 0$ is a non-constant α-subharmonic function for $\alpha \geq 0$. Then, there exist $x_0, x_1 \in X$ with $x_0 \sim x_1$ such that $u(x_1) > u(x_0) \geq 0$. Now, if $u(x_1) \geq u(y)$ for all

$y \sim x_1$, then

$$\begin{aligned}(\mathcal{L}+\alpha)u(x_1) &= \frac{1}{m(x_1)}\sum_{y\in X} b(x_1,y)(u(x_1)-u(y)) + \left(\frac{c(x_1)}{m(x_1)}+\alpha\right)u(x_1)\\ &> 0,\end{aligned}$$

which gives a contradiction to $(\mathcal{L}+\alpha)u(x_1) \leq 0$. Therefore, there exists $x_2 \sim x_1$ such that $u(x_1) < u(x_2)$.

Iterating this argument and using induction, we find an infinite path (x_n) such that $0 < u(x_n) < u(x_{n+1})$ for all $n \in \mathbb{N}$. We estimate, for any $p \in [1, \infty)$,

$$\sum_{x\in X} |u(x)|^p m(x) \geq \sum_{n=1}^{\infty} |u(x_n)|^p m(x_n) > |u(x_1)|^p \sum_{n=1}^{\infty} m(x_n) = \infty$$

by (M*). Therefore, $u \notin \ell^p(X, m)$ for any $p \in [1, \infty)$.

Combing the arguments above, we conclude that if (b, c) over (X, m) satisfies (M*) and u is a positive α-subharmonic function for $\alpha \geq 0$ and $u \in \ell^p(X, m)$, then u must be constant and, hence, zero. This proves the first statement. The second statement for α-harmonic functions follows directly from the first statement and Lemma 1.9. $\square$

Remark. We note that the result above does not hold for finite graphs for all $\alpha \geq 0$ (Exercise 8.5). Furthermore, the statement is not true for infinite graphs if we remove the assumption (M*). On the other hand, the statement is true for all graphs satisfying (M*) when $c \neq 0$ or $\alpha > 0$ (Exercise 8.6).

Remark. We note that the proof above works even when the graph is not connected but has at least one infinite connected component on which u is non-zero.

Remark. We have seen the case $p = \infty$ and $\alpha > 0$, i.e., the case of positive bounded α-subharmonic functions for $\alpha > 0$ in Chapter 7. In particular, Theorem 7.18 shows that such functions are trivial if and only if (b, c) over (X, m) is stochastically complete at infinity.

2. Uniqueness of the form and essential self-adjointness

In this section we discuss consequences of the ℓ^p-Liouville theorem for uniqueness of forms and operators. In particular, if a graph is connected over a measure space for which the measure of infinite paths is infinite, then we obtain uniqueness of associated forms as well as essential self-adjointness. Furthermore, we explicitly determine the domain of generators on ℓ^p.

We recall that a form Q with domain $D(Q)$ is said to be associated to a graph (b, c) over (X, m) if Q is a closed restriction of $\mathcal{Q}$ and $C_c(X) \subseteq D(Q)$, where $\mathcal{Q} = \mathcal{Q}_{b,c}$ is the energy form. Equivalently, we

can think of Q as being between $Q^{(D)}$ and $Q^{(N)}$, that is, Q is closed with
$$D(Q^{(D)}) \subseteq D(Q) \subseteq D(Q^{(N)})$$
and Q is a restriction of $Q^{(N)}$ to $D(Q)$. Here, $Q^{(D)}$ is the form with Dirichlet boundary conditions with domain
$$D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_Q}$$
and $Q^{(N)}$ is the form with Neumann boundary conditions with domain
$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m),$$
where $\mathcal{D}$ denotes the space of functions of finite energy.

If $D(Q^{(D)}) = D(Q^{(N)})$, then there is a unique such form associated to a graph. We presented some equivalent formulations of this property in Section 1. As a consequence of these equivalences and our Liouville theorem in the previous section, we get that (M*) implies that there is a unique associated form. Moreover, we recall that a positive self-adjoint restriction of $\mathcal{L}$ is called a Markov realization of $\mathcal{L}$ if the form associated to the restriction is a Dirichlet form and the domain of the form contains the finitely supported functions. When there is a unique such realization, then we say that $\mathcal{L}$ satisfies Markov uniqueness. This property was introduced and discussed in Section 3.

THEOREM 8.4 ((M*) implies form uniqueness). *If (b,c) is a connected graph over (X,m) which satisfies* (M*), *then*
$$Q^{(D)} = Q^{(N)}.$$
In particular, there exists a unique operator L associated to (b,c) which is the unique Markov realization of $\mathcal{L}$ and has domain
$$D(L) = \{f \in \mathcal{D} \cap \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}.$$

PROOF. By Theorem 3.2, $Q^{(D)} = Q^{(N)}$ if and only if every α-harmonic function for $\alpha > 0$ which is additionally in $D(Q^{(N)})$ is trivial. Therefore, the first part of the result follows directly from Theorem 8.3. Moreover, in Theorem 3.12 we have shown that Markov uniqueness is equivalent to form uniqueness and the domain of L is given in Corollary 3.3. This, gives the "in particular" statement. □

We now discuss the essential self-adjointness of the restriction of $\mathcal{L}$ to $C_c(X)$. That is, assuming that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, by restricting $\mathcal{L}$ to $C_c(X)$ we get a symmetric operator and we can ask when this operator has a unique self-adjoint extension. This property was discussed in Section 2.

We note that the assumption $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ is characterized in Theorem 1.29. In particular, $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ holds if we assume that the measure m satisfies $\inf_{y\sim x} m(y) > 0$ for all $x \in X$. This clearly holds if m satisfies (M). On the other hand, it is not always satisfied by graphs and measure spaces which satisfy (M*).

THEOREM 8.5 ((M*) implies essential self-adjointness). *Let* (b, c) *be a connected graph over* (X, m) *which satisfies* (M*). *Let* $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$. *Then, the restriction of* $\mathcal{L}$ *to* $C_c(X)$ *is essentially self-adjoint and the unique self-adjoint extension* L *has domain*

$$D(L) = \{f \in \ell^2(X, m) \mid \mathcal{L}f \in \ell^2(X, m)\}.$$

PROOF. By Theorem 3.6 the essential self-adjointness of the restriction of $\mathcal{L}$ to $C_c(X)$ is equivalent to the triviality of α-harmonic functions in $\ell^2(X, m)$ for $\alpha > 0$. Hence, the conclusion follows by Theorem 8.3. The statement about the domain of L follows from Theorem 3.6. $\square$

The associated operator L is the generator of the heat semigroup and resolvent on $\ell^2(X, m)$. We extended this semigroup and resolvent to all $\ell^p(X, m)$ spaces in Section 1. We denote the generators of these semigroups and resolvents by $L^{(p)}$. The domain of $L^{(p)}$ is defined via abstract theory which connects semigroups, resolvents and generators, as discussed in Section 1, see also Appendix D.

By Theorem 2.13, the action of $L^{(p)}$ coincides with the action of $\mathcal{L}$ on the domain of $L^{(p)}$. Below we also specify the domain of $L^{(p)}$ assuming condition (M*).

THEOREM 8.6 (Domain of $L^{(p)}$ given (M*)). *Let* (b, c) *be a connected graph over* (X, m) *which satisfies* (M*). *Let* $L^{(p)}$ *with domain* $D(L^{(p)})$ *be the generator of the semigroup on* $\ell^p(X, m)$ *for* $p \in [1, \infty)$. *Then,*

$$D(L^{(p)}) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\}.$$

PROOF. We have $D(L^{(p)}) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\}$ if and only if α-harmonic functions for $\alpha > 0$ in $\ell^p(X, m)$ are trivial by Theorem 3.8. Therefore, the conclusion follows by Theorem 8.3. $\square$

REMARK. We note that the results above hold for a slightly more general condition than (M*) which incorporates both the measure and the degree function (Exercise 8.7).

3. A spectral inclusion

We now turn to spectral consequences of lower bounds on the measure. We will see that the assumption that m is uniformly bounded from below implies that the spectrum of the Laplacian on $\ell^2(X, m)$ is included in the spectrum of the generators on $\ell^p(X, m)$ for all $p \in (1, \infty)$.

Excavation Exercise 8.1, which recalls some inclusions between $\ell^p(X, m)$ spaces under assumption (M), and Excavation Exercise 8.2, which recalls some facts about the spectrum of an operator and the adjoint of the operator in the case of Banach spaces, will be used in this section.

As a direct consequence of assumption (M) we get inclusions between $\ell^p(X, m)$ spaces. More specifically, (M) implies

$$\ell^p(X, m) \subseteq \ell^q(X, m)$$

for $1 \leq p \leq q < \infty$.

Furthermore, assumption (M) clearly implies (M*) when we have a graph (b,c) over (X,m). Thus, all of our previous results for (M*) also hold when assuming (M). In particular, by Theorem 8.6 it follows that (M) and connectedness of the graph imply

$$D(L^{(p)}) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\},$$

where $L^{(p)}$ is the generator of the semigroup on $\ell^p(X,m)$ for $p \in [1,\infty)$. Combined with the inclusions among the $\ell^p(X,m)$ spaces mentioned above, we obtain the following statement.

Lemma 8.7 (Domain inclusions under (M)). *Let (b,c) be a connected graph over (X,m) which satisfies* (M). *Let $L^{(p)}$ with domain $D(L^{(p)})$ be the generator of the semigroup on $\ell^p(X,m)$ for $p \in [1,\infty)$. Then,*

$$D(L^{(p)}) \subseteq D(L^{(q)})$$

for all $1 \leq p \leq q < \infty$. In particular, $L^{(q)}$ is an extension of $L^{(p)}$.

Proof. The inclusion of domains is immediate from Theorem 8.6, which gives $D(L^{(p)}) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}$ and the fact that $\ell^p(X,m) \subseteq \ell^q(X,m)$ for $1 \leq p \leq q < \infty$ whenever (M) holds. That $L^{(q)}$ is an extension of $L^{(p)}$ then follows from the fact that both operators act as $\mathcal{L}$ on their respective domains, see Theorem 2.13. □

We let $L = L^{(D)}$ denote the Laplacian on $\ell^2(X,m)$ and let $\sigma(L)$ denote the spectrum of L. We let $\sigma(L^{(p)})$ denote the spectrum of $L^{(p)}$. Let us emphasize that here we only deal with the real Banach spaces $\ell^p(X,m)$ and note that the spectra of the generators $L^{(p)}$ on the corresponding complex Banach spaces are not necessarily subsets of the real numbers. However, we are only concerned with the inclusion $\sigma(L) \subseteq \sigma(L^{(p)})$ in what follows and we have $\sigma(L) \subseteq [0,\infty)$.

We recall that if $p,q \in [1,\infty]$ are such that $1/p + 1/q = 1$, then the operators $L^{(p)}$ and $L^{(q)}$ are dual to each other. This is shown for the semigroup and resolvent in Theorems 2.9 and 2.11. The statement for the generators follows directly. In particular, this implies

$$\sigma(L^{(p)}) = \sigma(L^{(q)})$$

whenever $1/p + 1/q = 1$.

In order to prove our spectral inclusion, we will look at the resolvent $(L^{(p)} - \lambda)^{-1}$ for $\lambda \notin \sigma(L^{(p)})$. We start by showing that resolvents agree on the common part of their domains under assumption (M).

Lemma 8.8 (Consistency of the resolvents). *Let (b,c) be a connected graph over (X,m) which satisfies* (M). *Let $1 < p \leq q < \infty$. If $\lambda \notin \sigma(L^{(p)}) \cup \sigma(L^{(q)})$, then*

$$(L^{(p)} - \lambda)^{-1} f = (L^{(q)} - \lambda)^{-1} f$$

for all $f \in \ell^p(X,m) = \ell^p(X,m) \cap \ell^q(X,m)$.

Proof. As we assume (M) and connectedness of the graph, we get $D(L^{(p)}) \subseteq D(L^{(q)})$ from Lemma 8.7. Furthermore, (M) implies $\ell^p(X,m) \subseteq \ell^q(X,m)$ so that $\ell^p(X,m) = \ell^p(X,m) \cap \ell^q(X,m)$.

Let $f \in \ell^p(X,m)$ and $\lambda \notin \sigma(L^{(p)}) \cup \sigma(L^{(q)})$. Then, $(L^{(p)} - \lambda)^{-1} f \in D(L^{(p)}) \subseteq D(L^{(q)})$. Furthermore, as $L^{(q)}$ is an extension of $L^{(p)}$ by Lemma 8.7, we obtain

$$(L^{(q)} - \lambda)(L^{(p)} - \lambda)^{-1} f = (L^{(p)} - \lambda)(L^{(p)} - \lambda)^{-1} f = f.$$

Hence, $(L^{(p)} - \lambda)^{-1}$ is a right inverse for $(L^{(q)} - \lambda)$ on $\ell^p(X,m)$. Since right inverses are unique for invertible operators, $(L^{(p)} - \lambda)^{-1} = (L^{(q)} - \lambda)^{-1}$ on $\ell^p(X,m)$. $\square$

We will now combine the considerations above to prove the following spectral inclusion.

Theorem 8.9 (Spectral inclusion). *Let* (b,c) *be a connected graph over* (X,m) *which satisfies* (M). *Then,*

$$\sigma(L) \subseteq \sigma(L^{(p)})$$

for all $p \in (1,\infty)$.

Proof. Let $\lambda \in \mathbb{R}$ be such that $\lambda \notin \sigma(L^{(p)})$ for $p \in (1,\infty)$. We have to show that $\lambda \notin \sigma(L)$. As $\lambda \notin \sigma(L^{(p)})$, by duality it follows that $\lambda \notin \sigma(L^{(q)})$, where $q \in (1,\infty)$ satisfies $1/p + 1/q = 1$. Without loss of generality, we may assume that $1 < p \leq 2 \leq q < \infty$ as, otherwise, we merely interchange p and q. In particular, Lemma 8.7 implies

$$D(L^{(p)}) \subseteq D(L) \subseteq D(L^{(q)})$$

and that all operators agree on $D(L^{(p)})$.

By Lemma 8.8, as $\lambda \notin \sigma(L^{(p)}) \cup \sigma(L^{(q)})$ and $p \leq q$, $(L^{(p)} - \lambda)^{-1}$ and $(L^{(q)} - \lambda)^{-1}$ agree on $\ell^p(X,m)$. By the Riesz–Thorin interpolation theorem, Theorem E.21, boundedness of $(L^{(q)} - \lambda)^{-1}$ on both $\ell^p(X,m)$ and $\ell^q(X,m)$ and $1/p + 1/q = 1$, implies that $(L^{(q)} - \lambda)^{-1}$ is bounded on $\ell^2(X,m)$. We will show that $(L^{(q)} - \lambda)^{-1}$ is an inverse for $(L - \lambda)$ on $D(L)$ from which it follows that $\lambda \notin \sigma(L)$.

As $L^{(q)}$ is an extension of L, for $f \in D(L) \subseteq D(L^{(q)})$ we have

$$(L^{(q)} - \lambda)^{-1}(L - \lambda) f = (L^{(q)} - \lambda)^{-1}(L^{(q)} - \lambda) f = f.$$

On the other hand, let $f \in \ell^2(X,m)$ and let (f_n) be in $\ell^p(X,m) \subseteq \ell^2(X,m)$ such that $f_n \to f$ in $\ell^2(X,m)$. We note that this is possible as $C_c(X) \subseteq \ell^p(X,m)$ is dense in $\ell^2(X,m)$. As $(L^{(q)} - \lambda)^{-1}$ is bounded on $\ell^2(X,m)$ we have

$$(L^{(q)} - \lambda)^{-1} f_n \to (L^{(q)} - \lambda)^{-1} f$$

in $\ell^2(X,m)$ as $n \to \infty$. By Lemma 8.8, we also have $(L^{(q)} - \lambda)^{-1} f_n = (L^{(p)} - \lambda)^{-1} f_n \in D(L^{(p)}) \subseteq D(L)$ for all $n \in \mathbb{N}$ as $\lambda \notin \sigma(L^{(p)}) \cup \sigma(L^{(q)})$. Therefore,

$$(L-\lambda)(L^{(q)}-\lambda)^{-1} f_n = (L^{(q)}-\lambda)(L^{(q)}-\lambda)^{-1} f_n = f_n \to f$$

as $n \to \infty$ in $\ell^2(X,m)$.

Since L is a closed operator, we infer $(L^{(q)} - \lambda)^{-1} f \in D(L)$ and

$$(L-\lambda)(L^{(q)}-\lambda)^{-1} f = f.$$

Hence, $(L^{(q)} - \lambda)^{-1}$ is a bounded inverse for $(L - \lambda)$ and, therefore, $\lambda \notin \sigma(L)$. This completes the proof. □

REMARK. The reverse inclusion is true when the measure of the graph is finite and the weighted degree is bounded (Exercise 8.8).

4. The heat equation on ℓ^p

In this section we discuss the heat equation on $\ell^p(X,m)$. The general theory of strongly continuous semigroups found in Appendix D only gives existence of solutions for initial conditions in the domain of the generator. However, in the case of uniformly positive measure, we also have solutions for initial conditions in $\ell^p(X,m)$ for all $p \in [1,\infty]$.

We recall that a function $u\colon [0,\infty)\times X \longrightarrow \mathbb{R}$ is said to be a solution of the heat equation if $u(x,\cdot)$ is continuous on $[0,\infty)$, continuously differentiable on $(0,\infty)$ for every $x \in X$ and $u(\cdot,t) \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t) u_t(x) = 0$$

for all $x \in X$ and $t > 0$. The function u_0 is called the initial condition for the solution. We say that u is a solution in $\ell^p(X,m)$ for $p \in [1,\infty]$ if

$$\sup_{t \geq 0} \|u_t\|_p < \infty.$$

To generate such solutions, we recall that the semigroup e^{-tL} of L originally defined on $\ell^2(X,m)$ extends to a contraction Markov semigroup on $\ell^p(X,m)$ for $p \in [1,\infty]$ by Theorem 2.9. This extended semigroup is again denoted by e^{-tL}.

The following theorem is a direct consequence of Theorem 7.3 and the fact that every function in $\ell^p(X,m)$ is bounded in the case of uniformly positive measure.

THEOREM 8.10 (Existence of solutions of the heat equation on ℓ^p). *Let (b,c) be a graph over (X,m) which satisfies* (M). *For $f \in \ell^p(X,m)$ with $p \in [1,\infty]$ let*

$$u_t(x) = e^{-tL} f(x)$$

for $t \geq 0$ and $x \in X$. Then, u is a solution of the heat equation in $\ell^p(X,m)$ with initial condition f. Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

PROOF. Whenever the graph satisfies (M), we have $f \in \ell^\infty(X)$ for any $f \in \ell^p(X,m)$ with $p \in [1,\infty]$. Thus, u_t is a solution of the heat equation by Theorem 7.3. Furthermore, since e^{-tL} extends to a contraction Markov semigroup on $\ell^p(X,m)$ for $p \in [1,\infty]$ by Theorem 2.9, we have $\|u_t\|_p \leq \|f\|_p$ for all $f \in \ell^p(X,m)$, which gives that u is a solution on $\ell^p(X,m)$. Finally, the minimality statement is also included in Theorem 7.3. □

5. Graphs with standard weights

In this section we illustrate our results for graphs with standard weights. These are the examples most commonly appearing in the literature on graphs.

We recall that a graph (b,c) has standard weights if b takes values in $\{0,1\}$ and $c = 0$. In this case, we denote the set of edges by E, i.e.,

$$E = \{(x,y) \in X \times X \mid x \sim y\}.$$

Furthermore, the energy form $\mathcal{Q}$ acts as

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{(x,y)\in E} (f(x) - f(y))^2$$

with the space of functions of finite energy $\mathcal{D}$ given by

$$\mathcal{D} = \{f \in C(X) \mid \sum_{(x,y)\in E} (f(x) - f(y))^2 < \infty\}.$$

We note that $b(x,y) \in \{0,1\}$ for all $x,y \in X$ implies that b is locally finite so that $\mathcal{F} = C(X)$. The formal Laplacian in this case is denoted by $\widetilde{\Delta}$ and acts as

$$\widetilde{\Delta} f(x) = \sum_{y \in X, y\sim x} (f(x) - f(y))$$

for $f \in C(X)$. The choice of measure will then determine the specific Hilbert space and Laplacian. The two most prominent examples of measures in this setting are the counting measure and the normalizing measure.

Specifically, the counting measure $m = 1$ gives the number of vertices in a subset of X. In particular, the weighted degree function becomes $\deg(x) = \#\{y \mid y \sim x\}$, which is called the combinatorial vertex degree. We denote the Banach spaces $\ell^p(X,1)$ for $p \in [1,\infty]$ by

$\ell^p(X)$ and denote the Laplacian $L_{b,0,1}^{(D)}$ by Δ. By Theorem 1.6, Δ acts on $D(\Delta) \subseteq \ell^2(X)$ by

$$\Delta f(x) = \sum_{y \in X, y \sim x} (f(x) - f(y)).$$

By Corollary 1.31, Δ is a bounded operator if and only if deg is a bounded function on X.

It is clear that $m = 1$ satisfies (M) and, therefore, (M*). Furthermore, as b is locally finite, we have $\widetilde{\Delta}(C_c(X)) \subseteq \ell^2(X)$ by Theorem 1.29. Hence, as a direct consequence of our results we obtain the following statement.

Corollary 8.11 (Essential self-adjointness of Δ). *Let b be a connected graph with standard weights over (X, m) with $m = 1$. Then, there exists a unique form Q associated to b, the associated operator Δ is the unique self-adjoint extension of the restriction of $\widetilde{\Delta}$ to $C_c(X)$ and*

$$D(\Delta) = \{f \in \ell^2(X) \mid \widetilde{\Delta} f \in \ell^2(X)\}.$$

Proof. The uniqueness of the form associated to b follows from Theorem 8.4. Since $\widetilde{\Delta}(C_c(X)) \subseteq \ell^2(X)$ in the locally finite case by Theorem 1.29, the uniqueness of the self-adjoint extension of the restriction of $\widetilde{\Delta}$ to $C_c(X)$ and the statement about the domain $D(\Delta)$ follows from Theorem 8.5. □

We denote the generators of the semigroup of Δ on $\ell^p(X)$ by $\Delta^{(p)}$. By Theorem 2.13, the operators $\Delta^{(p)}$ are restrictions of $\widetilde{\Delta}$. We then obtain the following consequence concerning the domain of $\Delta^{(p)}$.

Corollary 8.12 (Domains of $\Delta^{(p)}$). *Let b be a connected graph with standard weights over (X, m) with $m = 1$. Then,*

$$D(\Delta^{(p)}) = \{f \in \ell^p(X) \mid \widetilde{\Delta} f \in \ell^p(X)\}$$

for all $p \in [1, \infty)$.

Proof. The statement follows immediately from Theorem 8.6. □

Furthermore, our spectral inclusion result reads as follows.

Corollary 8.13 (Spectral inclusion for Δ). *Let b be a connected graph with standard weights over (X, m) with $m = 1$. Then,*

$$\sigma(\Delta) \subseteq \sigma(\Delta^{(p)})$$

for all $p \in (1, \infty)$.

Proof. The statement follows from Theorem 8.9. □

Finally we get the existence of solutions of the heat equation in this case.

Corollary 8.14 (Existence of solutions of the heat equation on ℓ^p). *Let b be a connected graph with standard weights over (X, m) with $m = 1$. For $f \in \ell^p(X)$ with $p \in [1, \infty]$ let*

$$u_t(x) = e^{-t\Delta} f(x)$$

for $t \geq 0$ and $x \in X$. Then, u is a solution of the heat equation in $\ell^p(X)$ with initial condition f. Furthermore, if additionally $f \geq 0$, then u is the smallest positive supersolution of the heat equation with initial condition greater than or equal to f.

Proof. The result follows immediately from Theorem 8.10. □

Remark (Normalizing measure). We note that the other usual measure in the case of standard weights is the normalizing measure $n(x) = \deg(x) = \#\{y \mid y \sim x\}$ for $x \in X$. In this case, we denote the Laplacian $L^{(D)}_{b,0,n}$ associated to $Q^{(D)}_{b,0,n}$ by Δ_n and refer to it as the normalized Laplacian. As Δ_n is a bounded operator by Corollary 1.33, in this case the results above are trivially true.

Exercises

Excavation exercises.

Exercise 8.1 (ℓ^p inclusions under (M)). Let (X, m) be a measure space satisfying (M). Show that

$$\ell^p(X, m) \subseteq \ell^q(X, m)$$

for all $1 \leq p \leq q \leq \infty$.

Exercise 8.2 (Adjoint operators and their spectrum). Let E be a Banach space and let E^* be the dual space of E. For a densely defined operator $A\colon D(A) \longrightarrow E$ with $D(A) \subseteq E$, we define the adjoint A^* operator of A with domain $D(A^*)$ by

$$D(A^*) = \{\phi \in E^* \mid \text{ there exists a } \psi \in E^* \text{ extending } \phi \circ A\}$$

and

$$A^*\phi = \psi$$

for $\phi \in D(A^*)$.

(a) Let $\sigma(A)$ and $\sigma(A^*)$ denote the spectrum of A and A^*, respectively. Show that

$$\sigma(A) = \sigma(A^*).$$

(b) Let $L^{(p)}$ and $L^{(q)}$ be the generators of the semigroups on $\ell^p(X, m)$ and $\ell^q(X, m)$, respectively, where $1/p + 1/q = 1$. Show that

$$(L^{(p)})^* = L^{(q)}.$$

Note, in particular, that this shows $\sigma(L^{(p)}) = \sigma(L^{(q)})$.

(Hint: The material at the end of Appendix D might be useful for this exercise.)

Example exercises.

Exercise 8.3 (Non-trivial positive α-subharmonic functions). Give an example of a connected graph (b, c) over an infinite measure space (X, m) with a non-trivial positive α-subharmonic function in $\ell^p(X, m)$ for $p \in [1, \infty)$ and $\alpha > 0$.

Exercise 8.4 ((M*) but not $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$). Give an example of a graph which satisfies (M*) but not $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$.

Extension exercises.

EXERCISE 8.5 (Finite graphs and subharmonic functions). Show that there exist a non-trivial function $u \geq 0$ in $\ell^p(X, m)$ satisfying $\mathcal{L}u \leq 0$ when (b, c) is a graph over a finite set X with measure m. Classify all such functions.

EXERCISE 8.6 (Killing or strict positivity of α). Let (b, c) be a connected graph over (X, m) satisfying (M^*). Let $u \in \mathcal{F}$ satisfy $u \geq 0$ and $(\mathcal{L} + \alpha)u \leq 0$ for $\alpha \geq 0$. Show that if $u \in \ell^p(X, m)$ for $p \in [1, \infty)$ and either $c \neq 0$ or $\alpha > 0$, then $u = 0$.

EXERCISE 8.7 (More general summability criteria). Let (b, c) be a connected graph over (X, m). Recall the definition of the weighted degree as $\mathrm{Deg}(x) = (1/m(x))(\sum_{y \in X} b(x, y) + c(x))$.

(a) Show that if for every infinite path (x_n) the graph satisfies
$$\sum_{n=0}^{\infty} m(x_n) \prod_{k=0}^{n-1} \left(1 + \frac{1}{\mathrm{Deg}(x_k)}\right)^2 = \infty,$$
then $Q^{(D)} = Q^{(N)}$.

Show that if, additionally, $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$, then the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint.

(b) Let $p \in [1, \infty)$ and let $L^{(p)}$ denote the generator of the semigroup on $\ell^p(X, m)$. Show that if for every infinite path (x_n) the graph satisfies
$$\sum_{n=0}^{\infty} m(x_n) \prod_{k=0}^{n-1} \left(1 + \frac{1}{\mathrm{Deg}(x_k)}\right)^p = \infty,$$
then
$$D(L^{(p)}) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\}.$$

EXERCISE 8.8 ($\sigma(L^{(p)}) \subseteq \sigma(L)$). Let (b, c) be a graph over (X, m) such that $m(X) < \infty$ and $\mathrm{Deg}(x) = (1/m(x))(\sum_{y \in X} b(x, y) + c(x))$ is bounded. Let $p \in (1, \infty)$. Show that
$$\sigma(L^{(p)}) \subseteq \sigma(L).$$

Notes

The main observation for the proof of the Liouville result, Theorem 8.3, i.e., that positive non-constant α-harmonic functions for $\alpha > 0$ must strictly increase in some direction, goes back to the thesis [**Woj08**]. In [**Woj08**], this is used to show the essential self-adjointness of the Laplacian on graphs with standard weights and counting measure, which is presented here as Corollary 8.11. This result can also be found in [**Web10**].

For the general result on essential self-adjointness under the condition that infinite paths have infinite measure, Theorem 8.5, the first proof is found in [**KL12**]. Earlier statements which assume local finiteness and constant measure were asserted in [**Jor08**] and later proven in [**JP11**] and, independently, in [**TH10**]. A more general summability criterion for magnetic Schrödinger operators can be found in [**Gol14, GKS16**] and [**Sch20b**]. The statement on uniqueness of associated forms, Theorem 8.4, was shown in [**KL10**]. For examples where essential self-adjointness and the uniqueness of associated forms fail, see [**KL10, KL12, HKLW12**].

Let us mention that the essential self-adjointness of the Laplacian on graphs with standard weights and counting measure stands in contrast to the case of the adjacency operator. There, it is known that the adjacency operator might not be essentially self-adjoint. The first examples of such graphs were given in [**MO85, Mül87**]. See [**Gol10, GS11, GS13**] for further discussion of this problem and for some criteria for the essential self-adjointness of the adjacency operator.

In the case of the generator on ℓ^p spaces, the statement characterizing the domain, Theorem 8.6, appears in [**KL12**]. The spectral inclusion under uniform lower measure bounds, Theorem 8.9, is shown in [**BHK13**].

CHAMBER 9

Weak Spherical Symmetry

... I keep planets in orbit while I be coming with deeper and more ...
ODB.

In this chapter we discuss a class of graphs whose geometry has a weak spherical symmetry. We first introduce the notion of spherical symmetry that we wish to study and give several examples. We also introduce the idea of comparing an arbitrary graph to a weakly spherically symmetric graph, which will be a recurring theme in this chapter. We then characterize this geometric notion of symmetry in terms of the heat kernel in Section 1. Furthermore, we give heat kernel comparisons which immediately imply comparison results for the Green's function.

We then turn to spectral estimates in Section 2 and give an estimate for the bottom of the spectrum as well as criteria for the essential spectrum to be empty. At this point, we use the Agmon–Allegretto–Piepenbrink characterization of the bottom of the spectrum and essential spectrum. The final two sections, Sections 3 and 4, involve the study of recurrence and stochastic completeness, respectively. In these sections we first characterize recurrence and stochastic completeness for weakly spherically symmetric graphs in terms of geometric quantities, then give the corresponding comparison results for general graphs.

We let (b, c) be a connected graph over (X, m), $\mathcal{L} = \mathcal{L}_{b,c,m}$ be the formal graph Laplacian and $L = L^{(D)}_{b,c,m}$ be the Laplacian associated to the regular form $Q^{(D)}_{b,c,m}$ on $\ell^2(X, m)$. We recall that a graph is called locally finite if the sets $\{y \in X \mid y \sim x\}$ are finite for all $x \in X$. Many of the results in this chapter will involve assuming that a graph is locally finite. Furthermore, we denote by d the *combinatorial graph metric*, that is, the least number of edges in a path between two vertices.

Let O be a subset of X and define the distance to O by

$$d(O, x) = \min_{o \in O} d(o, x),$$

where $x \in X$. For most of our results below we will assume that O is a finite set.

We denote the *distance sphere of radius $r \in \mathbb{N}_0$ about O* by

$$S_r(O) = \{x \in X \mid d(O, x) = r\}.$$

For convenience, we let $S_{-1}(O) = \emptyset$. Moreover, we denote the *distance ball of radius* $r \in \mathbb{N}_0$ *about* O by

$$B_r(O) = \bigcup_{n=0}^{r} S_n(O) = \{x \in X \mid d(O,x) \leq r\}.$$

If O is a finite set and (b,c) is locally finite, then the sets $S_r(O)$ and $B_r(O)$ are finite for all $r \in \mathbb{N}_0$. Furthermore, connectedness of the graph is equivalent to the fact that $X = \bigcup_r B_r(O)$. As we will need to exhaust the graph via balls in various places, we will assume that all graphs in this chapter are connected.

We call a function $f \in C(X)$ *spherically symmetric (with respect to* O*)* if there exists a function $g\colon \mathbb{N}_0 \longrightarrow \mathbb{R}$ such that $f(x) = g(r)$ for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. With a slight abuse of notation, we then write

$$f(r) = f(x)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Although all of our notions involving symmetry depend on O, we will mostly omit this dependence in our notation and statements.

We next define the functions for which we will assume spherical symmetry. We let $k_\pm$ denote the *outer and inner degrees (with respect to* O*)*, which are functions $k_\pm\colon X \longrightarrow [0,\infty)$ defined via

$$k_\pm(x) = \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}(O)} b(x,y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Furthermore, we define the *potential* $q\colon X \longrightarrow [0,\infty)$ by

$$q(x) = \frac{c(x)}{m(x)}$$

for $x \in X$.

With these preparations we can now define the class of graphs which will be studied in this chapter.

Definition 9.1 (Weakly spherically symmetric graphs). We call a connected graph (b,c) over (X,m) *weakly spherically symmetric* with respect to a set $O \subseteq X$ if the outer and inner degrees $k_\pm$ and potential q are spherically symmetric with respect to O.

Remark. We call these graphs "weakly" spherically symmetric because we do not require any symmetry on the graph structure within the spheres. Stronger notions of spherical symmetry require that there exists a graph automorphism mapping one vertex to the other (or even interchanging the vertices) for any two vertices on the same sphere (Exercise 9.7).

Remark. Whenever O is finite, m is bounded below on $S_r(O)$ for all $r \in \mathbb{N}_0$ and the graph is weakly spherically symmetric, then all $S_r(O)$ are finite and the graph is locally finite (Exercise 9.8).

We now introduce two main classes of examples of weakly spherically symmetric graphs. The first class consists of trees and, in contrast, the second class of graphs we call anti-trees.

We call a sequence of vertices $(x_j)_{j=0}^n$ with $n \in \mathbb{N}$ a *cycle* if $x_j \sim x_{j+1}$ for all $j = 0, 1, \ldots, n-1$, $x_j \neq x_k$ for $j \neq k$ with $1 \leq j, k \leq n$ and $x_0 = x_n$. A connected graph with no cycles is called a *tree*.

Example 9.2 (Spherically symmetric trees). Let (b, c) be a connected graph over (X, m) with standard weights and counting measure, i.e., $b\colon X \times X \longrightarrow \{0, 1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$ and let $S_r(o) = S_r(O)$. We say that b is a *spherically symmetric tree with branching numbers k* if there exists a sequence $k\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ such that, for every vertex $x \in S_r(o)$ and every $r \in \mathbb{N}_0$,

$$k_+(x) = k(r), \qquad k_-(x) = 1$$

and $b|_{S_r(o) \times S_r(o)} = 0$.

We note that these graphs are indeed trees. Furthermore, we note that removing a single edge between spheres will disconnect any tree. This contrasts with anti-trees, which we now define.

Example 9.3 (Anti-trees). Let (b, c) be a connected graph over (X, m) with standard weights and counting measure, i.e., $b\colon X \times X \longrightarrow \{0, 1\}$, $c = 0$ and $m = 1$. Let $O = \{o\}$ for $o \in X$ and let $S_r(o) = S_r(O)$. Let $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ be given by $s(r) = \# S_r(o)$ for all $r \in \mathbb{N}_0$. We then say that b is an *anti-tree with sphere size s* if

$$k_\pm(x) = s(r) \qquad \text{for all } x \in S_{r \mp 1}(o) \text{ and } r \in \mathbb{N}_0.$$

See Figure 1 below for an example.

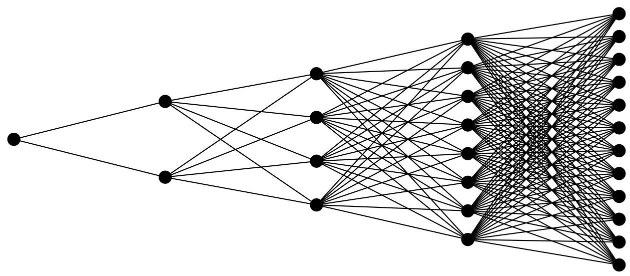

Figure 1. An anti-tree with $s(r) = 2^r$.

We note that the definition of an anti-tree implies

$$b|_{S_r(o) \times S_{r \pm 1}(o)} = 1.$$

In other words, every vertex in $S_r(o)$ is connected to all vertices in $S_{r+1}(o)$ for all $r \in \mathbb{N}_0$. Hence, to disconnect such a graph, we must remove all vertices between spheres. Furthermore, we note that we impose no restrictions on $b|_{S_r(o)\times S_r(o)}$.

To construct an anti-tree for a given sequence $s\colon \mathbb{N}_0 \longrightarrow \mathbb{N}$ of natural numbers with $s(0) = 1$, we partition the vertex set X into disjoint subsets U_r with $\#U_r = s(r)$ and let $b|_{U_r\times U_{r\pm 1}} = 1$ for $r \in \mathbb{N}_0$ with $b = 0$ otherwise, $m = 1$ and $c = 0$.

We will revisit spherically symmetric trees and anti-trees to illustrate our results in this chapter.

The following formulas will play a crucial role in the proofs of several results below. Hence we gather them together into one statement.

Lemma 9.4. *Let (b, c) be a weakly spherically symmetric graph over (X, m) with respect to $O \subseteq X$. Then,*

$$k_+(r)m(S_r(O)) = k_-(r + 1)m(S_{r+1}(O))$$

for all $r \in \mathbb{N}_0$, where both sides can be infinite. In particular, $m(S_r(O)) < \infty$ for all $r \in \mathbb{N}_0$ if and only if $m(O) < \infty$.

If f is a spherically symmetric function, then $f \in \mathcal{F}$ and $\mathcal{L}f$ is spherically symmetric with

$$\mathcal{L}f(x) = k_+(r)\big(f(r) - f(r + 1)\big) + k_-(r)\big(f(r) - f(r - 1)\big) + q(r)f(r)$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$.

Proof. The first formula follows by a simple computation using $k_+(r) = k_+(x)$ for all $x \in S_r(O)$, Fubini's theorem and the symmetry of b. Specifically, we have

$$\begin{aligned} k_+(r)m(S_r(O)) &= \sum_{x\in S_r(O)} k_+(x)m(x) \\ &= \sum_{x\in S_r(O)} \sum_{y\in S_{r+1}(O)} b(x, y) \\ &= \sum_{y\in S_{r+1}(O)} \sum_{x\in S_r(O)} b(y, x) \\ &= \sum_{y\in S_{r+1}(O)} k_-(y)m(y) \\ &= k_-(r + 1)m(S_{r+1}(O)). \end{aligned}$$

The "in particular" statement now follows by the formula and induction.

A spherically symmetric function f on a weakly spherically symmetric graph is clearly a bounded function on the neighbors of any vertex and, therefore, $f \in \mathcal{F}$. The second formula follows immediately from the definition of $\mathcal{L}$ and the assumption that f is spherically symmetric. $\square$

We now present a way to compare a weakly spherically symmetric graph and a general graph. Whenever we do so, we will use the superscript sym over the terms involving the spherically symmetric graph.

DEFINITION 9.5 (Stronger and weaker degree and potential growth). Let (b, c) be a connected graph over (X, m) and let $O \subseteq X$. Let $k_\pm$ denote the outer and inner degrees with respect to O and let q denote the potential of (b, c).

We say that (b, c) has *stronger* (respectively, *weaker*) *degree growth* than a weakly spherical symmetric graph $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$ if $m(O) = m^{\mathrm{sym}}(O^{\mathrm{sym}})$ and, for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$,

$$\begin{aligned} & k_+(x) \geq k_+^{\mathrm{sym}}(r) \quad \text{and} \quad k_-(x) \leq k_-^{\mathrm{sym}}(r) \\ \big(\text{respectively,} \qquad & k_+(x) \leq k_+^{\mathrm{sym}}(r) \quad \text{and} \quad k_-(x) \geq k_-^{\mathrm{sym}}(r)\big), \end{aligned}$$

where $k_\pm^{\mathrm{sym}}$ are the outer and inner degrees of $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to O^{sym}.

We say that (b, c) has *stronger* (respectively, *weaker*) *potential growth* than $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ if $m(O) = m^{\mathrm{sym}}(O^{\mathrm{sym}})$ and, for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$,

$$q(x) \geq q^{\mathrm{sym}}(r) \qquad (\text{respectively, } q(x) \leq q^{\mathrm{sym}}(r)),$$

where q^{sym} is the potential of $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over (X, m^{sym}).

In what follows, we will prove a series of results for weakly spherically symmetric graphs and then show the corresponding comparison results for graphs with a stronger or weaker degree/potential growth than a weakly spherically symmetric graph.

1. Symmetry of the heat kernel

In this section we are concerned with the symmetry of the heat kernel. In particular, we will show that the heat kernel yields a spherically symmetric function for locally finite weakly spherically symmetric graphs. We then turn to comparison theorems involving the heat kernel.

1.1. Symmetry of the kernel and Green's function.

In this subsection we establish the symmetry of the heat kernel and Green's function on a weakly spherically symmetric graph.

We first recall the definition of the heat kernel. The semigroup e^{-tL} on $\ell^2(X, m)$ for $t \geq 0$ gives rise to a kernel $p\colon [0, \infty) \times X \times X \longrightarrow \mathbb{R}$ via

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x, y) f(y) m(y)$$

for all $f \in \ell^2(X, m)$, $x \in X$ and $t \geq 0$. By the fact that the semigroup is positivity preserving, established in Corollary 1.22, we have $p_t(x, y) \geq 0$ for all $x, y \in X$ and $t \geq 0$ as $p_t(x, y) = e^{-tL}1_y(x)/m(y)$.

For a finite set $O \subseteq X$, we now define

$$\begin{aligned} p_t(x, O) &= \frac{1}{m(x)m(O)}\langle 1_x, e^{-tL}1_O\rangle \\ &= \frac{1}{m(O)}e^{-tL}1_O(x) \\ &= \frac{1}{m(O)}\sum_{o \in O} p_t(x, o)m(o) \end{aligned}$$

for $x \in X$ and $t \geq 0$. Thus, whenever O consists of a single vertex o, we recover the heat kernel

$$p_t(x, o) = p_t(x, \{o\})$$

for $x \in X$.

The first theorem of this subsection states that the function $p_t(\cdot, O)$ is spherically symmetric whenever the graph is weakly spherically symmetric with respect to the subset O.

Theorem 9.6 (Spherical symmetry of the heat kernel). *Let (b, c) over (X, m) be a locally finite graph. If (b, c) is weakly spherically with respect to a finite set $O \subseteq X$, then $p_t(\cdot, O)$ is a spherically symmetric function.*

We prove this theorem by capturing the geometric notion of weak spherical symmetry analytically. In fact, we will show that weak spherical symmetry of a locally finite graph is equivalent to an even stronger condition on the heat kernel, specifically, that the semigroup and an averaging operator introduced below commute. The fact that $p_t(\cdot, O)$ is spherically symmetric is then an immediate consequence.

To start the proof we introduce the *averaging operator* $\mathcal{A}\colon C(X) \longrightarrow C(X)$ on a locally finite graph with respect to a finite set $O \subseteq X$ by

$$\mathcal{A}f(x) = \frac{1}{m(S_r(O))}\sum_{y \in S_r(O)} f(y)m(y)$$

for $x \in S_r(O)$. With some additional care we could define $\mathcal{A}$ on non-locally finite graphs by making sure that the sums above converge absolutely.

We note that a function $f \in C(X)$ is spherically symmetric if and only if $\mathcal{A}f = f$. This will be used repeatedly below. We will denote the restriction of $\mathcal{A}$ to $\ell^2(X, m)$ by A, i.e.,

$$A = \mathcal{A}|_{\ell^2(X,m)}.$$

We now collect some basic facts about A.

Lemma 9.7 (Basic facts about A). *Let (b,c) be a locally finite connected graph over (X,m) and let $O \subseteq X$ be a finite set. Let $\mathcal{A}$ be the averaging operator with respect to O and let A be the restriction of $\mathcal{A}$ to $\ell^2(X,m)$. Then, A is a bounded self-adjoint operator on $\ell^2(X,m)$. More specifically, A is an orthogonal projection of $\ell^2(X,m)$ onto the subspace of spherically symmetric functions in $\ell^2(X,m)$.*

Proof. Let $f \in \ell^2(X,m)$. We note that $X = \bigcup_r S_r(O)$ from the assumption that (b,c) is connected. To show the boundedness of A, we use the Cauchy–Schwarz inequality as follows,

$$\begin{aligned}
\|Af\|^2 &= \sum_{x\in X}(Af)^2(x)m(x)\\
&= \sum_{r=0}^{\infty}\sum_{x\in S_r(O)}\left(\frac{1}{m(S_r(O))}\sum_{y\in S_r(O)}f(y)m(y)\right)^2 m(x)\\
&= \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)}f(y)m(y)\right)^2\\
&\leq \sum_{r=0}^{\infty}\frac{1}{m(S_r(O))}\left(\sum_{y\in S_r(O)}m(y)\right)\left(\sum_{y\in S_r(O)}f^2(y)m(y)\right)\\
&= \|f\|^2.
\end{aligned}$$

Hence, A is a bounded operator of norm 1 since $Af = f$ for any spherically symmetric function in $\ell^2(X,m)$.

Moreover, A is symmetric, and thus self-adjoint, by a direct calculation. As the range of A is included in the spherically symmetric functions and $A^2 = A$, the operator A is an orthogonal projection onto the spherically symmetric functions in $\ell^2(X,m)$. $\square$

The next lemma shows that weak spherical symmetry is equivalent to $\mathcal{A}$ and A commuting with the Laplacians $\mathcal{L}$ and L on suitable spaces. We recall that L is a restriction of $\mathcal{L}$ by Theorem 1.6 and the domain of L on a locally finite graph includes $C_c(X)$ by Theorem 1.29. Furthermore, since $O \subseteq X$ is assumed to be finite and (b,c) locally finite below, $S_r(O)$ is finite for all $r \in \mathbb{N}_0$ and A and L both map $C_c(X)$ to $C_c(X)$, i.e., $C_c(X)$ is invariant under both A and L.

Lemma 9.8 (Characterization of weak spherical symmetry). *Let (b,c) be a connected locally finite graph over (X,m) and let $O \subseteq X$ be finite. Then, the following statements are equivalent:*

(i) *The graph (b,c) is weakly spherically symmetric.*
(ii) *The operator $\mathcal{A}$ commutes with $\mathcal{L}$ on $C(X)$, i.e.,*

$$\mathcal{A}\mathcal{L} = \mathcal{L}\mathcal{A} \qquad \text{on } C(X).$$

(iii) *The operator A commutes with L on $C_c(X)$, i.e.,*

$$AL = LA \qquad \text{on } C_c(X).$$

PROOF. We denote $S_r(O)$ by S_r for $r \in \mathbb{N}_0$.

(i) $\Longrightarrow$ (ii): Obviously, multiplication by the spherically symmetric function q commutes with $\mathcal{A}$. Hence, we may assume that $q = 0$.

Since $\mathcal{A}f$ is spherically symmetric for $f \in C(X)$, by Lemma 9.4 we get, for $x \in S_r$,

$$\mathcal{L}\mathcal{A}f(x) = k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)).$$

On the other hand, by using Lemma 9.4 again, we get, for $x \in S_r$,

$$\begin{aligned}
\mathcal{A}\mathcal{L}f(x) &= \frac{1}{m(S_r)} \sum_{y \in S_r} \mathcal{L}f(y)m(y) \\
&= \frac{1}{m(S_r)} \sum_{y \in S_r} \sum_{z \in S_{r-1} \cup S_{r+1}} b(y,z)(f(y) - f(z)) \\
&= \frac{1}{m(S_r)} \sum_{y \in S_r} f(y) \sum_{z \in S_{r-1} \cup S_{r+1}} b(y,z) - \frac{1}{m(S_r)} \sum_{z \in S_{r-1} \cup S_{r+1}} f(z) \sum_{y \in S_r} b(y,z) \\
&= (k_+(r) + k_-(r))\mathcal{A}f(r) \\
&\qquad - \frac{k_+(r-1)}{m(S_r)} \sum_{z \in S_{r-1}} f(z)m(z) - \frac{k_-(r+1)}{m(S_r)} \sum_{z \in S_{r+1}} f(z)m(z) \\
&= (k_+(r) + k_-(r))\mathcal{A}f(r) \\
&\qquad - \frac{k_-(r)}{m(S_{r-1})} \sum_{z \in S_{r-1}} f(z)m(z) - \frac{k_+(r)}{m(S_{r+1})} \sum_{z \in S_{r+1}} f(z)m(z) \\
&= k_+(r)(\mathcal{A}f(r) - \mathcal{A}f(r+1)) + k_-(r)(\mathcal{A}f(r) - \mathcal{A}f(r-1)).
\end{aligned}$$

Thus we see $\mathcal{L}\mathcal{A}f = \mathcal{A}\mathcal{L}f$.

(ii) $\Longrightarrow$ (iii): This is clear as A and L are restrictions of $\mathcal{A}$ and $\mathcal{L}$.

(iii) $\Longrightarrow$ (i): Obviously, $A1_{S_r} = 1_{S_r}$ as 1_{S_r} is a spherically symmetric function. Furthermore, for $x \in S_{r\pm1}$, we have

$$L1_{S_r}(x) = -k_\mp(x)$$

by direct calculations. Thus, for $x \in S_{r\pm1}$, we have

$$\begin{aligned}
-k_\mp(x) &= LA1_{S_r}(x) \\
&= AL1_{S_r}(x) \\
&= -\frac{1}{m(S_{r\pm1})} \sum_{y \in S_{r\pm1}} k_\mp(y)m(y). \\
&= -Ak_\mp(r \pm 1).
\end{aligned}$$

Therefore, $k_\pm$ are spherically symmetric.

Similarly, for $x \in S_r$, we calculate $L1_{S_r}(x) = k_+(x) + k_-(x) + q(x)$. Therefore, for $x \in S_r$, we have

$$\begin{aligned} k_+(x) + k_-(x) + q(x) &= L1_{S_r}(x) \\ &= LA1_{S_r}(x) \\ &= AL1_{S_r}(x) \\ &= \frac{1}{m(S_r)} \sum_{y \in S_r} (k_+(y) + k_-(y) + q(y))m(y) \\ &= A(k_+ + k_- + q)(r). \end{aligned}$$

As we have already shown that $k_\pm$ are spherically symmetric, this shows that the potential q and, thus, the graph is weakly spherically symmetric. □

We next apply the general theory of reducing subspaces and commuting operators developed in Appendix 3. This result allows us to pass from the commutativity of the restriction of the Laplacian and the averaging operators to the finitely supported functions to commutativity on the entire domain of the Laplacian. Furthermore, it shows that this commutativity is equivalent to the averaging operator commuting with the semigroup. Specifically, we apply Corollary E.20 with $H = \ell^2(X, m)$ and $D_0 = C_c(X)$ to obtain the following lemma.

LEMMA 9.9. *Let (b, c) be a connected locally finite graph over (X, m) and let $O \subseteq X$ be finite. Then, the following statements are equivalent:*

(i) *$AL = LA$ on $C_c(X)$.*
(ii) *A maps $D(L)$ into $D(L)$ and $AL = LA$ on $D(L)$.*
(iii) *$Ae^{-tL} = e^{-tL}A$ on $\ell^2(X, m)$ for all $t \geq 0$.*

With these two lemmas, we can now show the desired symmetry of the heat kernel.

PROOF OF THEOREM 9.6. From Lemmas 9.8 and 9.9 we see that if (b, c) is a locally finite and weakly spherically symmetric graph with respect to a finite set O, then A and e^{-tL} commute. Hence, as $p_t(x, O) = e^{-tL}1_O(x)/m(O)$ for $x \in X$, we get

$$Ap_t(x, O) = \frac{1}{m(O)} Ae^{-tL}1_O(x) = \frac{1}{m(O)} e^{-tL}A1_O(x) = p_t(x, O)$$

for $x \in X$ and $t \geq 0$. Hence, $p_t(\cdot, O)$ is spherically symmetric. □

REMARK. If we denote the distance to O for a vertex $z \in X$ by $r_z = d(z, O)$, then combining Lemmas 9.8 and 9.9 we obtain that a locally finite graph is weakly spherically symmetric if and only if

$$\frac{1}{m(S_{r_x}(O))} \sum_{z \in S_{r_x}(O)} p_t(y, z)m(z) = \frac{1}{m(S_{r_y}(O))} \sum_{z \in S_{r_y}(O)} p_t(x, z)m(z)$$

for all $x, y \in X$ and $t \geq 0$. This implies $p_t(\cdot, O)$ is spherically symmetric (Exercise 9.9).

From the spherical symmetry of the heat kernel, we derive an immediate statement concerning the Green's function. We recall that in Chapter 6 we introduced the Green's function of a graph b as

$$G_m(x,y) = \int_0^\infty e^{-tL}1_y(x)dt = \int_0^\infty p_t(x,y)m(y)dt.$$

Furthermore, we call a connected graph b transient if and only if $G_m(x,y) < \infty$ for some (all) $x, y \in X$ and some (all) measure m. See Theorem 6.1 for various equivalent formulations of transience.

We now extend the definition of the Green's function by letting

$$G_m(x,O) = \frac{1}{m(O)} \sum_{o \in O} G_m(x,o)$$

whenever $O \subseteq X$ is finite and $x \in X$.

Corollary 9.10 (Spherical symmetry of the Green's function). *Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a finite set $O \subseteq X$. Assume that b is transient. Then, the Green's function $G_m(\cdot, O)$ is spherically symmetric.*

Proof. We calculate by the definitions above and Fubini's theorem

$$\begin{aligned} G_m(x,O) &= \frac{1}{m(O)} \sum_{o \in O} G_m(x,o) \\ &= \frac{1}{m(O)} \sum_{o \in O} \int_0^\infty p_t(x,o)m(o)dt \\ &= \int_0^\infty \frac{1}{m(O)} \sum_{o \in O} p_t(x,o)m(o)dt \\ &= \int_0^\infty p_t(x,O)dt \end{aligned}$$

for $x \in X$. By Theorem 9.6, $p_t(\cdot, O)$ is spherically symmetric. This completes the proof. □

1.2. Comparison theorems. We next turn to comparison results. That is, we will compare the heat kernel on a general graph to the heat kernel on a weakly spherically symmetric graph. As a consequence, we get comparisons for the Green's function.

By Theorem 9.6 we know the heat kernel $p^{\mathrm{sym}}(\cdot, O^{\mathrm{sym}})$ of a locally finite weakly spherically symmetric graph $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$ is a spherically symmetric function. Hence, we may write

$$p_t^{\mathrm{sym}}(r) = p_t^{\mathrm{sym}}(x, O^{\mathrm{sym}})$$

for all $x \in S_r(O^{\mathrm{sym}}), r \in \mathbb{N}_0$ and $t \geq 0$.

We will now compare the heat kernel on a weakly spherically symmetric graph and the heat kernel on a general graph. For this, we ultimately restrict to the case $c = 0$.

Theorem 9.11 (Heat kernel comparison with weakly spherically symmetric graphs). *Let b be a connected locally finite graph over (X, m) with heat kernel p and let $O \subseteq X$ be a finite set. If b over (X, m) has stronger (respectively, weaker) degree growth than a locally finite weakly spherically symmetric graph b^{sym} over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$ and heat kernel p^{sym}, then*

$$p_t(x, O) \leq p_t^{\mathrm{sym}}(r) \qquad (\textit{respectively, } p_t(x, O) \geq p_t^{\mathrm{sym}}(r))$$

for all $x \in S_r(O)$, $r \in \mathbb{N}_0$ and $t \geq 0$.

The proof will require several ingredients. One ingredient is the minimum principle for the heat equation, which we now recall. Specifically, if $U \subset X$ is a connected subset containing a vertex which is connected to a vertex outside of U, $T \geq 0$ and $u \colon [0, T] \times X \longrightarrow \mathbb{R}$ is such that $t \mapsto u_t(x)$ is continuously differentiable on $(0, T)$ for every $x \in U$, $u_t \in \mathcal{F}$ for all $t \in (0, T]$ and

- $(\mathcal{L} + \partial_t)u \geq 0$ on $(0, T) \times U$
- $u \wedge 0$ attains a minimum on $[0, T] \times U$
- $u \geq 0$ on $((0, T] \times X \setminus U) \cup (\{0\} \times U)$,

then

$$u \geq 0 \quad \text{on} \quad [0, T] \times U.$$

See Theorem 1.10 for a proof. We also recall that if $f \in \ell^2(X, m)$, then $u_t = e^{-tL} f$ is a solution of the heat equation with $u_0 = f$. If additionally $f \in D(L)$, then the solution extends to $t = 0$, see Theorem A.33 in Appendix A for a proof of the solution statements for $t > 0$ and Theorem D.6 in Appendix D for $t = 0$ in the case of $f \in D(L)$.

In order to utilize the minimum principle, we will exhaust the graph by finite subsets. To utilize the symmetry of the graph, we will exhaust by balls around O. As we will assume that O is finite and b is locally finite, it follows that $B_r(O)$ are finite sets for all $r \in \mathbb{N}_0$. The minimum principle will allow us to compare heat kernels locally. We will then pass from properties on subsets to properties on the entire graph. In order to do so, we now build upon some ideas from the exhaustion techniques found in Section 3.

We let (b, c) be a locally finite graph over (X, m) and let $O \subseteq X$ be finite. We denote $B_R(O)$ by B_R and the Dirichlet Laplacian on $\ell^2(B_R, m_R)$ by

$$L_R^{(D)} = L_{B_R}^{(D)},$$

where $m_R = m|_{B_R}$ for $R \in \mathbb{N}_0$. We recall that

$$L_R^{(D)} f(x) = \frac{1}{m(x)} \left(\sum_{y \in B_R} b(x,y)(f(x) - f(y)) + \big(d_R(x) + c(x)\big) f(x) \right)$$

for all $f \in \ell^2(B_R, m_R)$ and $x \in B_R$, where $d_R(x) = \sum_{y \notin B_R} b(x,y)$. We let $Q_R^{(D)}$ denote the associated form, so that

$$Q_R^{(D)}(f) = \langle L_R^{(D)} f, f \rangle$$

for all $f \in \ell^2(B_R, m_R)$.

We denote the heat kernel of $L_R^{(D)}$ on B_R by $p^{(R)}$ and refer to $p^{(R)}$ as the *Dirichlet restriction of the heat kernel*. We can then extend $p^{(R)}$ by zero to all of X. Then, we have

$$p_t^{(R)}(x, O) = \frac{1}{m(O)} e^{-tL_R^{(D)}} 1_O(x)$$

for $x \in B_R(O)$ and $p_t^{(R)}(x, O) = 0$ for $x \in X \setminus B_R(O)$ and all $t \geq 0$.

We now collect some basic properties of the Dirichlet restrictions of the heat kernel.

Lemma 9.12 (Dirichlet restrictions of the heat kernel). *Let (b, c) be a locally finite weakly spherically symmetric graph with respect to a finite set $O \subseteq X$. Let p be the heat kernel and let $p^{(R)}$ denote the Dirichlet restrictions of the heat kernel to $B_R(O)$ for $R \in \mathbb{N}_0$. Then, $p_t^{(R)}(\cdot, O)$ is spherically symmetric for every $R \in \mathbb{N}_0$ and $t \geq 0$ and*

$$\lim_{R \to \infty} p_t^{(R)}(x, O) = p_t(x, O)$$

for all $x \in X$ and $t \geq 0$.

Proof. As $B_R(O)$ is finite for all $R \in \mathbb{N}_0$, we obtain

$$\lim_{R \to \infty} p_t^{(R)}(x, O) = p_t(x, O)$$

for all $t \geq 0$ and $x \in X$ by Lemma 1.21.

To show that $p_t^{(R)}(\cdot, O)$ is spherically symmetric for every $R \in \mathbb{N}_0$ and $t \geq 0$ we will show that $p^{(R)}$ is the heat kernel of a locally finite weakly spherically symmetric graph. Specifically, we denote $S_R(O)$ by S_R, $B_R(O)$ by B_R and the restrictions of b and m to $B_R \times B_R$ and B_R by b_R and m_R, respectively. Then, $p^{(R)}$ is the heat kernel of the graph (b_R, c_R) over (B_R, m_R), where

$$c_R(x) = \begin{cases} c(x) & \text{if} \quad x \in B_{R-1} \\ c(x) + \sum_{y \in S_{R+1}} b(x,y) & \text{if} \quad x \in S_R. \end{cases}$$

Therefore, as $q = c/m$,

$$q_R(x) = \begin{cases} q(x) & \text{if} \quad x \in B_{R-1}(O) \\ q(x) + k_+(R) & \text{if} \quad x \in S_R(O), \end{cases}$$

so that both the outer and inner degrees and the potential of (b_R, c_R) over (B_R, m_R) are spherically symmetric. Hence, by Theorem 9.6, $p_t^{(R)}(\cdot, O)$ is a spherically symmetric function for every $t \geq 0$ and $R \in \mathbb{N}_0$. This completes the proof. $\square$

Having established basic properties of $p^{(R)}$ we now turn to another intuitively clear property, namely, the decay in space of $p_t^{(R)}(\cdot, O)$ for every $t \geq 0$. For the proof we assume additionally that $c = 0$.

As we have established that $p^{(R)}(\cdot, O)$ is spherically symmetric, we write $p_t^{(R)}(r)$ for $p_t^{(R)}(x, O)$ for all $x \in S_r(O)$ and $t \geq 0$.

LEMMA 9.13 (Heat kernel decay). *Let b be a locally finite weakly spherically symmetric graph over (X, m) with respect to a finite set $O \subseteq X$. Let p be the heat kernel and let $p^{(R)}$ be the Dirichlet restriction of p to $B_R(O)$ for $R \in \mathbb{N}_0$. Then, for all $t \geq 0$ and $r, R \in \mathbb{N}_0$,*

$$p_t^{(R)}(r) \geq p_t^{(R)}(r+1)$$

and thus

$$p_t(r) \geq p_t(r+1)$$

for $t \geq 0$ and $r \in \mathbb{N}_0$.

PROOF. It is clear that $p_t(r) \geq p_t(r+1)$ for all $t \geq 0$ and $r \in \mathbb{N}_0$ follows from $p_t^{(R)}(r) \geq p_t^{(R)}(r+1)$ and the convergence of the heat kernel shown in Lemma 9.12. Thus, we focus on proving $p_t^{(R)}(r) \geq p_t^{(R)}(r+1)$ for all $t \geq 0$ and $r, R \in \mathbb{N}_0$. We note that $p_t^{(R)}(r) \geq 0$ by positivity of the heat kernel while $p_t^{(R)}(r) = 0$ for $r > R$ by definition. Hence, we may focus on the case $0 \leq r \leq R$.

We fix $R \in \mathbb{N}_0$ and introduce the function $\varphi\colon [0, \infty) \longrightarrow \mathbb{R}$ via

$$\varphi(t) = \max_{0 \leq j < k \leq R} \left(p_t^{(R)}(k) - p_t^{(R)}(j) \right).$$

We note that it suffices to show $\varphi \leq 0$. As

$$p_0^{(R)}(r) = \begin{cases} 1/m(O) & \text{if} \quad r = 0 \\ 0 & \text{if} \quad r \in \mathbb{N}, \end{cases}$$

we directly obtain $\varphi(0) \leq 0$.

We now argue by contradiction. Specifically, we show the following claim:

Claim. If there exists a $t_0 > 0$ such that $\varphi(t_0) > 0$, then there is a neighborhood around t_0 such that φ is strictly monotone decreasing in this neighborhood.

Proof of the claim. Suppose there exists a $t_0 > 0$ such that $\varphi(t_0) > 0$. We can then choose an $r_0 < R_0$ such that $p_{t_0}^{(R)}(r_0)$ is a strict local minimum for $p_{t_0}^{(R)}(\cdot)$ and $p_{t_0}^{(R)}(R_0)$ is a strict local maximum for

$p_{t_0}^{(R)}(\cdot)$, i.e., one of the neighbors is strictly larger/smaller than the minimum/maximum, and

$$\varphi(t_0) = p_{t_0}^{(R)}(R_0) - p_{t_0}^{(R)}(r_0).$$

By the strict local maximality and minimality, we have

$$L_R^{(D)} p_{t_0}^{(R)}(R_0) > 0 \qquad \text{and} \qquad L_R^{(D)} p_{t_0}^{(R)}(r_0) < 0.$$

By the heat equation we conclude

$$\partial_t p_t^{(R)}(R_0)|_{t=t_0} < 0 \qquad \text{and} \qquad \partial_t p_t^{(R)}(r_0)|_{t=t_0} > 0$$

and, by the continuity of $\partial_t p_t^{(R)}$, we obtain that φ is strictly monotonically decreasing in a neighborhood of t_0. This proves the claim.

So, assume that there exists a $t_0 > 0$ such that $\varphi(t_0) > 0$ and let $t_1 \in [0, t_0]$ be such that $\varphi(t_1) = \max_{[0,t_0]} \varphi$. Then, $\varphi(t_1) > 0$ and, therefore, φ must be strictly monotone decreasing in a neighborhood of t_1 by the claim above. This is a contradiction to φ taking a strictly positive maximum at t_1 and $\varphi(0) \leq 0$. This completes the proof. $\square$

Remark. It is not hard to see that the inequalities for $p^{(R)}$ above are strict for $t > 0$ (Exercise 9.10).

Having assembled all of the necessary pieces for the proof, we now establish our heat kernel comparison.

Proof of Theorem 9.11. Let p^{sym} denote the heat kernel of the weakly spherically symmetric graph and let $p^{\mathrm{sym},(R)}$ be the Dirichlet restriction of p^{sym} to $B_R(O^{\mathrm{sym}}) \subseteq X^{\mathrm{sym}}$ for $R \in \mathbb{N}_0$. We define a spherically symmetric function $\varrho^{(R)} \colon [0,\infty) \times X \longrightarrow \mathbb{R}$ via

$$\varrho_t^{(R)}(x) = p_t^{\mathrm{sym},(R)}(r)$$

for $x \in S_r(O) \subseteq X$ and $t \geq 0$.

We assume that b has stronger degree growth than b^{sym}. By the heat kernel decay of weakly spherically symmetric graphs, Lemma 9.13, and the assumption of stronger degree growth, we get, for $x \in S_r(O)$ with $r \in \mathbb{N}_0$,

$$\begin{aligned}
\mathcal{L}\varrho_t^{(R)}(x) &= k_+(x)(p_t^{\mathrm{sym},(R)}(r) - p_t^{\mathrm{sym},(R)}(r+1)) \\
&\qquad\qquad + k_-(x)(p_t^{\mathrm{sym},(R)}(r) - p_t^{\mathrm{sym},(R)}(r-1)) \\
&\geq k_+^{\mathrm{sym}}(x)(p_t^{\mathrm{sym},(R)}(r) - p_t^{\mathrm{sym},(R)}(r+1)) \\
&\qquad\qquad + k_-^{\mathrm{sym}}(x)(p_t^{\mathrm{sym},(R)}(r) - p_t^{\mathrm{sym},(R)}(r-1)) \\
&= \mathcal{L}^{\mathrm{sym}} p_t^{\mathrm{sym},(R)}(r).
\end{aligned}$$

Hence,

$$(\mathcal{L} + \partial_t)\varrho_t^{(R)}(x) \geq (\mathcal{L}^{\mathrm{sym}} + \partial_t) p_t^{\mathrm{sym},(R)}(r) = 0$$

for $x \in X$ and $t \geq 0$.

We now let p denote the heat kernel of b and let $p^{(R)}$ denote the Dirichlet restriction of p to $B_R(O) \subseteq X$. We let

$$u_t(x) = \varrho_t^{(R)}(x) - p_t^{(R)}(x, O)$$

for $x \in X$ and $t \geq 0$. From the above, we obtain

$$(\mathcal{L} + \partial_t)u_t(x) \geq 0$$

on $[0, T] \times B_R(O)$ for arbitrary $T > 0$. By compactness and continuity, $u \wedge 0$ attains a minimum on $[0, T] \times B_R(O)$. Furthermore, as $\varrho_t^{(R)}(x) = p_t^{(R)}(x, O) = 0$ for $x \in X \setminus B_R(O)$, we have $u = 0$ on $[0, T] \times X \setminus B_R(O)$. Finally, as we assume $m^{\mathrm{sym}}(O^{\mathrm{sym}}) = m(O)$, we obtain, for $x \in O$,

$$\varrho_0^{(R)}(x) = p_0^{\mathrm{sym},(R)}(0) = \frac{1}{m^{\mathrm{sym}}(O^{\mathrm{sym}})} = \frac{1}{m(O)} = p_0^{(R)}(x, O)$$

and $\varrho_0^{(R)}(x) = p_0^{(R)}(x, O) = 0$ for $x \in X \setminus O$. Therefore, $u_0(x) = 0$ for all $x \in X$.

Thus, by the minimum principle for the heat equation, Theorem 1.10, $u_t(x) \geq 0$ on $[0, T] \times B_R(O)$. Therefore, for $x \in S_r(O)$ with $r \leq R$ and $t \in [0, T]$ we have

$$p_t^{\mathrm{sym},(R)}(r) = \varrho_t^{(R)}(x) \geq p_t^{(R)}(x, O).$$

The statement $p_t^{\mathrm{sym}}(r) \geq p_t(x, O)$ for $x \in S_r(O)$, $r \in \mathbb{N}_0$ and $t \geq 0$ then follows from the convergence of the Dirichlet restrictions given in Lemma 9.12. This completes the proof in the case of stronger degree growth. The proof for weaker degree growth follows in an analogous manner. □

We conclude this subsection with the corresponding comparison result for the Green's function. Recall that

$$G_m(x, O) = \frac{1}{m(O)} \sum_{o \in X} G_m(x, o)$$

for $x \in X$. Furthermore, by Corollary 9.10, $G_m(\cdot, O)$ is spherically symmetric for weakly spherically symmetric graphs so that we may write $G_m^{\mathrm{sym}}(r) = G_m^{\mathrm{sym}}(x, O)$ for $x \in S_r(O^{\mathrm{sym}})$ and $r \in \mathbb{N}_0$.

Theorem 9.14 (Green's function comparison with weakly spherically symmetric graphs). *Let b be a connected locally finite graph over (X, m) with Green's function G_m and let $O \subseteq X$ be finite. If b over (X, m) has stronger (respectively, weaker) degree growth than a locally finite weakly spherically symmetric graph b^{sym} over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$ and Green's function G^{sym}, then*

$$G_m(x, O) \leq G^{\mathrm{sym}}(r) \qquad (\textit{respectively, } G_m(O, x) \geq G^{\mathrm{sym}}(r))$$

for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$.

Proof. The statement follows immediately from Theorem 9.11, the definition of the Green's function, which gives

$$G_m(x,O) = \frac{1}{m(O)} \sum_{o\in X} G_m(x,o) = \frac{1}{m(O)} \sum_{o\in X} \int_0^\infty e^{-tL} 1_o(x)dt$$
$$= \int_0^\infty p_t(x,O)dt,$$

and the corresponding formula for $G^{\mathrm{sym}}(x)$. □

2. The spectral gap

In this section we study the bottom of the spectrum of the Laplacian. More specifically, we first give a criterion for the bottom of the spectrum to be strictly positive when the graph is weakly spherically symmetric. We then give comparison theorems for the bottom of the spectrum of the Laplacian on general graphs. We also prove a criterion for the spectrum of the Laplacian to be discrete.

We will use some basic facts about the essential spectrum from Appendix 2. Moreover, Excavation Exercise 9.1 recalls a basic trick involving logarithms and exponentials which will be used in the proof of Theorem 9.20.

We let $L = L^{(D)}_{b,c,m}$ denote the Laplacian associated to a graph (b,c) over (X,m), $\sigma(L)$ denote the spectrum of L and

$$\lambda_0(L) = \inf \sigma(L)$$

denote the bottom of the spectrum of L. As L arises from a positive form, by the variational characterization of the bottom of the spectrum we obtain $\lambda_0(L) \geq 0$, see Theorem E.8 in Appendix E.

In this section we will give criteria for the bottom of the spectrum to be strictly positive. In this context, the value $\lambda_0(L)$ is sometimes referred to as the spectral gap. The *discrete spectrum* $\sigma_{\mathrm{disc}}(L)$ of L consists of the isolated eigenvalues of finite multiplicity and the *essential spectrum* $\sigma_{\mathrm{ess}}(L)$ is the complement of $\sigma_{\mathrm{disc}}(L)$ in the spectrum, i.e., $\sigma_{\mathrm{ess}}(L) = \sigma(L) \setminus \sigma_{\mathrm{disc}}(L)$. Furthermore, we say that L has *purely discrete spectrum* or the *spectrum of L is discrete* if $\sigma(L) = \sigma_{\mathrm{disc}}(L)$, i.e., $\sigma_{\mathrm{ess}}(L) = \emptyset$, see Appendix 2 for more details.

We introduce the *boundary* ∂W of a set $W \subseteq X$ as

$$\partial W = (W \times (X \setminus W)) \cup ((X \setminus W) \times W).$$

We remark that there are many notions of boundaries when considering subsets of a graph. Here, we take a notion that is symmetric.

We will be particularly interested in the boundary of balls around a set and the total edge weight of this boundary. We therefore introduce the *area of the boundary of a ball* $B_r(O)$ for a set $O \subseteq X$ as

$$b(\partial B_r(O)) = \sum_{(x,y)\in\partial B_r(O)} b(x,y)$$

for $r \in \mathbb{N}_0$. We note that

$$b(\partial B_r(O)) = 2 \sum_{x\in S_r(O)} k_+(x)m(x)$$

so if the graph is weakly spherically symmetric, we obtain

$$b(\partial B_r(O)) = 2k_+(r)m(S_r(O))$$

for $r \in \mathbb{N}_0$.

With these notions, we will prove the following summability criterion for discreteness of the spectrum and positivity of the bottom of the spectrum for weakly spherically symmetric graphs.

THEOREM 9.15 (Area-volume ratio and spectrum). *Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a finite set $O \subseteq X$. If*

$$a = \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then

$$\lambda_0(L) \geq \frac{1}{2a}$$

and the spectrum of L is discrete.

The proof uses the Agmon–Allegretto–Piepenbrink theorem for the spectrum from Chapter 4, which we now recall. If $u \in \mathcal{F}$ and $\alpha \in \mathbb{R}$, then we say that u is α-harmonic if

$$(\mathcal{L} + \alpha)u = 0.$$

In the case of locally finite graphs, Theorem 4.14 states that there exists a strictly positive α-harmonic function if and only if

$$\alpha \geq -\lambda_0(L).$$

Hence, to show the strict positivity of the bottom of the spectrum, it suffices to show the existence of a strictly positive α-harmonic function for $\alpha < 0$.

To show this existence, we will use the following recursion formula for spherically symmetric solutions. This formula will also be used to prove criteria for recurrence and stochastic completeness presented later in this chapter, so we state it in a rather general form which involves a function instead of a constant α. We also recall that the potential $q = c/m$ is a spherically symmetric function when the graph is spherically symmetric.

Lemma 9.16 (Recursion formula for spherically symmetric solutions). *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a finite set $O \subseteq X$ and let $f \in C(X)$ be spherically symmetric. Then, a spherically symmetric function $u \in C(X)$ satisfies $(\mathcal{L} + f)u = 0$ if and only if*

$$u(r+1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} (q(n) + f(n)) m(S_n(O)) u(n)$$

for all $r \in \mathbb{N}_0$. In particular, u is uniquely determined by the choice of $u(0)$. Furthermore, if $u(0) > 0$ and $f > 0$, then $u(r+1) > u(r)$ for all $r \in \mathbb{N}_0$.

Proof. We will prove the recursion formula by induction. The uniqueness and monotonicity statements are then obvious from the recursion formula.

We will omit O from our notation below, writing B_r for $B_r(O)$ and S_r for $S_r(O)$. We recall that $b(\partial B_r) = 2k_+(r)m(S_r)$ for $r \in \mathbb{N}_0$. For $r = 0$, from $(\mathcal{L} + f)u(0) = 0$ we obtain

$$\begin{aligned} 0 &= k_+(0)(u(0) - u(1)) + (q(0) + f(0))u(0) \\ &= \frac{b(\partial B_0)}{2m(S_0)}(u(0) - u(1)) + (q(0) + f(0))u(0), \end{aligned}$$

which yields the formula after rearranging the terms.

Now, we assume that the recursion formula holds for $r - 1$, where $r \in \mathbb{N}$. From $(\mathcal{L} + f)u(r) = 0$ we obtain

$$k_+(r)(u(r) - u(r+1)) + k_-(r)(u(r) - u(r-1)) + (q(r) + f(r))u(r) = 0.$$

Therefore, by the induction hypothesis, $b(\partial B_r) = 2k_+(r)m(S_r)$ and $k_+(r-1)m(S_{r-1}) = k_-(r)m(S_r)$ proven in Lemma 9.4, we obtain

$$\begin{aligned} u(r+1) - u(r) =& \frac{k_-(r)}{k_+(r)}(u(r) - u(r-1)) + \frac{1}{k_+(r)}(q(r) + f(r))u(r) \\ =& \frac{k_-(r)}{k_+(r)} \left(\frac{2}{b(\partial B_{r-1})} \sum_{n=0}^{r-1} (q(n) + f(n)) m(S_n) u(n) \right) \\ &+ \frac{2}{b(\partial B_r)}(q(r) + f(r)) m(S_r) u(r) \\ =& \frac{k_-(r)}{k_+(r)} \left(\frac{1}{k_-(r) m(S_r)} \sum_{n=0}^{r-1} (q(n) + f(n)) m(S_n) u(n) \right) \\ &+ \frac{2}{b(\partial B_r)}(q(r) + f(r)) m(S_r) u(r) \\ =& \frac{2}{b(\partial B_r)} \sum_{n=0}^{r} (q(n) + f(n)) m(S_n) u(n). \end{aligned}$$

This proves the recursion formula and thus completes the proof. $\square$

We now use the recursion formula above to show that under the summability assumption found in Theorem 9.15, there exists a strictly positive α-harmonic function for $\alpha < 0$. This will prove the spectral gap via the Agmon–Allegretto–Piepenbrink theorem.

LEMMA 9.17. *Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a finite set $O \subseteq X$. If*

$$a = \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a strictly positive monotonically decreasing spherically symmetric function u which satisfies $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a}\right) u = 0.$$

PROOF. We will define a spherically symmetric function u with the required properties. We start by letting $u(0) = 1$. Then, by Lemma 9.16, u will satisfy $\left(\mathcal{L} - \frac{1}{2a}\right) u = 0$ if and only if u satisfies the recursion formula

$$u(r+1) - u(r) = -\frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n)$$

as we assume $c = 0$ and, thus, $q = 0$.

We will show that u is strictly monotonically decreasing and remains positive by using strong induction. More specifically, we will show that

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(r+1) < u(r)$$

for all $r \in \mathbb{N}_0$. The first inequality above is clear from the definition of a. For $r = 0$, the remaining inequalities follow directly from the recursion formula and $u(0) = 1$ as

$$u(1) - u(0) = -\frac{1}{a \cdot b(\partial B_0(O))} m(S_0(O)) = -\frac{m(O)}{a \cdot b(\partial O)} < 0$$

gives

$$1 - \frac{m(O)}{a \cdot b(\partial O)} = u(1) < 1 = u(0).$$

Now, assume that the inequalities hold up to $r - 1$, that is,

$$0 < 1 - \frac{1}{a} \sum_{n=0}^{k} \frac{m(B_n(O))}{b(\partial B_n(O))} \leq u(k+1) < u(k)$$

for $k = 0, 1, \ldots, r-1$. Therefore, $u(k) > 0$ for $k = 0, 1, \ldots, r$ and the recursion formula gives $u(r+1) - u(r) < 0$. Moreover, as u is then strictly decreasing up to r, we get $u(n) < u(0) = 1$ for all $n = 1, 2, \ldots, r$.

Hence, from the recursion formula and the inductive hypotheses we obtain

$$\begin{aligned}
u(r+1) &= u(r) - \frac{1}{a \cdot b(\partial B_r(O))} \sum_{n=0}^{r} m(S_n(O))u(n) \\
&> u(r) - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\
&\geq 1 - \frac{1}{a} \sum_{n=0}^{r-1} \frac{m(B_n(O))}{b(\partial B_n(O))} - \frac{m(B_r(O))}{a \cdot b(\partial B_r(O))} \\
&= 1 - \frac{1}{a} \sum_{n=0}^{r} \frac{m(B_n(O))}{b(\partial B_n(O))}.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 9.15. By Lemma 9.17 for

$$a = \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

there exists a strictly positive function u which satisfies

$$\left(\mathcal{L} - \frac{1}{2a}\right) u = 0.$$

Thus, $\lambda_0(L) \geq 1/2a$ follows from the Agmon–Allegretto–Piepenbrink theorem for the spectrum, Theorem 4.14.

For the statement concerning the essential spectrum consider the graph $b_R = b|_{X\setminus B_R(O)\times X\setminus B_R(O)}$ over $(X \setminus B_R(O), m_R)$, where $m_R = m|_{X\setminus B_R(O)}$ for $R \in \mathbb{N}$. Let $L_R = L^{(D)}_{X\setminus B_R(O)}$ be the Dirichlet Laplacian associated to b_R over $(X \setminus B_R(O), m_R)$. Since the graph is assumed to be locally finite and O is a finite subset, all balls $B_R(O)$ are finite and thus the operator L_R is a finite-dimensional, and thus compact, perturbation of the operator L for every $R \in \mathbb{N}$. Therefore, if we let $\lambda_0^{\mathrm{ess}}(L)$ denote the bottom of the essential spectrum

$$\lambda_0^{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L_R) \geq \inf \sigma(L_R) = \lambda_0(L_R)$$

for $R \in \mathbb{N}$, as follows by either Theorem 4.20 or Theorem E.7.

In order to estimate $\lambda_0(L_R)$, we let

$$a_R = \sum_{r=R+1}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < a$$

and let $\mathcal{L}_R$ be the formal Laplacian of the graph b_R. Since b_R is a weakly spherically symmetric graph over $(X \setminus B_r(O), m_R)$ with respect to O, there exists a strictly positive function u_R which satisfies

$$\left(\mathcal{L}_R - \frac{1}{2a_R}\right) u_R = 0$$

for every $R \in \mathbb{N}$ by Lemma 9.17. Hence, by the Agmon–Allegretto–Piepenbrink theorem for the spectrum, Theorem 4.14, and the inequalities above we get

$$\lambda_0^{\mathrm{ess}}(L) \geq \lambda_0(L_R) \geq \frac{1}{2a_R} \to \infty$$

as $R \to \infty$, cf. also Theorem 4.20. This shows $\sigma_{\mathrm{ess}}(L) = \emptyset$ and, therefore, L has purely discrete spectrum. □

We next illustrate Theorem 9.15 for spherically symmetric trees and anti-trees, i.e., for Examples 9.2 and 9.3. For trees, we get the following criterion for the spectral gap and discreteness of the spectrum.

Example 9.18 (Spherically symmetric trees and spectrum). Let b be a spherically symmetric tree with branching number k. If

$$a = \sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{2 \prod_{j=0}^{r} k(j)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ and the spectrum of L is discrete (Exercise 9.3).

For anti-trees we obtain the following criterion. We note, in particular, that this can be used to construct examples of graphs with strictly positive bottom of the spectrum and whose distance balls grow polynomially. We will have more to say about the bottom of the spectrum and volume growth in Chapter 13.

Example 9.19 (Anti-trees and spectrum). Let b be an anti-tree with sphere size s. If

$$a = \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{2s(r)s(r+1)} < \infty,$$

then $\lambda_0(L) \geq 1/2a$ and the spectrum of L is discrete (Exercise 9.4).

We will next prove a comparison result for spectral properties. For this, we recall that we compare the outer and inner degrees of a general graph to those of a weakly spherically symmetric graph. We also recall that we denote the corresponding quantities in the weakly spherically symmetric graph with a superscript sym.

Theorem 9.20 (Spectral comparison). *Let b be a connected locally finite graph over (X, m) and let $O \subseteq X$. If b has stronger (respectively, weaker) degree growth with respect to O than a locally finite weakly spherically symmetric graph b^{sym} over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$, then*

$$\lambda_0(L) \geq \lambda_0(L^{\mathrm{sym}}) \qquad (\textit{respectively, } \lambda_0(L) \leq \lambda_0(L^{\mathrm{sym}})).$$

Furthermore, if b^{sym} *over* $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ *satisfies*

$$a = \sum_{r=0}^{\infty} \frac{m^{\mathrm{sym}}(B_r(O^{\mathrm{sym}}))}{b^{\mathrm{sym}}(\partial B_r(O^{\mathrm{sym}}))} < \infty$$

and b *over* (X, m) *has stronger degree growth than* b^{sym} *over* $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$*, then*

$$\lambda_0(L) \geq \frac{1}{2a}$$

and the spectrum of L *is discrete.*

Proof. Assume that b has stronger degree growth than b^{sym}. By the heat kernel comparison, Theorem 9.11, we have

$$p_t(x, O) \leq p_t^{\mathrm{sym}}(r)$$

for all $x \in S_r(O) \subseteq X$, $r \in \mathbb{N}_0$ and $t \geq 0$. By definition

$$p_t(x, O) = \frac{1}{m(O)} \sum_{o \in O} p_t(x, o) m(o).$$

Therefore, we obtain

$$\frac{1}{m(O)} \sum_{o \in O} p_t(x, o) m(o) = p_t(x, O) \leq p_t^{\mathrm{sym}}(r) = p_t^{\mathrm{sym}}(x', o')$$

for all $x \in S_r(O)$, $o' \in O^{\mathrm{sym}}$, $x' \in S_r(O^{\mathrm{sym}})$ and $r \in \mathbb{N}_0$, where the last equality follows by the symmetry of the heat kernel of weakly spherically symmetric graphs established in Theorem 9.6.

Now, by the Theorem of Li, Theorem 5.6, we have

$$\lim_{t \to \infty} \frac{1}{t} \log p_t(x, y) = -\lambda_0(L)$$

for all $x, y \in X$ as the graph is connected. Hence, for $x, y \in O$ and $x', o' \in O^{\mathrm{sym}}$, we obtain

$$\begin{aligned} -\lambda_0(L) &= \lim_{t \to \infty} \frac{1}{t} \log p_t(x, y) \\ &= \lim_{t \to \infty} \frac{1}{t} \log \frac{1}{m(O)} \sum_{o \in O} p_t(x, o) m(o) \\ &\leq \lim_{t \to \infty} \frac{1}{t} \log p_t^{\mathrm{sym}}(x', o') \\ &= -\lambda_0(L^{\mathrm{sym}}). \end{aligned}$$

Therefore, $\lambda_0(L) \geq \lambda_0(L^{\mathrm{sym}})$ in the case of stronger degree growth. The proof of the statement for weaker degree growth follows analogously.

When b has stronger degree growth than b^{sym}, the estimate $\lambda_0(L) \geq 1/2a$ follows from the first statement and Theorem 9.15. For the statement about the discreteness of the spectrum, consider the Laplacians L_R and L_R^{sym} associated to the graphs

$$b_R = b|_{X \setminus B_R(O) \times X \setminus B_R(O)}$$

over $(X \setminus B_R(O), m_R)$, where $m_R = m|_{X\setminus B_R(O)}$ and

$$b_R^{\mathrm{sym}} = b^{\mathrm{sym}}|_{X^{\mathrm{sym}}\setminus B_R(O^{\mathrm{sym}})\times X^{\mathrm{sym}}\setminus B_R(O^{\mathrm{sym}})}$$

over $(X^{\mathrm{sym}}\setminus B_R(O^{\mathrm{sym}}), m_R^{\mathrm{sym}})$ where $m_R^{\mathrm{sym}} = m^{\mathrm{sym}}|_{X^{\mathrm{sym}}\setminus B_R(O^{\mathrm{sym}})}$ for $R \in \mathbb{N}$. Clearly, b_R also has stronger degree growth than b_R^{sym}. Therefore, by what we have proven above and the proof of Theorem 9.15, we have

$$\lambda_0(L_R) \geq \lambda_0(L_R^{\mathrm{sym}}) \geq \frac{1}{2a_R}$$

for

$$a_R = \sum_{r=R+1}^{\infty} \frac{m^{\mathrm{sym}}(B_r(O^{\mathrm{sym}}))}{b^{\mathrm{sym}}(\partial B_r(O^{\mathrm{sym}}))}$$

for all $R \in \mathbb{N}$.

Now, since L_R is a finite-dimensional and thus compact perturbation of L, we infer that the essential spectra and, in particular, the bottoms of the essential spectra $\lambda_0^{\mathrm{ess}}(L)$ and $\lambda_0^{\mathrm{ess}}(L_R)$ agree, see Corollary 4.19 or Theorem E.7. Hence,

$$\lambda_0^{\mathrm{ess}}(L) = \lambda_0^{\mathrm{ess}}(L_R) \geq \lambda_0(L_R) \geq \frac{1}{2a_R} \to \infty$$

as $R \to \infty$. Thus, the essential spectrum of L is empty and so L has purely discrete spectrum. This completes the proof. $\square$

3. Recurrence

In this section we present a characterization of recurrence for weakly spherically symmetric graphs. We then give a comparison result for general graphs.

We recall that for a graph b over (X, m) with associated Laplacian L, the Green's function $G\colon X \times X \longrightarrow [0, \infty]$ is defined as

$$G(x, y) = \int_0^\infty e^{-tL} 1_y(x) dt = \int_0^\infty p_t(x, y) m(y) dt.$$

A connected graph b is called recurrent if $G(x, y) < \infty$ for some (all) $x, y \in X$ and transient otherwise, see Theorem 6.1 in Chapter 6 for various other characterizations of this property. We also note that whenever b is connected, we have $G(x, y) > 0$ for all $x, y \in X$ as the heat semigroup is positivity improving by Theorem 1.26.

We note that recurrence is a measure-independent phenomenon in the sense that if b is recurrent for one measure m, then b is recurrent for all measures m. Hence, we either do not mention the measure or assume that $m = 1$ in the statements below. Furthermore, we recall that for a finite set $O \subseteq X$, we have defined

$$G(\cdot, O) = \frac{1}{m(O)} \sum_{o\in O} G(\cdot, o)$$

and have shown in Theorem 9.14 that $G(\cdot, O)$ is a spherically symmetric function in the case that the graph is weakly spherically symmetric with respect to O.

We will first give a characterization of recurrence for weakly spherically symmetric graphs. For this, we will use a characterization of recurrence in terms of superharmonic functions. We recall that a function $u \in \mathcal{F}$ is called superharmonic if $\mathcal{L}u \geq 0$. Theorem 6.1 (iv) states that a connected graph is transient if and only if there exists a positive non-constant superharmonic function. This will be used to prove the characterization below.

Finally, we recall that $\partial B_r(O)$ denotes the boundary of the ball of radius $r \in \mathbb{N}_0$ around O and $b(\partial B_r(O))$ denotes the total edge weight of the boundary of the ball, which we refer to as the area of the boundary of the ball. The following theorem gives a characterization of recurrence in terms of this quantity.

THEOREM 9.21 (Area ratio and recurrence). *Let b be a locally finite weakly spherically symmetric graph over X with respect to a finite set $O \subseteq X$. Then, b is recurrent if and only if*

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} = \infty.$$

PROOF. As noted above, the measure plays no role in recurrence. Hence, we let $m = 1$ be the counting measure on X. We also let $a > 0$ be a constant. By Lemma 9.16, the unique spherically symmetric function u with $u(0) = 1$ and

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

satisfies

$$\begin{aligned} u(r+1) - u(r) &= \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \frac{-1}{2a \cdot m(O)} 1_O(n) m(S_n(O)) u(n) \\ &= \frac{-1}{a \cdot b(\partial B_r(O))} \end{aligned}$$

for all $r \in \mathbb{N}_0$. Iterating this and using $u(0) = 1$, we obtain

$$u(r+1) = 1 - \frac{1}{a} \sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

for $r \in \mathbb{N}_0$. We will use this equality with different constants a for both implications in the proof.

First, if we assume that

$$a = \sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty,$$

then u is a non-constant strictly positive superharmonic function. Thus, b is transient by Theorem 6.1 (iv).

Conversely, assume that b is transient. Then, by Theorem 6.1 (xi), the Green's function is finite, i.e., $G(x,y) < \infty$ for all $x, y \in X$. Furthermore, by Theorem 6.26 (d), $G(\cdot, o)$ for $o \in O$ satisfies

$$\mathcal{L}G(\cdot, o) = 1_o.$$

Furthermore, $G(\cdot, o)$ is strictly positive as the graph is connected. By Corollary 9.10, the function

$$g_O(x) = \frac{1}{m(O)} \sum_{o \in O} G(x, o)$$

is spherically symmetric and from the above satisfies

$$\mathcal{L}g_O = \frac{1}{m(O)} \sum_{o \in O} \mathcal{L}G(\cdot, o) = \frac{1}{m(O)} 1_O = \frac{1}{g_O(o')m(O)} 1_O g_O,$$

where the last equality holds for all $o' \in O$ since g_O is spherically symmetric. Hence, if we let $u = g_O/g_O(o')$ for $o' \in O$, then u is strictly positive spherically symmetric and satisfies $u(0) = 1$ with

$$\left(\mathcal{L} - \frac{1}{2a \cdot m(O)} 1_O\right) u = 0$$

for $a = g_O(o')/2$. Now, by the consideration in the beginning of the proof, u must also satisfy

$$u(r+1) = 1 - \frac{1}{a} \sum_{k=0}^{r} \frac{1}{b(\partial B_k(O))}$$

and hence, then u is positive if

$$\sum_{r=0}^{\infty} \frac{1}{b(\partial B_r(O))} < \infty.$$

This completes the proof. $\square$

Remark. Another viewpoint on Theorem 9.21 is that the Green's function for weakly spherically symmetric graphs can be calculated explicitly as

$$G(x, o) = m(o) \sum_{n=r}^{\infty} \frac{1}{\frac{1}{2} b(\partial B_n(O))}$$

for $x \in S_r(O)$, $r \in \mathbb{N}$ and $o \in O$. In particular,

$$G(x, O) = \sum_{n=r}^{\infty} \frac{1}{\frac{1}{2} b(\partial B_n(O))}$$

for all $x \in S_r(O)$, $n \in \mathbb{N}$, from which Theorem 9.21 follows (Exercise 9.11).

We now illustrate the theorem above for our two main classes of examples, namely spherically symmetric trees and anti-trees from Examples 9.2 and 9.3. For trees the characterization of recurrence reads as follows.

EXAMPLE 9.22 (Spherically symmetric trees and recurrence). Let b be a spherically symmetric tree with branching number k. Then b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{n=0}^{r} k(n)} = \infty$$

(Exercise 9.3).

For anti-trees, rephrasing everything in terms of the sphere growth gives the following characterization.

EXAMPLE 9.23 (Anti-trees and recurrence). Let b be an anti-tree with sphere size s. Then, b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty$$

(Exercise 9.4).

Next, we give a comparison result for recurrence. We note that, intuitively speaking, stronger degree growth gives a larger push to infinity, which is needed for transience. This is made precise in the following result.

In order to evoke the definition of stronger and weaker degree growth, we need the presence of a measure. Hence, we will assume that the measure is the counting measure for both graphs. As part of the definition of stronger and weaker degree growth, this gives that the cardinalities of O and O^{sym} are the same.

THEOREM 9.24 (Recurrence comparison). *Let b be a locally finite graph over (X, m) with $m = 1$ and let $O \subseteq X$ be a finite set. If b has stronger (respectively, weaker) degree growth than a locally finite weakly spherically symmetric graph b^{sym} over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$, where $m^{\mathrm{sym}} = 1$ and b^{sym} is transient (respectively, recurrent), then b is transient (respectively, recurrent).*

PROOF. The statement follows immediately by Theorem 9.14 and the characterization of recurrence in terms of the finiteness of the Green's function, Theorem 6.1 (xi). $\square$

4. Stochastic completeness at infinity

In this section we investigate stochastic completeness at infinity. We first characterize this property for weakly spherically symmetric graphs and then give corresponding comparison results.

For the proof of the characterization we will need an elementary statement about the equivalence of the convergence of sums and products, which is recalled in Excavation Exercise 9.2.

We recall that a graph (b, c) over (X, m) with associated Laplacian $L = L^{(D)}_{b,c,m}$ is called stochastically complete at infinity if

$$e^{-tL}1 + \int_0^t e^{-sL}\frac{c}{m}\,ds = 1$$

for all $t \geq 0$. Here, e^{-tL} denotes the heat semigroup originally defined via the spectral theorem on $\ell^2(X, m)$ and then extended to $\ell^\infty(X)$ so that we may apply it to the constant function 1 and the integral term involves the semigroup extend to positive functions via the use of nets, see Sections 1 and 2 for details concerning these extensions.

Stochastic completeness at infinity has a number of equivalent formulations. We recall that a function $u \in \mathcal{F}$ is called α-harmonic for $\alpha \in \mathbb{R}$ if

$$(\mathcal{L} + \alpha)u = 0.$$

By Theorem 7.18, stochastic completeness at infinity is equivalent to the fact that every bounded α-harmonic function for $\alpha > 0$ is trivial. Furthermore, if $u \in \mathcal{F}$ satisfies $(\mathcal{L} + \alpha)u \geq 0$, then u is called α-superharmonic. The Khasminskii criterion for stochastic completeness states that if there exists a positive α-superharmonic function for $\alpha > 0$ which grows to infinity at infinity, then (b, c) is stochastically complete at infinity, see Theorem 7.31.

We first state a characterization of stochastic completeness at infinity for weakly spherically symmetric graphs. In particular, the criterion below compares the growth of the total measure and killing term of the ball to the area of the boundary.

Theorem 9.25 (Volume-area ratio and stochastic completeness). *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a finite set $O \subseteq X$. Then, (b, c) is stochastically complete at infinity if and only if*

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} = \infty.$$

In order to prove the theorem above, we first investigate the boundedness of α-harmonic functions for $\alpha > 0$ by using the recursion formula for solutions found in Lemma 9.16.

Lemma 9.26. *Let (b, c) be a locally finite weakly spherically symmetric graph over (X, m) with respect to a finite set $O \subseteq X$. Then, the following statements are equivalent:*

(i) *There exists $\alpha > 0$ and a non-trivial spherically symmetric α-harmonic function that is bounded.*

(ii) *For all $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded.*

(iii) *We have*

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} < \infty.$$

Proof. First of all, the recursion formula for spherically symmetric solutions, Lemma 9.16, gives that a spherically symmetric α-harmonic function u is uniquely determined by its value at 0. Thus, for a given $\alpha > 0$, all spherically symmetric α-harmonic functions are bounded if there exists a non-trivial α-harmonic function that is bounded.

Furthermore, for $\alpha > 0$, we let

$$a(\alpha) = \sum_{r=0}^{\infty} \frac{c(B_r(O)) + \alpha m(B_r(O))}{b(\partial B_r(O))}.$$

Obviously, the finiteness of $a(\alpha)$ for some $\alpha > 0$ is equivalent to the finiteness of $a(\alpha)$ for all $\alpha > 0$.

Thus, it remains to show that $a(\alpha) < \infty$ is equivalent to the existence of a non-trivial bounded spherically symmetric α-harmonic function.

By applying the recursion formula, Lemma 9.16, any spherically symmetric u with $(\mathcal{L} + \alpha)u = 0$ satisfies

$$u(r+1) - u(r) = \frac{2}{b(\partial B_r(O))} \sum_{n=0}^{r} \big(c(S_n(O)) + \alpha m(S_n(O))\big) u(n)$$

for all $r \in \mathbb{N}_0$, where we used $q(n)m(S_n(O)) = c(S_n(O))$, which follows from the spherical symmetry of q. Now, if $u(0) = 0$, then u is trivial, hence, we may assume that $u(0) \neq 0$ as we are interested in non-trivial solutions.

If we assume that $u(0) > 0$, then the recursion formula implies that u is monotonically increasing. In particular, $u(r) \geq u(0)$ for all $r \in \mathbb{N}_0$ and, thus,

$$u(r+1) - u(r) \geq \frac{2\,(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))} u(0)$$

for $r \in \mathbb{N}_0$. Therefore, if $a(\alpha) = \infty$, then

$$\begin{aligned} u(r) &= \sum_{n=0}^{r-1} (u(n+1) - u(n)) \\ &\geq \sum_{n=0}^{r-1} \frac{2\,(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))} u(0) \\ &\to \infty \end{aligned}$$

as $r \to \infty$. So, u is unbounded in this case. An analogous argument shows that $u(r) \to -\infty$ as $r \to \infty$ if $u(0) < 0$ and $a(\alpha) = \infty$.

On the other hand, if $u(0) > 0$, then u is strictly positive and monotonicity and iteration yield

$$\begin{aligned} u(r+1) &\leq \left(1 + \frac{2(c(B_r(O)) + \alpha m(B_r(O)))}{b(\partial B_r(O))}\right) u(r) \\ &\leq \prod_{n=0}^{r} \left(1 + \frac{2(c(B_n(O)) + \alpha m(B_n(O)))}{b(\partial B_n(O))}\right). \end{aligned}$$

Now, if $a(\alpha) < \infty$, then

$$\prod_{n=0}^{\infty} \left(1 + \frac{c(B_n(O)) + \alpha m(B_n(O))}{b(\partial B_n(O))}\right) < \infty$$

and the estimate above shows that u is bounded. A similar argument shows that u is strictly negative and bounded below if $u(0) < 0$ and $a(\alpha) < \infty$. This shows the equality between (i) and (iii).

Furthermore, since finiteness of $a(\alpha)$ for one $\alpha > 0$ is equivalent to finiteness of $a(\alpha)$ for all $\alpha > 0$, we get that (i) and (ii) are equivalent. This completes the proof. $\square$

Proof of Theorem 9.25. If

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} < \infty,$$

then there exists a non-trivial bounded α-harmonic function for $\alpha > 0$ by Lemma 9.26. Thus, the graph is stochastically incomplete at infinity by Theorem 7.18.

On the other hand, if the graph is stochastically incomplete at infinity, then there exists a positive non-trivial bounded function v which satisfies $(\mathcal{L}+\alpha)v = 0$ for $\alpha > 0$ by Theorem 7.18. We recall that $\mathcal{A}$ denotes the averaging operator defined by

$$\mathcal{A}f(x) = \frac{1}{m(S_r(O))} \sum_{y \in S_r(O)} f(y)m(y)$$

for $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Applying this to v gives that $u = \mathcal{A}v$ is a spherically symmetric function with

$$(\mathcal{L} + \alpha)u = \mathcal{L}\mathcal{A}v + \alpha\mathcal{A}v = \mathcal{A}(\mathcal{L} + \alpha)v = 0$$

since $\mathcal{L}$ and $\mathcal{A}$ commute by Lemma 9.8. Therefore, there exists a non-trivial bounded spherically symmetric function u which satisfies $(\mathcal{L} + \alpha)u = 0$ for $\alpha > 0$ and, thus,

$$\sum_{r=0}^{\infty} \frac{c(B_r(O)) + m(B_r(O))}{b(\partial B_r(O))} < \infty$$

by Lemma 9.26. This completes the proof. $\square$

We again illustrate the characterization of stochastic completeness at infinity for our two main classes of examples, namely, spherically symmetric trees and anti-trees from Example 9.2 and 9.3.

Example 9.27 (Spherically symmetric trees and stochastic completeness). Let b be a spherically symmetric tree with branching number k. Then b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{\prod_{j=0}^{r} k(j)} = \infty$$

(Exercise 9.3).

For anti-trees we obtain the following characterization. We note, in particular, that we can use this to construct examples of stochastically incomplete graphs whose balls grow polynomially. We will discuss stochastic completeness and volume growth further in Chapter 14.

Example 9.28 (Anti-trees and stochastic completeness). Let b be an anti-tree with sphere size s. Then, b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r)s(r+1)} = \infty$$

(Exercise 9.4).

Remark. We note that the volume growth criterion above does not hold for general graphs, as can be seen from examples (Exercise 9.5).

We can also link stochastic completeness and spectral properties for weakly spherically symmetric graphs. In order to apply our spectral results, we assume that $c = 0$. In this case, we speak of a graph as being stochastically complete if $e^{-tL}1 = 1$ for all $t \geq 0$. Given this, combining Theorem 9.25 with our previous spectral results we obtain the following connection.

Corollary 9.29. *Let b be a weakly spherically symmetric locally finite graph over (X, m) with respect to a finite set $O \subseteq X$. If b is stochastically incomplete, then*

$$\lambda_0(L) \geq \frac{1}{2a},$$

where

$$a = \sum_{r=0}^{\infty} \frac{m(B_r(O))}{b(\partial B_r(O))} < \infty$$

and the spectrum of L is discrete.

Proof. The corollary follows directly from Theorem 9.25 combined with Theorem 9.15. $\square$

Remark. We note that the conclusion of Corollary 9.29 does not hold for general graphs, as can be seen from examples (Exercise 9.6).

Finally, we present a comparison result for stochastic completeness at infinity. Here we note that while a larger degree growth gives a larger push to infinity, which is required for stochastic incompleteness at infinity, the potential can remove heat from the graph before it can be lost at infinity. Hence, in particular, we see that for stochastic incompleteness at infinity we need both a stronger degree growth and a weaker potential growth.

Theorem 9.30 (Stochastic completeness comparison). *Let (b, c) be a locally finite graph over (X, m) and let $O \subseteq X$ be a finite set. Let $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ over $(X^{\mathrm{sym}}, m^{\mathrm{sym}})$ be a locally finite weakly spherically symmetric graph with respect to a finite set $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$.*

(a) *Let (b, c) have stronger degree and weaker potential growth with respect to O than $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ with respect to O^{sym}. If $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ is stochastically incomplete at infinity, then (b, c) is stochastically incomplete at infinity.*
(b) *Let (b, c) have weaker degree and stronger potential growth with respect to O than $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ with respect to $O^{\mathrm{sym}} \subseteq X^{\mathrm{sym}}$. If $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ is stochastically complete at infinity, then (b, c) is stochastically complete at infinity.*

Proof. Let $\alpha > 0$ be fixed and let u be the spherically symmetric function on X^{sym} which satisfies $(\mathcal{L}^{\mathrm{sym}} + \alpha)u = 0$ with $u(0) = 1$, which is given by the recursion formula, Lemma 9.16. Note that u is strictly increasing since $\alpha > 0$. In particular, u is strictly positive. Given this u, we define a spherically symmetric function $v \in C(X)$ for $x \in S_r(O)$ and $r \in \mathbb{N}_0$ by

$$v(x) = u(r).$$

We now prove (a). Thus, assume that (b, c) has stronger degree and weaker potential growth than $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ and $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ is stochastically incomplete at infinity. Then, Theorem 9.25 and Lemma 9.26 imply that u, and therefore v, is bounded. Furthermore, since u is monotonically increasing, by using the assumptions of stronger degree growth and weaker potential growth, we infer, for $x \in S_r(O) \subseteq X$ and $r \in \mathbb{N}_0$,

$$\begin{aligned}
&(\mathcal{L} + \alpha)v(x) \\
&= k_+(x)(u(r) - u(r+1)) + k_-(x)(u(r) - u(r-1)) + q(x)u(r) \\
&\leq k_+^{\mathrm{sym}}(r)(u(r) - u(r+1)) + k_-^{\mathrm{sym}}(r)(u(r) - u(r-1)) + q^{\mathrm{sym}}(r)u(r) \\
&= (\mathcal{L}^{\mathrm{sym}} + \alpha)u(r) \\
&= 0.
\end{aligned}$$

Hence, there exists a strictly positive bounded α-superharmonic function v defined on X. Therefore, (b, c) is stochastically incomplete at infinity by Theorem 7.18.

To show (b), we assume that (b, c) has weaker degree and stronger potential growth than $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ and $(b^{\mathrm{sym}}, c^{\mathrm{sym}})$ is stochastically complete at infinity. By a similar argument to the above, we infer

$$(\mathcal{L} + \alpha)v(x) \geq (\mathcal{L} + \alpha)u(r) = 0$$

for all $x \in S_r(O)$ and all $r \in \mathbb{N}_0$. Furthermore, by Theorem 9.25 and Lemma 9.26, the function u must be unbounded. Hence, $v(x) \to \infty$ as $x \to \infty$, where $x \to \infty$ means that x tends to the point ∞ in the one point compactification $\widehat{X} = X \cup \{\infty\}$ of X. Hence, the graph (b, c) is stochastically complete by the Khasminskii criterion, Theorem 7.31. $\square$

Exercises

Excavation exercises.

Exercise 9.1 (Log-Sum-Exp formula).

(a) Let $s_j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$ and let $a = \max\{s_j\}_{j=1}^n$. Show that

$$\log \sum_{j=1}^{n} e^{s_j} = a + \log \sum_{j=1}^{n} e^{s_j - a}.$$

(b) Show that if $f_j \colon \mathbb{R} \longrightarrow (0, \infty)$ are functions with

$$\lim_{t\to\infty} \frac{\log f_j(t)}{t} = C$$

for all $j = 1, 2, \ldots, n$ and some constant C, then

$$\lim_{t\to\infty} \frac{1}{t} \log \sum_{j=1}^{n} f_j(t) = C.$$

Exercise 9.2 (Series and product convergence). Let (a_n) be a sequence of positive numbers. Show that the series $\sum_{n\geq 0} a_n$ converges if and only if the product $\prod_{n\geq 0}(1 + a_n)$ converges.

Example exercises.

Exercise 9.3 (Spherically symmetric trees). Let b be a spherically symmetric tree with branching numbers k. Show that

$$m(S_r(o)) = \prod_{j=0}^{r-1} k(j) \qquad \text{and} \qquad b(\partial B_r(o)) = 2m(S_{r+1}(o))$$

for all $r \in \mathbb{N}$.

(a) Show that if

$$a = \sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{2 \prod_{j=0}^{r} k(j)} < \infty,$$

then $\lambda_0(L) \geq 1/(2a)$ and the spectrum of L is discrete.

(b) Show that b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{\prod_{j=0}^{r} k(j)} = \infty.$$

(c) Show that b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{1 + \sum_{n=1}^{r} \prod_{j=0}^{n-1} k(j)}{\prod_{i=0}^{r} k(j)} = \infty.$$

Assume now that

$$k(j) = j^{\beta}$$

for $j \geq 0$ and $\beta > 0$. Determine the threshold for β in (a), (b), (c). What happens at the threshold?

EXERCISE 9.4 (Anti-trees). Let b be an anti-tree with sphere size s. Show that

$$m(S_r(o)) = s(r) \quad \text{and} \quad b(\partial B_r(o)) = 2s(r)s(r+1)$$

for $r \geq 0$.

(a) Show that if

$$a = \sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{2s(r)s(r+1)} < \infty,$$

then $\lambda_0(L) \geq 1/(2a)$ and the spectrum of L is discrete.

(b) Show that b is recurrent if and only if

$$\sum_{r=0}^{\infty} \frac{1}{s(r)s(r+1)} = \infty.$$

(c) Show that b is stochastically complete at infinity if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{n=0}^{r} s(n)}{s(r)s(r+1)} = \infty.$$

Assume now that

$$s(r) = (r+1)^{\beta}$$

for $r \geq 0$ and $\beta > 0$. Determine the threshold for β in (a), (b), (c). What happens at the threshold?

In particular, show that there exists an anti-tree such that $m(B_r(o))$ grows polynomially and the anti-tree is stochastically incomplete and has a spectral gap with purely discrete spectrum.

EXERCISE 9.5 (Counterexample for general graphs). Give an example of a stochastically incomplete graph b with standard weights and counting measure such that

$$\sum_{r=0}^{\infty} \frac{\# B_r(o)}{b(\partial B_r(o))} = \infty.$$

EXERCISE 9.6 (Stochastic incompleteness and spectrum for general graphs). Give an example of a stochastically incomplete graph such that the bottom of both the spectrum and the essential spectrum is zero.

(Hint: The stability results for stochastic completeness found in Subsection 8 may be useful for this.)

Extension exercises.

Exercise 9.7 (Spherically symmetric and strongly spherically symmetric graphs). Let (b, c) be a graph over (X, m). A graph automorphism is a bijective map $\pi\colon X \longrightarrow X$ such that

$$b \circ (\pi \times \pi) = b, \qquad c \circ \pi = c \qquad \text{and} \qquad m \circ \pi = m.$$

We say that a graph is *spherically symmetric* with respect to a set $O \subseteq X$ if for all $r \in \mathbb{N}_0$ and $x, y \in S_r(O)$ there exists a graph automorphism π such that $\pi(x) = y$ and *strongly spherically symmetric* if there exists a graph automorphism π such that $\pi(x) = y$ and $\pi(y) = x$.

(a) Show that every strongly spherically symmetric graph is spherically symmetric and every spherically symmetric graph is weakly spherically symmetric.
(b) Show that every spherically symmetric tree, Example 9.2, is strongly spherically symmetric.
(c) Show that every anti-tree, Example 9.3, is weakly spherically symmetric. Show, furthermore, that an anti-tree is spherically symmetric if and only if $b|_{S_r(o)\times S_r(o)}$ is a vertex transitive graph (i.e., for every two vertices $x, y \in S_r(o)$ there is a graph automorphism of $b|_{S_r(o)\times S_r(o)}$ mapping x to y). Give an example of a strongly spherically anti-tree and an anti-tree that is spherically symmetric but not strongly spherically symmetric.

Exercise 9.8 (Local finiteness and symmetry). Let (b, c) be a connected weakly spherically symmetric graph over (X, m) with respect to a set $O \subseteq X$. Suppose that $m(x) \geq C_r > 0$ for all $x \in S_r(O)$ and $r \in \mathbb{N}_0$. Show that the following statements are equivalent:

(i) O is finite.
(ii) $S_r(O)$ is finite for some $r \in \mathbb{N}_0$.
(iii) $S_r(O)$ is finite for all $r \in \mathbb{N}_0$.

In particular, show that any of the conditions above imply that the graph is locally finite.

Exercise 9.9 (Weak spherical symmetry and the heat kernel). Let (b, c) be a connected locally finite graph over (X, m) and let $O \subseteq X$ be finite. Let $r_z = d(z, O)$ denote the distance to O for $z \in X$. Show that (b, c) is weakly spherically symmetric with respect to O if and only if

$$\frac{1}{m(S_{r_x}(O))} \sum_{z \in S_{r_x}(O)} p_t(y, z) m(z) = \frac{1}{m(S_{r_y}(O))} \sum_{z \in S_{r_y}(O)} p_t(x, z) m(z)$$

for all $x, y \in X$ and $t \geq 0$. Show that this implies that

$$p_t(\cdot, O) = \frac{1}{m(O)} \sum_{o \in O} p_t(\cdot, o) m(o)$$

is a spherically symmetric function for all $t \geq 0$.

Exercise 9.10 (Strict decay of the Dirichlet kernel). Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a finite set $O \subseteq X$. Let p be the heat kernel and let $p^{(R)}$ be the Dirichlet restriction to $B_R(O)$ for $R \in \mathbb{N}_0$. Show that, for all $t>0$ and $R \geq r \in \mathbb{N}_0$,

$$p_t^{(R)}(r) > p_t^{(R)}(r+1).$$

Exercise 9.11 (Green's function and symmetry). Let b be a locally finite weakly spherically symmetric graph over (X,m) with respect to a finite set $O \subseteq X$. Show that

$$G(x,o) = m(o) \sum_{n=r}^{\infty} \frac{2}{b(\partial B_n(O))}$$

for $x \in S_r(O)$ and $o \in O$ is the Green's function, so that

$$G(x,O) = \sum_{n=r}^{\infty} \frac{2}{b(\partial B_n(O))}$$

for $x \in S_r(O)$. Give an alternate proof of Theorem 9.21 using this.

Notes

In large part, the material presented in this chapter is based on [**KLW13**], where the bulk of the results are presented for the case of $O = \{o\}$. The idea to extend these results to more general O can be found in [**BG15**].

Of course, there is a tremendous amount of work for regular trees or, more generally, spherically symmetric trees with standard weights, see, e.g., [**Bro91, Car72, CKW94, CY99, FTN91, Fuj96b, MS99, MS00, PP95, PW89, Ura97, Ura99**], among many other works.

The first appearance of anti-trees seems to be in the case of linear sphere growth, i.e., $s(r) = r + 1$ in the paper of Dodziuk/Karp [**DK88**]. In particular, as established there, this anti-tree is transient and the bottom of the spectrum of the Laplacian is 0. This same anti-tree appears in [**Web10**] as an example of a stochastically complete graph whose vertex degree is unbounded. The study of general anti-trees is then taken up in [**Woj11**], where stochastic completeness is characterized for spherically symmetric graphs with standard weights and counting measure. In particular, it is first shown there that anti-trees give examples of stochastically incomplete graphs of polynomial volume growth with respect to the combinatorial graph metric. This result is then used to establish the sharpness of volume growth criteria for stochastic completeness found in [**GHM12**].

The spectral theory of anti-trees is analyzed in [**BK13**], where it is shown that the spectrum consists mainly of eigenvalues with compactly supported eigenfunctions and a further spectral component which can be singular continuous. Anti-trees are also used as a counterexample to a conjecture of Golénia/Schumacher from [**GS11**] concerning the deficiency indices of the adjacency matrix, see [**GS13**].

The fact that weak spherical symmetry is equivalent to the semigroup and averaging operator commuting is Theorem 1 in [**KLW13**]. The heat kernel comparison, Theorem 9.11, is Theorem 2 in [**KLW13**]. For the case of standard edge weights and counting measure, this result appears for spherically symmetric trees in [**Woj08, Woj09**]. These results were inspired by corresponding result for Riemannian manifolds in [**CY81**]. The Green's function comparison, Theorem 9.14, is then an immediate consequence, see [**Ura97**] for comparisons of the discrete time Green's function of a graph with standard weights to a regular tree. The heat kernel decay, Lemma 9.13, which is a key step in the proof, goes back to Lemma 3.10 in [**Woj09**].

The estimate for the bottom of the spectrum and criterion for discreteness of the spectrum, Theorem 9.15, are Theorem 3 in [**KLW13**]. In the case of spherically symmetric manifolds, similar estimates for the bottom of the spectrum can be found in [**BPB06**].

The spectral comparison theorem, Theorem 9.20, is Theorem 4 in [**KLW13**]. Earlier work on graphs includes comparisons on the bottom of the spectrum with regular trees in the case of standard graph weights found in [**Bro91, Ura99, Ż99**]. For Riemannian manifolds, similar comparison theorems involve curvature quantities, see [**Che75, McK70**].

The characterization of recurrence for weakly spherically symmetric graphs, Theorem 9.21, can be found as Proposition 5.3 in [**Hua14a**] for the continuous time Green's function. For a discrete time Green's function, the result can already be found as Theorem 5.9 in the textbook [**Woe09**]. The fact that recurrence is independent of the choice of discrete or continuous time can be found in [**Sch17b**]. Comparison results for recurrence in the case of spherically symmetric Riemannian manifolds can be found in [**Ich82a**].

The characterization of stochastic completeness at infinity for weakly spherically symmetric graphs, Theorem 9.25, can be found as Theorem 5 in [**KLW13**]. For graphs with standard weights and counting measure, the result goes back to [**Woj11**] which, in turn, was inspired by the corresponding criterion for stochastic completeness of spherically symmetric Riemannian manifolds, see [**Gri99**]. Counterexamples for this criterion on general graphs are found in [**Hua11b**] and for manifolds in [**BB10**]. The counterpart to Corollary 9.29 linking stochastic incompleteness and discreteness of the spectrum, for spherically symmetric Riemannian manifolds, can be found in [**Har09**]. Finally, the comparison theorems for stochastic completeness, Theorem 9.30, can be found as Theorem 6 in [**KLW13**] and their counterparts in the manifold case in [**Ich82b**].

CHAMBER 10

Sparseness and Isoperimetric Inequalities

Couldn't peep it with a pair of bifocals I'm no joker, play me as a joker. Be on you like a house on fire, smoke ya.
ODB.

In this chapter we investigate what it means for a graph to have relatively few edges. This leads to the notions of weakly sparse, approximately sparse and sparse graphs, as well as graphs which satisfy a strong isoperimetric inequality. These notions are all introduced in Section 1. In Section 2 we prove the area and co-area formulas which will be a key tool in this chapter and will also play a role in the investigation of Cheeger inequalities later. We use these formulas to give connections between the notion of sparseness and form estimates for weakly sparse graphs in Section 3, for approximately sparse graphs in Section 4, for sparse graphs in Section 5 and for graphs satisfying a strong isoperimetric inequality in Section 6. This leads to characterizations of the discreteness of the spectrum of the Dirichlet Laplacian in terms of a weighted degree at infinity via the use of the min-max principle. In the case of purely discrete spectrum, we can also investigate eigenvalue asymptotics. This is done for the various notions of sparseness in the corresponding sections. Finally, we also give a criterion for a strong isoperimetric inequality in terms of a mean curvature-type quantity in Section 6.

1. Notions of sparseness

Loosely speaking, sparse graphs can be understood as graphs with relatively few edges. We discuss a hierarchy of notions of sparseness, namely, weakly sparse, approximately sparse and sparse graphs as well as graphs which satisfy a strong isoperimetric inequality. This hierarchy is illustrated in Figure 1 below.

Let us be more precise. For a subset $W \subseteq X$, we introduce the boundary of W as

$$\partial W = (W \times X \setminus W) \cup (X \setminus W \times W).$$

In the literature, the boundary of a set is often introduced with respect to a graph and consists of all of the edges emanating from the set. Here, we let the boundary be all pairs of vertices with one vertex inside W and the other outside of W. In this sense, ∂W consists of all possible edges emanating from W. Now, given a graph b, summing over ∂W with

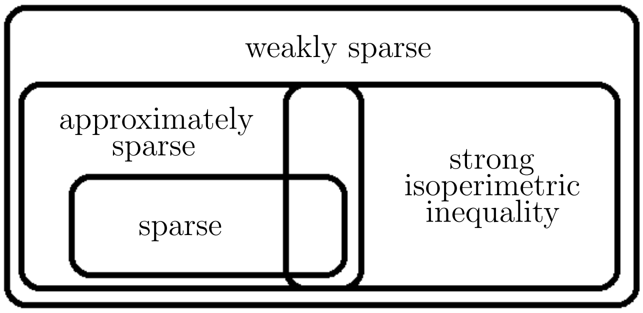

FIGURE 1. The hierarchy of sparse graphs.

respect to b singles out the edges of the graph b as $b(x, y) = 0$ if there is no edge between x and y. To this end, we let $b(A) = \sum_{(x,y)\in A} b(x, y)$ for $A \subseteq X \times X$.

DEFINITION 10.1 (Hierarchy of sparseness). Let (b, c) be a graph over (X, m).

(a) The graph is called *sparse* or, more specifically, *k-sparse* for $k \geq 0$ if
$$b(W \times W) \leq km(W)$$
for all finite sets $W \subseteq X$.

(b) The graph is called *approximately sparse* if for all $\varepsilon > 0$ there exists a $k_\varepsilon \geq 0$ such that
$$b(W \times W) \leq \varepsilon \left(\frac{1}{2} b(\partial W) + c(W) \right) + k_\varepsilon m(W)$$
for all finite sets $W \subseteq X$.

(c) The graph is called *weakly sparse* or, more specifically, *(a, k)-weakly sparse* for $a, k \geq 0$ if
$$b(W \times W) \leq a \left(\frac{1}{2} b(\partial W) + c(W) \right) + km(W)$$
for all finite sets $W \subseteq X$.

REMARK. Note that (b, c) is k-sparse if and only if (b, c) is $(0, k)$-weakly sparse. Furthermore, (b, c) is approximately sparse if for all $\varepsilon > 0$ there exists a $k_\varepsilon \geq 0$ such that (b, c) is $(\varepsilon, k_\varepsilon)$-weakly sparse.

The interpretation of sparseness as stating that the graph has only few edges becomes clear for graphs with standard weights and counting measure.

EXAMPLE 10.2 (Standard weights and sparseness). Let b be a graph with standard weights on X with counting measure, i.e., $b(x, y) \in \{0, 1\}$ for all $x, y \in X$, $c = 0$ and $m = 1$. The set of edges of the graph is given by $E = \{(x, y) \in X \times X \mid b(x, y) = 1\}$. For a set $W \subseteq X$, the set of edges in W is denoted by $E_W = E \cap (W \times W)$ and the edge boundary of W by $\partial_E W = \partial W \cap E$. Then for all finite $W \subseteq X$, a k-sparse graph satisfies

$$\#E_W \leq k \cdot \#W$$

while an (a, k)-weakly sparse graph satisfies

$$\#E_W \leq \frac{a}{2} \cdot \#\partial_E W + k \cdot \#W.$$

This shows that sparse graphs have few edges within a set when compared to the cardinality of the set.

We now discuss some specific examples which are left as an exercise.

Example 10.3 (Sparse graphs, Exercise 10.2). Let b be a graph with standard weights on X with the counting measure.

(a) If b is a tree, then b is 2-sparse.
(b) Using Euler's polyhedron formula one sees that if b is a planar graph, then b is 6-sparse.
(c) There exist graphs that are weakly sparse but not approximately sparse and graphs that are approximately sparse but not sparse.

We recall the definition of the normalizing measure n for a graph (b, c) over X given by $n(x) = \sum_{y \in X} b(x, y) + c(x)$ for $x \in X$. We note that for the choice of m as the normalizing measure n, the definition of sparseness becomes trivial for $k \geq 1$.

Example 10.4 (Normalizing measure). Let (b, c) be a graph over (X, n), where n is the normalizing measure. Then (b, c) is k-sparse for all $k \geq 1$.

Hence, the normalizing measure is not a suitable choice for the concept of sparseness for $k \geq 1$. However, for values of k between 0 and 1, this leads to another notion for graphs over X. This notion is referred to in the literature as a strong isoperimetric inequality. We first define this concept and show afterwards how it is related to sparseness and weak sparseness.

Definition 10.5 (Strong isoperimetric inequality). A graph (b, c) over X is said to satisfy a *strong isoperimetric inequality* with *isoperimetric constant* $\alpha > 0$ if

$$\alpha n(W) \leq \frac{1}{2} b(\partial W) + c(W)$$

for all finite sets $W \subseteq X$.

Remark. Note that $0 < \alpha \leq 1$ since otherwise the definition cannot be satisfied. Furthermore, if $\alpha = 1$, then $b = 0$.

In contrast to the notions of sparseness above, the concept of a strong isoperimetric inequality is independent of a measure. Let us revisit the example of graphs with standard weights.

Example 10.6 (Standard weights and isoperimetric inequality). Let b be a graph with standard weights over X. If b satisfies a strong isoperimetric inequality with isoperimetric constant α, then

$$\alpha \deg(W) \leq \frac{1}{2} \# \partial_E W.$$

Remark. There are examples of graphs that are sparse and either satisfy or do not satisfy a strong isoperimetric inequality (Exercise 10.3).

Remark. The maximal isoperimetric constant α is sometimes referred to as a Cheeger constant. Another version of Cheeger's constant is discussed in Chapter 13.

The lemma below clarifies the relation between weak sparseness and strong isoperimetric inequalities. For this equivalence the choice of the measure m is irrelevant. On the other hand, we relate k-sparseness with respect to the normalizing measure with $k \in (0,1)$ and strong isoperimetric inequalities.

Lemma 10.7 (Strong isoperimetric inequality and sparseness). *The following statements hold:*

(a) *A graph (b,c) over (X,m) satisfies a strong isoperimetric inequality with isoperimetric constant $\alpha > 0$ if and only if the graph is $((1-\alpha)/\alpha, 0)$-weakly sparse.*

(b) *A graph (b,c) over (X,n) satisfies a strong isoperimetric inequality with isoperimetric constant $\alpha > 0$ if and only if the graph is $(1-\alpha)$-sparse.*

Proof. For both (a) and (b) we use the identity

$$n(W) = b(W \times W) + \frac{1}{2} b(\partial W) + c(W)$$

for a finite set $W \subseteq X$.

For (a) we note that the identity implies that the inequality

$$\alpha n(W) \leq \frac{1}{2} b(\partial W) + c(W)$$

is equivalent to

$$\alpha b(W \times W) \leq (1-\alpha) \left(\frac{1}{2} b(\partial W) + c(W) \right),$$

which proves the statement.

For (b) we use the identity to get the equivalence of the inequalities

$$\alpha n(W) \leq \frac{1}{2} b(\partial W) + c(W) = n(W) - b(W \times W)$$

and

$$b(W \times W) \leq (1-\alpha) n(W),$$

which completes the proof. □

REMARK. It is also possible to give a relation between (a, k)-weakly sparse graphs for an arbitrary k and isoperimetric constants under further restrictions (Exercise 10.5).

2. Co-area formulae

In this section we present an area and a co-area formula in a general setting. These formulas will be used to derive spectral consequences.

We will present two formulas which involve the level sets of functions. For a function $f \in C(X)$ and $t \in \mathbb{R}$, we define the level sets

$$\Omega_t(f) = \{x \in X \mid f(x) > t\}.$$

For $w\colon X \times X \longrightarrow [0, \infty)$ and $U \subseteq X \times X$, we let

$$w(U) = \sum_{(x,y)\in U} w(x, y),$$

which may take the value ∞. We may think of w as a graph b, but we neither need the symmetry nor the summability assumptions.

The first formula relates the differences of a function to an integral over the boundary of the level sets. We refer to this as a co-area formula.

LEMMA 10.8 (Co-area formula). *Let* $w\colon X \times X \longrightarrow [0, \infty)$ *and* $f \in C(X)$. *Then,*

$$\sum_{x,y\in X} w(x, y)|f(x) - f(y)| = \int_{-\infty}^{\infty} w(\partial\Omega_t(f))dt,$$

where both sides may take the value ∞.

PROOF. For vertices $x, y \in X$ with $x \neq y$ we define the interval $I_{x,y}$,

$$I_{x,y} = [f(x) \wedge f(y), f(x) \vee f(y)),$$

and let $|I_{x,y}| = |f(x) - f(y)|$ be the length of $I_{x,y}$. Denote by $1_{x,y}$ the characteristic function of $I_{x,y}$. Then, for $t \in \mathbb{R}$ the inclusion $\{(x, y), (y, x)\} \subseteq \partial\Omega_t(f)$ holds if and only if $t \in I_{x,y}$. Therefore,

$$w(\partial\Omega_t(f)) = \sum_{x,y\in X} w(x, y)1_{x,y}(t).$$

Combining the considerations above, we calculate, using Tonelli's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} w(\partial\Omega_t(f))dt &= \int_{-\infty}^{\infty} \sum_{x,y\in X} w(x,y)1_{x,y}(t)dt \\ &= \sum_{x,y\in X} w(x,y) \int_{-\infty}^{\infty} 1_{x,y}(t)dt \\ &= \sum_{x,y\in X} w(x,y)|f(x)-f(y)|.\end{aligned}$$

This proves the statement. □

For the next formula, we assume that the function is positive and that there exists a measure on the space. The formula then relates the values of the function to the measure of the level sets associated to the function.

Lemma 10.9 (Area formula). *Let $m\colon X \longrightarrow [0,\infty)$ and $f\colon X \longrightarrow [0,\infty)$. Then,*

$$\sum_{x\in X} f(x)m(x) = \int_0^\infty m(\Omega_t(f))dt,$$

where both sides may take the value ∞.

Proof. We have $x \in \Omega_t(f)$ if and only if $1_{(t,\infty)}(f(x)) = 1$. We calculate, using Tonelli's theorem,

$$\begin{aligned}\int_0^\infty m(\Omega_t(f))dt &= \int_0^\infty \sum_{x\in\Omega_t(f)} m(x)dt \\ &= \int_0^\infty \sum_{x\in X} m(x)1_{(t,\infty)}(f(x))dt \\ &= \sum_{x\in X} m(x) \int_0^\infty 1_{(t,\infty)}(f(x))dt \\ &= \sum_{x\in X} m(x)f(x).\end{aligned}$$

This finishes the proof. □

3. Weak sparseness and the form domain

In this section we show that weak sparseness can be characterized by a functional inequality. In turn, this functional inequality allows us to explicitly determine the form domain $D(Q)$ for $Q = Q^{(D)}$ as an intersection of ℓ^2 spaces. A further consequence is a characterization of purely discrete spectrum for the Laplacian $L = L^{(D)}$ in terms of the weighted vertex degree.

For background material on the spectrum of multiplication operators, see Appendix A, for the essential spectrum and min-max principle, see Section 2 in Appendix E. Furthermore, Excavation Exercise 10.1, which characterizes discreteness of the spectrum of multiplication operators, will be used in this section.

We now explain some of the notation used below. Let (X, m) be a discrete measure space. Any function $f\colon X \longrightarrow [0,\infty)$ induces a symmetric form $q_f\colon \ell^2(X,m) \longrightarrow [0,\infty]$ defined by $q_f(g) = \langle g, fg\rangle$. With a slight abuse of notation, we will also write f for q_f. Furthermore, for two forms q, q' on a Hilbert space that both include a subspace D_0 in their domain, we write $q \leq q'$ on D_0 whenever $q(\varphi) \leq q'(\varphi)$ for all $\varphi \in D_0$.

We recall the notion of the weighted vertex degree for a graph (b, c) over (X, m) given as $\mathrm{Deg} = n/m$. When we speak about Deg as a form on $C_c(X)$ we always consider $C_c(X)$ as a subspace of $\ell^2(X, m)$, i.e., if $\varphi \in C_c(X)$, then

$$
\begin{aligned}
q_{\mathrm{Deg}}(\varphi) = \langle \varphi, \mathrm{Deg}\varphi\rangle_m &= \sum_{x\in X} \varphi^2(x)\mathrm{Deg}(x)m(x) \\
&= \sum_{x\in X} \varphi^2(x)n(x) = \langle \varphi, \varphi\rangle_n.
\end{aligned}
$$

As q_{Deg} is the quadratic form of the multiplication operator with respect to Deg on $\ell^2(X, m)$, this directly implies that the maximal form domain

$$D(q_{\mathrm{Deg}}) = \{f \in \ell^2(X,m) \mid \mathrm{Deg}^{1/2} f \in \ell^2(X,m)\}$$

of q_{Deg} satisfies

$$D(q_{\mathrm{Deg}}) = \ell^2(X,n) \cap \ell^2(X,m) = \overline{C_c(X)}^{\|\cdot\|_{q_{\mathrm{Deg}}}}$$

since the compactly supported functions are dense in every ℓ^2 space.

The theorem below provides a characterization of weakly sparse graphs in functional analytic terms. In particular, we see that the domain of the form q_{Deg} being equivalent to the domain of the Dirichlet form $D(Q)$ characterizes weak sparseness.

THEOREM 10.10 (Characterization of weak sparseness). *Let (b, c) be a graph over (X, m). Then, the following statements are equivalent:*

(i) *The graph is weakly sparse.*

(ii) *There exist $\tilde a \in (0, 1)$ and $\tilde k \geq 0$ such that on $C_c(X)$ we have*
$$(1 - \tilde a)\mathrm{Deg} - \tilde k \leq Q.$$

(iii) *There exist $\tilde a \in (0, 1)$ and $\tilde k \geq 0$ such that on $C_c(X)$ we have*
$$(1 - \tilde a)\mathrm{Deg} - \tilde k \leq Q \leq (1 + \tilde a)\mathrm{Deg} + \tilde k.$$

(iv) $D(Q) = \ell^2(X, n) \cap \ell^2(X, m)$.

The proof will be given below. The direction (iii) $\Longrightarrow$ (ii) is of course trivial. The direction (ii) $\Longrightarrow$ (i) follows directly by applying the inequality in (ii) to characteristic functions. To carry out a similar reasoning for the implication (i) $\Longrightarrow$ (iii) we have to reduce arbitrary functions to characteristic functions. This is done by virtue of the area and co-area formula above.

Before we come to details of the proof, we provide a corollary. This corollary gives a characterization of discreteness of the spectrum of L in terms of the weighted vertex degree tending to infinity.

Recall that an operator T on a Hilbert space is said to have purely discrete spectrum if the spectrum of T consists only of discrete eigenvalues of finite multiplicity. We define the weighted vertex degree at infinity by

$$\mathrm{Deg}_\infty = \sup_{K \subseteq X \text{ finite}} \inf_{x \in X \setminus K} \mathrm{Deg}(x).$$

This quantity can be understood as minimizing the vertex degree outside of larger and larger finite sets and taking the limit.

COROLLARY 10.11 (Discrete spectrum). *Let (b, c) be a weakly sparse graph over (X, m). Then, the spectrum of L is purely discrete if and only if* $\mathrm{Deg}_\infty = \infty$.

PROOF. Let $f_\pm = (1 \pm \tilde{a})\mathrm{Deg} \pm \tilde{k}$, where $\tilde{a}$ and $\tilde{k}$ are as in (ii) and (iii) of Theorem 10.10. By Theorem 10.10, weak sparseness is equivalent to $f_- \leq Q \leq f_+$ on $C_c(X)$. By a consequence of the min-max principle, see Theorem E.11 in Appendix 2.3, discreteness of the spectrum of L is now equivalent to discreteness of the spectrum of multiplication by Deg, which is equivalent to $\mathrm{Deg}_\infty = \infty$. □

The proof of Theorem 10.10 is divided into three lemmas and an argument which is essentially the closed graph theorem. The first lemma is the part where the area and co-area formula enter. It will also be used later in the case of sparse graphs to get sharper estimates.

LEMMA 10.12. *Let (b, c) be an (a, k)-weakly sparse graph over (X, m) for $a, k \geq 0$. Then, for all $\varphi \in C_c(X)$,*

$$\langle \varphi, (\mathrm{Deg} - k)\varphi \rangle \leq (1 + a) Q^{1/2}(\varphi)(2\langle \varphi, \mathrm{Deg}\varphi \rangle - Q(\varphi))^{1/2}.$$

PROOF. Let $\varphi \in C_c(X)$ and denote the level sets of φ^2 by $\Omega_t = \{x \in X \mid \varphi^2(x) > t\}$ for $t \geq 0$. Then, $\mathrm{Deg} = n/m$ yields

$$\langle \varphi, (\mathrm{Deg} - k)\varphi \rangle = \sum_{x \in X} \varphi^2(x) n(x) - k \sum_{x \in X} \varphi^2(x) m(x).$$

The area formula, Lemma 10.9, gives

$$\ldots = \int_0^\infty (n(\Omega_t) - k m(\Omega_t))\, dt.$$

Applying the identity $n(W) = b(W \times W) + \frac{1}{2}b(\partial W) + c(W)$ for finite sets $W \subseteq X$ and the (a, k)-weak sparseness, we get

$$\begin{aligned}\ldots &= \int_0^\infty \left(b(\Omega_t \times \Omega_t) + \frac{1}{2}b(\partial(\Omega_t)) + c(\Omega_t) - km(\Omega_t) \right) dt \\ &\leq (1+a) \int_0^\infty \left(\frac{1}{2}b(\partial(\Omega_t)) + c(\Omega_t) \right) dt.\end{aligned}$$

Now, employing the co-area formula, Lemma 10.8 for the first term and the area formula, Lemma 10.9 for the second term, we arrive at

$$\ldots = \frac{1+a}{2} \sum_{x,y \in X} b(x,y)|\varphi^2(x) - \varphi^2(y)| + (1+a) \sum_{x \in X} c(x)\varphi^2(x)$$

$$= \frac{1+a}{2} \left(\sum_{x,y \in X} b(x,y)|\varphi(x) - \varphi(y)||\varphi(x) + \varphi(y)| + 2 \sum_{x \in X} c(x)\varphi^2(x) \right).$$

For the next step we add a point x_∞ to X, i.e., let $\widetilde{X} = X \cup \{x_\infty\}$ and define $\widetilde{b}$ as $\widetilde{b} = b$ on $X \times X$ and $\widetilde{b}(x, x_\infty) = \widetilde{b}(x_\infty, x) = c(x)$. Furthermore, we extend φ to $\widetilde{X}$ by letting $\varphi(x_\infty) = 0$. Then we continue using the Cauchy–Schwarz inequality, which yields

$$\ldots = \frac{1+a}{2} \sum_{x,y \in \widetilde{X}} \widetilde{b}(x,y)|\varphi(x) - \varphi(y)||\varphi(x) + \varphi(y)|$$

$$\leq \frac{1+a}{\sqrt{2}} Q^{1/2}(\varphi) \left(\sum_{x,y \in X} b(x,y)(\varphi(x) + \varphi(y))^2 + 2 \sum_{x \in X} c(x)\varphi^2(x) \right)^{1/2}$$

$$= (1+a)Q^{1/2}(\varphi) \left(2\langle \varphi, \mathrm{Deg}\varphi \rangle - Q(\varphi)\right)^{1/2},$$

where we use $(\varphi(x) + \varphi(y))^2 = 2(\varphi^2(x) + \varphi^2(y)) - (\varphi(x) - \varphi(y))^2$ in the last equality. This finishes the proof. $\square$

We use the lemma above to show that weak sparseness implies the form inequality.

Lemma 10.13 (Weak sparseness implies form inequality). *Let (b, c) be a weakly sparse graph over (X, m). Then, there exist $\tilde{a} \in [0, 1)$ and $\tilde{k} \geq 0$ such that on $C_c(X)$ we have*

$$(1 - \tilde{a})\mathrm{Deg} - \tilde{k} \leq Q \leq (1 + \tilde{a})\mathrm{Deg} + \tilde{k}.$$

If the graph is (a, k)-weakly sparse for $a, k \geq 0$, then $\tilde{a}$, $\tilde{k}$ can be chosen as

$$\tilde{a} = \frac{\sqrt{a^2 + 2a + (a^2 \wedge \frac{1}{4})}}{1+a} \quad \textit{and} \quad \tilde{k} = (2k(1 - \tilde{a})) \vee \left(\frac{k\left(\left(\frac{1}{a} - a\right) \vee \frac{3}{2}\right)}{2(1+a)} \right).$$

If the graph satisfies a strong isoperimetric inequality with isoperimetric constant $\alpha > 0$, then we can choose $\tilde{a} = \sqrt{1 - \alpha^2}$ and $\tilde{k} = 0$.

Proof. Let $\varphi \in C_c(X)$. Assume first that $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle \geq 0$. Then, the estimate in Lemma 10.12 can be squared and we obtain

$$(1+a)^2 Q^2(\varphi) - 2(1+a)^2\langle \varphi, \mathrm{Deg}\varphi\rangle Q(\varphi) + \langle \varphi, (\mathrm{Deg} - k)\varphi\rangle^2 \leq 0.$$

Letting

$$r = \langle \varphi, \mathrm{Deg}\varphi\rangle^2 - (1+a)^{-2}\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle^2$$

and resolving the quadratic inequality above for $Q(\varphi)$, we arrive at

$$\langle \varphi, \mathrm{Deg}\varphi\rangle - r^{1/2} \leq Q(\varphi) \leq \langle \varphi, \mathrm{Deg}\varphi\rangle + r^{1/2}.$$

In the case of a strong isoperimetric inequality, we have that (b, c) is (a, k)-weakly sparse with $a = (1-\alpha)/\alpha$ and $k = 0$ by Lemma 10.7. In particular, we have $r = (1-\alpha^2)\langle \varphi, \mathrm{Deg}\varphi\rangle^2$ and, therefore, we obtain the statement with $\tilde{a} = \sqrt{1-\alpha^2}$ and $\tilde{k} = 0$.

We proceed to estimate r for the weakly sparse case in general. We let $0 < \lambda < 1$ and use the inequality $\xi\zeta \leq (\frac{1}{2}\xi + \frac{1}{2}\zeta)^2$ with $\xi = \lambda\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle$ and $\zeta = \frac{1}{\lambda}k\langle \varphi, \varphi\rangle$ to get

$$\begin{aligned}
(1+a)^2 r =& (a^2+2a)\langle \varphi, \mathrm{Deg}\varphi\rangle^2 + k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle \\
\leq& (a^2+2a)\langle \varphi, \mathrm{Deg}\varphi\rangle^2 + \left(\lambda\langle \varphi, \mathrm{Deg}\varphi\rangle + \frac{k}{2}\left(\frac{1}{\lambda} - \lambda\right)\langle \varphi, \varphi\rangle\right)^2 \\
\leq& \left(\left(a^2+2a+\lambda^2\right)^{1/2}\langle \varphi, \mathrm{Deg}\varphi\rangle + \frac{k}{2}\left(\frac{1}{\lambda} - \lambda\right)\langle \varphi, \varphi\rangle\right)^2.
\end{aligned}$$

Setting $\lambda = a \wedge 1/2$,

$$\tilde{a} = \frac{\left(a^2+2a+\left(a^2 \wedge \frac{1}{4}\right)\right)^{\frac{1}{2}}}{1+a} \quad \text{and} \quad \tilde{k} = \frac{k\left(\left(\frac{1}{a} - a\right) \vee \frac{3}{2}\right)}{2(1+a)}$$

we obtain $r^{1/2} \leq \tilde{a}\langle \varphi, \mathrm{Deg}\varphi\rangle + \tilde{k}\langle \varphi, \varphi\rangle$ with $\tilde{a} < 1$. This yields the desired inequality for $\varphi \in C_c(X)$ with $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle \geq 0$.

For the case of φ with $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle < 0$, we choose $\tilde{a}$ as above and let $\tilde{k} = 2k(1-\tilde{a})$. The lower bound follows immediately since

$$\langle \varphi, ((1-\tilde{a})\mathrm{Deg} - \tilde{k})\varphi\rangle = (1-\tilde{a})\langle \varphi, (\mathrm{Deg} - 2k)\,\varphi\rangle \leq 0 \leq Q(\varphi)$$

as $\tilde{a} < 1$. For the upper bound we check

$$\begin{aligned}
\langle \varphi, ((1+\tilde{a})\mathrm{Deg} + \tilde{k})\varphi\rangle =& \langle \varphi, ((1+\tilde{a})\mathrm{Deg} + 2k(1-\tilde{a}))\,\varphi\rangle \\
&\geq (3-\tilde{a})\langle \varphi, \mathrm{Deg}\varphi\rangle \\
&> 2\langle \varphi, \mathrm{Deg}\varphi\rangle \\
&\geq Q(\varphi),
\end{aligned}$$

where the last inequality follows from $Q \leq 2\mathrm{Deg}$, which is proven in Theorem 1.27. This finishes the proof. □

We next show the opposite direction, i.e., that the form inequality implies weak sparseness.

Lemma 10.14 (Lower form inequality implies weak sparseness). *Let (b,c) be a graph over (X,m) and let $\tilde{a} \in (0,1)$ and $\tilde{k} \geq 0$ be such that on $C_c(X)$ we have*

$$(1-\tilde{a})\mathrm{Deg} - \tilde{k} \leq Q.$$

Then, (b,c) over (X,m) is (a,k)-weakly sparse with

$$a = \frac{\tilde{a}}{1-\tilde{a}} \qquad \textit{and} \qquad k = \frac{\tilde{k}}{1-\tilde{a}}.$$

Proof. Let $W \subseteq X$ be finite and let 1_W be the characteristic function of W. We will use the equalities $n(W) = b(W \times W) + b(\partial W)/2 + c(W)$ and $Q(1_W) = b(\partial W)/2 + c(W)$. The assumed inequality applied with $\varphi = 1_W$ yields

$$(1-\tilde{a})\left(b(W\times W) + \frac{1}{2}b(\partial W) + c(W)\right) - \tilde{k}m(W) \leq \frac{1}{2}b(\partial W) + c(W).$$

Rearranging the terms, we infer

$$b(W \times W) \leq \frac{\tilde{a}}{1-\tilde{a}}\left(\frac{1}{2}b(\partial W) + c(W)\right) + \frac{\tilde{k}}{(1-\tilde{a})}m(W),$$

which completes the proof. □

We now have all the ingredients to prove Theorem 10.10.

Proof of Theorem 10.10. The implication (i) $\Longrightarrow$ (iii) follows from Lemma 10.13. The implication (iii) $\Longrightarrow$ (iv) follows from the abstract definition of the form domain. Assume (iv), which is $D(Q) = \ell^2(X,n) \cap \ell^2(X,m) = D(q_{\mathrm{Deg}})$. By the closed graph theorem, the canonical embedding of $D(Q)$ into $\ell^2(X,n+m) = \ell^2(X,n) \cap \ell^2(X,m)$ is bounded. This gives the implication (iv) $\Longrightarrow$ (ii). Finally, the implication (ii) $\Longrightarrow$ (i) follows from Lemma 10.14. □

Remark. It is also possible to prove the implication (ii) $\Longrightarrow$ (iii) of Theorem 10.10 directly (Exercise 10.6).

4. Approximate sparseness and first order eigenvalue asymptotics

In this section we study approximately sparse graphs. Analogous to the case of weakly sparse graphs in the previous section, we characterize approximate sparseness via inequalities on the form. We also show discreteness of the spectrum for the Laplacian $L = L^{(D)}$ associated to the form $Q = Q^{(D)}$ when the degree function goes to infinity uniformly.

We recall that approximate sparseness means that for every $\varepsilon > 0$ there exists a $k_\varepsilon \geq 0$ such that the graph is $(\varepsilon, k_\varepsilon)$-weakly sparse. Hence, the constant ε controlling the weight on the edges on the boundary can be made small at the expense of a larger k_ε, which is the constant controlling the measure of the set. When ε tends to zero it is possible that

k_ε does not give an upper bound. In this case, the graph is approximately sparse but not sparse.

For approximately sparse graphs we get analogous inequalities to Theorem 10.10 but now for an arbitrary small $\tilde{a}$ at the expense of a larger $\tilde{k}$. The proof is a rather immediate consequence of the explicit estimates on the mutual dependence of a, k and $\tilde{a}$, $\tilde{k}$ proven in the lemmas of the previous section.

THEOREM 10.15 (Characterization of approximate sparseness). *Let (b, c) be a graph over (X, m). Then, the following statements are equivalent:*

(i) *The graph is approximately sparse.*

(ii) *For every $\tilde{\varepsilon} > 0$, there exists a $\tilde{k}_\varepsilon \geq 0$ such that on $C_c(X)$ we have*
$$(1 - \tilde{\varepsilon})\mathrm{Deg} - \tilde{k}_\varepsilon \leq Q.$$

(iii) *For every $\tilde{\varepsilon} > 0$, there exists a $\tilde{k}_\varepsilon \geq 0$ such that on $C_c(X)$ we have*
$$(1 - \tilde{\varepsilon})\mathrm{Deg} - \tilde{k}_\varepsilon \leq Q \leq (1 + \tilde{\varepsilon})\mathrm{Deg} + \tilde{k}_\varepsilon.$$

PROOF. (i) $\Longrightarrow$ (iii): Assume that the graph is $(\varepsilon, k_\varepsilon)$-weakly sparse with ε arbitrarily small. By Lemma 10.13 we see that the graph then satisfies
$$(1 - \tilde{\varepsilon})\mathrm{Deg} - \tilde{k}_\varepsilon \leq Q \leq (1 + \tilde{\varepsilon})\mathrm{Deg} + \tilde{k}_\varepsilon$$
on $C_c(X)$, where
$$\tilde{\varepsilon} = \sqrt{\frac{2\varepsilon}{1 + \varepsilon}} \quad \text{and} \quad \tilde{k}_\varepsilon = \frac{k_\varepsilon(1 - \varepsilon)}{2\varepsilon}$$
for ε small enough. This gives the statement.

(iii) $\Longrightarrow$ (ii): This is trivial.

(ii) $\Longrightarrow$ (i): Given $\varepsilon > 0$, there exists $0 < \tilde{\varepsilon} < 1$ such that $\varepsilon = \tilde{\varepsilon}/(1 - \tilde{\varepsilon})$. If $\tilde{k}_\varepsilon$ is as given by (ii), then the graph is $(\varepsilon, k_\varepsilon)$-weakly sparse with $k_\varepsilon = \tilde{k}_\varepsilon/(1 - \tilde{\varepsilon})$ by Lemma 10.14. $\square$

REMARK. The proof above actually shows that $\tilde{\varepsilon}$ can be bounded by $\sqrt{2\varepsilon}$ and $\tilde{k}_\varepsilon$ can be bounded by $k_\varepsilon/(2\varepsilon)$.

We now give some spectral consequences. As for the case of weak sparseness, we show that the spectrum of approximately sparse graphs is discrete if and only if the degree function goes to infinity uniformly. Furthermore, in the case of discrete spectrum, we provide asymptotics of the eigenvalues $\lambda_n(L)$ of L which are counted in increasing order with multiplicity. To this end, when $\mathrm{Deg}_\infty = \infty$, we enumerate the vertices (x_n) of X so that $\mathrm{Deg}(x_n) \leq \mathrm{Deg}(x_{n+1})$ for $n \in \mathbb{N}_0$. We let
$$d_n = \mathrm{Deg}(x_n)$$
for $n \in \mathbb{N}_0$ and observe that d_n are the eigenvalues of the multiplication operator by Deg on $\ell^2(X, m)$.

Corollary 10.16 (Eigenvalue asymptotics). *Let (b, c) be an approximately sparse graph over (X, m). Then the spectrum of L is discrete if and only if* $\mathrm{Deg}_\infty = \infty$. *In this case,*

$$\lim_{n\to\infty} \frac{\lambda_n(L)}{d_n} = 1.$$

Proof. By Theorem 10.15 we have, for all $\varphi \in C_c(X)$ with $\|\varphi\| = 1$ and $\varepsilon > 0$,

$$f_-(\mathrm{Deg}(\varphi)) \leq Q(\varphi) \leq f_+(\mathrm{Deg}(\varphi)),$$

with continuous and monotonically increasing functions $f_\pm\colon [0, \infty) \longrightarrow \mathbb{R}$ given by

$$f_\pm(t) = (1 \pm \varepsilon)t \pm k_\varepsilon,$$

where k_ε is chosen according to Theorem 10.15. Thus, the characterization of discrete spectrum via $\mathrm{Deg}_\infty = \infty$ follows immediately from a consequence of the min-max principle, Theorem E.11.

Furthermore, for $\mathrm{Deg}_\infty = \infty$, Theorem E.11 also readily gives

$$(1 - \varepsilon)d_n - k_\varepsilon \leq \lambda_n(L) \leq (1 + \varepsilon)d_n + k_\varepsilon.$$

As $\varepsilon > 0$ can be chosen arbitrarily small, the statement follows. $\square$

5. Sparseness and second order eigenvalue asymptotics

In this section we derive spectral consequence for sparse graphs. First, we show an even stronger non-linear form estimate. This estimate allows us to estimate the spectrum and to prove bounds on the second order of the eigenvalue asymptotics.

We start with an estimate for the form of a sparse graph. As usual, we let $Q = Q^{(D)}$.

Theorem 10.17 (Sparseness implies form inequality). *Let (b, c) be a k-sparse graph over (X, m) for some $k \geq 0$. Then, for all $\varphi \in C_c(X)$, we have*

$$\begin{aligned}\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle - &\left|k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right|^{1/2} \leq Q(\varphi)\\ &\leq \langle \varphi, (\mathrm{Deg} + k)\varphi\rangle + \left|k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right|^{1/2}.\end{aligned}$$

If, additionally, $\mathrm{Deg} \geq k$, *then, for all $\varphi \in C_c(X)$,*

$$\begin{aligned}\langle \varphi, \mathrm{Deg}\varphi\rangle - &\left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2} \leq Q(\varphi)\\ &\leq \langle \varphi, \mathrm{Deg}\varphi\rangle + \left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2}.\end{aligned}$$

Proof. Let $\varphi \in C_c(X)$. We apply the estimate of Lemma 10.12 for k-sparse graphs which are $(0, k)$-weak sparse. This gives

$$\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle \leq Q^{1/2}(\varphi)(2\langle \varphi, \mathrm{Deg}\varphi\rangle - Q(\varphi))^{1/2}.$$

If $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle \geq 0$, then we obtain

$$Q^2(\varphi) - 2\langle \varphi, \mathrm{Deg}\varphi\rangle Q(\varphi) + \langle \varphi, (\mathrm{Deg} - k)\varphi\rangle^2 \leq 0,$$

which, after resolving the quadratic inequality for $Q(\varphi)$, results in

$$\begin{aligned}\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle &- \left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2} \\ &\leq \langle \varphi, \mathrm{Deg}\varphi\rangle - \left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2} \leq Q(\varphi) \\ &\qquad \leq \langle \varphi, \mathrm{Deg}\varphi\rangle + \left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2} \\ &\qquad \leq \langle \varphi, (\mathrm{Deg} + k)\varphi\rangle + \left(k\langle \varphi, \varphi\rangle\langle \varphi, (2\mathrm{Deg} - k)\varphi\rangle\right)^{1/2}.\end{aligned}$$

This implies the inequality for all $\varphi \in C_c(X)$ with $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle \geq 0$. In particular, this shows the second set of inequalities.

For $\langle \varphi, (\mathrm{Deg} - k)\varphi\rangle < 0$, the lower bound follows immediately from $Q(\varphi) \geq 0$ and the upper bound is implied by the fact that $Q \leq 2\mathrm{Deg}$ from Theorem 1.27 in Section 5. □

Remark. Notably, the theorem above is not stated as an equivalence. However, the inequality in case $\mathrm{Deg} \geq k$ in the theorem above implies that the graph is approximately sparse (Exercise 10.4).

We get an immediate spectral estimate for $L = L^{(D)}$ from the second set of inequalities in the theorem above. Let $d = \inf_{x\in X} \mathrm{Deg}(x)$ and assume that $D = \sup_{x\in X} \mathrm{Deg}(x)$ is finite.

Corollary 10.18 (Sparseness and spectral estimates). *Let (b, c) be a k-sparse graph over (X, m) for some $k \geq 0$ such that $d \geq k$ and $D < \infty$. Then,*

$$\sigma(L) \subseteq [d - \sqrt{k(2D - k)}, D + \sqrt{k(2D - k)}].$$

Proof. The conclusion follows from the additional statement in Theorem 10.17 as $\mathrm{Deg} \geq d$ and $\sigma(L) \subseteq [\inf_{\|\varphi\|=1} Q(\varphi), \sup_{\|\varphi\|=1} Q(\varphi)]$, see Theorem E.8, Theorem 1.27 and the fact that $\sup \sigma(L) = \|L\|$. □

Since a d-regular tree with standard weights and counting measure is 2-sparse by Exercise 10.2, we obtain that $\sigma(\Delta) \subseteq [d - 2\sqrt{d-1}, d + 2\sqrt{d-1}]$ from the corollary above. It is well known, in fact, that equality holds and, therefore, the estimate of the corollary is sharp.

Next, we come to the second order asymptotics of eigenvalues $\lambda_n(L)$ of L in the case where the weighted vertex degree grows to infinity. Recall that in this case we denote by d_n the eigenvalues of the multiplication operator by Deg. The theorem below gives a rigorous form of the inequality

$$d_n - \sqrt{2kd_n} \lesssim \lambda_n(L) \lesssim d_n + \sqrt{2kd_n}$$

for large n.

COROLLARY 10.19 (Second order eigenvalue asymptotics). *Let* (b, c) *be a* k*-sparse graph over* (X, m)*. Then the spectrum of* L *is discrete if and only if* $\mathrm{Deg}_\infty = \infty$*. In this case,*

$$\lim_{n\to\infty} \frac{\lambda_n(L)}{d_n} = 1$$

and

$$-\sqrt{2k} \leq \liminf_{n\to\infty} \frac{\lambda_n(L) - d_n}{\sqrt{d_n}} \leq \limsup_{n\to\infty} \frac{\lambda_n(L) - d_n}{\sqrt{d_n}} \leq \sqrt{2k}.$$

PROOF. First of all we notice that if a graph is sparse, then it is weakly sparse. Hence, the characterization of discreteness of the spectrum follows from Corollary 10.11. Moreover, by Theorem 10.17 we have, for $\varphi \in C_c(X)$ with $\|\varphi\| = 1$,

$$g_-(\mathrm{Deg}(\varphi)) \leq Q(\varphi) \leq g_+(\mathrm{Deg}(\varphi)),$$

with continuous functions $g_\pm \colon [0, \infty) \longrightarrow \mathbb{R}$ given by

$$g_\pm(t) = t \pm k \pm |k(2t - k)|^{1/2}.$$

To apply the consequence of the min-max principle, Theorem E.11, we additionally require monotonicity of g_- and g_+, which does not necessarily hold for small t. However, there are clearly monotonically increasing functions $f_- \leq g_-$ and $f_+ \geq g_+$ which agree with g_- and g_+, respectively, for large enough values.

Thus, we obtain from Theorem E.11

$$f_-(d_n) \leq \lambda_n(L) \leq f_+(d_n).$$

Assuming $\mathrm{Deg}_\infty = \infty$, we can enumerate the eigenvalues of Deg by d_n, which tend to infinity. As $f_\pm = g_\pm$ for large arguments, we therefore obtain for large n

$$d_n - k - |k(2d_n - k)|^{1/2} \leq \lambda_n(L) \leq d_n + k + |k(2d_n - k)|^{1/2}.$$

Hence, the statements follow. □

6. Isoperimetric inequalities and Weyl asymptotics

In this section we characterize graphs with a strong isoperimetric inequality in terms of form estimates. In turn, this allows for estimates on the Weyl asymptotics as a corollary. Afterwards we present a sufficient criteria for a strong isoperimetric inequality in terms of a mean curvature.

6.1. Main theorem and corollaries. In this subsection we show that for graphs that satisfy a strong isoperimetric inequality we get a similar characterization as in the case of weakly sparse graphs and approximately sparse graphs. This is immediate from previous considerations. We then discuss spectral asymptotics in this case.

THEOREM 10.20 (Characterization of strong isoperimetric inequality). *Let* (b, c) *be a graph over* (X, m). *Then, the following statements are equivalent:*

(i) *The graph satisfies a strong isoperimetric inequality with isoperimetric constant* $\alpha > 0$.

(ii) *There exists an* $a \in (0, 1)$ *such that on* $C_c(X)$ *we have*
$$(1 - a)\mathrm{Deg} \leq Q.$$

(iii) *There exists an* $a \in (0, 1)$ *such that on* $C_c(X)$ *we have*
$$(1 - a)\mathrm{Deg} \leq Q \leq (1 + a)\mathrm{Deg}.$$

Furthermore, the constant a *in* (ii) *and* (iii) *can be chosen to be* $a = \sqrt{1 - \alpha^2}$.

PROOF. The implication (i) $\Longrightarrow$ (iii) follows from Lemma 10.13 with $a = \sqrt{1 - \alpha^2}$. The implication (iii) $\Longrightarrow$ (ii) is trivial. Finally, the implication (ii) $\Longrightarrow$ (i) follows by combining Lemma 10.14 and Lemma 10.7 with $\alpha = 1 - a$. □

Similarly to the case of sparse graphs, we get an immediate spectral estimate from the theorem above. Let
$$d = \inf_{x \in X} \mathrm{Deg}(x) \quad \text{and} \quad D = \sup_{x \in X} \mathrm{Deg}(x).$$

COROLLARY 10.21 (Fujiwara's theorem). *Let* (b, c) *be a graph over* X *satisfying a strong isoperimetric inequality with isoperimetric constant* $\alpha > 0$. *Then,*
$$\sigma(L) \subseteq [d(1 - \sqrt{1 - \alpha^2}), D(1 + \sqrt{1 - \alpha^2})],$$
where the upper bound of the interval is ∞ *if* $D = \infty$.

PROOF. The statement follows from the theorem above and the spectral inclusion $\sigma(L) \subseteq [\inf_{\|\varphi\|=1} Q(\varphi), \sup_{\|\varphi\|=1} Q(\varphi)]$, see Theorem E.8, Theorem 1.27 and the fact that $\sup \sigma(L) = \|L\|$. □

Again, it is well known that the estimate is sharp for regular trees with standard weights.

REMARK. In the case of bounded degree one can even characterize $\alpha = 0$ by $\inf \sigma(L) = 0$ (Exercise 10.7).

Next, we come to Weyl asymptotics. For a positive self-adjoint operator A on a Hilbert space and $\lambda < \lambda_0^{\mathrm{ess}}(A)$, we let $N_\lambda(A)$ be the number of eigenvalues less than λ counted with multiplicity. The next corollary states that we can characterize purely discrete spectrum for graphs that satisfy a strong isoperimetric inequality. Furthermore, we can determine the first order of the Weyl asymptotics, which is the asymptotics of $N_\lambda(L)$ as λ tends to infinity in the case of discrete spectrum.

COROLLARY 10.22 (Weyl asymptotics). *Let (b, c) be a graph over X that satisfies a strong isoperimetric inequality with isoperimetric constant $\alpha > 0$. Then, the spectrum of L is discrete if and only if $\mathrm{Deg}_\infty = \infty$. In this case,*

$$N_{\lambda/(1+a)}(\mathrm{Deg}) \leq N_\lambda(L) \leq N_{\lambda/(1-a)}(\mathrm{Deg}),$$

where $a = \sqrt{1 - \alpha^2}$.

PROOF. As graphs which satisfy a strong isoperimetric inequality are weakly sparse by Lemma 10.7, the characterization of discreteness of the spectrum follows by the corresponding statement for weakly sparse graphs, Corollary 10.11. Then, Theorem 10.20 above and the consequence of the min-max principle, Theorem E.11, give

$$(1 - a)d_n \leq \lambda_n(L) \leq (1 + a)d_n,$$

where d_n denote the eigenvalues of Deg in increasing order whenever $\mathrm{Deg}_\infty = \infty$. Hence, we have, for all $\lambda \geq 0$,

$$\{n \mid (1 + a)d_n \leq \lambda\} \subseteq \{n \mid \lambda_n(L) \leq \lambda\} \subseteq \{n \mid (1 - a)d_n \leq \lambda\}.$$

Since the cardinalities of these sets coincide with $N_\lambda((1 + a)\mathrm{Deg})$, $N_\lambda(L)$ and $N_\lambda((1 - a)\mathrm{Deg})$, we have

$$N_{\lambda/(1+a)}(\mathrm{Deg}) = N_\lambda((1 + a)\mathrm{Deg}) \leq N_\lambda(L) \leq N_\lambda((1 - a)\mathrm{Deg}) = N_{\lambda/(1-a)}(\mathrm{Deg}).$$

This finishes the proof. □

REMARK. One can also give criteria for the discreteness of the spectrum using a so-called isoperimetric constant at infinity (Exercise 10.8).

6.2. A mean curvature criterion. In this subsection we present a sufficient criterion for a strong isoperimetric inequality. This criterion is given in terms of a mean curvature.

Let (b, c) be a graph over (X, m). For a fixed vertex $o \in X$, let S_r be the sphere of radius r about o with respect to the combinatorial graph distance d. Define the quantities $b_\pm \colon X \longrightarrow [0, \infty)$ by

$$b_\pm(x) = \sum_{y \in S_{r\pm1}} b(x, y)$$

for $x \in S_r$. These quantities are a measure-independent version of the quantities $k_\pm$ from Chapter 9. Furthermore, we define a function $K \colon X \longrightarrow \mathbb{R}$, which can be seen as a mean curvature, by

$$K = \frac{b_- - b_+}{n},$$

where n is the normalizing measure. Here, we suppress the dependence of K on o in the notation.

With the normalizing measure, the curvature K arises by taking the Laplacian of the combinatorial distance function $d(o, \cdot)$ from o, which, by analogy to the Riemannian setting, gives a notion of curvature.

LEMMA 10.23. *Let b be a graph over (X,n) and $o \in X$. The function $d(o,\cdot)$ is in $\mathcal{F}$ and*

$$\mathcal{L}d(o,\cdot) = K.$$

PROOF. The function $d(o,\cdot)$ is bounded on the combinatorial neighborhood of every vertex. Hence, $d(o,\cdot) \in \mathcal{F}$. We calculate for $x \in S_r$

$$\begin{aligned}
&\mathcal{L}d(o,x)n(x) \\
&= \sum_{y\in S_{r-1}} b(x,y)(d(o,x)-d(o,y)) + \sum_{y\in S_{r+1}} b(x,y)(d(o,x)-d(o,y)) \\
&= b_-(x) - b_+(x).
\end{aligned}$$

This finishes the proof. $\square$

We now relate lower bounds on the mean curvature to an isoperimetric inequality for a graph over a space with the normalizing measure.

THEOREM 10.24 (Mean curvature and isoperimetric inequality). *Let (b,c) be graph over (X,n). If there exists a $C>0$ such that $-K + c/n \geq C$, then the graph satisfies a strong isoperimetric inequality with isoperimetric constant at least C.*

PROOF. Denote by $\mathcal{L}_0$ the formal Laplacian for the graph $(b,0)$ over (X,n). Given a finite set $W \subseteq X$, we estimate, using the assumption $-K + c/n \geq C$ and employing Lemma 10.23,

$$\begin{aligned}
Cn(W) - c(W) &\leq -\sum_{x\in W} K(x)n(x) \\
&= -\sum_{x\in W} \mathcal{L}_0 d(o,x)n(x) \\
&= -\sum_{x\in X} 1_W(x)\mathcal{L}_0 d(o,x)n(x).
\end{aligned}$$

Since W is finite, $1_W \in C_c(X)$ and $d(o,\cdot) \in \mathcal{F}$, we proceed using Green's formula, Proposition 1.5,

$$\begin{aligned}
\ldots &= -\frac{1}{2}\sum_{x,y\in X} b(x,y)(d(o,x)-d(o,y))(1_W(x)-1_W(y)) \\
&\leq \frac{1}{2}\sum_{x,y\in X} b(x,y)|d(o,x)-d(o,y)||1_W(x)-1_W(y)| \\
&\leq \frac{1}{2}\sum_{x,y\in X} b(x,y)|1_W(x)-1_W(y)| \\
&= \frac{1}{2}b(\partial W).
\end{aligned}$$

This yields the statement. $\square$

Exercises

Excavation exercises.

Exercise 10.1 (Multiplication operators and spectrum). Let (b, c) be a graph over (X, m). Show that the multiplication operator defined by multiplying by Deg on $\ell^2(X, m)$ has purely discrete spectrum if and only if $\mathrm{Deg}_\infty = \infty$, where

$$\mathrm{Deg}_\infty = \sup_{K \subseteq X \text{ finite}} \inf_{x \in X \setminus K} \mathrm{Deg}(x),$$

by determining all eigenvalues and eigenfunctions of the multiplication operator.

Example exercises.

Exercise 10.2 (Trees and planar graphs are sparse). Consider graphs with standard weights and counting measure.

(a) Show that trees are 2-sparse.

(b) Show that planar graphs are 6-sparse.
(Hint: Use Euler's polyhedron formula.)

(c*) Show that there exist graphs that are weakly sparse but not approximately sparse and graphs that are approximately sparse but not sparse.
(Hint: Let (β_n) be a sequence of natural numbers. Denote by b_0 a spherically symmetric tree (for the definition, see Chapter 9) with standard weights, root called o and vertex degree β_n in the n-th sphere. Denote by S_n the vertices of the distance sphere of radius n with respect to the combinatorial graph distance (and observe that $\#S_n = \beta_0 \prod_{j=1}^{n-1}(\beta_j - 1)$). Now choose a sequence of natural numbers (γ_n) such that there exists a γ_n-regular graph over S_n with standard weights, i.e., every vertex in S_n has exactly γ_n neighbors, $n \in \mathbb{N}$. We denote these graphs by b_n. (Show that such graphs exist for every $n \in \mathbb{N}$ whenever $\gamma_n \beta_0 \prod_{j=1}^{n-1}(\beta_j - 1)$ is even and $\gamma_n < \beta_0 \prod_{j=1}^{n-1}(\beta_j - 1)$, $n \geq 1$.) Let

$$b = \sum_{n=0}^{\infty} b_n$$

and

$$a = \limsup_{n \to \infty} \frac{\gamma_n}{\beta_n},$$

which may take the value ∞. If $a' > a$, then there exists a $k \geq 0$ such that the graph b is (a', k)-weakly sparse. On the other hand, if the graph b is (a', k)-weakly sparse for some $k \geq 0$, then $a' \geq a$. Hence, b is approximately sparse if and only if $a = 0$. In this case, the graph b is sparse if and only if $\limsup_{n \to \infty} \gamma_n < \infty$.)

Exercise 10.3 (Sparseness and strong isoperimetric inequality).

(a) Present an example of a graph that is sparse and satisfies a strong isoperimetric inequality.
(b) Present an example of a graph that is sparse but does not satisfy a strong isoperimetric inequality.
(c) Present an example of a graph that is not sparse but satisfies a strong isoperimetric inequality.
(Hint: Use Theorem 10.24.)

Exercise 10.4 (Approximate sparseness and Q). Let (b, c) be a graph over (X, m). If

$$Q^2(\varphi) - 2\langle \varphi, \mathrm{Deg}\varphi \rangle Q(\varphi) + \langle \varphi, (\mathrm{Deg} - k)\varphi \rangle^2 \leq 0$$

for all $\varphi \in C_c(X)$, show that (b, c) is approximately sparse.

Extension exercises.

Exercise 10.5 (Weak sparseness and isoperimetric inequality). Show that an (a, k)-weakly sparse graph (b, c) over (X, m) satisfies an isoperimetric inequality with isoperimetric constant

$$\alpha = \frac{d_0 - k}{d_0(1 + a)}$$

with $d_0 = \inf_{x \in X}(n/m)(x)$ whenever $d_0 > k$.

Exercise 10.6 (Form inequalities). Show the implication (ii) $\Longrightarrow$ (iii) of Theorem 10.10 directly.

Exercise 10.7 ($\alpha = 0$). Let (b, c) be a graph over (X, m) with associated Laplacian $L = L^{(D)}$. Let $D = \sup_{x \in X} \mathrm{Deg}(x)$. Show that if $D < \infty$, then

$$\inf_{W \subseteq X, W \text{finite}} \frac{\frac{1}{2}b(\partial W) + c(W)}{n(W)} = 0$$

if and only if $\lambda_0(L) = 0$.

Exercise 10.8 (Isoperimetric constant and spectrum). For a connected graph (b, c) over X consider the isoperimetric constant defined via

$$\alpha = \inf_{W \subseteq X} \frac{\frac{1}{2}b(\partial W) + c(W)}{n(W)}$$

and the isoperimetric constant at infinity defined via

$$\alpha_\infty = \sup_{K \subseteq X \text{ finite}} \inf_{W \subseteq X \setminus K} \frac{\frac{1}{2}b(\partial W) + c(W)}{n(W)}.$$

Let m be a uniformly positive measure, i.e., $\inf_{x\in X} m(x) > 0$ and

$$\mathrm{Deg}_\infty = \sup_{K\subseteq X \text{ finite}} \inf_{x\in X\setminus K} \mathrm{Deg}(x).$$

(a) Assume that $\alpha_\infty > 0$. Show that the operator L has purely discrete spectrum if and only if $\mathrm{Deg}_\infty = \infty$.
(Hint: Show that for all $\varepsilon > 0$ there exists a $C_\varepsilon \geq 0$ such that the inequalities

$$\begin{aligned}(1-\varepsilon)(1-\sqrt{1-\alpha_\infty})\mathrm{Deg} - C_\varepsilon &\leq Q \\ &\leq (1+\varepsilon)(1-\sqrt{1-\alpha_\infty})\mathrm{Deg} + C_\varepsilon\end{aligned}$$

hold.)

(b) Assume that $D_\infty = \infty$. Show that $\alpha > 0$ if and only if $\alpha_\infty > 0$.
(Hint: Consider the operator L_n with respect to the graph (b, c) over (X, n) and show that $\lambda_0^{\mathrm{ess}}(L_n) > 0$ if and only if $\alpha_\infty > 0$ and $\lambda_0(L_n) = 0$ if and only if $\alpha = 0$.)

Notes

Historically, sparseness and isoperimetric estimates are well-known concepts that have appeared in rather disjoint contexts. The unified treatment of the two topics and their connection to spectral estimates that is presented here goes back to material in [**BGK15**] for graphs with standard weights.

In numerics, sparseness of a matrix classically means that it has relatively few non-zero entries. This notion found its way into graph theory in works such as [**EGS76, Lor79**], where it translates into having relatively few edges compared to the number of vertices. We note that there exists a multitude of different definitions for sparseness, too many to cover completely here, so we only cite [**Bre07, LS08, AABL12**] to give an idea of the range of notions. The first connection to spectral graph theory is due to Mohar [**Moh13, Moh15**], where the number of large eigenvalues of the adjacency matrix of a finite graph is related to the sparseness of the graph.

The first explicit relation between isoperimetric inequalities and an estimate on the bottom of the spectrum of the Laplacian on manifolds is found in the work of Cheeger [**Che70**]. However, the general idea of relating isoperimetric estimates and Sobolev inequalities can be traced back to Federer/Fleming [**FF60**] and Maz′ja [**Maz60**]. These results are proven in the continuum setting. The first results for Laplacians on graphs were independently shown by Dodziuk [**Dod84**] and Alon/Milman [**AM85**]. These results were preceded by works of Fiedler, which include similar ideas for the adjacency matrix of a finite graph [**Fie73**]. The specific setting of [**Dod84, AM85**] are graphs with standard weights and counting measure. Later, a corresponding result was proven for the normalizing measure by Dodziuk/Kendall [**DK86**]. Since then various versions have been proven, see e.g. [**Ger88, Moh88, Moh91, Fuj96b, CGY00, Dod06**]. For finite graphs such results are found in the textbooks [**Big93, Chu97**].

The first use of isoperimetric inequalities to show discreteness of the spectrum can be traced back to the work of Donnelly/Li [**DL79**] in the case of manifolds. For graphs with standard weights and normalizing measure, a result of Fujiwara [**Fuj96b**] gives conditions under which the essential spectrum consists of one point. For the counting measure, discreteness of the spectrum was shown in [**Kel10, Woj08, Woj09**]. This was generalized for weighted graphs in [**KL10**] and the estimates proven there were used by Golénia [**Gol14**] to derive eigenvalue asymptotics in terms of the degree.

The results as they are presented in this chapter are an extension of [**BGK15**] from the setting of standard weights with counting measure to general graphs. Furthermore, the second term in the eigenvalue asymptotics for sparse graphs in Theorem 10.17 is an extension of [**BGK21**]. The mean curvature criterion in Subsection 6.2 appeared in [**DK88**] for graphs with standard weights and is used in [**Woj09**] to show discreteness of the spectrum.

Finally, let us mention that isoperimetric inequalities play an essential role in applications such as parallel computing. Here one is interested in dividing a large data set into subsets such that the connection between these subsets is minimal. While finding these sets is computationally hard, spectral computations are easier in numerics. Thus, the applications perspective is somewhat reverse to ours, see[**BH09, ST96**] and references therein for a first look into these considerations.

Part 3

Geometry and Intrinsic Metrics

Synopsis

The concept of a metric is a most fundamental one in geometry. In this part of the book we introduce the notion of intrinsic metrics for a graph in Chapter 11. We then use intrinsic metrics to give a series of Caccioppoli inequalities which are useful for studying harmonic functions, form uniqueness, recurrence and the spectrum in Chapter 12. Next, we use intrinsic metrics to study the bottom of the spectrum in Chapter 13. More specifically, we give both a lower bound via a Cheeger inequality involving an isoperimetric constant and an upper bound via a Brooks estimate involving volume growth. Finally, we use intrinsic metrics to establish a uniqueness class result for the heat equation and to give a volume growth criterion for stochastic completeness in Chapter 14.

CHAMBER 11

Intrinsic Metrics: Definition and Basic Facts

Off on a natural charge, bon-voyage.
ODB.

In this chapter we introduce the notion of an intrinsic metric. Section 1 is devoted to definitions and motivations. An important class of examples are so-called path metrics, which we discuss in Section 2. In this section we prove a Hopf–Rinow theorem, which characterizes metric completeness. This theory is interesting for its own sake but can be skipped by the reader who is only interested in specific applications. Section 3 discusses relevant examples and relates intrinsic metrics to other metrics on graphs that appear in the literature. Finally, in Section 4 we discuss several assumptions on the metric and introduce some cutoff functions which we will use in the forthcoming chapters.

1. Definition and motivation

In this section we introduce intrinsic metrics and motivate the definition via several facts from the setting of Riemannian manifolds. We also give a characterization of intrinsic metrics in terms of a Lipschitz property.

For the theory developed in this chapter we do not need distances to be non-degenerate. That is, we allow for different vertices to be distance zero apart. This yields a so-called pseudo metric. Specifically, a *pseudo metric* is a symmetric map $\varrho\colon X \times X \longrightarrow [0, \infty)$ with zero diagonal that satisfies the triangle inequality.

DEFINITION 11.1 (Intrinsic metric). A pseudo metric ϱ is called an *intrinsic metric* for a graph b over (X, m) if

$$\sum_{y \in X} b(x, y)\varrho^2(x, y) \leq m(x)$$

for all $x \in X$. We call a pseudo metric ϱ an *intrinsic metric* for a graph (b, c) over (X, m) if ϱ is intrinsic for b over (X, m).

We stress that we speak of intrinsic metrics although we only assume that the corresponding objects are pseudo metrics. This is common in the literature and we follow this convention for the sake of brevity.

Let us discuss the motivation of this definition from the perspective of Riemannian geometry. The reader familiar with Riemannian manifolds can fill in the details. Otherwise, the statements below can be taken as facts to motivate our approach. Let M be a connected Riemannian manifold and let

$$A_1(M) = \{f \in L^1_{\mathrm{loc}}(M) \mid f \text{ is weakly differentiable and } |\nabla_M f|^2 \leq 1\},$$

where ∇_M is the Riemannian gradient. Then, the Riemannian distance d_M can be recovered via

$$\begin{aligned} d_M(x,y) &= \sup\{f(x) - f(y) \mid f \in A_1(M)\} \\ &= \sup\{\delta(x,y) \mid \delta \text{ is a metric with } \delta(o,\cdot) \in A_1(M) \text{ for all } o \in M\} \end{aligned}$$

for $x, y \in M$.

From these equalities, one can deduce $d_M(o,\cdot) \in A_1(M)$, which is equivalent to

$$\mathrm{Lip}_1(M) = A_1(M),$$

where $\mathrm{Lip}_1(M)$ is the set of Lipschitz continuous functions with respect to d_M with Lipschitz constant 1.

To discuss how our definition of intrinsic metrics is related to these facts for manifolds, we have to introduce the concept of the norm of a gradient first. Let b be a graph over (X, m) and let $f \in C(X)$. Then, we define the norm of the gradient $|\nabla f| = |\nabla f|_{b,m}$ of f as

$$|\nabla f|(x) = \left(\frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))^2 \right)^{1/2}$$

for $x \in X$.

In order to motivate this definition, we observe that the directional difference of a function $f \in C(X)$ gives rise to a function $\nabla f \colon X \times X \longrightarrow \mathbb{R}$ via

$$\nabla f(x,y) = f(x) - f(y).$$

We then take the scalar product of ∇f times ∇f over the fiber $\{x\} \times X$ with respect to b considered as a measure, i.e.,

$$\langle \nabla f, \nabla f \rangle_b(x) = \sum_{y \in X} b(x,y)(f(x) - f(y))^2,$$

which takes values in $[0, \infty]$. This gives rise to a function and, by extending to subsets, a measure $\langle \nabla f, \nabla f \rangle_b$ on X.

Now, we consider the Radon–Nikodym derivative of the measure $\langle \nabla f, \nabla f \rangle_b$ with respect to the measure m. Taking the square root of this Radon–Nikodym derivative yields the notion of the norm of the gradient, as introduced above. That is, $|\nabla f| = |\nabla f|_{b,m}$ is given by

$$|\nabla f|(x) = \left(\frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))^2 \right)^{1/2}$$

for $x \in X$. Then, as $c = 0$, we have, for all $f \in C(X)$,

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x \in X} \langle \nabla f, \nabla f \rangle_b(x) = \frac{1}{2} \sum_{x \in X} |\nabla f|^2(x) m(x).$$

With these notions we define

$$A_1(X) = \{f \in C(X) \mid |\nabla f|^2 \leq 1\}.$$

Given $A_1(X)$, we can define a metric σ on X in analogy to the manifold case as

$$\sigma(x, y) = \sup\{f(x) - f(y) \mid f \in A_1(X)\}.$$

It is immediate that

$$A_1(X) \subseteq \mathrm{Lip}_{1,\sigma}(X),$$

where $\mathrm{Lip}_{C,\delta}(X)$ is the set of Lipschitz continuous functions on X with respect to a pseudo metric δ having Lipschitz constant C. We will refer to such functions as C-Lipschitz for short.

However, in general, we do not have an equality even for finite graphs. In fact, in general, the space $A_1(X)$ will not be the space of Lipschitz functions for any metric. This is due to the fact that, while the space of Lipschitz functions is closed under taking suprema, the space $A_1(X)$ may not be. This is illustrated in the following example.

Example 11.2 ($A_1(X) \neq \mathrm{Lip}_{1,\delta}(X)$). Let $X = \{0, 1, 2\}$ and let b be symmetrically given by $b(0, 1) = b(1, 2) = 1$, $b(0, 2) = 0$ and $m = 1$. Then, the functions f and g with $f(0) = g(2) = 1$ and 0 otherwise are both in $A_1(X)$. However, the function $f \vee g$ is not in $A_1(X)$ since

$$|\nabla(f \vee g)|^2(1) = (f(1) - f(0))^2 + (g(1) - g(2))^2 = 2 > 1 = m(1).$$

On the other hand for any metric δ, it is always the case that $f \vee g \in \mathrm{Lip}_{1,\delta}(X)$ if $f, g \in \mathrm{Lip}_{1,\delta}(X)$. So, we conclude that $A_1(X)$ cannot be equal to $\mathrm{Lip}_{1,\delta}$ for any metric δ.

The example shows that, in general, the function $\sigma(o, \cdot)$ is not in $A_1(M)$. Thus, σ is not an intrinsic metric in this case, as we will see below. Indeed, we see that our definition of intrinsic metrics coincides with the fact that the 1-Lipschitz functions are included in $A_1(X)$ or, equivalently, that the gradient of the pseudo metric with one variable fixed has norm less than one.

Lemma 11.3 (Characterization intrinsic metrics). *Let b be a graph over (X, m) and let ϱ be a metric. Then, the following statements are equivalent:*

(i) *ϱ is an intrinsic metric.*
(ii) $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$.
(iii) $|\nabla \varrho(o, \cdot)|^2 \leq 1$, *i.e.*, $\varrho(o, \cdot) \in A_1(X)$ *for all $o \in X$.*

In particular, if $\eta \in C(X)$ is C-Lipschitz with respect to an intrinsic metric ϱ and $C \geq 0$, then

$$|\nabla \eta|^2 \leq C^2.$$

Proof. (i) $\Longrightarrow$ (ii): Let $f \in \mathrm{Lip}_{1,\varrho}(X)$, where ϱ is an intrinsic metric. Then,

$$\frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))^2 \leq \frac{1}{m(x)} \sum_{y \in X} b(x,y)\varrho^2(x,y) \leq 1$$

so that $f \in A_1(X)$.

(ii) $\Longrightarrow$ (iii): Note that $\varrho(o,\cdot) \in \mathrm{Lip}_{1,\varrho}(X)$ for every $o \in X$ since $|\varrho(o,x) - \varrho(o,y)| \leq \varrho(x,y)$. As $\mathrm{Lip}_{1,\varrho}(X) \subseteq A_1(X)$, we conclude $\varrho(o,\cdot) \in A_1(X)$.

(iii) $\Longrightarrow$ (i): Note that since $|\nabla \varrho(o,\cdot)|^2 \leq 1$ for all $o \in X$, we get

$$\begin{aligned} |\nabla \varrho(o,o)|^2 &= \frac{1}{m(o)} \sum_{y \in X} b(o,y)(\varrho(o,o) - \varrho(o,y))^2 \\ &= \frac{1}{m(o)} \sum_{y \in X} b(o,y)\varrho^2(o,y) \leq 1. \end{aligned}$$

Hence,

$$\sum_{y \in X} b(o,y)\varrho^2(o,y) \leq m(o)$$

for all $o \in X$, so that ϱ is intrinsic.

The "in particular" statement follows immediately as C-Lipschitz means that

$$|\eta(x) - \eta(y)| \leq C\varrho(x,y)$$

for $x, y \in X$. □

Remark. The definition of $|\nabla f|$ also appears in the setting of the so-called Bakry–Émery calculus. The Bakry–Émery calculus starts with a Laplace operator, which in our case is $\mathcal{L}$, in order to define the norm of a gradient square via

$$\Gamma(f) = -\frac{1}{2}(\mathcal{L}f^2 - 2f\mathcal{L}f).$$

A direct calculation shows

$$\Gamma(f) = \frac{1}{2}|\nabla f|^2.$$

2. Path metrics and a Hopf–Rinow theorem

In this section we define path metrics on discrete spaces. The main goal is a Hopf–Rinow theorem, which states that metric completeness, geodesic completeness and finiteness of balls are all equivalent for path metrics on locally finite graphs. Along the way, we show that the assumptions of discreteness and metric completeness already yield that any two vertices can be connected by a geodesic.

Let X be a discrete space. We call a symmetric map $w\colon X \times X \longrightarrow [0, \infty]$ with

$$w(x, y) = 0 \qquad \text{if and only if} \qquad x = y$$

a *weight* over X. Weights are a slightly different notion than graphs as we allow for the value infinity, ask for a non-vanishing off-diagonal and do not assume a summability condition about the vertices. Nevertheless, we use the same terminology as in the case of graphs to speak about vertices, neighbors, paths and connectedness.

More specifically, we call the elements of X vertices and say that x and y are *neighbors* if $w(x, y) < \infty$, in which case we write $x \sim y$. Thus, in contrast to graphs, the lack of a connection between vertices is indicated by w being infinite. Furthermore, we call a sequence (x_n) of pairwise different elements of X a *path* if all subsequent elements are neighbors. We say that w is *connected* if every two elements of X can be connected by a path. We note that every vertex can be connected to itself by a path which consists of that vertex alone.

We let $\Pi_{x,y}$ denote the set of all paths from x to y and call the sum of the weights along a path the *length* of a path. That is, if $(x_k)_{k=0}^n \in \Pi_{x,y}$, then the length $l_w((x_k))$ of (x_k) is

$$l_w((x_k)) = \sum_{k=0}^{n-1} w(x_k, x_{k+1}).$$

Definition 11.4 (Path metric). Let w be a weight over X. We define the *path (pseudo) metric* δ_w with respect to w by

$$\delta_w(x, y) = \inf_{(x_k) \in \Pi_{x,y}} l_w((x_k)),$$

where we let $\inf \emptyset = \infty$. Moreover, if a metric δ can be realized by a path metric δ_w with respect to a weight w, then we say that δ is *induced* by w. We call any such δ a *path metric* and call (X, δ) a *path metric space.*

Remark. A path metric also often gives a weight on the graph. Furthermore, the path metric induced by this weight is equal to the original path metric (Exercise 11.4).

Similar to the definition of intrinsic metrics, we only assume that δ is a pseudo metric; however, we suppress the term "pseudo" unless it is relevant for our discussion.

An important example is the combinatorial graph distance. Here, we start with a graph b over X.

EXAMPLE 11.5 (Combinatorial graph distance). Let b be a graph over X. Let $w_b(x,y) = 1$ if $b(x,y) > 0$ and ∞ otherwise. Then, the path metric with respect to w_b is the combinatorial graph distance $\delta_{w_b} = d$.

Obviously, the combinatorial graph distance is actually a metric and not only a pseudo metric. The next example shows that a path metric is not necessarily non-degenerate and that the topology induced by a path metric is not necessarily Hausdorff.

EXAMPLE 11.6 (A non-Hausdorff space). Let $X = \mathbb{N}_0 \cup \{\infty\}$ and define w by $w(0,n) = w(n,0) = w(n,\infty) = w(\infty,n) = 1/n$ for $n \in \mathbb{N}$ and ∞ otherwise. Let δ_w be the path pseudo metric induced by w. Then, $\delta_w(0,\infty) = 2\inf_{n\in N}(1/n) = 0$.

Analogous to graphs, we now define the notion of local finiteness.

DEFINITION 11.7 (Locally finite weight). We call a weight w over X *locally finite* if

$$\#\{y \in X \mid w(x,y) < \infty\} < \infty$$

for all $x \in X$. If the weight that induces a path metric δ is locally finite, then we call (X,δ) a *locally finite path metric space.*

It turns out that path pseudo metrics for locally finite weights are, in fact, metrics. Indeed, the statement is even stronger in that such a metric induces the discrete topology on X.

LEMMA 11.8 (Locally finite implies discrete). *Let (X,δ) be a locally finite path metric space. Then, (X,δ) is a discrete metric space. In particular, δ is a metric and compact sets are finite.*

PROOF. Let w be the weight that induces δ and let $x,y \in X$ with $x \neq y$. Then, any path from x to y must pass through one of the finitely many neighbors $y_1,\ldots,y_N$ of y. Let $\min_{i=1,\ldots,N} w(y,y_i) = w_0 > 0$. Then,

$$\delta_w(x,y) = \inf_{(x_k)\in\Pi_{x,y}} l_w((x_k)) \geq w_0.$$

This implies the statements. □

Next, we turn to the study of paths and geodesics for a connected weight w over X. We first introduce the set Π_o of one-sided paths starting at a vertex $o \in X$, that is,

$$\Pi_o = \{(x_n) \mid (x_n) \text{ is a path with } x_0 = o\}.$$

We equip Π_o with the metric

$$\gamma_o((x_n),(y_n)) = \min\{1/(r+1) \mid x_0 = y_0, x_1 = y_1, \ldots, x_r = y_r\}$$

if $(x_n) \neq (y_n)$ and 0 otherwise.

Observe that, for a finite path $(x_0, \ldots, x_k)$, the equalities for terms of index strictly larger than k are not defined and, therefore, not satisfied. Hence, the distance between $(x_0, \ldots, x_k)$ and any different path is at least $1/(k+1)$. It can be directly seen that γ_o is even an ultra metric, i.e.,

$$\gamma_o((x_n),(y_n)) \leq \gamma_o((x_n),(z_n)) \vee \gamma_o((z_n),(y_n))$$

for all $(x_n), (y_n), (z_n) \in \Pi_o$.

The following observation is crucial for the considerations below and, in particular, for the proof of the Hopf–Rinow theorem.

Proposition 11.9 (Compactness of Π_o). *Let (X, δ) be a locally finite path metric space. Then, for all $o \in X$, the metric space (Π_o, γ_o) of all paths starting at o is compact.*

Proof. We will show that (Π_o, γ_o) is totally bounded and complete, which is equivalent to (Π_o, γ_o) being compact.

We first show total boundedness. For $n \in \mathbb{N}$, let Π_o^n be the paths in Π_o with n vertices. Furthermore, for $p \in \Pi_o$, denote by Π_p the set of all paths that start with the path p. Let $B_r(p)$ be the ball of radius r about $p \in \Pi_o$ with respect to γ_o. By the definition of the metric γ_o, we see that, for all $p \in \Pi_o^n$,

$$\Pi_p \subseteq B_{1/n}(p).$$

Therefore, for $\varepsilon > 0$ and $n \geq 1/\varepsilon$, we have

$$\Pi_o = \bigcup_{k=0}^{n} \bigcup_{p \in \Pi_o^k} B_\varepsilon(p)$$

since every path $q \in \Pi_o$ which has n or fewer vertices is obviously included in $B_\varepsilon(q)$ and every path $q \in \Pi_o$ with more than n vertices is $1/n$ close to a path with n vertices. Now, by local finiteness, the sets Π_o^k are finite for all $k \in \mathbb{N}$. This shows that (Π_o, γ_o) is totally bounded.

To show completeness, consider a Cauchy sequence $x^{(k)} = (x_n^{(k)})$ of paths in Π_o. By the definition of γ_o, elements in the sequence for large k have to coincide in their first vertices. Thus, we can define a path $(x_n) \in \Pi_o$ such that $x_n = x_n^{(k)}$ for all k large enough. This finishes the proof. □

Next, we turn to a study of geodesics for a weight w over X.

Definition 11.10 (Geodesic). Let w be a weight over X. We call a path (x_n) a *geodesic* with respect to w if $\delta_w(x_0, x_k) = w(x_0, x_1) + \ldots + w(x_{k-1}, w_k)$ for all k. If, for every two vertices x and y, there is a geodesic connecting x and y, then we call a path metric space (X, δ) a *geodesic space* with respect to w.

Note that, if (x_n) is a geodesic, then $\delta_w(x_j, x_k) = w(x_j, x_{j+1}) + \ldots + w(x_{k-1}, x_k)$ for all j and k.

Example 11.6 shows that geodesics may not exist if the space is not locally finite. The example below shows that even for locally finite spaces geodesics need not exist.

EXAMPLE 11.11 (A non-geodesic space). Let $X = \mathbb{N}_0$ and let w be such that

$$
\begin{aligned}
w(2(n-1), 2n) = w(2n, 2(n-1)) &= 2^{-n} \\
w(2n-1, 2n+1) = w(2n+1, 2n-1) &= 2^{-n} \\
w(2n, 2n+1) = w(2n+1, 2n) &= 2^{2-n}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and ∞ otherwise. This space can be visualized as an infinite ladder where there are two infinite paths along $2\mathbb{N}_0$ and $2\mathbb{N}_0 + 1$ that are connected at each level by a single edge of decreasing length. Obviously, the space is locally finite.

The vertices 0 and 1 are connected by infinitely many paths

$$p_k = (0, 2, \ldots, 2k, 2k+1, 2k-1, \ldots, 1)$$

for $k \in \mathbb{N}$ which have length

$$
\begin{aligned}
&\sum_{j=1}^{k} w(2(j-1), 2j) + w(2k, 2k+1) + \sum_{j=1}^{k} w(2j-1, 2j+1) \\
&\quad = 2(1 - 2^{-k}) + 2^{2-k} \\
&\quad = 2 - 2^{-k+1} + 2^{2-k}.
\end{aligned}
$$

Hence, $\delta_w(0, 1) = 2$ but none of the paths p_k realize this length.

One aspect of the example above is that it has infinite paths of finite length. These paths form Cauchy sequences that do not converge. We recall that a metric space is called *complete* if every Cauchy sequence converges. Below we show that discreteness of the space along with completeness implies that infinite paths must have infinite length.

LEMMA 11.12 (Discrete and complete implies infinite length). *Let (X, δ) be a path metric space with weight w. If $\delta = \delta_w$ is discrete and complete, then for any infinite path (x_n) we have*

$$\sum_{k=0}^{\infty} w(x_k, x_{k+1}) = \infty.$$

PROOF. If $\sum_{k=0}^{\infty} w(x_k, x_{k+1}) < \infty$, then (x_n) is a Cauchy sequence which has a limit in X by the assumption of completeness. This, however, is a contradiction to the discreteness of the space. $\square$

With the help of this lemma and the proposition above we can show the following criterion for a path metric space to be a geodesic space.

Proposition 11.13 (Discrete and complete imply geodesic). *Let (X, δ) be a connected path metric space with weight w. If $\delta = \delta_w$ is discrete and complete, then (X, δ) is a geodesic space with respect to w.*

Proof. Let w be the weight inducing δ and let $o, o' \in X$. By connectedness, there is a path $o = x_0 \sim \ldots \sim x_n = o'$. Let r be the length of the path, that is,

$$r = \sum_{k=0}^{n-1} w(x_k, x_{k+1}).$$

Consider the subset $\Pi^r_{o,o'} \subseteq \Pi_o$ of all paths starting at o, ending at o' and having length less than or equal to r. If the set $\Pi^r_{o,o'}$ is infinite, then $\Pi^r_{o,o'}$ contains a sequence of pairwise distinct paths $(x_n^{(k)})$ that converges to a path $(x_n) \in \Pi_o$ with respect to γ_o due to the compactness of (Π_o, γ_o) shown in Proposition 11.9. By convergence, we deduce that for an arbitrary $\varepsilon > 0$ there exists a $k_\varepsilon \in \mathbb{N}$ such that $x_j^{(k_\varepsilon)} = x_j$ for all $j \leq 1/\varepsilon$. Hence, as the $(x_n^{(k)})$ are pairwise distinct, (x_n) is an infinite sequence which satisfies, for all n and $\varepsilon = 1/n$,

$$\sum_{j=0}^{n-1} w(x_j, x_{j+1}) = \sum_{j=0}^{n-1} w(x_j^{(k_\varepsilon)}, x_{j+1}^{(k_\varepsilon)}) \leq r.$$

Hence, (x_n) is an infinite sequence of finite length. This, however, is impossible in a discrete and complete path metric space by Lemma 11.12. Thus, the set $\Pi^r_{o,o'}$ is finite and, therefore, contains a path from o to o' of minimal length. Hence, there exists a geodesic from o to o'. $\square$

We need one final ingredient to present a Hopf–Rinow theorem. This is the notion of geodesic completeness.

Definition 11.14 (Geodesic completeness). A path metric space (X, δ) with weight w is called *geodesically complete* if every infinite geodesic (x_n) with respect to w has infinite length, i.e., $\sum_{n=0}^{\infty} w(x_n, x_{n+1}) = \infty$.

The following proposition gives a criterion for a path metric space to be geodesic in terms of local finiteness and geodesic completeness.

Proposition 11.15 (Locally finite and geodesically complete imply geodesic). *Let (X, δ) be a connected path metric space with weight w. If $\delta = \delta_w$ is locally finite and geodesically complete, then (X, δ) is a geodesic space with respect to w.*

Proof. Let $x, y \in X$ such that $x \neq y$. Our aim is to find a geodesic from x to y. To this end, let (p_n) be a sequence of paths in $\Pi_{x,y}$ such that

$$\lim_{n\to\infty} l_w(p_n) = \delta(x,y).$$

Since $(\Pi_{x,y}, \gamma_x)$ is compact due to local finiteness by Proposition 11.9, the sequence (p_n) has a convergent subsequence. Without loss of generality we assume that (p_n) converges to a path p. We distinguish two cases.

Case 1. The path p is finite. Since all finite paths are discrete points in Π_x, so $p = p_n$ for n large enough. Thus, $p \in \Pi_{x,y}$ and p is a geodesic.

Case 2. The path p is infinite. We show that $p = (x_k)$ is a geodesic of finite length in this case, which contradicts the assumption of geodesic completeness. We fix $k \in \mathbb{N}$. Since $l_w(p_n) \to \delta(x,y)$ as $n \to \infty$ there exists, for all $\varepsilon > 0$, an $N \in \mathbb{N}$ such that for all $n \geq N$ the paths $p_n \in \Pi_{x,y}$ start with $(x_0, \ldots, x_k)$ and $l_w(p_n) \leq \delta(x,y) + \varepsilon$. We estimate

$$\begin{aligned}
\delta(x,y) &\leq \delta(x_0, x_k) + \delta(x_k, y)\\
&\leq l_w((x_0, \ldots, x_k)) + \delta(x_k, y)\\
&\leq l_w(p_n)\\
&\leq \delta(x,y) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we see that all inequalities turn into equalities and, therefore,

$$\delta(x_0, x_k) + \delta(x_k, y) = l_w((x_0, \ldots, x_k)) + \delta(x_k, y),$$

which implies

$$\delta(x_0, x_k) = l_w((x_0, \ldots, x_k)).$$

Hence, p is a geodesic which has finite length. This contradicts the assumption of geodesic completeness. $\square$

We are now in position to state and prove a Hopf–Rinow theorem.

THEOREM 11.16 (Hopf–Rinow theorem). *Let (X, δ) be a locally finite connected path metric space. Then, the following statements are equivalent:*

(i) *The distance balls in (X, δ) are compact (i.e., finite).*
(ii) *(X, δ) is complete as a metric space.*
(iii) *(X, δ) is geodesically complete.*

Furthermore, under any of the equivalent conditions (X, δ) is a geodesic space.

PROOF. First, we observe that local finiteness implies the discreteness of the space (X, δ) by Lemma 11.8.

(i) $\Longrightarrow$ (ii): This follows directly from the discreteness of (X, δ).

(ii) $\Longrightarrow$ (iii): If (X, δ) is discrete and complete, then (X, δ) does not contain any infinite paths of finite length by Lemma 11.12.

(iii) $\Longrightarrow$ (i): We will show this by contradiction. Assume that (X, δ) is geodesically complete and that there exists a distance ball of radius r about a vertex $o \in X$ that has infinite cardinality. We will show that there then exists an infinite geodesic of finite length. Observe that by Proposition 11.15, the path metric space (X, δ) is geodesic and, therefore, geodesics exist.

Let $\Gamma_{o,r}$ be the set of geodesics (x_n) starting at o that have length less than or equal to r, i.e., $x_0 = o$ and $\delta(x_0, x_n) \leq r$ for all $n \in \mathbb{N}_0$. We first observe that $\Gamma_{o,r}$ is a closed subset of Π_o with respect to the metric γ_o. This follows since if $(x_n^{(k)})$ is a sequence of elements in $\Gamma_{o,r}$ converging to (x_n) with respect to γ_o, then for every N there exists an M such that $x_0^{(k)} = x_0, \ldots, x_N^{(k)} = x_N$ for all $k \geq M$. Hence, (x_n) is a geodesic of length less than or equal to r. Thus, $\Gamma_{o,r}$ is closed and since Π_o is compact by Proposition 11.9, $\Gamma_{o,r}$ is compact.

Moreover, by local finiteness, for every $k \in \mathbb{N}$ there is at least one element in $\Gamma_{o,r}$ that has at least k vertices, say $(x_n^{(k)})$. This follows since, if not, then there exists a $k \in \mathbb{N}$ such that all geodesics in $\Gamma_{o,r}$ have fewer than k vertices. Hence, if $\delta(x, o) \leq r$, then $d(x, o) \leq k$, where d is the combinatorial graph distance. By local finiteness, the set of vertices which are within distance k of o with respect to the combinatorial graph distance is finite. But this contradicts the assumption that the ball of radius r about o with respect to the metric δ is infinite.

By the compactness of $\Gamma_{o,r}$, the sequence of geodesics $(x_n^{(k)})$ has a convergent subsequence in $\Gamma_{o,r}$. Hence, there exists an infinite geodesic of finite length, which completes the proof of the implication and the equivalences.

For the "furthermore" statement, we note that under condition (ii) (X, δ) is discrete and complete and therefore a geodesic space by Proposition 11.13. This completes the proof. $\square$

In the remark below we discuss the relevance of the assumption of local finiteness in the theorem above.

Remark. It is well known that for general metric spaces we always have the implication (i) $\Longrightarrow$ (ii). Furthermore, for any discrete path metric space which is a geodesic space, we have the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). The other implications do not hold in general, as discussed in the exercises found at the end of the chapter. In particular, Exercise 11.1 shows that (ii) does not always imply (i) and Exercise 11.2 shows that (iii) does not always imply (ii).

To end this section we show how finiteness of balls implies that we have a geodesic space even if we are not in a locally finite situation.

Proposition 11.17 (Compact balls imply geodesic space). *Let (X, δ) be a path metric space such that all distance balls are finite. Then, (X, δ) is a geodesic space.*

PROOF. Let $x, y \in X$ with $x \neq y$. Then, the distance $r = \delta(x, y)$ can be calculated by taking the infimum over all paths from x to y that do not leave the ball $B_{2r}(x)$. By assumption, the ball $B_{2r}(x)$ is finite and, therefore, there are only finitely many such paths. Hence, the infimum is assumed along one such path, which must therefore be a geodesic. □

3. Examples and relations to other metrics

In this section we present examples of intrinsic metrics and relate them to other metrics that appear in the literature. In particular, we discuss relations to the combinatorial graph distance and the resistance metric.

3.1. The degree path metric. In this subsection we show that for every graph b over (X, m) there exists a non-trivial intrinsic metric.

Let b be a graph over (X, m). We first recall the definition of the weighted degree Deg as

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)$$

for $x \in X$. Furthermore, for any pseudo metric ϱ on X we call the value

$$s = \sup_{x \sim y} \varrho(x, y)$$

the *jump size* of ϱ. If $s < \infty$, then we say that the metric has *finite jump size*.

We now use the weighted degree to construct an intrinsic metric for a given graph. Furthermore, we introduce the notion of cutting a weight from above by a strictly positive number. This can be used to give metrics with finite jump size.

DEFINITION 11.18 (Degree path metrics ϱ_s). For a graph b over (X, m) we call the path metric $\varrho = \delta_w$ with weight

$$w(x, y) = \left(\frac{1}{\mathrm{Deg}(x)} \wedge \frac{1}{\mathrm{Deg}(y)} \right)^{1/2}$$

for $x \sim y \in X$ and ∞ otherwise the *degree path metric*.

If $s \in (0, \infty]$ and we let

$$w_s(x, y) = \left(\frac{1}{\mathrm{Deg}(x)} \wedge \frac{1}{\mathrm{Deg}(y)} \right)^{1/2} \wedge s$$

for $x \sim y \in X$ and ∞ otherwise, then we call the resulting path metric $\varrho_s = \delta_{w_s}$ the *degree path metric with jump size bounded by s*.

We note that $\varrho_\infty = \varrho$ for degree path metrics. We now show that the pseudo metrics ϱ_s are intrinsic for all $s \in (0, \infty]$.

LEMMA 11.19 (ϱ_s is intrinsic). *For a graph b over (X,m) and $s \in (0,\infty]$ the degree path metric ϱ_s is an intrinsic metric.*

PROOF. Let $x \in X$. Then,

$$\sum_{y\in X} b(x,y)\varrho_s^2(x,y) \le \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)\vee \mathrm{Deg}(y)} \le \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)} = m(x).$$

This proves the statement $\square$

We now illustrate degree path metrics on graphs with standard weights and on graphs with normalizing measure.

EXAMPLE 11.20 (Standard weights). Let b be a graph over X with standard weights, i.e., $b(x,y) \in \{0,1\}$ for $x,y \in X$. In this case, the graph is locally finite and, therefore, by Lemma 11.8, the pseudo metric ϱ_s is a discrete metric for any measure m. In the case of the counting measure $m = 1$, the metric ϱ_s is given via the weight

$$w_s(x,y) = \left(\frac{1}{\deg(x)} \wedge \frac{1}{\deg(y)}\right)^{1/2} \wedge s,$$

where deg is the combinatorial degree.

EXAMPLE 11.21 (Normalizing measure). For a graph b over X, recall that the normalizing measure n is given by $n(x) = \sum_{y\in X} b(x,y)$. In this case, the weighted vertex degree becomes $\mathrm{Deg} = 1$. Therefore, $\varrho_s = \varrho_1$ for $s \ge 1$ and $\varrho_1 = d$ is the combinatorial graph distance. The case when the combinatorial graph metric is intrinsic is characterized in the next subsection.

Next, we present an intuition for the definition of ϱ_s in the case that $s = \infty$. It can be seen that $\varrho = \varrho_\infty$ measures distances by the traveling time of the heat along edges. This is discussed in detail below.

REMARK (Probabilistic interpretation). In Section 5 we discussed that heat can be modeled by the Markov process $(\mathbb{X}_t)_{t\ge 0}$ associated to the semigroup e^{-tL} via

$$e^{-tL}f(x) = \mathbb{E}_x(1_{\{t<\zeta\}}f(\mathbb{X}_t))$$

for $x \in X$ whenever $c = 0$, where $\mathbb{E}_x$ is the expected value conditioned on the process starting at x and ζ is the explosion time. The "heat particle" modeled by this process jumps from a vertex x to a neighbor y with probability $b(x,y)/\sum_{z\in X} b(x,z)$. Moreover, the probability of not having left x at time t is given by

$$\mathbb{P}_x(\mathbb{X}_s = x, 0 \le s \le t) = e^{-\mathrm{Deg}(x)t}.$$

Qualitatively, this indicates that the larger the value of $\mathrm{Deg}(x)$ is, the faster the "heat particle" leaves x.

Looking at the definition of $\varrho(x,y)$, the larger the weighted degree of either x or y is, the closer the two vertices are with respect to this distance. Combining these two observations, we see that the faster the random walker jumps along an edge, the shorter the edge is with respect to ϱ. Of course, the jumping time along an edge connecting x to y is not symmetric and depends on whether one jumps from x to y or from y to x as the weighted degrees of x and y can be very different. In order to get a symmetric function, ϱ favors the vertex with the larger degree and the faster jumping time.

There is a direct analogy to the Riemannian setting in terms of mean exit times of small balls. Consider a d-dimensional Riemannian manifold. The first order term of the mean exit time of a small ball of radius r is $r^2/2d$.

On a locally finite graph, for a vertex x a "small open" ball with respect to ϱ can be thought to have radius

$$r = \inf_{y\sim x} \varrho_\infty(x,y) = \left(\frac{1}{\mathrm{Deg}(x)} \wedge \frac{1}{\max_{y\sim x}\mathrm{Deg}(y)}\right)^{1/2}$$

namely, this ball contains only the vertex itself. Computing the mean exit time of this ball, i.e., $\mathbb{X}_t$ leaving x, gives

$$\int_0^\infty e^{-s\mathrm{Deg}(x)}ds = \frac{1}{\mathrm{Deg}(x)} \geq r^2$$

and equality holds whenever $\mathrm{Deg}(x) = \max_{y\sim x}\mathrm{Deg}(y)$.

3.2. The combinatorial graph distance. In this subsection we consider the combinatorial graph metric and characterize when it is equivalent to an intrinsic metric.

We recall that given a graph b over (X,m) the combinatorial metric d is the path metric induced by the weight w given by $w(x,y) = 1$ if $b(x,y) > 0$ and $w(x,y) = \infty$ if $b(x,y) = 0$ for $x,y \in X$. In the lemma below we show that d is equivalent to an intrinsic metric if and only if Deg or, equivalently, L is bounded. Furthermore, this is equivalent to having a uniform lower bound on the distance between neighbors with respect to any intrinsic metric.

Lemma 11.22 (When is the combinatorial graph distance intrinsic). *Let b be a graph over (X,m). Then, the following statements are equivalent:*

(i) *The combinatorial graph distance d is equivalent to the degree path metric ϱ_s for some $s \in (0,\infty]$.*
(ii) *The combinatorial graph distance d is equivalent to an intrinsic metric.*
(iii) *There exists an intrinsic metric ϱ such that $\varrho(x,y) \geq C > 0$ for all $x \sim y$.*
(iv) Deg *is a bounded function.*
(v) *L is a bounded operator.*

In particular, all of the conditions hold if $m \geq n$, where n is the normalizing measure.

PROOF. (i) $\Longrightarrow$ (ii): This is clear as ϱ_s is intrinsic for all s.

(ii) $\Longrightarrow$ (iii): This is also clear as $d(x, y) = 1$ if $x \sim y$.

(iii) $\Longrightarrow$ (iv): Let ϱ be an intrinsic metric such that $0 < C \leq \varrho(x, y)$ for $x \sim y$. Then,

$$C^2 \sum_{y \in X} b(x, y) \leq \sum_{y \in X} b(x, y)\varrho^2(x, y) \leq m(x)$$

for all $x \in X$. Hence, $\mathrm{Deg} \leq 1/C^2$.

(iv) $\Longleftrightarrow$ (v): This follows from Theorem 1.27.

(iv) $\Longrightarrow$ (i): Assume that $\mathrm{Deg} \leq C$ and consider the degree path metric ϱ_1 from Definition 11.18. Clearly, $\varrho_1 \leq d$. On the other hand, by $\mathrm{Deg} \leq C$, we immediately obtain $\varrho_1 \geq (C^{-1/2} \wedge 1)d$. This completes the proof. $\square$

REMARK. In the case of trees, the combinatorial graph distance can be related to the metric σ defined in Section 1 (Exercise 11.5).

In the next example we discuss another class of metrics which are a generalization of the combinatorial graph distance.

EXAMPLE 11.23. For a graph b over X and $q \in (0, \infty]$ we consider the path metric induced by the weight

$$w_{b,q}(x, y) = \frac{1}{b(x, y)^{1/q}}$$

for $x \sim y$ and ∞ otherwise. We denote this metric by d_q. For $q = \infty$, $1/q = 0$, so the metric d_∞ is the combinatorial graph distance and d_∞ is intrinsic whenever $m \geq n$. For $q = 2$, the metric d_2 is intrinsic whenever $m(x) \geq \#\{y \mid y \sim x\}$ (Exercise 11.3).

3.3. The resistance metric. In this subsection we relate intrinsic metrics to the free effective resistance metric which first appeared in Section 4 and then in Section 1 and plays a prominent role in the theory of electric networks.

We start with a pseudo metric that is the square root of the free effective resistance metric and which is related to σ introduced via the space $A_1(X)$ in Section 1. We let

$$r(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

for $x, y \in X$. Obviously, if $m \geq 1$, then the supremum is taken over a smaller set than $A_1(X)$ and, hence, $r \leq \sigma$.

REMARK. We note that it can be shown that the supremum in the definition is actually a maximum (Exercise 11.6). Additionally, it can be seen that the quantity referred to as the free effective resistance r^2

is also a pseudo metric, which is referred to as the resistance metric (Exercise 6.9).

The theorem below shows how to recover r from intrinsic metrics.

THEOREM 11.24 (Resistance and intrinsic metrics). *Let b be a graph over X. Then,*

$$r = \sup\{\varrho \mid \varrho \text{ is intrinsic for } b \text{ over } (X, m) \text{ with } m(X) \leq 2\}.$$

REMARK. The obscure number 2 in the statement of theorem above arises since we do not include a factor of $1/2$ in the definition of $|\nabla f|^2$.

PROOF. Denote the function on the right-hand side of the claimed equality by r^*.

We start by showing that $r \leq r^*$. Let $f \in \mathcal{D}$. Then, ϱ_f defined by

$$\varrho_f(x, y) = |f(x) - f(y)|$$

for $x, y \in X$ is an intrinsic pseudo metric with respect to the measure m_f given by

$$m_f(x) = \sum_{y \in X} b(x, y)(f(x) - f(y))^2.$$

Obviously, $m_f(X) = 2\mathcal{Q}(f)$. Hence, $\varrho_f(x, y) \leq r^*(x, y)$ for all $f \in \mathcal{D}$ with $\mathcal{Q}(f) \leq 1$. Since

$$r = \sup\{\varrho_f \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

this implies $r \leq r^*$.

We now show that $r \geq r^*$. Let ϱ be an intrinsic metric with respect to a measure m with $m(X) \leq 2$. For $o \in X$, we define $f(x) = \varrho(o, x)$. Then, by the triangle inequality and Fubini's theorem, we get

$$\begin{aligned}
\mathcal{Q}(f) &= \frac{1}{2} \sum_{x, y \in X} b(x, y)(\varrho(o, x) - \varrho(o, y))^2 \\
&\leq \frac{1}{2} \sum_{x \in X} \sum_{y \in X} b(x, y) \varrho^2(x, y) \\
&\leq \frac{1}{2} \sum_{x \in X} m(x) \\
&\leq 1.
\end{aligned}$$

Therefore, $r(x, y) \geq |f(x) - f(y)| = |\varrho(o, x) - \varrho(o, y)|$ for all $o \in X$. Letting $o = y$, we get $r(x, y) \geq \varrho(x, y)$ so that $r \geq \varrho$ for all intrinsic metrics ϱ with $m(X) \leq 2$. Therefore, $r \geq r^*$. $\square$

EXAMPLE 11.25 (Trees). If b is a tree, then the metric d_1 introduced in Example 11.23 above is equal to the free effective resistance r^2 (Exercise 11.3).

4. Geometric assumptions and cutoff functions

In this section we discuss some important geometric assumptions that we will use in the rest of the book. These geometric assumptions guarantee the existence of cutoff functions with certain properties. We also show that the existence of certain cutoff functions is equivalent to completeness properties of the graph.

We start with a property of pseudo metrics concerning distance balls. This assumption is inspired by property (i) of the Hopf–Rinow theorem, Theorem 11.16. For a given pseudo metric ϱ we define the distance ball $B_r = B_r(o)$ about $o \in X$ with radius $r \geq 0$ by

$$B_r = \{y \in X \mid \varrho(o, y) \leq r\}.$$

We say that a pseudo metric ϱ admits *finite balls* if the following holds:

(B) The distance balls $B_r(o)$ are finite for all $o \in X$ and $r \geq 0$.

Remark. This property can be characterized for trees with standard weights and bounded combinatorial degree (Exercise 11.7).

Next, we come to a somewhat weaker assumption. We say that the weighted vertex degree is *bounded on balls* whenever the following holds:

(B*) The restriction of Deg to $B_r(o)$ is bounded for all $o \in X$ and $r \geq 0$.

Remark. Clearly, (B) implies (B*). However, the reverse implication does not necessarily hold (Exercise 11.8).

We can characterize the condition (B*) as follows:

Lemma 11.26 (Characterization of (B*)). *Let b be a graph over (X, m). Then, the following statements are equivalent:*

(i) *The weighted vertex degree is bounded on balls (B*).*
(ii) *The restriction of $\mathcal{L}$ to functions in $\mathcal{F}$ supported on any ball is a bounded operator.*
(iii) *The Radon–Nikodym derivative of the normalizing measure n with respect to the measure m is bounded on balls.*

Proof. The equivalence (i) $\iff$ (ii) follows from Theorem 1.27. For the equivalence (i) $\iff$ (iii) notice that the Radon–Nikodym derivative of the normalizing measure n with respect to the measure m is given by the weighted vertex degree Deg. $\square$

Philosophically, the assumptions (B) and (B*) can be understood as a condition bounding a pseudo metric from below in a certain sense. Next, we introduce an assumption which may be understood as an upper bound. We have already seen the concept of the jump size in connection with the degree path metric.

We say a pseudo metric ϱ has finite jump size if the following holds:

(J) The jump size $s = \sup\{\varrho(x,y) \mid x, y \in X,\ x \sim y\}$ is finite.

EXAMPLE 11.27. The metrics ϱ_s defined in Subsection 3.1 have jump size at most s. Hence, they satisfy (J) whenever $s < \infty$.

Combining the assumptions (B) and (J) forces the graph to be locally finite.

LEMMA 11.28 ((B) and (J) imply local finiteness). *Let b be a graph over (X, m) and let ϱ be a pseudo metric. If ϱ satisfies* (B) *and* (J), *then the graph is locally finite.*

PROOF. If there is a vertex with infinitely many neighbors, then there exists a distance ball containing all of them by the finite jump size (J). However, this is impossible by (B). $\square$

We now come to the construction of basic cutoff functions. We have already seen in Lemma 11.3 that we can estimate the gradient squared of Lipschitz functions with respect to an intrinsic metric. This will now be explored in more detail.

Given a pseudo metric ϱ on X, a subset $A \subseteq X$ and $R > 0$, we define $\eta = \eta_{A,R} \colon X \longrightarrow [0, \infty)$ by

$$\eta(x) = \left(1 - \frac{\varrho(x, A)}{R}\right)_+,$$

where

$$\varrho(x, A) = \inf_{y \in A} \varrho(x, y)$$

for $x \in X$ and $s_+ = s \vee 0$ for all $s \in \mathbb{R}$.

We recall the definition of the gradient squared of a function $f \in C(X)$ as

$$|\nabla f(x)|^2 = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))^2$$

for $x \in X$. The lemma below collects basic properties of η.

PROPOSITION 11.29 (Basic properties of cutoff functions). *Let b be a graph over (X, m) and let ϱ be a metric. Let $A \subseteq X$, $R > 0$ and $\eta = \eta_{A,R}$ be the cutoff function defined above. Then,*

(a) $1_A \leq \eta \leq 1_{B_R(A)}$ *and* $\eta \nearrow 1$ *as* $R \to \infty$.

(b) *If ϱ is an intrinsic metric with jump size $s \in [0, \infty]$, then*

$$|\nabla \eta|^2(x) \leq \frac{1}{R^2} 1_{B_{R+s}(A) \cap B_s(X \setminus A)}(x)$$

for all $x \in X$, where the characteristic function on the right-hand side is equal to 1 if $s = \infty$.

PROOF. Statement (a) is obvious from the definition of η.

To see (b), we first observe from (a) that the function

$$(x, y) \mapsto \eta(x) - \eta(y)$$

is zero on $A \times A$ and $(X \times X) \setminus (B_R(A) \times B_R(A))$. Moreover, by the assumption that the jump size is s, the map b is zero on pairs of vertices of distance larger than s. Hence, the map

$$(x, y) \mapsto b(x, y)(\eta(x) - \eta(y))^2$$

and thus $|\nabla\eta|^2$ is supported on $U = (B_{R+s}(A) \cap B_s(X \setminus A))^2$.

As η is $1/R$-Lipschitz with respect to the intrinsic metric ϱ we obtain

$$|\nabla\eta|^2 \leq \frac{1}{R^2}$$

by the "in particular" statement in Lemma 11.3. This completes the proof of (b). □

We now discuss some consequences of the assumptions of finite balls (B) and finite jump size (J) for the cutoff function η. We use the convention $B_r(o) = \emptyset$ for $r < 0$ and $o \in X$.

COROLLARY 11.30. *Let b be a graph over (X, m) and let ϱ be a metric. Let $o \in X$, $r, R > 0$ and $\eta = \eta_{B_r(o),R}$ be the cutoff function defined above with $A = B_r(o)$.*

(a) *If ϱ admits finite balls* (B)*, then $\eta \in C_c(X)$.*
(b) *If ϱ is an intrinsic metric with jump size $s \in [0, \infty]$, then $f|\nabla\eta|^2 \in \ell^1(X, m)$ for all f such that $f1_{B_{R+r+s}(o)\setminus B_{r-s}(o)} \in \ell^1(X, m)$, where the characteristic function is equal to 1 if $s = \infty$. In particular,*

$$\sum_{x\in X} |f(x)||\nabla\eta|^2(x)m(x) \leq \frac{1}{R^2} \sum_{x\in B_{R+r+s}(o)\setminus B_{r-s}(o)} |f(x)|m(x),$$

where the sum on the right-hand side is over X if $s = \infty$.

PROOF. Statement (a) follows from part (a) of Proposition 11.29. Statement (b) follows by part (b) of Proposition 11.29. □

Finally, we show that the existence of certain cutoff functions is equivalent to completeness properties of the graph. Recall that by the Hopf–Rinow theorem, Theorem 11.16, finiteness of distance balls for an intrinsic path metric is equivalent to metric and geodesic completeness whenever the graph is locally finite.

THEOREM 11.31 (Characterization of intrinsic metric with (B)). *Let b be a graph over (X, m). Then, the following statements are equivalent:*

(i) *There exists an intrinsic metric with finite distance balls* (B).
(ii) *There exists a sequence (χ_k) in $C_c(X)$ such that $\chi_k \nearrow 1$ pointwise and $\||\nabla\chi_k|\|_\infty \to 0$ as $k \to \infty$.*

PROOF. (i) $\Longrightarrow$ (ii): Let $o \in X$ and let $\chi_k = \eta_{B_k(o),k}$ for $k \in \mathbb{N}$, where η is as defined above. Then, $\chi_k \nearrow 1$ pointwise as $k \to \infty$. Moreover, by Corollary 11.30 (b) applied with $f = 1_x$ for $x \in X$ we obtain

$$|\nabla \chi_k|(x) \leq \frac{1}{k} \to 0$$

as $k \to \infty$. As the estimate does not depend on x, this completes the proof.

(ii) $\Longrightarrow$ (i): Let (χ_k) be a sequence as assumed in (ii). By passing to a subsequence we can assume without loss of generality that (χ_k) satisfies for all $x \in X$

$$\sum_{k=1}^{\infty} (1 - \chi_k(x))^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \| |\nabla \chi_k|^2 \|_\infty \leq 1.$$

We define $\varrho(x, y)$, for $x, y \in X$, by

$$\begin{aligned} \varrho(x,y) &= \left(\sum_{k=1}^{\infty} (\chi_k(x) - \chi_k(y))^2 \right)^{1/2} \\ &\leq \left(\sum_{k=1}^{\infty} (1 - \chi_k(x))^2 \right)^{1/2} + \left(\sum_{k=1}^{\infty} (1 - \chi_k(y))^2 \right)^{1/2} < \infty. \end{aligned}$$

Clearly, ϱ is also non-negative, vanishes at $x = y$ and satisfies the triangle inequality. Therefore, ϱ is a pseudo metric. Furthermore, ϱ is intrinsic since, for all $x \in X$,

$$\begin{aligned} \sum_{y \in X} b(x,y) \varrho^2(x,y) &= \sum_{y \in X} b(x,y) \sum_{k=1}^{\infty} (\chi_k(x) - \chi_k(y))^2 \\ &= m(x) \sum_{k=1}^{\infty} |\nabla \chi_k|^2(x) \\ &\leq m(x). \end{aligned}$$

To show the finiteness of distance balls (B), let $R \geq 0$ and $o \in X$. Let $i \in \mathbb{N}_0$ be such that $\chi_i(o) \geq 1/2$. We show that $B_R(o) \subseteq \operatorname{supp} \chi_{j-1}$ whenever $j > 4R^2 + i$. Fix such a j and assume x is not in the support of χ_{j-1}. Since, $\chi_k \nearrow 1$ as $k \to \infty$ we have $\chi_k(o) \geq 1/2$ for $i \leq k \leq j-1$ and $\chi_k(x) = 0$ for $k \leq j - 1$. Thus,

$$\varrho^2(x, o) \geq \sum_{k=i}^{j-1} (\chi_k(x) - \chi_k(o))^2 \geq \frac{j - i}{4} > R^2.$$

Thus, we infer $x \in X \setminus B_R(o)$ and, therefore, $B_R(o) \subseteq \operatorname{supp} \chi_{j-1}$ for $j > 4R^2 + i$. Hence, (B) follows from $\chi_j \in C_c(X)$. $\square$

REMARK. In certain situations a much weaker condition on the functions (χ_k) suffices to guarantee the existence of an intrinsic metric satisfying (B). Specifically, we recall the functions $\varphi_x\colon X \longrightarrow [0,\infty)$ defined by $\varphi_x(y) = b(x,y)/m(y)$ for $x \in X$, which we used to characterize $C_c(X) \subseteq D(L)$ and $\ell^2(X,m) \subseteq \mathcal{F}$ in Theorem 1.29. If these functions can be uniformly approximated by compactly supported functions, then (i) and (ii) above are equivalent to

(iii) There exists a sequence (χ_k) in $C_c(X)$ such that $\chi_k \nearrow 1$ as $k \to \infty$ pointwise and $(\||\nabla\chi_k|\|_\infty)$ is bounded.

This follows by applying general theory (Exercise 11.9).

Exercises

Example exercises.

EXERCISE 11.1 (Metric completeness does not imply finiteness of balls). Give an example of a metrically complete path metric space which has an infinite distance ball.

EXERCISE 11.2 (Metric incompleteness with geodesic completeness). Give an example of a metrically incomplete path metric space which has no infinite geodesics and which is, therefore, geodesically complete.

EXERCISE 11.3 (Examples of path metrics). For a connected graph b over X and $q \in (0, \infty)$ we consider the path pseudo metric d_q given by the weights $w = w_{b,q}$ with

$$w_{b,q}(x, y) = \frac{1}{b(x, y)^{1/q}}$$

for $x \sim y$ and ∞ otherwise.

(a) Show that d_q is a metric for all $q \in (0, \infty)$.

(b) Show that if the graph is locally finite, then

$$d_q(x, y) \to d(x, y)$$

as $q \to \infty$ for all $x, y \in X$, where d is the combinatorial graph distance.

(b*) Is the convergence of (b) still valid if one drops the local finiteness assumption?

(c) Show that, for $q = 1$, we have

$$r^2 \leq d_1,$$

where r^2 is the free effective resistance metric introduced in Subsection 3.3.

(c*) Show that, if b is a tree, then $r^2 = d_1$.

(d) Show that, for $q = 2$, the metric d_2 is intrinsic whenever m is bounded below by the combinatorial degree, that is,

$$m(x) \geq \#\{y \in X \mid y \sim x\}$$

for all $x \in X$.

Extension exercises.

EXERCISE 11.4 (Stability of path metrics). Let $\delta = \delta_w$ be the path metric induced by a weight w. When is δ a weight? Show that in case δ is a weight, for the path metric δ_δ induced by δ we have

$$\delta = \delta_\delta.$$

Exercise 11.5 (The combinatorial graph metric for trees). Let b be a tree with standard weights over (X, m) and let m be the counting measure. Let

$$\sigma(x, y) = \sup\{f(x) - f(y) \mid f \in A_1(X)\},$$

where $A_1(X) = \{f \in C(X) \mid |\nabla f|^2 \leq 1\}$ and let d denote the combinatorial graph distance. Show that $\sigma = d/2$.

Exercise 11.6 (Free effective resistance). Let b be a graph over X. Recall that

$$r(x, y) = \sup\{f(x) - f(y) \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

for $x, y \in X$. Show that the supremum is actually a maximum, i.e.,

$$r(x, y) = \max\{f(x) - f(y) \mid f \in \mathcal{D}, \mathcal{Q}(f) \leq 1\}$$

for $x, y \in X$.

Exercise 11.7 (Trees and intrinsic metrics with finite distance balls). Let b be a tree with standard weights over X with bounded combinatorial degree. Characterize the set of measures such that there exists an intrinsic metric satisfying (B).

Exercise 11.8 ((B*) and not (B)). Recall that (B) means that all distance balls are finite while (B*) means that the weighted vertex degree is bounded on balls. Give an example of a graph that allows for an intrinsic metric that satisfies (B*) but does not satisfy (B) for all intrinsic metrics.

Exercise 11.9 (Characterization of finite distance balls). Let b be a graph over (X, m) and define $\varphi_x \colon X \longrightarrow [0, \infty)$ by

$$\varphi_x(y) = \frac{1}{m(y)} b(x, y)$$

for $x \in X$. Assume, for every $x \in X$, there exists a sequence (ψ_n) in $C_c(X)$ such that $\|\varphi_x - \psi_n\|_\infty \to 0$ as $n \to \infty$. Show that the following statements are equivalent:

(i) There exists an intrinsic metric with finite distance balls (B).
(ii) There exists a sequence (χ_k) in $C_c(X)$ such that $\chi_k \nearrow 1$ as $k \to \infty$ pointwise and $(\||\nabla \chi_k|\|_\infty)$ is bounded.

(Hint 1: What is the dual space of the uniform closure of $C_c(X)$? Use this to show that the boundedness and pointwise convergence yield weak convergence of $|\nabla \chi_k| \to 0$ as $k \to \infty$.)

(Hint 2: Mazur's lemma says that for every weakly convergent sequence in a Banach space there is a sequence of convex combinations of its members that converges strongly to the same limit. Use this to show that 0 belongs to the uniform closure of the convex hull of $\{|\nabla\chi_k| \mid k \geq n\}$ for every $n \in \mathbb{N}$. From this show that there is a sequence (χ_k') of convex combinations of members of χ_k such that $\||\nabla\chi_k'|\|_\infty \to 0$ as $k \to \infty$.)

Notes

The use of intrinsic metrics in the context of Dirichlet forms goes back to Sturm's fundamental work [**Stu94**]. In this paper, Sturm introduces the concept of an intrinsic metric for a strongly local Dirichlet form. Furthermore, in this and subsequent works an impressive amount of spectral geometry is generalized from manifolds to all spaces with a strongly local Dirichlet form with the help of intrinsic metrics. This includes various volume growth criteria for global properties on manifolds which were discussed in the influential survey article of Grigor′yan [**Gri99**].

The extension of this concept to Dirichlet forms which are not strongly local seems to not have been pursued for some time until it appeared independently and rather simultaneously in various works: This includes papers of Folz [**Fol11, Fol14b**] centered around heat kernel bounds and stochastic completeness for graphs, where the corresponding metrics appear under the name of adapted metrics; investigations of Masamune/Uemura [**MU11**] as well as Grigor′yan/Huang/Masamune [**GHM12**] dealing with stochastic completeness featuring a slightly more general class of metrics for jump processes; and the work Frank/Lenz/Wingert [**FLW14**].

Here, we follow the article [**FLW14**]. Indeed, the framework developed there has served as a basis for applications in various works. We will have more to say about applications to graphs in later chapters. Here, we already mention the Habilitationsschrift [**Kel14**] and survey [**Kel15**] of Keller, as these works expand on the specialization of the framework of [**FLW14**] to graphs. The article [**FLW14**] itself presents a theory of intrinsic metrics for general regular Dirichlet forms including a Rademacher theorem which was new even in the strongly local case. The article also presents both sufficient and necessary criteria for a metric to be an intrinsic metric for a purely non-local Dirichlet form. When restricted to graphs, these criteria give the concept of intrinsic metric discussed in Section 1. The (counter)example, Example 11.2, is a reformulation of Example 6.2 from [**FLW14**] which proves a slightly different point, namely, that the supremum of intrinsic metrics is in general not an intrinsic metric. The implication from (i) to (ii) in Lemma 11.3 is a direct application of the Rademacher theorem of [**FLW14**] to the case of graphs and the equivalence between (i) and (iii) follows from the necessary and sufficient criteria given in [**FLW14**]. For graphs, this part of the material is also discussed explicitly in [**Kel14, Kel15**] and Keller/Lenz/Schmidt/Wirth [**KLSW15**].

The original Hopf–Rinow theorem is found in [**HR31**]. The discrete version in Section 2 can be seen as an elaboration on corresponding parts of Huang/Keller/Masamune/Wojciechowski [**HKMW13**]. In particular, Theorem 11.16 can already be found in [**HKMW13**]; however, the approach given above is different. In particular, the essential Proposition 11.9 and its consequence Proposition 11.13 seem not to have appeared in print before. Note that a part of Theorem 11.16 also appears in the proof of Theorem 1.5 in a paper of Milatovic [**Mil11**]. The argument given there is based on length spaces in the sense of Burago/Burago/Ivanov [**BBI01**] and, while not mentioned explicitly, it seems that the length spaces in question are metric graphs associated to discrete graphs. A systematic treatment of the Hopf–Rinow theorem on discrete path spaces even beyond the locally finite case can be found in [**KM19**].

The material of Section 3 goes back to various sources. The metric ϱ_s for $s = 1$ was introduced in the thesis of Huang [**Hua11a**] as an example for the general class of metrics discussed in [**FLW14**]. For the first order term of the mean exit time of a small ball as discussed in Subsection 3.1 in the context of Riemannian manifolds, see [**Pin85**]. A discussion of the relationship between intrinsic metrics and combinatorial metrics can be found in various places. In particular, the equivalence between (ii) and (iv) in Lemma 11.22 is a special case of the inequalities derived in Section 14.2 of [**FLW14**] and is stated explicitly in [**HKMW13, KLSW15**]. Parts of Lemma 11.22 have also appeared in [**Kel15**]. The case $q = 2$ for the family of metrics discussed in Example 11.23 was used in [**CdVTHT11**]. The relationship to the free resistance metric presented in Theorem 11.24 was established in [**GHK$^+$15**].

An early manifestation of the fact that the combinatorial graph metric is not the non plus ultra can be found in an article by Davies [**Dav93**]. In this article, a variety of metrics are introduced incorporating ideas from non-commutative geometry. Although none of the metrics proposed there are intrinsic for graphs with unbounded degree, there are parallels in the underlying ideas.

Finally, the properties discussed in Section 4 have been used in some form in any article using intrinsic metrics for graphs as a tool. The particular discussion as it is presented here is found in [**Kel15**], with the exception of Theorem 11.31, which is taken from Appendix A in [**LSW21**]. This theorem is important since some of the literature suggests that the existence of certain cutoff functions, sometimes referred to as χ-completeness, is a less restrictive condition than the existence of an intrinsic metric with finite distance balls. This is not the case, as Theorem 11.31 shows and is consistent with the situation for Riemannian manifolds [**BGL14**].

CHAMBER 12

Harmonic Functions and Caccioppoli Theory

I'm just swingin swords strictly based on keyboards,
unbalanced like elephants and ants on see-saws.
GZA.

In this chapter we develop techniques to analyze solutions of the equation

$$\mathcal{L}u = \lambda u$$

for $\lambda \in \mathbb{R}$ and u in a suitable function space which is included in $\mathcal{F}$. Here, the operator $\mathcal{L} = \mathcal{L}_{b,c,m}$ is the formal Laplacian of a graph (b, c) over (X, m) and $\mathcal{F}$ is the formal domain of $\mathcal{L}$. We have encountered such functions before. They are called α-harmonic functions, i.e., functions which satisfy $(\mathcal{L} + \alpha)u = 0$, where $\alpha = -\lambda$. The reason why we write the equation differently in this chapter is that we ultimately want to think of such functions as generalized eigenfunctions in the context of the Shnol′ theorems.

For $\lambda \leq 0$, we establish Liouville theorems. Such results give criteria for the absence of non-constant solutions to $\mathcal{L}u = \lambda u$ and include the case $\lambda = 0$, when such functions are called harmonic. In Section 2 we establish criteria for the constancy of harmonic functions in $\ell^p(X, m)$ for $p \in (1, \infty)$. Furthermore, as an application of the Liouville theorems, we give criteria for uniqueness of the form, essential self-adjointness and recurrence in Section 3.

For $\lambda \geq 0$, we prove two versions of a Shnol′ theorem in Section 4. This theorem gives a criterion for λ to be in the spectrum of the Laplacian associated to a graph via a growth condition on the function u. The growth here is measured in terms of an intrinsic metric which has finite distance balls and jump size.

The key tool for all of these results are variants of the Caccioppoli inequality which are established in Section 1. Roughly speaking, such inequalities allow us to estimate the energy of u times a cutoff function by u times the energy of the cutoff function. In applications, we use cutoff functions defined via intrinsic metrics. To guarantee that these cutoff functions have the required properties, we assume some boundedness conditions on balls. Along the way, we use these assumptions to establish a Green's formula, which is a crucial step in the proof of the most involved Caccioppoli inequalities.

1. Caccioppoli inequalities

In this section we introduce versions of the Caccioppoli inequality in order of increasing complexity. We also present first applications of these inequalities via a Gaffney result which proves form uniqueness and essential self-adjointness for metrically complete graphs. Furthermore, we establish a Green's formula.

When dealing with operators and forms associated to graphs we often encounter expressions of the form $u\mathcal{L}u$ or $(\nabla u)^2$ and weighted sums of these functions over the entire space. Here, we define

$$\nabla_{x,y}f = f(x) - f(y)$$

for $f \in C(X)$ and $x, y \in X$. In applications, it is very useful to localize these functions, i.e., to consider expressions of the form $\varphi^2 u\mathcal{L}u$, $(\nabla_{x,y}(\varphi u))^2$ or $\varphi^2(\nabla u)^2$ for a function φ which is often finitely supported or linked to the metric. Such functions φ are then referred to as cutoff functions. Of course, the questions arises as to how the original terms are related to the localized terms. It turns out that we can establish inequalities between these terms when they are summed over the entire space. In particular, under suitable assumptions, it is possible to get rid of the differences of u altogether. Instead, only differences of the cutoff functions appear. This is the topic of Caccioppoli theory.

In order to illustrate the discussion above, we consider the simple case when u is harmonic and $c = 0$. A Caccioppoli inequality then takes the form

$$\sum_{x,y \in X} b(x,y)\varphi^2(x)(\nabla_{x,y}u)^2 \leq C \sum_{x,y \in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2$$

for some $C > 0$ whenever u and φ are in suitably chosen function spaces. Thus we see that we trade differences of the solution u for differences of the cutoff function φ.

After this general discussion, we now outline the content of the following subsections. As a first step, in Subsection 1.1, we prove a general estimate and derive some easy Caccioppoli inequalities for $\varphi \in C_c(X)$ and $u \in \mathcal{F}$ satisfying $\mathcal{L}u = \lambda u$. Although basic, these inequalities already have immediate consequences concerning the uniqueness of the form and essential self-adjointness when balls with respect to an intrinsic metric are finite. These uniqueness results turn out to be special cases of later considerations, however, we can already demonstrate the basic idea of our approach without too many technical details.

In Subsection 1.2 we prove a Green's formula which will be key in establishing subsequent Caccioppoli inequalities. In Subsection 1.3 we then show a refined Caccioppoli inequality for φ satisfying a boundedness condition and $u \in \ell^2(X,m)$ which satisfies $\mathcal{L}u = \lambda u$. We then show a similar, though more complicated, inequality for $u \in \ell^p(X,m)$ for $p \in (1,\infty)$ satisfying $u \geq 0$ and $\mathcal{L}u \leq \lambda u$ in Subsection 1.4.

1.1. Basic Caccioppoli inequalities and consequences. In this subsection we prove two simple Caccioppoli-type inequalities. We then demonstrate the use of such inequalities by deriving various consequences. In particular, we prove the uniqueness of associated forms as well as the essential self-adjointness of the Laplacian when we assume that the graph is metrically complete. We refer to these results as Gaffney theorems.

We first discuss how $\mathcal{F}$ is invariant under multiplication with bounded functions and how $\mathcal{L}(u\varphi)$ is related to $\varphi\mathcal{L}u$.

LEMMA 12.1. *Let (b,c) be a graph over (X,m). Let $u \in \mathcal{F}$ and $\varphi \in \ell^\infty(X)$. Then, $u\varphi \in \mathcal{F}$ and for $x \in X$*

$$\mathcal{L}(u\varphi)(x) = \varphi(x)\mathcal{L}u(x) + \frac{1}{m(x)}\sum_{y\in X} b(x,y)u(y)\nabla_{x,y}\varphi.$$

PROOF. That $u\varphi \in \mathcal{F}$ is obvious. As for the formula, we note that

$$\nabla_{x,y}(u\varphi) = \varphi(x)\nabla_{x,y}u + u(y)\nabla_{x,y}\varphi$$

holds for all $x,y \in X$ by a direct calculation, see Lemma 2.25. From this we obtain the formula after we multiply by $b(x,y)$, summing and dividing by $m(x)$. □

Now, we can present our basic estimate. It deals with the energy $\mathcal{Q}(u\varphi)$, i.e., the energy of u which is localized by φ.

LEMMA 12.2 (The basic cutoff inequality). *Let (b,c) be a graph over (X,m). Let $\varphi \in C_c(X)$ and $u \in \mathcal{F}$. Then,*

$$\mathcal{Q}(u\varphi) \leq \sum_{x\in X}\varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

REMARK. We note that the case $\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2 = \infty$ is possible in the situation of the lemma. In later applications, however, this case will not occur.

PROOF. As $u\varphi \in C_c(X)$, using Green's formula, Proposition 1.5, we find

$$\mathcal{Q}(u\varphi) = \sum_{x\in X}\mathcal{L}(u\varphi)(x)(u\varphi)(x)m(x).$$

From Lemma 12.1 we then get

$$\mathcal{Q}(u\varphi) = \sum_{x\in X}(u\varphi)(x)m(x)\left(\varphi(x)\mathcal{L}u(x) + \frac{1}{m(x)}\sum_{y\in X} b(x,y)u(y)\nabla_{x,y}\varphi\right).$$

Now, clearly the sums on the right-hand side are absolutely convergent as φ has finite support. Thus, we obtain

$$\mathcal{Q}(u\varphi) = \sum_{x\in X} \varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \frac{1}{2}\sum_{x,y\in X} b(x,y)u(x)u(y)(\nabla_{x,y}\varphi)^2.$$

Note that in this formula all sums are absolutely convergent. Now, we can use the inequality $|u(x)u(y)| \leq (u^2(x) + u^2(y))/2$ along with symmetry to estimate the second term on the right hand side and obtain the statement of the lemma. $\square$

REMARK. We note that we can rephrase the statement of the lemma as

$$\mathcal{Q}(u\varphi) \leq \mathcal{Q}(u, u\varphi^2) + \frac{1}{2}\sum_{x,y\in X} b(x,y)u(x)u(y)(\nabla_{x,y}\varphi)^2$$

whenever u belongs to $\mathcal{D}$.

The basic cutoff inequality given in the preceding lemma already has some direct applications, as we discuss next. When $u \in \mathcal{F}$ satisfies $\mathcal{L}u = \lambda u$ we can use the lemma to obtain an estimate on the energy $\mathcal{Q}(u\varphi)$ which does not contain any differences of the function u.

THEOREM 12.3 (First Caccioppoli-type inequality). *Let (b,c) be a graph over (X,m). Let $\varphi \in C_c(X)$ and $u \in \mathcal{F}$ such that $\mathcal{L}u = \lambda u$ for $\lambda \in \mathbb{R}$. Then,*

$$\mathcal{Q}(u\varphi) \leq \lambda\|u\varphi\|^2 + \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

PROOF. This is immediate from Lemma 12.2 and the assumption that u satisfies $\mathcal{L}u = \lambda u$. $\square$

Alternatively, we can also use the basic cutoff inequality to estimate sums involving $u\mathcal{L}u$ irrespective of whether u is a generalized eigenfunction or not.

THEOREM 12.4 (Second Caccioppoli-type inequality). *Let (b,c) be a graph over (X,m). If $u \in \mathcal{F}$ and $\varphi \in C_c(X)$, then*

$$-\sum_{x\in X} \mathcal{L}u(x)u(x)\varphi^2(x)m(x) \leq \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

In particular, if u additionally satisfies $\mathcal{L}u = \lambda u$ for $\lambda \in \mathbb{R}$, then

$$-\lambda\|u\varphi\|^2 \leq \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

PROOF. This follows directly from $\mathcal{Q} \geq 0$ and Lemma 12.2. $\square$

REMARK. The preceding result will also be used in the proof of the volume growth criterion for stochastic completeness found in Chapter 14.

We can also use the basic cutoff inequality to deal with localized versions of the energy, i.e., to treat suitable sums of functions of the form $(\varphi\nabla u)^2$. This requires an additional step relating $(\varphi\nabla u)^2$ to $(\nabla(u\varphi))^2$. Details are given in the next lemma. Note that the lemma holds for all functions u, φ on X.

LEMMA 12.5. *Let (b, c) be a graph over (X, m). Then, for all $u, \varphi \in C(X)$, we have*

$$\sum_{x,y\in X} b(x,y)\varphi^2(x)(\nabla_{x,y}u)^2 \leq$$
$$2\sum_{x,y\in X} b(x,y)(\nabla_{x,y}(u\varphi))^2 + 2\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2,$$

where either side may take the value ∞.

PROOF. By a direct calculation, see Lemma 2.25, the Leibniz rule says

$$\nabla_{x,y}(\varphi u) = \varphi(x)\nabla_{x,y}u + u(y)\nabla_{x,y}\varphi.$$

Combining this with the inequality $(s+t)^2 \leq 2s^2 + 2t^2$ for $s, t \in \mathbb{R}$ we find

$$\begin{aligned}(\varphi(x)\nabla_{x,y}u)^2 &= (\nabla_{x,y}(u\varphi) - u(y)\nabla_{x,y}\varphi)^2 \\ &\leq 2(\nabla_{x,y}(u\varphi))^2 + 2u^2(y)(\nabla_{x,y}\varphi)^2.\end{aligned}$$

Now, the statement follows after multiplication by b and summation, where we use additionally that $u^2(y)$ in the last term can be replaced by $u^2(x)$ due to symmetry. □

We now derive the following result, which gives an estimate on the energy of a function which is localized. Here we assume that the function is in the domain of the formal Laplacian and the cutoff function is finitely supported.

THEOREM 12.6 (Caccioppoli inequality – $\mathcal{F}$ version). *Let (b, c) be a graph over (X, m). Let $\varphi \in C_c(X)$ and $u \in \mathcal{F}$. Then,*

$$\frac{1}{4}\sum_{x,y\in X} b(x,y)\varphi^2(y)(\nabla_{x,y}u)^2 + \sum_{x\in X} c(x)(u\varphi)^2(x)$$
$$\leq \sum_{x\in X}\varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

If u additionally satisfies $\mathcal{L}u = \lambda u$ for $\lambda \in \mathbb{R}$, then the first term on the right-hand side simplifies to $\lambda\|u\varphi\|^2$.

REMARK. The theorem does not claim that the right-hand side is necessarily finite, neither does it claim that the left-hand side is necessarily finite.

Proof. This is a direct consequence of Lemma 12.5 and Lemma 12.2. Specifically, using Lemma 12.5 we can estimate the terms on the left-hand side,

$$\mathrm{LHS} = \frac{1}{4}\sum_{x,y} b(x,y)\varphi^2(y)(\nabla_{x,y}u)^2 + \sum_{x\in X} c(x)(u\varphi)^2(x)$$

from above by

$$\frac{1}{2}\sum_{x,y} b(x,y)(\nabla_{x,y}(u\varphi))^2 + \frac{1}{2}\sum_{x,y} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2 + \sum_{x\in X} c(x)(u\varphi)^2(x).$$

From the definition of the energy form this is equivalent to

$$\mathcal{Q}(u\varphi) + \frac{1}{2}\sum_{x,y} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

So, an application of Lemma 12.2 gives the desired estimate, namely,

$$\mathrm{LHS} \leq \sum_{x\in X} \varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

This finishes the proof. □

We will now apply Theorem 12.4 to the case when φ is a cutoff function involving an intrinsic metric. As this case is of fundamental importance, we first discuss some background and basic ideas before turning to the actual statement of the theorem.

We will use a cutoff function based on an intrinsic metric. More specifically, for an intrinsic metric ϱ, a vertex $o \in X$ and a radius $r > 0$, we let $B_r = B_r(o)$ denote the ball of radius r about o with respect to ϱ and let η be the cutoff function defined by

$$\eta(x) = \left(1 - \frac{\varrho(x, B_r)}{2r}\right)_+.$$

As η is $1/2r$-Lipschitz with respect to ϱ, we obtain

$$\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2 \leq \frac{1}{4r^2}m(x)$$

for all $x \in X$, compare Lemma 11.3 and Proposition 11.29. Furthermore, as η is supported on B_{3r}, the condition of finite balls with respect to ϱ implies that $\eta \in C_c(X)$.

The result below gives a condition for form and, hence, Markov uniqueness and essential self-adjointness. This is a first version of a Gaffney theorem.

Theorem 12.7 (Gaffney theorem – finite distance balls). *Let (b, c) be a graph over (X, m). If there exists an intrinsic metric ϱ such that the distance balls are finite* (B)*, then*

$$Q^{(D)} = Q^{(N)}.$$

Moreover, if $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint and

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}.$$

Proof. By Theorems 3.2 and 3.6 it suffices to show that α-harmonic functions in $\ell^2(X,m)$ for $\alpha > 0$ are trivial. Hence, let $u \in \ell^2(X,m) \cap \mathcal{F}$ satisfy $(\mathcal{L}+\alpha)u = 0$ for $\alpha > 0$. For $r > 0$, we let B_r denote the ball of radius r about a vertex o and let $\eta(x) = (1 - \varrho(x,B_r)/2r)_+$ be the cutoff function discussed above. By (B), $\eta \in C_c(X)$ and clearly $\eta = 1$ on B_r. Moreover, $\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2 \leq m(x)/4r^2$ for all $x \in X$. Invoking this estimate and using Theorem 12.4 with $\varphi = \eta$ and $\lambda = -\alpha$ we obtain

$$\begin{aligned}\alpha\|u1_{B_r}\|^2 \leq \alpha\|u\eta\|^2 = -\lambda\|u\eta\|^2 &\leq \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\eta)^2 \\ &= \frac{1}{2}\sum_{x\in X} u^2(x)\left(\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2\right) \\ &\leq \frac{1}{8r^2}\|u\|^2.\end{aligned}$$

Hence, letting $r \to \infty$ and using $\alpha > 0$, we infer $u = 0$. This completes the proof. □

As a second formulation of a Gaffney result, we state a criterion for essential self-adjointness and form and Markov uniqueness under a metric completeness assumption. For this, we have to restrict to locally finite graphs and path metrics.

Theorem 12.8 (Gaffney theorem – metric completeness). *Let (b,c) be a locally finite graph over (X,m). If there exists an intrinsic path metric ϱ such that (X,ϱ) is geodesically complete, then*

$$Q^{(D)} = Q^{(N)}$$

and the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint with

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}.$$

Proof. The Hopf–Rinow theorem, Theorem 11.16, gives that distance balls in locally finite graphs are finite under the completeness condition. Moreover, Theorem 1.29 implies $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ for locally finite graphs. Given these two facts, the theorem is a direct consequence of Theorem 12.7. □

1.2. A Green's formula. In this subsection we establish a variant of Green's formula which uses intrinsic metrics. This will be a fundamental tool in the proof of the Caccioppoli inequalities that follow.

In general, the proof of Green's formulas is a direct calculation once we know that all of the involved sums are absolutely convergent. We will first prove an auxiliary lemma that will establish the required convergence. To this end we define the *combinatorial neighborhood* $N(U)$ of a set $U \subseteq X$ as the set of all vertices which are in U or are connected by an edge to a vertex in U, i.e.,

$$N(U) = U \cup \{y \in X \mid y \sim x \text{ for some } x \in U\}.$$

In what follows U will typically be the support of some cutoff function $\varphi \in C(X)$, so it is natural to use the notation

$$N(\varphi) = N(\operatorname{supp} \varphi).$$

Moreover, given a pseudo metric ϱ, we recall the notation $B_\varepsilon(U)$ for the ε-ball about U with respect to ϱ, i.e.,

$$B_\varepsilon(U) = \{y \in X \mid \text{ there exists an } x \in U \text{ with } \varrho(x, y) \leq \varepsilon\}.$$

The cutoff functions we use in the following are bounded and the weighted vertex degree is bounded on some neighborhood of the support. Specifically, we say a function $\varphi \in \ell^\infty(X)$ has property (C) with respect to ϱ if the following condition is satisfied:

(C) The weighted vertex degree Deg is bounded on $B_\varepsilon(\operatorname{supp} \varphi)$ for some $\varepsilon > 0$.

With these notions, we can state our lemma concerning the summability of certain functions. This result will be used for the proof of Green's formula as well as in the proof of subsequent Caccioppoli inequalities.

Lemma 12.9. *Let (b, c) be a graph over (X, m) and let ϱ be an intrinsic metric. Let $\varphi \in \ell^\infty(X)$ satisfy* (C) *with respect to ϱ. For all $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ and $f, g \in C(X)$ such that $f 1_{N(\varphi)} \in \ell^p(X, m)$ and $g 1_{N(\varphi)} \in \ell^q(X, m)$ each of the following sums*

$$\sum_{x,y \in X} b(x, y)|f(x)g(x)\varphi(x)|, \sum_{x,y \in X} b(x, y)|f(x)g(x)\varphi(y)|,$$
$$\sum_{x,y \in X} b(x, y)|f(x)g(y)\varphi(x)|, \sum_{x,y \in X} b(x, y)|f(y)g(x)\varphi(x)|$$

is finite. In fact, each sum is bounded by $C\|f 1_{N(\varphi)}\|_p \|g 1_{N(\varphi)}\|_q \|\varphi\|_\infty$ with $C = \varepsilon^{-2} \vee \sup_{B_\varepsilon(\operatorname{supp} \varphi)} \mathrm{Deg}$.

Proof. Let $U = \operatorname{supp} \varphi$ and

$$b' = b - b 1_{X \setminus U \times X \setminus U}, \qquad c' = c 1_U$$

be the graph for which all edge weights and killing terms completely outside of U are set to zero. We denote the corresponding weighted degree by Deg'. We first show that Deg' is bounded.

Claim. $\mathrm{Deg}' \leq (\varepsilon^{-2} \vee \sup_{B_\varepsilon(U)} \mathrm{Deg})$.

Proof of the claim. Clearly, $\mathrm{Deg}' \le \mathrm{Deg}$ and, therefore, Deg' is bounded on $B_\varepsilon(U)$ by property (C). Furthermore, $\mathrm{Deg}' = 0$ on $X \setminus N(U)$. Hence, it remains to show that Deg' is bounded by ε^{-2} on $N(U) \setminus B_\varepsilon(U)$.

Let $x \in N(U) \setminus B_\varepsilon(U)$. We observe that $c'(x) = 0$, $b'(x, y) = 0$ for all $y \notin U$ while $b'(x, y) = b(x, y)$ and $\varrho(x, y) > \varepsilon$ for all $y \in U$. Since ϱ is an intrinsic metric, we get

$$\varepsilon^2 \mathrm{Deg}'(x) = \frac{1}{m(x)} \sum_{y\in U} b'(x, y)\varepsilon^2 \le \frac{1}{m(x)} \sum_{y\in U} b(x, y)\varrho^2(x, y) \le 1.$$

Hence, $\mathrm{Deg}' \le (\varepsilon^{-2} \vee \sup_{B_\varepsilon(U)} \mathrm{Deg})$, as claimed.

After these preparations we now come to the actual estimate. We only treat the sum $\sum_{x,y\in X} b(x, y)|f(x)g(x)\varphi(y)|$. The other sums can be estimated similarly.

By the assumption $\operatorname{supp} \varphi = U$ we have

$$b(x, y)\varphi(y) \neq 0$$

only for $y \in U$ and $x \in N(U)$, in which case $b(x, y) = b'(x, y)$. In particular, we have

$$\sum_{x,y\in X} b(x, y)|\varphi(y)|a(x, y) = \sum_{x,y\in N(U)} b'(x, y)|\varphi(y)|a(x, y)$$

for any $a\colon X \times X \longrightarrow [0, \infty)$. Combining this observation with Hölder's inequality with b' considered as a measure and using the claim, we estimate

$$\begin{aligned}
&\sum_{x,y\in X} b(x, y)|f(x)||g(y)||\varphi(y)| \\
&\le \|\varphi\|_\infty \sum_{x,y\in N(U)} b'(x, y)|f(x)||g(y)| \\
&\le \|\varphi\|_\infty \left(\sum_{x,y\in N(U)} b'(x, y)|f(x)|^p \right)^{1/p} \left(\sum_{x,y\in N(U)} b'(x, y)|g(y)|^q \right)^{1/q} \\
&\le \|\varphi\|_\infty \left(\sum_{x\in N(U)} \mathrm{Deg}'(x)|f(x)|^p m(x) \right)^{1/p} \left(\sum_{y\in N(U)} \mathrm{Deg}'(y)|g(y)|^q m(y) \right)^{1/q} \\
&\le \|\mathrm{Deg}'\|_\infty \|f 1_{N(U)}\|_p \|g 1_{N(U)}\|_q \|\varphi\|_\infty.
\end{aligned}$$

Since Deg' is bounded by $\varepsilon^{-2} \vee \sup_{B_\varepsilon(U)} \mathrm{Deg}$, the statement follows. $\square$

We now use the lemma above along with some basic estimates to establish the absolute convergence of sums involved in the following variant of Green's formula.

PROPOSITION 12.10 (Green's formula). *Let* (b,c) *be a graph over* (X,m) *and let* ϱ *be an intrinsic metric. Let* $U \subseteq X$ *and assume that* Deg *is bounded on* $B_\varepsilon(U)$ *for some* $\varepsilon > 0$. *Let* $p, q \in [1,\infty]$ *satisfy* $1/p + 1/q = 1$ *and* $f, g \in C(X)$ *be such that* $f1_{N(U)} \in \ell^p(X,m) \cap \mathcal{F}$ *and* $g \in \ell^q(X,m)$ *with* $\operatorname{supp} g \subseteq U$. *Then, we have*

$$\sum_{x \in X} \mathcal{L}f(x)g(x)m(x) = \frac{1}{2}\sum_{x,y \in X} b(x,y)\nabla_{x,y}f\nabla_{x,y}g + \sum_{x \in X} c(x)f(x)g(x),$$

where all sums converge absolutely.

PROOF. The formula follows by a direct calculation once we establish the absolute convergence of all terms involved.

The absolute convergence of the sums on the right-hand side involving b follows from the previous lemma applied with $\varphi = 1_U$ as $\operatorname{supp} g \subseteq U$. The absolute convergence of the sum involving c on the right-hand side can be shown directly by using the assumptions and Hölder's inequality as follows,

$$\begin{aligned}\sum_{x \in X} c(x)|f(x)||g(x)| &\leq \|\mathrm{Deg} 1_U\|_\infty \sum_{x \in U} |f(x)||g(x)|m(x) \\ &\leq \|\mathrm{Deg} 1_U\|_\infty \|f1_U\|_p \|g1_U\|_q.\end{aligned}$$

Finally, we note that also the sum on the left-hand side is absolutely convergent. Indeed, the term $g(\mathcal{L}f)m$ can be written out using the definition of $\mathcal{L}$ as

$$\sum_{x \in X} g(x)\left(\sum_{y \in X} b(x,y)(f(x) - f(y)) + c(x)f(x)\right).$$

Now, we can argue as in the case of the treatment of the right-hand side. This completes the proof. □

1.3. An ℓ^2-Caccioppoli inequality. In this subsection prove a Caccioppoli inequality for functions in $\ell^2(X,m)$.

We consider the following situation ensuring that the Green's formula of the preceding subsection can be applied: For a graph (b,c) over (X,m) with an intrinsic metric ϱ, we consider a bounded function φ such that Deg is bounded on an ε-neighborhood of its support for some $\varepsilon > 0$. This was referred to as φ satisfying (C) with respect to ϱ. Moreover, we consider $u \in \mathcal{F}$ with $u1_{N(\varphi)} \in \ell^2(X,m)$, where $N(\varphi)$ is the combinatorial neighborhood of the support of φ.

We note that in this situation the assumptions of Proposition 12.10 and Lemma 12.9 are satisfied for $f = u\varphi$ and $g = u\varphi$ as well as for $f = u$ and $g = u\varphi^2$. Moreover, it is obvious that $u\varphi \in \mathcal{F}$ for $u \in \mathcal{F}$ and $\varphi \in \ell^\infty(X)$. Given this, we can prove the analogue of the basic cutoff inequality in our situation by a straightforward adaption of the argument used above.

Lemma 12.11 (The basic cutoff inequality in ℓ^2). *Let (b,c) be a graph over (X,m) and let ϱ be an intrinsic metric. Let $\varphi \in \ell^\infty(X)$ satisfy* (C) *with respect to ϱ and let $u \in \mathcal{F}$ with $u1_{N(\varphi)} \in \ell^2(X,m)$. Then,*

$$\mathcal{Q}(u\varphi) \leq \sum_{x\in X} \varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \frac{1}{2}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.$$

Proof. This follows by virtually the same proof as Lemma 12.2 with the following two modifications: The use of the Green's formula from Proposition 1.5 is replaced by the use of the Green's formula from Proposition 12.10 applied to $f = u\varphi$ and $g = u\varphi$ and the necessary absolute convergence of the sums is ensured by Lemma 12.9. □

Using condition (C) we can now establish every result we obtained in Subsection 1.1 by replacing the use of Lemma 12.2 with the previous lemma. In particular, we obtain the following version of the Caccioppoli inequality which is a counterpart to Theorem 12.6.

Theorem 12.12 (Caccioppoli inequality – ℓ^2 version). *Let (b,c) be a graph over (X,m) and let ϱ be an intrinsic metric. Let $\varphi \in \ell^\infty(X)$ satisfy* (C) *with respect to ϱ and let $u \in \mathcal{F}$ with $u1_{N(\varphi)} \in \ell^2(X,m)$. Then,*

$$\begin{aligned}\frac{1}{4}\sum_{x,y\in X} b(x,y)\varphi^2(y)(\nabla_{x,y}u)^2 &+ \sum_{x\in X} c(x)(u\varphi)^2(x)\\ &\leq \sum_{x\in X}\varphi^2(x)\mathcal{L}u(x)u(x)m(x) + \sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.\end{aligned}$$

If u additionally satisfies $\mathcal{L}u = \lambda u$ on $\operatorname{supp}\varphi$ *for $\lambda \in \mathbb{R}$, then*

$$\begin{aligned}\frac{1}{4}\sum_{x,y\in X} b(x,y)\varphi^2(y)(\nabla_{x,y}u)^2 &+ \sum_{x\in X} c(x)(u\varphi)^2(x)\\ &\leq \lambda\|u\varphi\|^2 + \sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\varphi)^2.\end{aligned}$$

1.4. An ℓ^p-Caccioppoli inequality. In this subsection we will prove a Caccioppoli inequality for functions in $\ell^p(X,m)$ for $p \in (1,\infty)$. This inequality will be used in the proof of Yau's Liouville theorem concerning subharmonic functions. Along the way, we prove a lemma which will be used in the proof of Karp's Liouville theorem.

The version of the Caccioppoli inequality we prove in this subsection is the most sophisticated one. This version involves positive subsolutions as opposed to solutions. Therefore, the ℓ^p-Caccioppoli inequality does not reduce to the ℓ^2-version presented in the last subsection.

We start with a key lemma which will be used in the proof of the inequality as well as in the proof of Karp's theorem, Theorem 12.17.

For this reason, we state and prove it separately. For the statement, we recall that a function is said to satisfy condition (C) if the weighted vertex degree is bounded on some ball about the support of the function.

LEMMA 12.13. *Let (b,c) be a graph over (X,m) and let ϱ be an intrinsic metric. Let $\varphi \in \ell^\infty(X)$ satisfy* (C) *with respect to ϱ. Let $u \in \mathcal{F}$ satisfy $u \geq 0$, $u1_{N(\varphi)} \in \ell^p(X,m)$ for some $p \in (1,\infty)$ and $\mathcal{L}u \leq \lambda u$ on $\operatorname{supp}\varphi$ for $\lambda \in \mathbb{R}$. Then,*

$$C \sum_{x,y\in X} b(x,y)\varphi^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x\in X} c(x)(u^p\varphi^2)(x)$$
$$\leq \lambda\|u\varphi^{2/p}\|_p^p - \sum_{x,y\in X} b(x,y)\varphi(y)u^{p-1}(x)\nabla_{x,y}u\nabla_{x,y}\varphi,$$

where $C = (1\wedge(p-1))/2$ and $(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 = 0$ if $u(x) = u(y) = 0$.

PROOF. We note from the assumptions that $\varphi^2u^{p-1} \in \ell^q(X,m)$ for $1/p + 1/q = 1$. Hence, from $\mathcal{L}u \leq \lambda u$ with $u \geq 0$ and the Green's formula, Proposition 12.10, applied to $f = u$, $g = \varphi^2u^{p-1}$ and $U = \operatorname{supp}\varphi$, we get

$$\begin{aligned}
&\lambda\|u\varphi^{2/p}\|_p^p - \sum_{x\in X} c(x)(u^p\varphi^2)(x)\\
&= \sum_{x\in X}\lambda(u^p\varphi^2)(x)m(x) - \sum_{x\in X} c(x)(u^p\varphi^2)(x)\\
&\geq \sum_{x\in X}\mathcal{L}u(x)(u^{p-1}\varphi^2)(x)m(x) - \sum_{x\in X} c(x)(u^p\varphi^2)(x)\\
&= \frac{1}{2}\sum_{x\in X} b(x,y)\nabla_{x,y}u\nabla_{x,y}(u^{p-1}\varphi^2).
\end{aligned}$$

By the first Leibniz rule found in Lemma 2.25 we have

$$\nabla_{x,y}(u^{p-1}\varphi^2) = u^{p-1}(x)\nabla_{x,y}\varphi^2 + \varphi^2(y)\nabla_{x,y}u^{p-1}$$

and thus

$$\ldots = \frac{1}{2}\sum_{x\in X} b(x,y)\nabla_{x,y}u\left(u^{p-1}(x)\nabla_{x,y}\varphi^2 + \varphi^2(y)\nabla_{x,y}u^{p-1}\right).$$

Furthermore, a direct calculation gives $\nabla_{x,y}\varphi^2 = 2\varphi(y)\nabla_{x,y}\varphi + (\nabla_{x,y}\varphi)^2$ so that

$$\ldots = \frac{1}{2}\sum_{x,y\in X} b(x,y)\nabla_{x,y}u$$
$$\cdot\Big(2u^{p-1}(x)\varphi(y)\nabla_{x,y}\varphi + u^{p-1}(x)(\nabla_{x,y}\varphi)^2 + \varphi^2(y)\nabla_{x,y}u^{p-1}\Big).$$

One can separate the sum above into three sums which are absolutely convergent by applying Lemma 12.9 repeatedly with $f = u$ and $g = u^{p-1}$. For the second of these sums, we use symmetry to observe

$$\sum_{x,y\in X} b(x,y)u^{p-1}(x)(\nabla_{x,y}u)(\nabla_{x,y}\varphi)^2 = \frac{1}{2}\sum_{x,y\in X} b(x,y)(\nabla_{x,y}u^{p-1})(\nabla_{x,y}u)(\nabla_{x,y}\varphi)^2 \geq 0$$

as $\nabla_{x,y}u^{p-1}\nabla_{x,y}u \geq 0$ for $p > 1$, which holds since $\nabla_{x,y}u^{p-1}$ and $\nabla_{x,y}u$ have the same sign. As a result of this positivity, we may drop this term when we estimate the sum from below. For the third term in the sum, we apply the mean value inequality for the function u^{p-1} found in Lemma 2.28 (c) to obtain

$$\nabla_{x,y}u\nabla_{x,y}u^{p-1} = |\nabla_{x,y}u||\nabla_{x,y}u^{p-1}| \geq 2C(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2$$

with $C = (1 \wedge (p-1))/2$. We note that for the case when $u(x) = 0$ or $u(y) = 0$ the inequality is trivial and in the case where both are 0 we use the convention $(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 = 0$ as assumed. Putting these estimates together we arrive at

$$\lambda\|u\varphi^{2/p}\|_p^p - \sum_{x\in X} c(x)(u^p\varphi^2)(x) \geq \sum_{x,y\in X} b(x,y)u^{p-1}(x)\varphi(y)(\nabla_{x,y}u)(\nabla_{x,y}\varphi) + C\sum_{x,y\in X} b(x,y)\varphi^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2.$$

The statement follows by rearranging the terms. $\square$

Using the lemma above, we now state and prove a Caccioppoli inequality for positive subsolutions in $\ell^p(X,m)$ for $p \in (1,\infty)$.

THEOREM 12.14 (Caccioppoli inequality – ℓ^p version). *Let (b,c) be a graph over (X,m) and let ϱ be an intrinsic metric. Let $\varphi \in \ell^\infty(X)$ satisfy* (C) *with respect to ϱ. Let $u \in \mathcal{F}$ satisfy $u \geq 0$, $u1_{N(\varphi)} \in \ell^p(X,m)$ for some $p \in (1,\infty)$ and $\mathcal{L}u \leq \lambda u$ on $\operatorname{supp}\varphi$ for $\lambda \in \mathbb{R}$. Then,*

$$C\sum_{x,y\in X} b(x,y)\varphi^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x\in X} c(x)(u^p\varphi^2)(x) \leq \lambda\|u\varphi^{2/p}\|_p^p + \frac{1}{4C}\sum_{x,y\in X} b(x,y)u^p(x)(\nabla_{x,y}\varphi)^2,$$

where $C = (1\wedge(p-1))/4$ and $(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 = 0$ if $u(x) = u(y) = 0$.

Proof. Let $C_0 = (1 \wedge (p-1))/2$. By applying Lemma 12.13 we have the estimate

$$\begin{aligned}
&C_0 \sum_{x,y \in X} b(x,y)\varphi^2(y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y} u)^2 + \sum_{x \in X} c(x)(u^p \varphi^2)(x) \\
&\leq \lambda \|u\varphi^{2/p}\|_p^p - \sum_{x,y \in X} b(x,y)\varphi(y) u^{p-1}(x) \nabla_{x,y} u \nabla_{x,y}\varphi \\
&\leq \lambda \|u\varphi^{2/p}\|_p^p + \sum_{x,y \in X} b(x,y)\varphi(y)(u(x) \vee u(y))^{p-1} |\nabla_{x,y} u||\nabla_{x,y}\varphi|.
\end{aligned}$$

We now employ the inequality $|\alpha\beta| \leq \frac{1}{2C_0}\alpha^2 + \frac{C_0}{2}\beta^2$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha = (u(x) \vee u(y))^{p/2}|\nabla_{x,y}\varphi|$ and $\beta = \varphi(y)(u(x) \vee u(y))^{p/2-1}|\nabla_{x,y} u|$ to estimate

$$\begin{aligned}
\ldots \leq \lambda \|u\varphi^{2/p}\|_p^p &+ \frac{1}{2C_0} \sum_{x,y \in X} b(x,y)(u(x) \vee u(y))^p (\nabla_{x,y}\varphi)^2 \\
&+ \frac{C_0}{2} \sum_{x,y \in X} b(x,y)\varphi^2(y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y} u)^2.
\end{aligned}$$

Subtracting the third term on the right-hand side from both sides of the inequality yields

$$\begin{aligned}
&\frac{C_0}{2} \sum_{x,y \in X} b(x,y)\varphi^2(y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y} u)^2 + \sum_{x \in X} c(x)(u^p \varphi^2)(x) \\
&\qquad \leq \lambda \|u\varphi^{2/p}\|_p^p + \frac{1}{2C_0} \sum_{x,y \in X} b(x,y)(u(x) \vee u(y))^p (\nabla_{x,y}\varphi)^2.
\end{aligned}$$

Finally, invoking the inequality $(u(x) \vee u(y))^p \leq u^p(x) + u^p(y)$, which holds since $u \geq 0$, and using symmetry, we obtain the desired statement. □

2. Liouville theorems

In this section we prove two Liouville theorems. The classic Liouville theorem states that every bounded harmonic function is constant. We will present two variants of such a result, each arising by replacing the assumption of boundedness by an ℓ^p bound.

We will now use the ℓ^p version of the Caccioppoli inequality to study harmonic and positive subharmonic functions. Recall that $u \in \mathcal{F}$ is called harmonic if $\mathcal{L}u = 0$ and subharmonic if $\mathcal{L}u \leq 0$. Liouville theorems give conditions for such functions to be constant.

We have already seen one such result, specifically, that every positive subharmonic function in $\ell^p(X,m)$ for $p \in [1,\infty)$ must be 0 whenever the graph is connected and the measure of every infinite path is infinite, see Theorem 8.3 in Section 1. In this section, instead of a condition on the measure, we assume a condition on the geometry.

Specifically, we assume the existence of an intrinsic metric such that the weighted degree is bounded on balls (B*). This will allow us to construct suitable cutoff functions and utilize the Caccioppoli inequalities. Our first result concerns positive subharmonic functions.

THEOREM 12.15 (Yau's Liouville theorem). *Let (b, c) be a connected graph over (X, m). If there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls* (B*), *then every positive subharmonic function $u \in \ell^p(X, m)$ for some $p \in (1, \infty)$ is constant.*

PROOF. Let $r > 0$, $o \in X$ and $B_r = B_r(o)$ be the ball of radius r around o defined with respect to the intrinsic metric ϱ. For $R > 0$, we define the cutoff function

$$\eta(x) = \eta_{r,R}(x) = \left(1 - \frac{\varrho(x, B_r)}{R}\right)_+,$$

where $x \in X$. We note that $1_{B_r} \leq \eta \leq 1_{B_{R+r}}$. Moreover, by definition η is 1/R-Lipschitz, i.e., satisfies $|\eta(x) - \eta(y)| \leq \frac{1}{R}\varrho(x, y)$. As ϱ is intrinsic, this implies

$$\sum_{y \in X} b(x, y)(\nabla_{x,y}\eta)^2 \leq \frac{1}{R^2} m(x)$$

for $x \in X$.

Using these properties and the ℓ^p-Caccioppoli inequality, Theorem 12.14, with $\varphi = \eta$ and $\lambda = 0$ we get the following inequality, for some constant $C > 0$,

$$\begin{aligned}
&\sum_{x \in X} \sum_{y \in B_r} b(x, y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x \in B_r} c(x)u^p(x) \\
&\leq \sum_{x,y \in X} b(x, y)\eta^2(y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x \in X} c(x)(u^p\eta^2)(x) \\
&\leq C \sum_{x,y \in X} b(x, y)u^p(x)(\nabla_{x,y}\eta)^2 \\
&\leq \frac{C}{R^2}\|u\|_p^p.
\end{aligned}$$

We recall our convention that $(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 = 0$ if $u(x) = u(y) = 0$.

Since $u \in \ell^p(X, m)$, the right-hand side is finite and letting $R \to \infty$ yields

$$\sum_{x \in X} \sum_{y \in B_r} b(x, y)(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x \in B_r} c(x)u^p(x) = 0.$$

Since this holds for all $r > 0$, Fatou's lemma leads to

$$\sum_{x,y \in X} b(x, y)(u(x) \vee u(y))^{p-2}(u(x) - u(y))^2 + \sum_{x \in X} c(x)u^p(x) = 0.$$

By connectedness, we infer that u is constant. □

REMARK. If the graph is not connected, we obtain that u is constant on every connected component of the graph. Furthermore, we note that the proof yields $u = 0$ whenever $c \neq 0$.

Since the positive and negative parts of a harmonic function are subharmonic, we get the following immediate corollary for harmonic functions.

COROLLARY 12.16 (Yau's Liouville theorem for harmonic functions). *Let (b, c) be a connected graph over (X, m). If there exists an intrinsic metric ϱ such that* Deg *is bounded on balls* (B*)*, then every harmonic function $u \in \ell^p(X, m)$ for some $p \in (1, \infty)$ is constant.*

PROOF. If u is harmonic, then the positive and negative parts $u_\pm$ of u are positive subharmonic functions by Lemma 1.9. Moreover, $\|u_\pm\|_p \leq \|u\|_p$. Hence, the statement follows from Theorem 12.15 and the fact that $u = u_+ - u_-$. □

REMARK. We note that in the theorem and corollary the case $p = 1$ is excluded. Indeed, the corresponding statement for $p = 1$ does not hold in general (Exercise 12.1). On the other hand, stochastic completeness implies that there are no non-zero positive superharmonic functions in $\ell^1(X, m)$ (Exercise 12.7).

Next, we present a Liouville theorem which has a weaker assumption on the subharmonic function but a stronger geometric assumption. Specifically, we require only that the ℓ^p norm of the function does not grow too fast but we add the assumption of finite jump size. We recall that the jump size s of an intrinsic metric ϱ is defined as

$$s = \sup_{x \sim y} \varrho(x, y)$$

and we say the intrinsic metric ϱ has finite jump size (J) if $s < \infty$. We denote the balls with respect to ϱ around a vertex $o \in X$ by $B_r = B_r(o)$.

THEOREM 12.17 (Karp's Liouville theorem). *Let (b, c) be a connected graph over (X, m). Suppose that there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls* (B*) *and ϱ has finite jump size* (J)*. A positive subharmonic function u is constant if*

$$\int_{r_0}^{\infty} \frac{r}{\|u 1_{B_r}\|_p^p} dr = \infty$$

for some $p \in (1, \infty)$ and some $r_0 \geq 0$ with $u 1_{B_{r_0}} \neq 0$.

REMARK. If for one $r_0 \geq 0$ and $o \in X$ with $u 1_{B_{r_0}(o)} \neq 0$ the integral

$$\int_{r_0}^{\infty} \frac{r}{\|u 1_{B_{r_0}(o)}\|_p^p} \, dr$$

diverges, then the integral will diverge for each $r \geq 0$ and $o' \in X$ with $u 1_{B_r(o')} \neq 0$. Furthermore, if u does not admit an $r > 0$ with $u 1_{B_r} \neq 0$, then u is equal to 0 and hence also constant.

PROOF. Let $u \in \mathcal{F}$ be a positive subharmonic function. We may assume $u1_{B_r} \in \ell^p(X,m)$ for all $r \geq 0$ since otherwise $\int_{r_1}^\infty r/\|u1_{B_r}\|_p^p dr = 0$ for some $r_1 \geq 0$ and this gives the contradiction $\int_{r_0}^\infty r/\|u1_{B_r}\|_p^p dr < \infty$.

Furthermore, for $R > r > s$, where s denotes the jump size of ϱ, we let η be the cutoff function defined by

$$\eta(x) = \eta_{r,R}(x) = \left(1 - \frac{\varrho(x, B_r)}{R-r}\right)_+ .$$

We note that $1_{B_r} \leq \eta \leq 1_{B_R}$. Additionally, by the fact that ϱ has finite jump size s, we get that the mapping $(x,y) \mapsto b(x,y)\nabla_{x,y}\eta$ is supported on $(B_{R+s} \setminus B_{r-s}) \times (B_{R+s} \setminus B_{r-s})$.

Taking these properties into account and squaring both sides of Lemma 12.13 with $\varphi = \eta$ and $\lambda = 0$, yields

$$\begin{aligned}
&\left(\sum_{x,y\in X} b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2\right)^2 \\
&\leq \left(\sum_{x,y\in X} b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 + \sum_{x\in X} c(x)(u^p\eta^2)(x)\right)^2 \\
&\leq C\left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)u^{p-1}(x)\eta(y)\nabla_{x,y}u\nabla_{x,y}\eta\right)^2 \\
&\leq C\left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)(u(x)\vee u(y))^{p-1}\eta(y)\nabla_{x,y}u\nabla_{x,y}\eta\right)^2
\end{aligned}$$

for some constant $C > 0$ which may change from line to line. We note that in the last line we used $p > 1$ to estimate $u^{p-1}(x)$ by $(u(x) \vee u(y))^{p-1}$. We next apply the Cauchy–Schwarz inequality and $(u(x) \vee u(y))^p \leq u^p(x) \vee u^p(y) \leq u^p(x) + u^p(y)$ to obtain

$$\begin{aligned}
\ldots \leq\, & C\left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)(u(x)\vee u(y))^p(\nabla_{x,y}\eta)^2\right) \\
&\cdot \left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2\right) \\
\leq\, & C\left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)u^p(x)(\nabla_{x,y}\eta)^2\right) \\
&\cdot \left(\sum_{x,y\in B_{R+s}\setminus B_{r-s}} b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2\right).
\end{aligned}$$

As η is a cutoff function defined with respect to an intrinsic metric ϱ we find $\sum_y b(x,y)(\nabla_{x,y}\eta)^2 \leq m(x)/(R-r)^2$ for $x \in B_{R+s} \setminus B_{r-s}$. We use this to estimate the first sum to obtain

$$\begin{aligned}\ldots \leq & \frac{C}{(R-r)^2}\|u1_{B_{R+s}\setminus B_{r-s}}\|_p^p \\ & \cdot \left(\sum_{x,y \in B_{R+s}\setminus B_{r-s}} b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 \right).\end{aligned}$$

Letting $F(x,y) = b(x,y)\eta^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2$ and putting everything together we have shown

$$\left(\sum_{x,y\in X} F(x,y) \right)^2 \leq \frac{C}{(R-r)^2}\|u1_{B_{R+s}\setminus B_{r-s}}\|_p^p \sum_{x,y\in B_{R+s}\setminus B_{r-s}} F(x,y)$$

for some positive constant C.

We next iterate this estimate over a sequence of radii (R_j) and a sequence of cutoff functions (φ_j) in place of $\eta_{r,R}$. We let $R_0 \geq 3s$ and let

$$\begin{aligned} R_j &= 2^j R_0 && \text{for } j \in \mathbb{N}, \\ \varphi_j &= \eta_{R_j+s, R_{j+1}-s} && \text{for } j \in \mathbb{N}_0. \end{aligned}$$

Letting

$$\delta_{j+1} = (R_{j+1}-s) - (R_j + s) = R_j - 2s$$

and using $\operatorname{supp}\varphi_j \subseteq B_{R_{j+1}-s}$, the estimate above yields

$$\begin{aligned} & \left(\sum_{x,y\in B_{R_{j+1}}} b(x,y)\varphi_j^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 \right)^2 \\ \leq & \frac{C}{\delta_{j+1}^2}\|u1_{B_{R_{j+1}}\setminus B_{R_j}}\|_p^p \left(\sum_{x,y\in B_{R_{j+1}}\setminus B_{R_j}} b(x,y)\varphi_j^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 \right). \end{aligned}$$

We let

$$\begin{aligned} U_j &= \|u1_{B_{R_j}}\|_p^p \\ Q_{j+1} &= \sum_{x,y\in B_{R_{j+1}}} b(x,y)\varphi_j^2(y)(u(x)\vee u(y))^{p-2}(\nabla_{x,y}u)^2 \end{aligned}$$

for $j \in \mathbb{N}_0$. We note that the estimate above implies

$$Q_{j+1}^2 \leq C\frac{U_{j+1}}{\delta_{j+1}^2}(Q_{j+1} - Q_j),$$

where we use $\varphi_{j-1} \leq \varphi_j$ and the fact that the sum in Q_j is over a smaller set than the sum which is actually subtracted from Q_{j+1}.

If $Q_1 = 0$, then as R_0 can be chosen arbitrarily large and $\varphi_0 = 1$ on B_{R_0}, we conclude

$$(u(x) \vee u(y))^{p-2}(\nabla_{x,y}u)^2 = 0$$

for all $x, y \in X$ with $x \sim y$. Therefore, connectedness implies that u is constant.

It remains to show that $Q_1 = 0$. Assume not, that is, $Q_1 > 0$. Since $\varphi_j \leq \varphi_{j+1}$ and $u \geq 0$ we infer $Q_j \leq Q_{j+1}$ for all $j \in \mathbb{N}_0$ and, therefore,

$$Q_j Q_{j+1} \leq Q_{j+1}^2 \leq C\frac{U_{j+1}}{\delta_{j+1}^2}(Q_{j+1} - Q_j)$$

for $j \in \mathbb{N}_0$. As $Q_j \neq 0$ for all $j \in \mathbb{N}$, we may rearrange the terms above to yield

$$\frac{1}{C}\frac{\delta_{j+1}^2}{U_{j+1}} + \frac{1}{Q_{j+1}} \leq \frac{1}{Q_j}.$$

Hence, starting at $j = 1$ and iterating this estimate over j, gives

$$\frac{1}{C}\sum_{j=1}^{\infty}\frac{\delta_{j+1}^2}{U_{j+1}} \leq \frac{1}{Q_1} < \infty.$$

However, as $R_j = 2^j R_0$, we note that

$$\frac{1}{2}\int_{R_j}^{R_{j+1}} \frac{r}{\|u1_{B_r}\|_p^p}dr \leq \frac{1}{2}(R_{j+1} - R_j)\frac{R_{j+1}}{\|u1_{B_{R_j}}\|_p^p} = \frac{R_j^2}{\|u1_{B_{R_j}}\|_p^p}.$$

Therefore, the assumption $\int_{r_0}^{\infty} r/\|u1_{B_{r_j}}\|_p^p dr = \infty$ yields the divergence of
$\sum_{j\geq r_0} R_j^2/\|u1_{B_{R_j}}\|_p^p$ and thus of $\sum_{j\geq r_0}\delta_j^2/U_j$ by the limit comparison test. This contradiction completes the proof. □

Remark. The result above can be formulated in terms of the growth of a positive subharmonic function (Exercise 12.8).

As in the case of Yau's Liouville theorem, we immediately get the following version of Karp's theorem for harmonic functions.

Corollary 12.18 (Karp's Liouville theorem for harmonic functions). *Let (b, c) be a connected graph over (X, m). Suppose that there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls* (B*) *and ϱ has finite jump size* (J). *Let B_r denote the distance ball of radius r about a vertex with respect to ϱ. Then, every harmonic function u such that*

$$\int_{r_0}^{\infty}\frac{r}{\|u1_{B_r}\|_p^p}dr = \infty$$

for some $r_0 \geq 0$ with $u1_{B_{r_0}} \neq 0$ and some $p \in (1, \infty)$ is constant.

PROOF. If u is harmonic, then the positive and negative parts $u_\pm$ of u are positive subharmonic functions by Lemma 1.9. Moreover, $\|u_\pm\|_p \leq \|u\|_p$. Hence, the statement follows from Theorem 12.17 and the fact that $u = u_+ - u_-$. $\square$

We finish this section with a series of applications which are left as exercises.

REMARK. Let (b, c) be a connected graph over (X, m) with an intrinsic metric ϱ such that the weighted vertex degree is bounded on distance balls (B*) and has finite jump size (J). Moreover, let

$$\varrho_1(\cdot) = \varrho(o, \cdot) \vee 1$$

for some $o \in X$. Show that:

(a) Every positive subharmonic function in $\ell^p(X, \varrho_1^{-2} m)$ for some $p \in (1, \infty)$ is constant (Exercise 12.9).

(b) If $\varrho_1^q \in \ell^1(X, m)$ for $q \in \mathbb{R}$, then every positive subharmonic function u such that, for some $\varepsilon > 0$ and $C \geq 0$,

$$u \leq C\varrho_1^{q+2-\varepsilon}$$

is constant. In particular, for $q > -2$ every bounded subharmonic function is constant (Exercise 12.10).

(c) If $m(X) < \infty$, then every positive subharmonic function u such that, for some $\varepsilon > 0$ and $C \geq 0$,

$$u \leq C\varrho_1^{2-\varepsilon}$$

is constant. In particular, every bounded subharmonic function u is constant (Exercise 12.11).

(d) If, for some $\beta > 0$,

$$\limsup_{r\to\infty} \frac{1}{r^\beta} \log m(B_{r+1} \setminus B_r) < 0,$$

then every positive subharmonic function u such that there exist $p > 0$ and $C \geq 0$ with

$$u \leq C\varrho_1^p$$

is constant (Exercise 12.12).

3. Applications of the Liouville theorems

In this section we derive consequences of the Liouville theorems. In particular, we use Yau's Liouville theorem to show several uniqueness properties for forms and operators. Moreover, we use Karp's Liouville theorem to give a criterion for recurrence.

We now harvest various consequences of the Liouville theorems established in Section 2. More specifically, we will show form uniqueness, Markov uniqueness and essential self-adjointness whenever the weighted degree is bounded on distance balls for some intrinsic metric by applying Yau's theorem in Subsection 3.1. We will then use Yau's theorem again to characterize the domain of the generators on ℓ^p in Subsection 3.2. Finally, we will apply Karp's theorem to establish a volume growth criterion for recurrence in Subsection 3.3.

The Liouville theorems presented in the previous section only establish the constancy of positive subharmonic functions. However, to prove form uniqueness and essential self-adjointness, we need to show the triviality of α-harmonic functions for $\alpha > 0$. The following basic lemma allows us to pass from the constancy of positive subharmonic functions to triviality of α-harmonic functions. It will be used repeatedly in the results that follow.

LEMMA 12.19. *Let (b, c) be a graph over (X, m) and let $p \in [1, \infty]$. If every positive subharmonic function in $\ell^p(X, m)$ is constant, then every α-harmonic function in $\ell^p(X, m)$ is trivial for $\alpha > 0$.*

PROOF. By Lemma 1.9, both the positive and negative parts $u_\pm$ of an α-harmonic function u are positive α-subharmonic functions and hence subharmonic if $\alpha > 0$. Therefore, u_+ and u_- are constant by assumption and so is $u = u_+ - u_-$. As $(\mathcal{L} + \alpha)u = (c/m + \alpha)u = 0$ and $\alpha > 0$, it follows that $u = 0$. $\square$

3.1. Form uniqueness and essential self-adjointness. In this subsection we discuss the uniqueness of the form and essential self-adjointness of the Laplacian when there exists an intrinsic metric such that the weighted degree is bounded on balls.

We first discuss form uniqueness. We recall that $Q^{(D)}$ is the restriction of $\mathcal{Q}$ to the form closure of the restriction to $C_c(X)$ while $Q^{(N)}$ is the restriction of $\mathcal{Q}$ to $\mathcal{D} \cap \ell^2(X, m)$. In Theorem 3.12 we have shown that form uniqueness is equivalent to Markov uniqueness of the operator. We start by showing that these two form domains are equal whenever condition (B*) holds by applying Yau's Liouville theorem.

THEOREM 12.20 ((B*) implies $Q^{(D)} = Q^{(N)}$). *Let (b, c) be a graph over (X, m). If there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls* (B*)*, then*

$$Q^{(D)} = Q^{(N)}.$$

In particular, there exists a unique operator L associated to (b, c) which is the unique Markov realization of $\mathcal{L}$ and has domain

$$D(L) = \{f \in \mathcal{D} \cap \ell^2(X, m) \mid \mathcal{L}f \in \ell^2(X, m)\}.$$

PROOF. Theorem 3.2 gives that $Q^{(D)} = Q^{(N)}$ if and only if every α-harmonic function for $\alpha > 0$ in $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$ is trivial. Hence, the result follows immediately by applying Theorem 12.15 and

Lemma 12.19 to connected components of the graph. The "in particular" statement follows immediately from Corollary 3.3 and Theorem 3.12. □

Next, we discuss essential self-adjointness. We recall that essential self-adjointness means that the restriction of $\mathcal{L}$ to $C_c(X)$ has a unique self-adjoint extension in $\ell^2(X,m)$. To be able to even state this property, we additionally need the assumption that $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$. This assumption is characterized in Theorem 1.29.

Theorem 12.21 ((B*) implies essential self-adjointness). *Let (b,c) be a graph over (X,m). If $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ and there exists an intrinsic metric ϱ that such that* Deg *is bounded on distance balls (B*), then the restriction of $\mathcal{L}$ to $C_c(X)$ is essentially self-adjoint and the unique self-adjoint extension L has domain*

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}.$$

Proof. By Theorem 3.6, essential self-adjointness of the restriction of $\mathcal{L}$ to $C_c(X)$ is equivalent to the triviality of α-harmonic functions in $\ell^2(X,m)$ for $\alpha > 0$. Hence, essential self-adjointness follows immediately by combining Theorem 12.15 and Lemma 12.19 over connected components of the graph. The statement about the domain of L follows from Theorem 3.6 as $L = L^{(D)}$ in the essentially self-adjoint case. □

Remark. We note that as finiteness of balls (B) clearly implies that the weighted degree function is bounded on balls (B*), it follows that the results above generalize the Gaffney theorems, Theorems 12.7 and 12.8, found in Subsection 1.1.

3.2. Domains of the generators. In this subsection we give consequences of the Liouville theorems for the domains of the generators on $\ell^p(X,m)$.

We recall that the generators $L^{(p)}$ of semigroups and resolvents on $\ell^p(X,m)$ for $p \in (1,\infty)$ were introduced in Section 1 by extending the semigroup on the Hilbert space $\ell^2(X,m)$ to all $\ell^p(X,m)$ for $p \in [1,\infty]$. In particular, we note that by Theorem 2.13, the action of $L^{(p)}$ is given by $\mathcal{L}$.

Theorem 12.22 (Domain of $L^{(p)}$ under (B*)). *Let (b,c) be a graph over (X,m). Let $D(L^{(p)})$ denote the domain of $L^{(p)}$, the generator of the semigroup on $\ell^p(X,m)$ for $p \in (1,\infty)$. If there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls (B*), then*

$$D(L^{(p)}) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}.$$

Proof. From Theorem 3.8 we get that $D(L^{(p)}) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}$ if and only if α-harmonic functions for $\alpha > 0$ in

$\ell^p(X,m)$ are trivial. Hence, the statement follows immediately by Theorem 12.15 and Lemma 12.19 applied to connected components of the graph. $\square$

3.3. Recurrence. We now show that if the measure of distance balls does not grow too rapidly, then a graph is recurrent.

We recall that in Chapter 6 we discussed the notion of recurrence for connected graphs. One characterization of recurrence is that all bounded superharmonic functions, i.e., bounded functions u satisfying $\mathcal{L}u \geq 0$, are constant. We use this characterization along with Karp's theorem to show recurrence under a growth condition on the measure of balls with respect to an intrinsic metric.

THEOREM 12.23 (Volume growth and recurrence). *Let b be a connected graph over (X,m). Suppose that there exists an intrinsic metric ϱ such that* Deg *is bounded on distance balls* (B*) *and ϱ has finite jump size* (J). *Let B_r denote the distance ball of radius r about a vertex o defined with respect to ϱ. If*

$$\int_{r_0}^{\infty} \frac{r}{m(B_r)} dr = \infty$$

for some $r_0 \geq 0$, then b is recurrent.

REMARK. It is not hard to see that $\int_{r_0}^{\infty} r/m(B_r)dr = \infty$ for all $r_0 \geq 0$ and all $o \in X$ if the integral diverges for one $r_0 \geq 0$ and one $o \in X$.

PROOF. By Theorem 6.1 (iv.c) the graph is recurrent if and only if every bounded superharmonic function is constant. Let v be a nontrivial bounded superharmonic function. Then, as $c = 0$, $u = -v + \|v\|_\infty$ is a positive bounded subharmonic function. We may assume that there exists an $r_1 > r_0$ such that $u1_{B_{r_1}} \neq 0$. By assumption, for $p \in (1,\infty)$, we have

$$\int_{r_1}^{\infty} \frac{r}{\|u1_{B_r}\|_p^p} dr \geq \frac{1}{\|u\|_\infty^p} \int_{r_0}^{\infty} \frac{r}{m(B_r)} dr = \infty.$$

By Karp's Liouville theorem, Theorem 12.17, we infer that u is constant, which implies that v is constant as well. This finishes the proof. $\square$

We finish this section with a series of remarks.

REMARK. Theorem 12.23 is optimal in the sense that there exist examples of transient graphs satisfying (B*) and (J) with $m(B_r)$ growing like $r^{2+\varepsilon}$ for any $\varepsilon > 0$ (Exercise 12.2). It is also possible to give a characterization of recurrence involving the finiteness of the measure of the space (Exercise 12.13).

REMARK. We will prove a criterion for stochastic completeness which allows for a much stronger volume growth, see Theorem 14.11 in Section 3.

4. Shnol′ theorems

In this section we use Caccioppoli inequalities to study the spectrum of the Laplacian via generalized eigenfunctions. In particular, we prove two versions of a Shnol′ result which states that whenever generalized eigenfunctions do not grow too fast, the generalized eigenvalue is in the spectrum.

We denote the spectrum of the Laplacian $L = L^{(D)}$ associated to a graph (b, c) over (X, m) by $\sigma(L)$. We refer to a non-trivial function $u \in \mathcal{F}$ which satisfies

$$\mathcal{L}u = \lambda u$$

for $\lambda \in \mathbb{R}$ as a *generalized eigenfunction*. We prove two versions of a Shnol′ theorem for which we make the notion of growing not too fast precise. In the first version, we consider subexponentially bounded generalized eigenfunctions. In the second, more general, version, we assume that the ratio of norms of the generalized eigenfunction on the boundary divided by the interior tends to zero.

We first make the notion of a subexponentially bounded function precise.

DEFINITION 12.24 (Subexponentially bounded function). Let ϱ be an intrinsic metric. A function $u \in C(X)$ is said to be *subexponentially bounded* with respect to ϱ if, for some $o \in X$ and all $\alpha > 0$,

$$e^{-\alpha\varrho(o,\cdot)}u \in \ell^2(X, m).$$

REMARK. The definition above raises the question if the notion depends on the choice of $o \in X$. In fact, if $e^{-\alpha\varrho(o,\cdot)}u \in \ell^2(X, m)$ for some $o \in X$ and some $\alpha > 0$, then $e^{-\beta\varrho(x,\cdot)}u \in \ell^2(X, m)$ for all $x \in X$ and all $\beta \geq \alpha$ (Exercise 12.14).

REMARK. We have seen the notion of graphs satisfying a strong isoperimetric inequality. In this case, if the weighted degree is bounded, the constant functions are not subexponentially bounded (Exercise 12.15).

For both versions of the Shnol′ theorem we require that balls defined with respect to an intrinsic metric are finite and that the metric has finite jump size. We note that under these assumptions, the graph is locally finite by Lemma 11.28 and there exists a unique operator L associated to (b, c) by Theorem 12.20.

THEOREM 12.25 (Shnol′ theorem). *Let (b, c) be a graph over (X, m) and let ϱ be an intrinsic metric with finite distance balls* (B) *and finite jump size* (J). *If $\lambda \in \mathbb{R}$ and $u \in \mathcal{F}$ satisfy*

$$\mathcal{L}u = \lambda u$$

and $u \neq 0$ is subexponentially bounded, then $\lambda \in \sigma(L)$.

REMARK. From the theorem above we can conclude that if the measure of balls grows subexponentially, then 0 is in the spectrum of L if $c = 0$ (Exercise 12.16).

We will deduce the first version of the Shnol′ theorem from the more general version presented below. To state this second version, we require the notion of an annulus about a set. For $U \subseteq X$ and $R \geq 0$, we let

$$A_R(U) = B_R(U) \cap B_R(X \setminus U),$$

which can be understood as an annulus of radius R about the boundary of U.

THEOREM 12.26 (General Shnol′ theorem). *Let (b, c) be a graph over (X, m) and let ϱ be an intrinsic metric with finite distance balls* (B) *and finite jump size* (J) *denoted by s. Let $\lambda \in \mathbb{R}$ and let $u \in \mathcal{F}$ be such that $\mathcal{L}u = \lambda u$ and $u \neq 0$. If there exists a sequence of finite sets (U_n) and $R > 2s$ with*

$$\frac{\|u 1_{A_R(U_n)}\|}{\|u 1_{U_n}\|} \to 0$$

as $n \to \infty$, then $\lambda \in \sigma(L)$.

The key estimate for the proof of the general Shnol′ theorem is the Shnol′ inequality presented below.

LEMMA 12.27 (Shnol′ inequality). *Let (b, c) be a graph over (X, m) and let ϱ be an intrinsic metric with finite distance balls* (B) *and finite jump size* (J) *denoted by s. Let U be a finite set and let $u \in \mathcal{F}$ and $\lambda \in \mathbb{R}$ satisfy $\mathcal{L}u = \lambda u$. Let $\varepsilon > 0$ and define $\eta = \eta_{U,\varepsilon}$ by*

$$\eta(x) = \eta_{U,\varepsilon}(x) = \left(1 - \frac{\varrho(x, U)}{\varepsilon}\right)_+ .$$

Then, there exists a $C \geq 0$ such that, for all $v \in \mathcal{D} \cap \ell^2(X, m)$,

$$\left|(\mathcal{Q} - \lambda)(u\eta^2, v)\right| \leq C \|v\|_{\mathcal{Q}} \|u 1_{A_{\varepsilon+2s}(U)}\|.$$

REMARK. As U is finite, so is $B_\varepsilon(U)$ due to the assumption of finiteness of distance balls. In particular, $\eta \in C_c(X)$ as η is supported on $B_\varepsilon(U)$. Hence, $u\eta^2$ belongs to $\mathcal{D}$ and we can form $\mathcal{Q}(u\eta^2, v)$ in the statement of the lemma.

PROOF. Due to the finiteness of U and the finiteness of distance balls, the set $B_\delta(U)$ is finite for all $\delta > 0$. In particular, the sets $A_{\varepsilon+s}(U)$ and $D_{\varepsilon,s}$, which will be introduced below, are finite as they are subsets of $B_\delta(U)$ for suitable δ. For the same reason, all cutoff functions encountered in the proof belong to $C_c(X)$. Finally, we note that $N(U) \subseteq B_s(U)$ for the combinatorial neighborhood $N(U)$ of U so that $N(U)$ is finite. These finiteness properties will be used tacitly in the proof.

By Green's formula, Proposition 1.4, applied to $u \in \mathcal{F}$, which satisfies $\mathcal{L}u = \lambda u$ and $\eta^2 v \in C_c(X)$, we have

$$\begin{aligned}\lambda\langle u\eta^2, v\rangle &= \sum_{x\in X}(\lambda u)(x)\eta^2(x)v(x)m(x) = \sum_{x\in X}\mathcal{L}u(x)(\eta^2 v)(x)m(x)\\ &= \frac{1}{2}\sum_{x,y\in X} b(x,y)\nabla_{x,y}u\nabla_{x,y}(\eta^2 v) + \sum_{x\in X} c(x)(u\eta^2 v)(x).\end{aligned}$$

On the other hand, by the definition of $\mathcal{Q}$, we get

$$\mathcal{Q}(u\eta^2, v) = \frac{1}{2}\sum_{x,y\in X} b(x,y)\nabla_{x,y}(u\eta^2)\nabla_{x,y}v + \sum_{x\in X} c(x)(u\eta^2 v)(x).$$

Hence, combining these equalities gives

$$(\mathcal{Q}-\lambda)(u\eta^2, v) = \frac{1}{2}\sum_{x,y\in X} b(x,y)\left(\nabla_{x,y}(u\eta^2)\nabla_{x,y}v - \nabla_{x,y}u\nabla_{x,y}(\eta^2 v)\right).$$

Now, applying the first two Leibniz rules in Lemma 2.25 gives

$$\begin{aligned}\ldots = \frac{1}{2}\sum_{x,y\in X} b(x,y)\big(&u(x)\nabla_{x,y}\eta^2\nabla_{x,y}v + \eta^2(y)\nabla_{x,y}u\nabla_{x,y}v\\ &- v(x)\nabla_{x,y}u\nabla_{x,y}\eta^2 - \eta^2(y)\nabla_{x,y}u\nabla_{x,y}v\big).\end{aligned}$$

We observe that the second and fourth terms above cancel. Applying the equality $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$ for $\alpha, \beta \in \mathbb{R}$ to the remaining terms yields

$$\begin{aligned}\ldots = \frac{1}{2}\sum_{x,y\in X} b(x,y)\big(&u(x)(\eta(x)+\eta(y))\nabla_{x,y}\eta\nabla_{x,y}v\\ &- v(x)(\eta(x)+\eta(y))\nabla_{x,y}\eta\nabla_{x,y}u\big).\end{aligned}$$

Now, by construction, η is supported in $B_\varepsilon(U)$ and constant on U. Thus, we have $\nabla_{x,y}\eta = 0$ for $x \sim y$ if either x or y is not in

$$D_{\varepsilon,s}(U) = B_{\varepsilon+s}(U) \cap B_s(X \setminus U),$$

which is a subset of $A_{\varepsilon+s}(U)$. Furthermore, $0 \le \eta \le 1$, so $\eta(x)+\eta(y) \le 2$. Hence,

$$\begin{aligned}|(\mathcal{Q}-\lambda)(u\eta^2, v)| \le &\sum_{x,y\in X} b(x,y)|u(x)||\nabla_{x,y}\eta||\nabla_{x,y}v|\\ &+\frac{1}{2}\sum_{x,y\in D_{\varepsilon,s}(U)} b(x,y)|v(x)(\eta(x)+\eta(y))\nabla_{x,y}\eta||\nabla_{x,y}u|.\end{aligned}$$

Using the Cauchy–Schwarz inequality, the inequality $(\alpha+\beta)^2 \le 2\alpha^2 + 2\beta^2$ for $\alpha, \beta \in \mathbb{R}$ and symmetry yields

$$\begin{aligned}\ldots \le \mathcal{Q}^{1/2}(v)&\left(2\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\eta)^2\right)^{1/2}\\ &+\left(\sum_{x,y\in D_{\varepsilon,s}(U)} b(x,y)\eta^2(x)(\nabla_{x,y}u)^2\right)^{1/2}\\ &\cdot\left(\sum_{x,y\in D_{\varepsilon,s}(U)} b(x,y)v^2(x)(\nabla_{x,y}\eta)^2\right)^{1/2}.\end{aligned}$$

We now discuss the terms appearing in the preceding estimate: The first term in the first product is part of our ultimate estimate. As for the second term of the first product, we note from the discussion above that $\nabla_{x,y}\eta = 0$ if either x or y is not in $A_{\varepsilon+s}(U)$ and $x \sim y$. Moreover, by definition, the function η is $1/\varepsilon$-Lipschitz continuous with respect to ϱ. So, as ϱ is intrinsic we get by a direct computation, see Lemma 11.3, the estimate

$$\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2 \le \frac{m(x)}{\varepsilon^2}1_{A_{\varepsilon+s}(U)}(x).$$

Hence, the term in question can be estimated by

$$\begin{aligned}\sum_{x,y\in X} b(x,y)u^2(x)(\nabla_{x,y}\eta)^2 &= \sum_{x\in X} u^2(x)\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2\\ &\le \frac{1}{\varepsilon^2}\|u1_{A_{\varepsilon+s}(U)}\|^2.\end{aligned}$$

We now focus on the second product. For the first term, namely

$$\sum_{x,y\in D_{\varepsilon,s}(U)} b(x,y)\eta^2(x)(\nabla_{x,y}u)^2,$$

we want to apply the Caccioppoli inequality, Theorem 12.6. Here, a slight adjustment is necessary for the following reason: We want to end up with the restriction of u to an annulus, specifically to $A_{\varepsilon+2s}(U)$. Now, the sum is taken over a type of annulus $D_{\varepsilon,s}(U)$ which is contained in $A_{\varepsilon+s}(U)$. However, the cutoff function η appearing in the sum is not supported on an annulus but rather on $B_\varepsilon(U)$. In order to remedy this, we will replace η by a function χ which agrees with η on $D_{\varepsilon,s}(U)$ and at the same time is supported on an annulus. More precisely, we define

$$\chi = \left(1 - \frac{\varrho(\cdot, U \cap B_s(X \setminus U))}{\varepsilon}\right)_+$$

and note that χ agrees with η on $D_{\varepsilon,s}(U)$, is supported on $A_{\varepsilon+s}(U)$ and satisfies $\nabla_{x,y}\chi \neq 0$ only if both x and y belong to $A_{\varepsilon+2s}(U)$ for $x \sim y$. Now, we are ready to apply Theorem 12.6 in the following chain of estimates,

$$\begin{aligned}
&\sum_{x,y \in D_{\varepsilon,s}(U)} b(x,y)\eta^2(x)(\nabla_{x,y}u)^2 \\
&\qquad = \sum_{x,y \in D_{\varepsilon,s}(U)} b(x,y)\chi^2(x)(\nabla_{x,y}u)^2 \\
&\qquad \leq \sum_{x,y \in X} b(x,y)\chi^2(x)(\nabla_{x,y}u)^2 \\
&\qquad \leq C\left(\lambda\|\chi u\|^2 + \sum_{x,y \in X} b(x,y)u^2(x)(\nabla_{x,y}\chi)^2\right) \\
&\qquad \leq C\left(\lambda + \frac{1}{\varepsilon^2}\right)^2 \|u 1_{A_{\varepsilon+2s}(U)}\|^2.
\end{aligned}$$

Here, the last inequality follows from the fact that χ is supported on $A_{\varepsilon+s}(U)$ combined with the estimate

$$\sum_{y \in X} b(x,y)(\nabla_{x,y}\chi)^2 \leq \frac{m(x)}{\varepsilon^2} 1_{A_{\varepsilon+2s}(U)}(x).$$

Finally, the second term of the second product can be estimated as

$$\begin{aligned}
\sum_{x,y \in D_{\varepsilon,s}(U)} b(x,y)v^2(x)(\nabla_{x,y}\eta)^2 &\leq \sum_{x \in X} v^2(x)\left(\sum_{y \in X} b(x,y)(\nabla_{x,y}\eta)^2\right) \\
&\leq \frac{1}{\varepsilon^2}\|v\|^2.
\end{aligned}$$

Putting these three estimates into the chain of inequalities above we conclude the result since $\|v\|_{\mathcal{Q}}^2 = \mathcal{Q}(v) + \|v\|^2$. $\square$

REMARK. We comment on the structural content of the preceding lemma in a series of remarks:

(a) The argument given in the above proof actually provides an estimate for the term

$$T(u,v,\eta) = \left|\frac{1}{2}\sum_{x,y \in X} b(x,y)\left(\nabla_{x,y}(u\eta^2)\nabla_{x,y}v - \nabla_{x,y}u\nabla_{x,y}(\eta^2 v)\right)\right|$$

for arbitrary $u \in \mathcal{F}$ and $v \in \mathcal{D}$. More specifically, it shows that $T(u, v, \eta)$ is bounded above by $(2 + \frac{1}{\varepsilon})$ times

$$\mathcal{Q}^{1/2}(v)\|u1_{A_{\varepsilon+s}(U)}\| + \|v1_{A_{\varepsilon+s}(U)}\| \left(\|1_{A_{\varepsilon+2s}(U)}u\mathcal{L}u\|_1 + \|u1_{A_{\varepsilon+2s}(U)}\|^2\right)^{1/2}.$$

(b) In the lemma, the function u satisfies additionally $\mathcal{L}u = \lambda u$ and $v \in \ell^2(X, m)$. These conditions allow one to simplify the estimate in part (a): The ℓ^2 condition on v makes it possible to consider v instead of the restriction of v to the annulus in the ℓ^2-norm. The equation $\mathcal{L}u = \lambda u$ makes it possible to replace the product $u\mathcal{L}u$ by λu^2. This equation is also used at the beginning of the proof to show that the expression of interest, namely $|(\mathcal{Q} - \lambda)(u\eta^2, v)|$), agrees with $T(u, v, \eta)$.

(c) The term $T(u, v, \eta)$ can be understood as

$$\sum_{x\in X} \mathcal{L}(\eta^2 u)(x)v(x)m(x) - \sum_{x\in X} \mathcal{L}u(x)\eta^2(x)v(x)m(x)$$

by Green's formula. Moreover, if u actually belongs to $\mathcal{D}$ and not only $\mathcal{F}$ it can also be understood as

$$\mathcal{Q}(\eta^2 u, v) - \mathcal{Q}(u, \eta^2 v).$$

This ends the remark.

Having established the Shnol′ inequality, we can prove the general version of the Shnol′ theorem. The proof uses the concept of Weyl sequences from Appendix 2. In particular, Theorem E.3 gives a characterization for λ to be in the spectrum in terms of a sequence whose energy goes to zero.

Proof of Theorem 12.26. Let $u \in \mathcal{F}$ be such that $\mathcal{L}u = \lambda u$. Let (U_n) be a sequence of finite sets with

$$\frac{\|u1_{A_R(U_n)}\|}{\|u1_{U_n}\|} \to 0$$

as $n \to \infty$, where $R > 2s$. Let $\varepsilon = R - 2s > 0$. By assumption, the ε-ball about every finite set U is finite and, therefore, $\eta_{U_n,\varepsilon} \in C_c(X)$ for all $n \in \mathbb{N}$. Hence,

$$u_n = u\eta^2_{U_n,\varepsilon} \in C_c(X) \subseteq D(Q),$$

which we now show to be a Weyl sequence. Since $\|u_n\| \geq \|u1_{U_n}\|$, by applying the Shnol′ inequality, Lemma 12.27, we get for all $v \in D(Q)$ with $\|v\|_Q = 1$

$$\frac{|(\mathcal{Q} - \lambda)(u_n, v)|}{\|u_n\|^2} \leq C\frac{\|u1_{A_{\varepsilon+2s}(U_n)}\|^2}{\|u_n\|^2} \leq C\frac{\|u1_{A_{\varepsilon+2s}(U_n)}\|^2}{\|u1_{U_n}\|^2} \to 0$$

as $n \to \infty$. Thus, $(u_n/\|u_n\|)$ is a Weyl sequence and, therefore, $\lambda \in \sigma(L)$ by Theorem E.3. □

To conclude the Shnol′ theorem from the more general version we need the following auxiliary lemma.

LEMMA 12.28. *Let $g\colon [0,\infty) \longrightarrow [0,\infty)$ such that for every $\alpha > 0$ there exists a constant $C_\alpha \geq 0$ with*

$$g(r) \leq C_\alpha e^{\alpha r}$$

for all $r \geq 0$. Then, for all $\varepsilon > 0$ and $\delta > 0$, there exists an unbounded monotonically increasing sequence (r_k) in $(0,\infty)$ with

$$g(r_k + \varepsilon) \leq e^{\delta} g(r_k)$$

for all $k \in \mathbb{N}$.

PROOF. Assume the contrary. Then there exists an $r_0 \geq 0$ such that $g(r_0) > 0$ and $g(r+\varepsilon) > e^{\delta} g(r)$ for all $r \geq r_0$. By induction we get

$$g(r_0 + n\varepsilon) \geq e^{\delta n} g(r_0)$$

for $n \in \mathbb{N}$. Now, by the assumption $g(r) \leq C_\alpha e^{\alpha r}$ applied to $r = r_0 + n\varepsilon$ and $\alpha = \delta/(2\varepsilon)$, we get

$$g(r_0) \leq g(r_0 + n\varepsilon) e^{-\delta n} \leq C_\alpha e^{\alpha(r_0 + n\varepsilon)} e^{-\delta n} = C_\alpha e^{\alpha r_0} e^{-\delta n/2} \to 0$$

as $n \to \infty$. This contradicts $g(r_0) > 0$. $\square$

PROOF OF THEOREM 12.25. Let u satisfy $\mathcal{L}u = \lambda u$ and be subexponentially bounded. Furthermore, let $u_r = u1_{B_r}$ for $r > 0$. Then, using that u is subexponentially bounded, we get for $\alpha > 0$

$$\|u_r\|^2 = \sum_{x \in B_r} (e^{\alpha\varrho(o,x)} e^{-\alpha\varrho(o,x)} u(x))^2 m(x) \leq e^{2\alpha r} \|e^{-\alpha\varrho(o,\cdot)} u\|^2.$$

Hence, $g\colon [0,\infty) \longrightarrow [0,\infty)$ given by $g(r) = \|u_r\|^2$ satisfies the assumption of the lemma above with $C_\alpha = \|e^{-\alpha\varrho(o,\cdot)} u\|^2$. Thus, for every $n \in \mathbb{N}$ and $\varepsilon = 2R$, there exists a sequence $(\tilde{r}_k(n))$ such that

$$\|u_{\tilde{r}_k(n)+2R}\|^2 \leq e^{1/n} \|u_{\tilde{r}_k(n)}\|^2.$$

Letting $r_n = \tilde{r}_n(n) + R$, we get

$$\|u_{r_n+R}\|^2 \leq e^{1/n} \|u_{r_n-R}\|^2.$$

We notice that $A_R(B_{r_n}) = B_{r_n+R} \setminus B_{r_n-R}$. We use this and the inequality above to estimate

$$\begin{aligned}
\frac{\|u1_{A_R(B_{r_n})}\|^2}{\|u_{r_n}\|^2} &= \frac{\|u_{r_n+R}\|^2 - \|u_{r_n-R}\|^2}{\|u_{r_n}\|^2} \\
&\leq (e^{1/n} - 1) \frac{\|u_{r_n-R}\|^2}{\|u_{r_n}\|^2} \\
&\leq (e^{1/n} - 1) \to 0
\end{aligned}$$

as $n \to \infty$. Hence, $\lambda \in \sigma(L)$ follows from Theorem 12.26. $\square$

Exercises

Example exercises.

Exercise 12.1 (Failure of ℓ^1-Liouville). Show that there exists a connected graph b over (X,m) with an intrinsic metric ϱ such that:

- the weighted vertex degree is bounded on distance balls (B*)
- there exists a non-constant harmonic function in $\ell^1(X,m)$.

(Hint: Consider a graph over $\mathbb{Z}$ with exponentially decaying edge weights and measure such that the quotient of the two is not square summable.)

Exercise 12.2 (Volume growth and recurrence). Show that for any $\varepsilon > 0$ there exists a connected graph b over (X,m) with an intrinsic metric ϱ and the following properties:

- the weighted vertex degree is bounded on distance balls (B*) and ϱ has finite jump size (J)
- the measure $m(B_r)$ of the distance balls B_r grows like $r^{2+\varepsilon}$
- the graph is transient.

(Hint: Consider a graph over $\mathbb{N}$ with $b(x,y) \neq 0$ if and only if $|x-y| = 1$ and apply Theorem 9.21.)

Exercise 12.3 (Subexponentially bounded functions in $\mathbb{Z}^n$). Let $X = \mathbb{Z}^n$ for $n \in \mathbb{N}$ and b be the graph with standard weights where $b(x,y) = 1$ if and only if $|x-y| = 1$ and let $m = 1$. Denote by B_r the ball of radius r about $0 \in \mathbb{Z}^n$ with respect to the combinatorial graph metric. Show that a function u is subexponentially bounded if and only if

$$\limsup_{r\to\infty} \frac{1}{r} \log \|u1_{B_r}\|_\infty \leq 0.$$

In particular, show that every polynomial u is subexponentially bounded. Here a polynomial is a function in the linear hull of $\{(x_1,\ldots,x_n) \mapsto x_1^{k_1}\cdots x_n^{k_n} \mid k_1,\ldots,k_n \in \mathbb{N}_0\}$.

Exercise 12.4 (Subexponentially bounded functions on regular trees). Let b be a k-regular tree with standard weights, which is a graph without cycles where every vertex has $k+1$ neighbors and $m = 1$. Denote by S_r the sphere of radius r about an arbitrary vertex with respect to the combinatorial graph metric. Show that a function u is subexponentially bounded if

$$\limsup_{r\to\infty} \frac{1}{r} \log \|u1_{S_r}\|_\infty \leq -\frac{1}{2}\log k.$$

Exercise 12.5 (Subexponentially bounded functions on spherically symmetric trees*). Let b be a spherically symmetric tree with standard weights where the number of forward neighbors in the r-th sphere with respect to the combinatorial graph distance is a monotonically increasing function, see Example 9.2 for the definition of a spherically symmetric tree. Give a criterion for a function to be subexponentially bounded in the spirit of Exercise 12.4 in the following two situations:

(a) Consider the normalizing measure and the combinatorial graph distance.
(b) Consider the counting measure and the degree path metric from Definition 11.18.

Challenge!

Exercise 12.6 (Subexponentially bounded functions on anti-trees*). Let b be an anti-tree with standard weights where the r-sphere has $s(r) = (r+1)^\alpha$ vertices for $\alpha > 0$, see Example 9.3 for the definition of an anti-tree. Give a criterion for a function to be subexponentially bounded in the spirit of Exercise 12.4 in the following two situations:

(a) Consider the normalizing measure and the combinatorial graph distance.
(b) Consider the counting measure and the degree path metric from Definition 11.18.

Challenge!

Extension exercises.

Exercise 12.7 (ℓ^1-Liouville theorem). Let b be a connected stochastically complete graph over (X, m). Show that every positive superharmonic function in $\ell^1(X, m)$ is constant.

(Hint 1: Compare a positive superharmonic function to the Green's function $G(o, \cdot)$ for some fixed $o \in X$.)

(Hint 2: Show that the Green's function $G(o, \cdot)$ is not in $\ell^1(X, m)$ for all $o \in X$.)

Exercise 12.8 (Growth and subharmonic functions). Let (b, c) be a graph with an intrinsic metric ϱ such that Deg is bounded on distance balls (B*) and has finite jump size (J). Suppose that u is a positive subharmonic function which satisfies

$$\limsup_{r\to\infty} \frac{1}{r^2 \log r} \|u 1_{B_r}\|_p^p < \infty.$$

Show that u is constant.

For Exercises 12.9 to 12.12 let (b, c) be a connected graph over (X, m) with an intrinsic metric ϱ be such that the weighted vertex degree is bounded on distance balls (B*) and has finite jump size (J). Furthermore, define

$$\varrho_1(\cdot) = \varrho(o, \cdot) \vee 1$$

for some $o \in X$.

EXERCISE 12.9. Show that, for $p \in (1, \infty)$, every positive subharmonic function in $\ell^p(X, \varrho_1^{-2} m)$ is constant.

EXERCISE 12.10. Let $q \in \mathbb{R}$. Assume that $\varrho_1^q \in \ell^1(X, m)$. Show that every positive subharmonic function u such that for some $\varepsilon > 0$ and $C \geq 0$

$$u \leq C \varrho_1^{q+2-\varepsilon}$$

is constant. In particular, show that if $q > -2$, then every bounded subharmonic function u is constant.

EXERCISE 12.11. Assume that $m(X) < \infty$. Show that every positive subharmonic function u such that for some $\varepsilon > 0$ and $C \geq 0$

$$u \leq C \varrho_1^{2-\varepsilon}$$

is constant. In particular, show that every bounded subharmonic function u is constant.

EXERCISE 12.12. Let B_r for $r \geq 0$ be the distance balls about some vertex defined with respect to ϱ and assume that for some $\beta > 0$

$$\limsup_{r \to \infty} \frac{1}{r^\beta} \log m(B_{r+1} \setminus B_r) < 0.$$

Then, every positive subharmonic function u such that there exists $p > 0$ and $C \geq 0$ such that

$$u \leq C \varrho_1^p$$

is constant.

EXERCISE 12.13 (Characterization of recurrence). Let b be a connected graph over (X, m). Show that b is recurrent if and only if there exists a finite measure m and an intrinsic metric ϱ such that all distance balls are finite (B) and ϱ has finite jump size (J).

(Hint: See Exercise 6.13 and use the f given there to define the measure $m_f(x) = \sum_{y \in X} b(x, y)(f(x) - f(y))^2$ and metric $d_f(x, y) = |f(x) - f(y)|$.)

Exercise 12.14 (Subexponentially bounded function). Let (b, c) be a graph over (X, m) and let ϱ be an intrinsic metric. Let $f \in C(X)$ and assume that $e^{-\alpha\varrho(o,\cdot)} f \in \ell^2(X, m)$ for some $\alpha > 0$ and $o \in X$. Show that $e^{-\beta\varrho(x,\cdot)} f \in \ell^2(X, m)$ for all $\beta \geq \alpha$ and all $x \in X$.

Exercise 12.15 (Isoperimetric inequality and growth). Let b be a graph over (X, m) with bounded weighted vertex degree which satisfies a strong isoperimetric inequality with constant $\alpha > 0$, see Definition 10.5. Show that the constant functions are not subexponentially bounded.

Exercise 12.16 (Subexponential volume growth and spectrum). Let b be a graph over (X, m) and let ϱ be an intrinsic metric with finite balls (B) and finite jump size (J). Let B_r denote the distance balls about a vertex for $r \geq 0$ and assume that

$$\limsup_{r \to \infty} \frac{1}{r} \log m(B_r) \leq 0.$$

Show that 0 is in the spectrum of the unique associated operator L.

Notes

Caccioppoli inequalities are a well-known tool in geometric analysis, see the volumes [**Cac63a, Cac63b**] for the original work of Caccioppoli. We focus in these notes on the graph case but also give some references for manifolds and Dirichlet forms.

The Caccioppoli-type inequality presented as Theorem 12.4 can be found in [**HKMW13**], as well as [**TH10, CdVTHT11, Mil11**]. In the special case of the normalized Laplacian, the Caccioppoli inequality for ℓ^2, Theorem 12.12, appears in [**CG98, LX10**], and for ℓ^p for $p \in (1,\infty)$, Theorem 12.14, in [**HS97, HJ14, RSV97**]. For general graph Laplacians, the p-Caccioppoli inequality, Theorem 12.14, appears in [**HK14**]. For Dirichlet forms, the Caccioppoli inequality for $p = 2$ is shown for strongly local Dirichlet forms in [**BdMLS09**] and for general Dirichlet forms in [**FLW14**].

Yau's theorem, Theorem 12.15, was proven by Yau for manifolds in [**Yau76**] and is a corollary of Karp's theorem shown in [**Kar82**]. This was later extended to strongly local Dirichlet forms by Sturm in [**Stu94**]. For graphs, the general statement is proven by Hua/Keller in [**HK14**], improving upon results from [**HS97, RSV97, Mas09, HJ14**].

The results on form uniqueness and essential self-adjointness, Theorems 12.20 and 12.21, can be found in [**HKMW13**] and give an analogue to results of Roelcke [**Roe60**], Chernoff [**Che73**] and Strichartz [**Str83**] for Riemannian manifolds. The result on Markov uniqueness in Theorem 12.20 can be found in [**Sch20b**] and gives an analogue to a theorem of Gaffney, see [**Gaf51, Gaf54**], and also [**Mas05, GM13**]. For prior and related work on graphs, see [**CdVTHT11, Mil11, Mil12, MT14, MT15**].

The volume test for recurrence, Theorem 12.23, can be found for Riemannian manifolds in [**Kar82**], see also [**Gri83, Gri85, Var83**]. An extension to strongly local Dirichlet forms is given in [**Stu94**]. For graphs, this was shown in [**HK14**], improving previous criteria for graphs found in [**DK88, RSV97, Woe00**] and for jump processes in [**MUW12**]. There is also a similar criterion in terms of the boundary of balls proven by Nash-Williams in [**NW59**], see also the presentation in [**Gri18**]. The characterization of recurrence in terms of intrinsic metrics which appears in the exercises is from [**Sch20b**].

The classical Shnol′ theorem can be found in [**Sno57**] for Euclidean space and was rediscovered by Simon in [**Sim81**], see the discussion in [**CFKS87**]. For an accessible reference for the original proof of Shnol′, see [**Gla66**]. For strongly local Dirichlet forms, this result can be found in [**BdMLS09**]. A first version for graphs appears in [**HK11**]. The proof given here follows [**FLW14**], which gives a proof for general

Dirichlet forms. For a different approach relating the growth of solutions to the ground state, see [**BP20, BD19**].

Shnol′ theorems show that generalized eigenvalues belong to the spectrum once they admit generalized eigenfunctions which do not grow too fast. There is also interest in converse results showing that for, in some sense, most points in the spectrum there exists a generalized eigenfunction which does not grow to fast. This is often discussed in the context of expansion in terms of generalized eigenfunctions. A treatment for graphs can be found in [**LT16**]. For strongly local Dirichlet forms, see [**BdMS03**]. We refer to these works for further references. We also note that both Shnol′ results and their converses are also discussed for special graphs based on lattices with additional long range interaction terms in [**Han19**].

Finally, we note that the spelling of Shnol′ in the non-Russian literature shows some degree of variation. Here, we follow the spelling given in articles in the Russian Mathematical Surveys on the occasion of the 70th birthday of Shnol′[**ABG+99**] and upon his passing away [**AAA+17**].

CHAMBER 13

Spectral Bounds

C... rules everything around me.
Method Man.

This chapter is dedicated to the study of the bottom of the spectrum and the bottom of the essential spectrum

$$\lambda_0(L) = \inf \sigma(L) \quad \text{and} \quad \lambda_0^{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L)$$

of the Laplacian $L = L^{(D)}$ associated to a graph. We will give lower and upper bounds of the form

$$\frac{h^2}{2} \leq \lambda_0(L) \leq \lambda_0^{\mathrm{ess}}(L) \leq \frac{\mu^2}{8},$$

where h is an isoperimetric or Cheeger constant and μ is an exponential volume growth rate. We establish the lower bound in Section 1 and the upper bound in Section 2. Along the way, we also prove lower bounds for the bottom of the spectrum of the Neumann Laplacian $L^{(N)}$ in Subsection 1.2 and upper bounds in Section 2.

1. Cheeger constants and lower spectral bounds

In this section we prove an analogue to Cheeger's famous theorem on Riemannian manifolds. This result relates an isoperimetric constant, called the Cheeger constant, to the bottom of the spectrum. We will first introduce the Cheeger constant and then prove a Cheeger inequality for the Laplacian $L = L^{(D)}$ associated to a graph in Subsection 1.1. Afterwards, we turn to estimates for the Neumann Laplacian and the case of finite measure in Subsection 1.2.

We first give the definition of the Cheeger constant. To this end, recall the definition of the boundary of a set $W \subseteq X$, which is given by

$$\partial W = (W \times X \setminus W) \cup (X \setminus W \times W),$$

and the concept of an intrinsic metric, which is a pseudo metric ϱ satisfying

$$\sum_{y \in X} b(x,y)\varrho^2(x,y) \leq m(x)$$

for all $x \in X$.

Definition 13.1 (Cheeger constant). Let b be a graph over (X,m) and let ϱ be an intrinsic metric. For a finite set $W \subseteq X$, we let the *area of the boundary* be given by

$$A_{b\varrho}(\partial W) = \frac{1}{2} \sum_{(x,y)\in\partial W} b(x,y)\varrho(x,y) = \frac{1}{2}(b\varrho)(\partial W).$$

We define the *Cheeger constant* $h = h_{b\varrho,m}$ by

$$h = \inf_{W\subseteq X, W \text{ finite}} \frac{A_{b\varrho}(\partial W)}{m(W)}.$$

Remark. Using the assumption that ϱ is intrinsic, it follows that the area of the boundary of any finite set is finite (Exercise 13.1).

Remark. In Section 1 we introduced an isoperimetric constant α which depends only on the graph b over X. More specifically, we considered

$$\alpha = \inf_{W\subseteq X, W \text{ finite}} \frac{\frac{1}{2}b(\partial W)}{n(W)}.$$

In the spectral estimates for the Laplacian L associated to a graph b over a measure space (X,m) given in Section 6, the measure m entered the estimates via the weighted vertex degree. In this section we prove a lower spectral bound which depends on h only. The measure m then appears in the definition of the intrinsic metric.

1.1. A Cheeger inequality. In this subsection we state and prove a Cheeger inequality for L.

In order to carry out the proof, we recall the area and co-area formula from Section 2. For a function $f \in C(X)$ we denote the level sets of f, for $t \in \mathbb{R}$, by

$$\Omega_t(f) = \{x \in X \mid f(x) > t\}.$$

The co-area formula, Lemma 10.8, applied with the weight $w = \frac{1}{2}b\varrho$ says

$$\frac{1}{2}\sum_{x,y\in X} b(x,y)\varrho(x,y)|f(x) - f(y)| = \int_{-\infty}^{\infty} A_{b\varrho}(\partial\Omega_t(f))dt.$$

Furthermore, the area formula, Lemma 10.9, for positive $f \geq 0$ gives

$$\sum_{x\in X} m(x)f(x) = \int_0^{\infty} m(\Omega_t(f))dt,$$

where in both formulae the value ∞ is allowed. However, as we will apply these formulae for $f = \varphi^2$ and $\varphi \in C_c(X)$, the value is always finite.

THEOREM 13.2 (Cheeger inequality). *Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Then,*

$$\lambda_0(L) \geq \frac{h^2}{2}.$$

PROOF. Let $\varphi \in C_c(X)$ and denote the level sets of φ^2 by $\Omega_t = \Omega_t(\varphi^2)$. Then, by the area and co-area formulae, Lemmas 10.9 and 10.8 and the definition of the Cheeger constant, we derive

$$\begin{aligned} h\|\varphi\|^2 &= h \sum_{x \in X} \varphi^2(x) m(x) \\ &= h \int_0^\infty m(\Omega_t) dt \\ &\leq \int_0^\infty A_{b\varrho}(\partial \Omega_t) dt \\ &= \frac{1}{2} \sum_{x,y \in X} b(x,y)\varrho(x,y)|\varphi^2(x) - \varphi^2(y)| \\ &= \frac{1}{2} \sum_{x,y \in X} b(x,y)\varrho(x,y)|\varphi(x) - \varphi(y)||\varphi(x) + \varphi(y)|. \end{aligned}$$

An application of the Cauchy–Schwarz inequality then yields

$$\ldots \leq \frac{1}{\sqrt{2}} Q^{1/2}(\varphi) \left(\sum_{x,y \in X} b(x,y)\varrho^2(x,y)(\varphi(x) + \varphi(y))^2 \right)^{1/2}.$$

Now, Young's inequality $(\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2$ for $\alpha, \beta \in \mathbb{R}$, symmetry of both b and ϱ and the intrinsic metric property give

$$\ldots \leq \sqrt{2} Q^{1/2}(\varphi)\|\varphi\|.$$

Putting everything together then yields

$$\frac{h}{\sqrt{2}} \leq \frac{Q^{1/2}(\varphi)}{\|\varphi\|}.$$

By squaring both sides, this gives the statement by the variational characterization of $\lambda_0(L)$, see Theorem E.8. □

REMARK. The proof of the theorem above is very similar to the proof of Theorem 10.20 and, in particular, to Lemma 10.12. However, the presence of the intrinsic metric ϱ is an essential ingredient in estimating the sums in the proof above.

REMARK. When ϱ satisfies a uniform lower bound, one also has an upper bound on the bottom of the spectrum via h (Exercise 13.2). We recall that when ϱ satisfies a uniform lower bound, Deg is a bounded function and thus L is a bounded operator on $\ell^2(X, m)$ by Lemma 11.22. Furthermore, in this case, we can characterize the vanishing of h via the equality of the bottom of the spectra of L and of the generator of the semigroup on $\ell^1(X,m)$ (Exercise 13.3).

1.2. The Neumann Laplacian and finite measure. In this subsection we first give an estimate for the bottom of the spectrum of the Neumann Laplacian in terms of a slightly modified Cheeger constant. Afterwards, we consider the case of finite measure and focus on the bottom of the spectrum of the Neumann Laplacian restricted to the orthogonal complement of the constant functions.

We first introduce a Cheeger constant for the Neumann Laplacian. Recall that $L^{(N)}$, the Neumann Laplacian, is the operator associated to the form $Q^{(N)}$, which is the restriction of $\mathcal{Q}$ to $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$. We define the *Neumann Cheeger constant* by

$$h^{(N)} = \inf_{W \subseteq X,\, m(W)<\infty} \frac{A_{b\varrho}(\partial W)}{m(W)}.$$

We have the following estimate, which is analogous to Theorem 13.2.

THEOREM 13.3 (Cheeger for Neumann – λ_0 version). *Let b be a graph over (X,m) and let ϱ be an intrinsic metric. Then,*

$$\lambda_0(L^{(N)}) \geq \frac{{h^{(N)}}^2}{2}.$$

PROOF. We follow verbatim the proof of Theorem 13.2 with functions in $D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X,m)$ instead of functions in $C_c(X)$. Since such functions are in $\ell^2(X,m)$, the level sets of their squares have finite measure, which allows us to carry over the arguments. □

We now present a Cheeger inequality for graphs of finite measure, that is, for a graph b over (X,m) which satisfies $m(X) < \infty$. As above, we consider the Neumann Laplacian which has zero bottom of the spectrum if $m(X) < \infty$. This follows as the constant functions are in $D(L^{(N)})$ and are eigenfunctions corresponding to the eigenvalue zero, see Theorem 5.8. If a function $f \in D(Q^{(N)})$ is orthogonal to the constant functions in $\ell^2(X,m)$, we write $f \perp 1$. The following spectral quantity measures the spectral gap of $L^{(N)}$ in this case,

$$\lambda_1(L^{(N)}) = \inf_{f \in D(Q^{(N)}),\, f\perp 1,\, \|f\|=1} Q^{(N)}(f).$$

We will estimate $\lambda_1(L^{(N)})$ for graphs with finite measure by the following isoperimetric constant

$$h_1^{(N)} = \inf_{W \subseteq X,\, m(W) \leq m(X)/2} \frac{A_{b\varrho}(\partial W)}{m(W)}.$$

More specifically, we prove the following estimate.

THEOREM 13.4 (Cheeger for Neumann – λ_1 version). *Let b be a graph over (X,m) with $m(X)<\infty$ and let ϱ be an intrinsic metric. Then,*

$$\lambda_1(L^{(N)}) \geq \frac{{h_1^{(N)}}^2}{2}.$$

PROOF. The proof utilizes the same basic ideas as the proofs above. However, in order to use the isoperimetric constant $h_1^{(N)}$ we have to restrict our attention to functions which are supported on sets with at most half of the full measure. To this end we need some preparation.

For a number $t\in\mathbb{R}$ and a function $f\in C(X)$ we use the notation

$$m(f>t)=m(\{x\in X\mid f(x)>t\})$$

with analogous definitions for $m(f\geq t)$, $m(f<t)$ and $m(f\leq t)$. We start with a claim which yields a quantity generalizing the notion of a median of a function on finite sets to functions on infinite sets.

Claim. There exists an $M\in\mathbb{R}$ such that

$$m(f>M)\leq\frac{m(X)}{2}\qquad\text{and}\qquad m(f<M)\leq\frac{m(X)}{2}.$$

Proof of the claim. Let $F\colon\mathbb{R}\longrightarrow\mathbb{R}$ be given by

$$F(t)=m(f\leq t)-\frac{m(X)}{2}.$$

It is easy to see that F is a monotone increasing right continuous function such that $\lim_{t\to\pm\infty}F(t)=\pm m(X)/2$. By the nested interval principle and the right continuity, one finds an $M\in\mathbb{R}$ such that

$$F(M)\geq 0\qquad\text{and}\qquad F(t)\leq 0 \text{ for } t<M.$$

From $F(M)\geq 0$, we conclude

$$m(f>M)=m(X)-m(f\leq M)=\frac{m(X)}{2}-F(M)\leq\frac{m(X)}{2}.$$

Moreover, from $F(t)\leq 0$ for $t<M$, we deduce

$$m(f<M)\leq\sup_{t<M}m(f\leq t)=\sup_{t<M}\left(F(t)+\frac{m(X)}{2}\right)\leq\frac{m(X)}{2}.$$

This proves the claim.

As $m(X)<\infty$, we have that 1 is an eigenfunction corresponding to the eigenvalue 0, see Theorem 5.8. Let $f\in D(L^{(N)})$ with $f\perp 1$. Furthermore, set $g=f-M$, where M is as in the claim and observe that $g\in D(L^{(N)})$. Then, $m(\operatorname{supp}g_\pm)\leq m(X)/2$. Then, by the same line of argument as in the proof of Theorem 13.2, we obtain

$$\frac{{h_1^{(N)}}^2}{2}\|g_\pm\|^2\leq Q^{(N)}(g_\pm).$$

To combine these two estimates we notice that, since $Q^{(N)}$ is a Dirichlet form, we have

$$Q^{(N)}(g_+ + g_-) = Q^{(N)}(|g|) \leq Q^{(N)}(g) = Q^{(N)}(g_+ - g_-)$$

and, therefore, using bilinearity,

$$Q^{(N)}(g_+, g_-) \leq 0$$

which readily gives

$$Q^{(N)}(g_+) + Q^{(N)}(g_-) \leq Q^{(N)}(g).$$

Furthermore, since $f \perp M$, we obtain

$$\|f\|^2 \leq \|f\|^2 + \|M\|^2 = \|f - M\|^2 = \|g\|^2 = \|g_+\|^2 + \|g_-\|^2$$

and, hence,

$$\begin{aligned}\frac{{h_1^{(N)}}^2}{2}\|f\|^2 &\leq \frac{{h_1^{(N)}}^2}{2}\left(\|g_+\|^2 + \|g_-\|^2\right) \leq Q^{(N)}(g_+) + Q^{(N)}(g_-) \\ &\leq Q^{(N)}(g) = Q^{(N)}(f).\end{aligned}$$

This proves the theorem. □

2. Volume growth and upper spectral bounds

In this section we prove upper bounds for the bottom of the spectrum and the bottom of the essential spectrum in terms of exponential volume growth. A particular consequence of our theorems is that a graph with subexponential volume growth has bottom of the spectrum at zero.

We will prove our upper bounds for the Neumann Laplacian $L^{(N)} = L^{(N)}_{b,0,m}$ of a graph. In previous chapters, we have established various criteria for $L^{(N)}$ and L to agree. Hence, we get results for L as corollaries by applying these results.

2.1. The Brooks–Sturm theorem. In this subsection we first state a theorem which we refer to as the Brooks–Sturm theorem. This theorem connects the bottom of the spectrum with volume growth. We then state some consequences of this result.

Let b be a graph over (X, m) and let ϱ be an intrinsic metric. The distance balls $B_r(o)$ of radius r about a vertex $o \in X$ with respect to ϱ are given as

$$B_r(o) = \{x \in X \mid \varrho(o, x) \leq r\}.$$

We define the *exponential volume growth rate with variable center* μ_0 by

$$\mu_0 = \liminf_{r \to \infty} \inf_{o \in X} \frac{1}{r} \log \frac{m(B_r(o))}{m(B_1(o))}.$$

Furthermore, we define the *exponential volume growth rate* μ by

$$\mu = \inf_{o \in X} \liminf_{r \to \infty} \frac{1}{r} \log \frac{m(B_r(o))}{m(B_1(o))}.$$

Obviously,

$$\mu_0 \leq \mu.$$

In fact, taking the infimum over $o \in X$ in the definition of μ is not necessary since the limit inferior does not depend on o, as the next lemma shows.

Lemma 13.5. *Let b be a connected graph over (X, m). Then,*

$$\mu = \liminf_{r \to \infty} \frac{1}{r} \log m(B_r(x))$$

for all $x \in X$.

Proof. For $z \in X$, let

$$\mu_0(z) = \liminf_{r \to \infty} \frac{1}{r} \log m(B_r(z)) = \liminf_{r \to \infty} \frac{1}{r} \log \frac{m(B_r(z))}{m(B_1(z))},$$

where the second equality follows from $r^{-1} \log m(B_1(z)) \to 0$ as $r \to \infty$. Let $x, y \in X$. As ϱ takes values in $[0, \infty)$, for $t = \varrho(x, y) < \infty$ and $r \geq t$,

$$B_{r-t}(y) \subseteq B_r(x).$$

Consider a sequence (r_k) such that the limit inferior in the definition of $\mu_0(x)$ is realized and observe that

$$\frac{1}{r_k} \log m(B_{r_k - t}(y)) \leq \frac{1}{r_k} \log m(B_{r_k}(x))$$

for all large k. Since $(r_k - t)/r_k \to 1$ as $k \to \infty$, we infer

$$\mu_0(y) \leq \liminf_{k \to \infty} \frac{1}{r_k} \log m(B_{r_k - t}(y)) \leq \liminf_{k \to \infty} \frac{1}{r_k} \log m(B_{r_k}(x)) = \mu_0(x).$$

Reversing the roles of x and y then yields $\mu_0(x) = \mu_0(y)$. $\square$

In the following subsections we will prove the following connection between the volume growth and the bottom of the (essential) spectrum.

Theorem 13.6 (Theorem of Brooks–Sturm). *Let b be a connected graph over (X, m) and let ϱ be an intrinsic metric. Then,*

$$\lambda_0(L^{(N)}) \leq \frac{\mu_0^2}{8}.$$

If, furthermore, $m(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L^{(N)}) \leq \frac{\mu^2}{8}.$$

In the case of jump size bounded from above and from below we get another estimate. In this case the Laplacian is bounded, Lemma 11.22 and $L = L^{(D)} = L^{(N)}$.

THEOREM 13.7. *Let b be a graph over (X, m) and let ϱ be an intrinsic metric such that $\varrho(x, y) \in [\delta, 1]$ for $x \sim y$ and $\delta > 0$. Then,*

$$\lambda_0(L) \leq \frac{(e^{\mu_0/2} - 1)^2}{1 + \delta^2 e^{\mu_0}}.$$

If, furthermore, $m(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L) \leq \frac{(e^{\mu/2} - 1)^2}{1 + \delta^2 e^{\mu}}.$$

REMARK. If one computes the Taylor expansion of the right-hand side as a function of μ_0 (μ, respectively) at $\mu_0 = 0$ ($\mu = 0$, respectively), then one sees that the first order term is exactly the bound which is obtained in Theorem 13.6 if $\delta = 1$.

Before we come to the proof, let us discuss a result corresponding to Theorem 13.6 for the Dirichlet Laplacian $L^{(D)}$. In the setting of Theorem 13.7, the operators $L^{(D)}$ and $L^{(N)}$ agree as they are bounded operators. However, there are other situations when these two operators agree. We highlight some such situations below.

THEOREM 13.8. *Let b be a connected graph over (X, m) and let ϱ be an intrinsic metric such that one of the following conditions is satisfied:*

(B) *All distance balls with respect to ϱ are finite.*
(B*) *The weighted degree* Deg *is bounded on all distance balls with respect to ϱ.*
(D) *The weighted degree* Deg *is bounded.*
(M) $\inf_{x \in X} m(x) > 0$.
(M*) *All infinite paths have infinite measure.*

Then, $L^{(D)} = L^{(N)}$ so that

$$\lambda_0(L^{(D)}) \leq \frac{\mu_0^2}{8}$$

and if $m(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L^{(D)}) \leq \frac{\mu^2}{8}.$$

The proof of the theorem above is a direct consequence of Theorem 13.6 and the equality of $L^{(N)}$ and $L^{(D)}$. However, we will also discuss a direct proof in Subsection 2.3 under any of the conditions (B), (D) or (M).

REMARK. It is obvious that (B) $\Longrightarrow$ (B*) and (M) $\Longrightarrow$(M*).

We now provide a corollary of Theorem 13.7 in the case where the jump size is exactly 1. Recall the definition of the normalizing measure given by $n(x) = \sum_{y \in X} b(x,y)$ for $x \in X$.

COROLLARY 13.9 (Fujiwara's theorem). *Let b be a connected graph over (X, n) and consider the volume growth with respect to the combinatorial graph distance. Then, for the operator $L = L^{(D)} = L^{(N)}$,*

$$\lambda_0(L) \leq 1 - \frac{1}{\cosh(\mu_0/2)}$$

and, if $n(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L) \leq 1 - \frac{1}{\cosh(\mu/2)}.$$

PROOF. By Theorem 1.27, the Laplacian $L^{(D)}$ is bounded and $L^{(D)} = L^{(N)}$ since $\mathrm{Deg} = 1$ in the case of the normalizing measure. Furthermore, by Example 11.21, the combinatorial graph distance d is an intrinsic metric in this case. As $d(x,y) = 1$ for all $x \sim y$, Theorem 13.7 gives

$$\lambda_0(L) \leq \frac{(e^{\mu_0/2} - 1)^2}{(1 + e^{\mu_0})} = 1 - \frac{2e^{\mu_0/2}}{1 + e^{\mu_0}} = 1 - \frac{1}{\cosh(\mu_0/2)}.$$

Similarly,

$$\lambda_0^{\mathrm{ess}}(L) \leq 1 - \frac{1}{\cosh(\mu/2)}$$

if $n(X) = \infty$. This completes the proof. $\square$

2.2. The test functions. Next, we introduce the test functions for the proof of the Brooks–Sturm theorem.

For the parameters $r \geq 0$, $o \in X$ and $\beta > 0$, we define the function $f_{r,o,\beta} \colon X \longrightarrow [0, \infty)$ by

$$f_{r,o,\beta}(x) = \left(\left(e^{\beta r} \wedge e^{\beta(2r - \varrho(o,x))}\right) - 1\right)_+ .$$

We observe the following basic properties of $f = f_{r,o,\beta}$, where $B_r = B_r(o)$,

- $f|_{B_r} = e^{\beta r} - 1$
- $f|_{B_{2r} \setminus B_r} = e^{\beta(2r - \varrho(o,\cdot))} - 1$
- $f|_{X \setminus B_{2r}} = 0$.

We will use f as a test function to estimate the bottom of the spectrum and the bottom of the essential spectrum.

The important conceptual feature of the functions $f = f_{r,o,\beta}$ is the exponential decay on $B_{2r} \setminus B_r$. For $\beta > \mu_0/2$, the functions f^2 decay faster than the volume grows. Furthermore, the function is cut off twice. The first cutoff consists of the subtraction by 1 and the second is the cutoff at distance $2r$, which ensures that the functions are supported on distance balls. In this way, we can guarantee that the functions are square summable under appropriate hypotheses. The other cutoff on B_r is not needed for the estimate of λ_0 but only for λ_0^{ess}. The important feature of this cutoff is that the normalization of f converges weakly to zero as $r \to \infty$.

Furthermore, for the parameters $r \geq 0$, $o \in X$ and $\beta > 0$, we define the auxiliary functions $g_{r,o,\beta}\colon X \longrightarrow [0,\infty)$ by

$$g_{r,o,\beta} = (f_{r,o,\beta} + 2)\, 1_{B_{2r}(o)}.$$

We observe the following basic properties for $g = g_{r,o,\beta}$

- $g|_{B_r} = e^{\beta r} + 1$
- $g|_{B_{2r}\setminus B_r} = e^{\beta(2r-\varrho(o,\cdot))} + 1$
- $g|_{X\setminus B_{2r}} = 0.$

We prove the following further properties of the functions above.

Lemma 13.10 (Test functions). *Let b be a graph over (X,m) and let ϱ be an intrinsic metric. Let $o \in X$, $r \geq 0$ and $\beta > 0$.*

(a) *For all $x, y \in X$, $f = f_{r,o,\beta}$ and $g = g_{r,o,\beta}$,*

$$(f(x) - f(y))^2 \leq C(\beta)\left(g^2(x) + g^2(y)\right)\varrho^2(x,y),$$

with $C(\beta) = \beta^2/2$. If, furthermore, $\varrho(x,y) \leq 1$ for all $x \sim y$, then

$$C(\beta) = C_{x,y}(\beta) = \frac{(e^{\beta}-1)^2}{1+\varrho^2(x,y)e^{2\beta}}.$$

In particular, the functions $f_{r,o,\beta}$ are Lipschitz continuous with respect to ϱ.

If $\mu_0 < \infty$, then there exist sequences (o_k) in X and (r_k) in $[0,\infty)$ such that:

(b) *$f_{r_k,o_k,\beta}, g_{r_k,o_k,\beta} \in \ell^2(X,m)$ for all k.*

(c) *If $\beta > \mu_0/2$,*

$$\lim_{k\to\infty} \frac{\|g_{r_k,o_k,\beta}\|}{\|f_{r_k,o_k,\beta}\|} = 1.$$

If $\mu < \infty$, then there exists an $o \in X$, such that:

(d) *$f_{r,o,\beta}, g_{r,o,\beta} \in \ell^2(X,m)$ for all $r \geq 0$ and $\beta > 0$.*

(e) *If $\beta > \mu_0/2$, then there exists a sequence (r_k) such that*

$$\lim_{k\to\infty} \frac{\|g_{r_k,o,\beta}\|}{\|f_{r_k,o,\beta}\|} = 1.$$

(f) *If $m(X) = \infty$, then*

$$\frac{f_{r,o,\beta}}{\|f_{r,o,\beta}\|} \to 0 \qquad \textit{weakly as } r \to \infty.$$

Proof. (a) Let $x, y \in X$ and let

$$t = r \wedge (2r - \varrho(o,x)) \quad \text{and} \quad s = r \wedge (2r - \varrho(o,y)).$$

We consider the case $x, y \in B_{2r}(o)$ first. Then, by Lemma 2.29 (a),

$$|f_{r,o,\beta}(x) - f_{r,o,\beta}(y)| = |e^{\beta t} - e^{\beta s}| \leq \frac{\beta}{2}(e^{\beta t} + e^{\beta s})|t - s|.$$

We use $|t-s| \leq |\varrho(o,x) - \varrho(o,y)|$, which can be seen by checking cases, $e^{\beta t} \leq g_{r,o,\beta}(x)$, $e^{\beta s} \leq g_{r,o,\beta}(y)$ and the triangle inequality to get

$$\ldots \leq \frac{\beta}{2}(g_{r,o,\beta}(x) + g_{r,o,\beta}(y))\varrho(x,y).$$

Squaring both sides and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ finishes the proof in the case that $x, y \in B_{2r}(o)$.

Since $f_{r,o,\beta}$ is supported on $X \setminus B_{2r}(o)$, the statement is clear for $x, y \in X \setminus B_{2r}(o)$. For $x \in B_{2r}(o)$, $y \in X \setminus B_{2r}(o)$, we have again by Lemma 2.29 (a),

$$\begin{aligned} |f_{r,o,\beta}(x) - f_{r,o,\beta}(y)| &= |e^{\beta t} - 1| \\ &= |e^{\beta t} - e^{\beta \cdot 0}| \\ &\leq \frac{\beta}{2}(e^{\beta t} + 1)t \\ &= \frac{\beta}{2}(g_{r,o,\beta}(x) + g_{r,o,\beta}(y))t. \end{aligned}$$

Since $\varrho(0,y) \geq 2r$, we have

$$t = r \wedge (2r - \varrho(o,x)) \leq 2r - \varrho(o,x) \leq \varrho(o,y) - \varrho(o,x) \leq \varrho(x,y),$$

which finishes the proof of the first statement of (a).

The second statement follows along the same lines using Lemma 2.29 (b) and noting that

$$\gamma \mapsto \frac{\gamma}{(1+\gamma^2 e^{2\beta})^{1/2}}$$

is an increasing function for $\gamma \geq 0$.

The Lipschitz continuity of f with respect to ϱ is now clear as g is bounded.

(b) If $\mu_0 < \infty$, then there exist sequences (o_k) in X and $r_k \to \infty$ such that

$$\mu_0 = \lim_{k\to\infty} \frac{1}{2r_k} \log \frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))}.$$

In particular, $m(B_{2r_k}(o_k)) < \infty$ and, therefore, $f_k = f_{r_k,o_k,\beta}$ and $g_k = g_{r_k,o_k,\beta} \in \ell^2(X,m)$ for $\beta > 0$, since f_k, g_k are bounded functions supported on $B_{2r_k}(o_k)$.

(c) Let $\beta > \mu_0/2$ and let $\varepsilon < 1$ such that $0 < \varepsilon < \beta - \mu_0/2$. Then there exists a k_0 such that, for $k \geq k_0$,

$$\frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))} \leq e^{(2\mu_0+\varepsilon)r_k}.$$

By definition $g_k = (f_k + 2)1_{B_{2r_k}(o_k)}$, so we estimate using Cauchy–Schwarz and Young's inequality $(s+t)^2 \leq (1-\varepsilon)^{-1}s^2 + \varepsilon^{-1}t^2$,

$$\|g_k\|^2 \leq \left(\|f_k\| + 2\sqrt{m(B_{2r_k}(o_k))}\right)^2 \leq \frac{1}{1-\varepsilon}\|f_k\|^2 + \frac{4}{\varepsilon}m(B_{2r_k}(o_k)).$$

Now, there exist $k_1 \geq k_0$ and a constant $c > 0$ such that for all $k \geq k_1$,

$$\|f_k\|^2 \geq m(B_{r_k}(o_k))(e^{\beta r_k} - 1)^2 \geq cm(B_{r_k}(o_k))e^{2\beta r_k}.$$

Thus, for all $k \geq k_1$,

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{4}{\varepsilon c}e^{-2\beta r_k}\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))}.$$

Now, given ε as above, there exists $k_2 \geq k_1$ such that for all $k \geq k_2$,

$$\frac{m(B_{r_k}(o_k))}{m(B_1(o_k))} \geq \inf_{o\in X}\frac{m(B_{r_k}(o))}{m(B_1(o))} \geq e^{(\mu_0-\varepsilon)r_k}.$$

Therefore,

$$\frac{m(B_{2r_k}(o_k))}{m(B_{r_k}(o_k))} = \frac{m(B_{2r_k}(o_k))}{m(B_1(o_k))}\frac{m(B_1(o_k))}{m(B_{r_k}(o_k))} \leq e^{(\mu_0+2\varepsilon)r_k}.$$

Since $0 < \varepsilon < \beta - \mu_0/2$, we can combine this with the estimate above to conclude

$$\frac{\|g_k\|^2}{\|f_k\|^2} \leq \frac{1}{1-\varepsilon} + \frac{4}{\varepsilon c}e^{(\mu_0-2\beta+2\varepsilon)r_k} \to \frac{1}{1-\varepsilon}$$

as $k \to \infty$. Since ε was chosen arbitrarily and $0 \leq f_k \leq g_k$, statement (c) follows.

(d) and (e): Let $\mu < \infty$ and let $o \in X$. Then by Lemma 13.5, for every $\varepsilon > 0$ there exist $r_k \to \infty$ such that

$$m(B_{2r_k}(o)) \leq e^{(2\mu+\varepsilon)r_k}.$$

Now, the proof of statements (d) and (e) follows along the same lines as the proofs of (b) and (c) with o_k replaced by o.

For (f) let $\varphi \in \ell^2(X,m)$ with $\|\varphi\| = 1$. For every $\varepsilon > 0$, there exists an $r > 0$ such that

$$\|\varphi 1_{X\setminus B_r(o)}\| \leq \frac{\varepsilon}{2}.$$

Moreover, since $m(X) = \infty$, there exists an $R \geq r$ such that

$$m(B_r(o)) \leq \frac{\varepsilon^2}{4}m(B_R(o))$$

and, therefore,

$$\begin{aligned}\|f_{r,o,\beta}1_{B_r(o)}\|^2 &= (e^{\beta r}-1)^2 m(B_r(o)) \leq \frac{\varepsilon^2}{4}(e^{\beta r}-1)^2 m(B_R(o)) \\ &\leq \frac{\varepsilon^2}{4}\|f_{r,o,\beta}1_{B_R(o)}\|^2 \leq \frac{\varepsilon^2}{4}\|f_{r,o,\beta}\|^2.\end{aligned}$$

Putting this together with the estimate for $\|\varphi 1_{X\setminus B_r(o)}\|$ for r chosen as above and using the Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned}\langle \varphi, f_{r,o,\beta}\rangle &\leq \langle \varphi 1_{B_r(o)}, f_{r,o,\beta}1_{B_r(o)}\rangle + \langle \varphi 1_{X\setminus B_r(o)}, f_{r,o,\beta}\rangle \\ &\leq \|\varphi\|\|f_{r,o,\beta}1_{B_r(o)}\| + \|\varphi 1_{X\setminus B_r(o)}\|\|f_{r,o,\beta}\| \\ &\leq \varepsilon\|f_{r,o,\beta}\|.\end{aligned}$$

This implies (f). $\square$

The first part of the lemma above provides the tools needed to establish the next proposition estimating the energy of f by the norm of g. These estimates will be crucial for the proof of the main theorems.

Proposition 13.11. *Let b be a graph over (X,m) and let ϱ be an intrinsic metric. Then, for the functions $f_{r,o,\beta}, g_{r,o,\beta}$, $r \geq 0$, $o \in X$, $\beta > 0$, we have*

$$\mathcal{Q}(f_{r,o,\beta}) \leq \frac{\beta^2}{2}\|g_{r,o,\beta}\|^2.$$

If $\varrho \in [\delta, 1]$ for $x \sim y$ and $\delta > 0$, then

$$\mathcal{Q}(f_{r,o,\beta}) \leq \frac{(e^\beta - 1)^2}{(1+\delta^2 e^{2\beta})}\|g_{r,o,\beta}\|^2.$$

Proof. By the first inequality of Lemma 13.10 (a) above and the fact that ϱ is an intrinsic metric, we estimate

$$\begin{aligned}\mathcal{Q}(f_{r,o,\beta}) &= \frac{1}{2}\sum_{x,y\in X} b(x,y)(f_{r,o,\beta}(x) - f_{r,o,\beta}(y))^2 \\ &\leq \frac{\beta^2}{4}\sum_{x,y\in X} b(x,y)\left(g^2_{r,o,\beta}(x) + g^2_{r,o,\beta}(y)\right)\varrho^2(x,y) \\ &= \frac{\beta^2}{2}\sum_{x\in X} g^2_{r,o,\beta}(x)\sum_{y\in X} b(x,y)\varrho^2(x,y) \\ &\leq \frac{\beta^2}{2}\sum_{x\in X} g^2_{r,o,\beta}(x)m(x) \\ &= \frac{\beta^2}{2}\|g_{r,o,\beta}\|^2.\end{aligned}$$

If $\delta \leq \varrho(x,y) \leq 1$ for $x \sim y$, then we use the inequality of Lemma 13.10 (a) with $C(\beta) = (e^\beta - 1)^2/(1+\varrho^2(x,y)e^{2\beta})$ as above to estimate

$$
\begin{aligned}
\mathcal{Q}(f_{r,o,\beta}) &\le \frac{1}{2}\sum_{x,y\in X} b(x,y)\frac{(e^{\beta}-1)^2}{(1+\varrho(x,y)^2e^{2\beta})}\left(g^2_{r,o,\beta}(x)+g^2_{r,o,\beta}(y)\right)\varrho^2(x,y)\\
&\le \frac{(e^{\beta}-1)^2}{(1+\delta^2e^{2\beta})}\sum_{x\in X} g^2_{r,o,\beta}(x)\sum_{y\in X} b(x,y)\varrho^2(x,y)\\
&\le \frac{(e^{\beta}-1)^2}{(1+\delta^2e^{2\beta})}\|g_{r,o,\beta}\|^2.
\end{aligned}
$$

This completes the proof. □

2.3. Proofs of the theorems. We now give the proofs of the Brooks–Sturm theorem and its variants, Theorems 13.6, 13.7 and 13.8.

Proof of Theorem 13.6. Let the sequences (o_k) and (r_k) be taken from Lemma 13.10 (b) and set $f_{k,\beta} = f_{r_k,o_k,\beta}$ and $g_{k,\beta} = g_{r_k,o_k,\beta}$ for $\beta > 0$. By the first inequality of Proposition 13.11 and Lemma 13.10 (b) we have $f_{k,\beta} \in \ell^2(X,m)$ and $\mathcal{Q}(f_{k,\beta}) < \infty$ for all $k \in \mathbb{N}_0$ and $\beta > 0$. Hence, $f_{k,\beta} \in \ell^2(X,m) \cap \mathcal{D} = D(Q^{(N)})$. By the variational characterization of $\lambda_0(L^{(N)})$, Theorem E.8, we get

$$
\begin{aligned}
\lambda_0(L^{(N)}) &= \inf_{f\in D(Q^{(N)})}\frac{Q^{(N)}(f)}{\|f\|^2}\\
&\le \inf_{k\in\mathbb{N}_0,\beta>\mu_0/2}\frac{Q^{(N)}(f_{k,\beta})}{\|f_{k,\beta}\|^2}\\
&\le \inf_{\beta>\mu_0/2}\frac{\beta^2}{2}\lim_{k\to\infty}\frac{\|g_{k,\beta}\|^2}{\|f_{k,\beta}\|^2}\\
&= \frac{\mu_0^2}{8},
\end{aligned}
$$

where we used Lemma 13.10 (c) in the last equality.

For the proof of the inequality for $\lambda_0^{\mathrm{ess}}(L^{(N)})$, we assume that $m(X) = \infty$ and consider the functions $f_{k,\beta} = f_{r_k,o,\beta}$ with (r_k) and o chosen as in Lemma 13.10 (d)–(f) for $\beta > \mu/2$. Again by virtue of Proposition 13.11 and Lemma 13.10 (d), we have $f_{k,\beta} \in D(Q^{(N)})$. Furthermore, by (f) of Lemma 13.10 the functions $f_{k,\beta}/\|f_{k,\beta}\|$ converge weakly to 0 as $k \to \infty$. Thus, by the Persson theorem, Theorem E.12, Proposition 13.11 and Lemma 13.10 (d) we obtain

$$
\lambda_0^{\mathrm{ess}}(L^{(N)}) \le \liminf_{k\in\mathbb{N}_0}\frac{Q^{(N)}(f_{k,\beta})}{\|f_{k,\beta}\|^2} \le \frac{\beta^2}{2}\liminf_{k\to\infty}\frac{\|g_{k,\beta}\|^2}{\|f_{k,\beta}\|^2} = \frac{\beta^2}{2}.
$$

Since the estimate holds for all $\beta > \mu/2$ we conclude the statement. □

Proof of Theorem 13.7. The proof is analogous to that of Theorem 13.6 but we use the second inequality of Proposition 13.11 instead of the first one. □

Proof of Theorem 13.8. Obviously, (B) $\Longrightarrow$ (B*) and (M) $\Longrightarrow$ (M*). In the previous chapters we have proven that any of the conditions (B*), (D) and (M*) imply $L^{(D)} = L^{(N)}$. In particular, the implication for (B*) follows from Theorem 12.20, the implication for (D) follows from Theorem 1.27 and the implication for (M*) follows from Theorem 8.4.

However, let us sketch a direct proof under any of the stronger assumptions (B), (D) or (M). In any of these cases, one has to show that the test functions $f = f_{r,o,\beta}$ are in $D(Q^{(D)})$ for appropriate r, o and β. Lemma 13.10 already gives that $f \in \ell^2(X, m)$. In the case of (D), the form $\mathcal{Q}$ is bounded and, therefore, $D(Q^{(D)}) = \ell^2(X, m)$. In the case of (B) the distance balls are finite and as $\mathrm{supp} f \subseteq B_{2r}(o)$, we have $f \in C_c(X) \subseteq D(Q^{(D)})$. Finally, consider the case (M). If $\mu_0 = \infty$ or $\mu = \infty$, then there is nothing to prove. So assume the contrary. Then, there exist distance balls of finite measure. But (M) implies that these balls must be finite. Since f is supported on balls, we conclude $f \in C_c(X)$ and, thus, $f \in D(Q^{(D)})$. $\square$

Remark. If one is only interested in a weaker result such as

$$\lambda_0(L^{(N)}) \leq \frac{\overline{\mu}_0^2}{8}$$

involving an upper exponential volume growth defined by

$$\overline{\mu}_0 = \inf_{o \in X} \limsup_{r \to \infty} \frac{1}{r} \log m(B_r(o)),$$

then it suffices to consider the test functions $f_\beta = e^{-\beta \varrho(o,\cdot)}$ with $\beta > \overline{\mu}_0/2$ (Exercise 13.4).

Exercises

Extension Exercises.

Exercise 13.1 (Finite boundary area). Let b be a graph over (X,m) and let ϱ be an intrinsic metric. Show that for all finite sets the area of the boundary is finite. More specifically, if W is a finite subset of X, then

$$A_{b\varrho}(\partial W) \leq (mn)^{1/2}(W),$$

where n denotes the normalizing measure $n(x) = \sum_{y\in X} b(x,y)$.

Exercise 13.2 (Upper bound via h). Let b be a graph over (X,m). Let ϱ be an intrinsic metric such that $\varrho(x,y) \geq C > 0$ for all $x \sim y$. Show that for the Cheeger constant h we have

$$\lambda_0(L) \leq \frac{h}{C}.$$

Exercise 13.3 ($\lambda_0(L) = \lambda_0(L^{(1)})$). Let b be a graph over (X,m) with associated operator $L = L^{(D)}$ and denote by $\lambda_0(L)$ the bottom of the spectrum of L. Let $L^{(1)}$ denote the generator of the semigroup on $\ell^1(X,m)$ and let

$$\lambda_0(L^{(1)}) = \inf_{f\in D(L^{(1)}), \|f\|_1=1} \|L^{(1)}f\|_1.$$

Let ϱ be an intrinsic metric such that $\varrho(x,y) \geq C > 0$ for all $x \sim y$. Show that

$$\lambda_0(L) = \lambda_0(L^{(1)})$$

if and only if $h = 0$.

Exercise 13.4 (Upper exponential growth). Let b be a graph over (X,m). Let ϱ be an intrinsic metric. Let the upper exponential volume growth be defined by

$$\overline{\mu}_0 = \inf_{o\in X} \limsup_{r\to\infty} \frac{1}{r} \log m(B_r(o)),$$

where $B_r(o)$ is defined with respect to ϱ. Show that

$$\lambda_0(L^{(N)}) \leq \frac{\overline{\mu}_0^2}{8}$$

using the test functions $f_\beta = e^{-\beta\varrho(o,\cdot)}$ for $\beta > \overline{\mu}_0/2$.

Notes

The (pre)-history of Cheeger's inequality was already extensively discussed in the notes for Chapter 10, so we mention here only the seminal work of Cheeger [**Che70**] for manifolds, the works of Dodziuk [**Dod84**] and Dodziuk/Kendall [**DK86**] for infinite graphs and Alon/Milman [**AM85**] for finite graphs. The result as it is discussed here, Theorem 13.2 in Section 1, appeared in [**BKW15**]. We note that this theorem allows us to recover the results of [**Dod84, DK86**] as these papers deal with a bounded situation in which case the combinatorial graph metric is equivalent to an intrinsic metric. Furthermore, an extension to the p-Laplacian is found in [**KM16**]. In particular [**KM16**] includes Theorem 13.3 as a special case. Although Theorem 13.4 seems not to have been published before, it is proven by well-known techniques.

The history of the upper spectral bound in terms of exponential volume growth which we refer to as Brooks–Sturm theorem in Section 2 has its beginning in the work of Brooks [**Bro81**]. In this paper, an estimate for λ_0^{ess} is found in the context of Riemannian manifolds. This was later generalized to the context of strongly local Dirichlet forms in an unpublished manuscript by Notarantonio [**Not98**]. For strongly local Dirichlet forms, Sturm proved an estimate for λ_0 in [**Stu94**]. For more subtle estimates for λ_0 in the case of Riemannian manifolds we refer the reader to Li/Wang [**LW01, LW10**].

In the case of graphs and the normalized Laplacian, corresponding results were obtained and successively improved upon by various authors, for examples see [**DK88, OU94, Fuj96a, Hig03**]. Theorems 13.6, 13.7 and 13.8 were proven in [**HKW13**] in the more general context of regular Dirichlet forms. A slightly less general version of Theorem 13.7 was independently proven by Folz [**Fol14a**]. Fujiwara's Theorem, Corollary 13.9, can be found in [**Fuj96a**].

CHAMBER 14

Volume Growth Criterion for Stochastic Completeness and Uniqueness Class

Raw I'mma give to ya, with no trivia, raw like ...
U-God.

In this chapter we present a volume growth criterion for stochastic completeness. More specifically, we show that the measure of finite balls defined with respect to an intrinsic metric must grow superexponentially in order for a graph to be stochastically incomplete.

Let b be a graph over (X, m). In Chapter 7 we studied the phenomenon of stochastic completeness. This property has many equivalent formulations. In particular, stochastic completeness is equivalent to the preservation of the constant function 1 by the semigroup or to uniqueness of bounded solutions of the heat equation.

More precisely, if $L = L^{(D)}$ is the Laplacian associated to b over (X, m), then we say that b over (X, m) is stochastically complete if

$$e^{-tL}1 = 1$$

for all $t \geq 0$, where e^{-tL} is the semigroup associated to L. Furthermore, we recall that a function $u\colon [0, \infty) \times X \longrightarrow \mathbb{R}$ is called a solution of the heat equation with initial condition $u_0 \in C(X)$ if $u(\cdot, 0) = u_0$, $t \mapsto u_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ for all $x \in X$, $u_t \in \mathcal{F}$ for all $t > 0$ and

$$(\mathcal{L} + \partial_t)u_t(x) = 0$$

for all $x \in X$ and $t > 0$, where $\mathcal{L}$ is the formal Laplacian of b over (X, m). In Theorem 7.2 we have shown for connected graphs that stochastic completeness is equivalent to the fact that the only bounded solution u of the heat equation with $u_0 = 0$ is the zero function.

We now state the volume growth criterion for stochastic completeness that we will establish in this chapter. Let ϱ be an intrinsic metric with finite balls B_r around a fixed vertex and let $\log^{\#} = 1 \vee \log$. If

$$\int_0^\infty \frac{r}{\log^{\#}(m(B_r))} dr = \infty,$$

then the graph is stochastically complete. This is shown as Theorem 14.11 in Section 3. We note that the statement is incorrect when using the combinatorial graph metric as we have seen in Chapter 9 that there exists stochastically incomplete graphs whose balls in the combinatorial graph metric grow polynomially.

We will actually prove a more general theorem, namely, Theorem 14.2 in Section 1, which we refer to as a uniqueness class theorem. It says that if the ℓ^2 norm on distance balls of a solution u of the heat equation with $u_0 = 0$ does not grow too rapidly, then $u = 0$. However, this theorem does not hold for general graphs but only for graphs which we call globally local. The globally local condition means that we can control the jump size of the metric outside of large distance balls.

We then deduce the volume growth criterion for stochastic completeness from the uniqueness class theorem by considering so-called refinements, which we discuss in Section 2. A refinement is a modified graph where additional vertices are "inserted" in existing edges and, as a result, the jump size becomes smaller. Thus, refined graphs can be made to satisfy the globally local condition. We can then infer stochastic completeness of the refined graph from the uniqueness class statement. It is left to show that stochastic completeness of the original graph follows from the stochastic completeness of the refined graph. This is achieved in Theorem 14.8 with the help of the Omori–Yau maximum principle.

There is one more rather technical twist in the argument. Specifically, globally local graphs with finite distance balls are always locally finite. Furthermore, the concept of refinements can only be applied to locally finite graphs. However, for non-locally finite graphs with finite distance balls, there are infinitely many neighbors which are far away from all vertices. In order to complete the proof, we will also show that removing edges corresponding to large distances has no impact on the stochastic completeness of the graph. This is carried out in Section 3.

1. Uniqueness class

In this section we prove a uniqueness class criterion for solutions of the heat equation. More specifically, we show that if a solution of the heat equation with zero initial conditions on a globally local graph does not grow too fast, then the solution must be zero.

We will first introduce the class of globally local graphs. Roughly speaking, for such graphs the jump size outside of a ball decays as we take larger and larger balls. When the metric is additionally assumed to be intrinsic, we will prove a condition under which there exists a unique solution of the heat equation for such graphs.

If b is a graph over X with a pseudo metric ϱ, we recall that the jump size of ϱ is given by

$$s = \sup\{\varrho(x, y) \mid x, y \in X,\ x \sim y\}.$$

We say that a graph has finite jump size if $s < \infty$ and denote this condition by (J). In what follows we consider the jump size outside of distance balls. More specifically, we define the *jump size at distance larger or equal to* r for $r \in \mathbb{R}$ by

$$s_r = \sup\{\varrho(x, y) \mid x, y \in X,\ x \sim y,\ \varrho(x, o) \wedge \varrho(y, o) \geq r\},$$

where $o \in X$ is a fixed vertex. We note that although the choice of the vertex o plays a role in the definition of s_r, this choice will ultimately play no role in our results. Furthermore, we note that $s = s_0$ is the jump size of the graph.

We now define the notion of a graph and a pseudo metric to be globally local.

Definition 14.1 (Globally local graphs). A graph b over X with a pseudo metric ϱ is *globally local* with respect to a monotonically increasing function $f\colon (0, \infty) \longrightarrow (0, \infty)$ if there exist constants $A > 1$, $B > 0$ and $r_0 > 0$ such that

$$s_r \leq \frac{Br}{f(Ar)}$$

for all $r \geq r_0$ and $s_0 = s < \infty$.

Remark. We note that $r \mapsto Br/f(Ar)$ does not need to be monotone decreasing. Since $s_r \leq s < \infty$ for all $r \geq 0$, we immediately see that being globally local is only a restriction whenever $f(r) > r$ for r large. Indeed, we often think of f as satisfying $\lim_{r\to\infty} r/f(r) = 0$ as, for example, when $f(r) = r \log r$ for $r > 0$.

We recall that a pseudo metric ϱ is called intrinsic if

$$\sum_{y\in X} b(x, y)\varrho^2(x, y) \leq m(x)$$

for all $x \in X$. We also recall our geometric condition (B) that all distance balls with respect to an intrinsic metric ϱ are finite, that is,

(B) The distance balls $B_r(o)$ are finite for all $o \in X$ and $r \geq 0$.

With these definitions we can now formulate our uniqueness class criterion.

Theorem 14.2 (Uniqueness class). *Let b be a graph over (X, m) and let ϱ be an intrinsic metric with finite balls* (B). *Assume that b is globally local with respect to a monotonically increasing function $f\colon (0, \infty) \longrightarrow (0, \infty)$ such that*

$$\int_1^\infty \frac{r}{f(r)} dt = \infty.$$

Let $0 < T \leq \infty$. *If a solution of the heat equation* $u\colon [0,T) \times X \longrightarrow \mathbb{R}$ *with initial condition* $u_0 = 0$ *satisfies*

$$\int_0^T \|u_t 1_{B_r}\|^2 dt \leq e^{f(r)}$$

for all $r \geq 0$, *then* $u = 0$.

Remark. By definition, globally local graphs have finite jump size (J). Therefore, together with the assumption (B) the graphs considered in the theorem are locally finite, see Lemma 11.28. We note that it is possible to give a uniqueness class statement for graphs satisfying only (B) and (J) (Exercise 14.2).

The proof of this theorem is rather involved and will be given at the end of this section. We first give a basic estimate for solutions of the heat equation multiplied by a finitely supported function. This estimate involves an application of a Caccioppoli-type inequality. To state it we recall the notion of the norm of the gradient squared of a function $f \in C(X)$, given by

$$|\nabla f|^2(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(\nabla_{x,y} f)^2,$$

where $\nabla_{x,y} f = f(x) - f(y)$ for $x, y \in X$.

Lemma 14.3 (Basic estimate). *Let* b *be a graph over* (X,m). *Let* $0 < T \leq \infty$ *and let* $u\colon [0,\infty) \times X \longrightarrow \mathbb{R}$ *be a solution of the heat equation. Let* $K \subseteq X$ *be finite and* $\varphi\colon [0,T) \times X \longrightarrow \mathbb{R}$ *satisfy*

- $\operatorname{supp} \varphi_t \subseteq K$ *for all* $t \in [0,T)$
- $t \mapsto \varphi_t(x)$ *is continuous on* $[0,T)$ *for all* $x \in X$
- $t \mapsto \varphi_t(x)$ *is continuously differentiable on* $(0,T)$ *for all* $x \in X$.

Then, for all $0 \leq \delta \leq t < T$,

$$\|u_t\varphi_t\|^2 - \|u_{t-\delta}\varphi_{t-\delta}\|^2 \leq \int_{t-\delta}^t \sum_{x \in X} u_\tau^2(x) \left(\partial_\tau \varphi_\tau^2 + |\nabla \varphi_\tau|^2\right)(x) m(x) d\tau.$$

Proof. We compute, using the fundamental theorem of calculus,

$$\begin{aligned}
\|u_t\varphi_t\|^2 - \|u_{t-\delta}\varphi_{t-\delta}\|^2 &= \int_{t-\delta}^t \partial_\tau \|u_\tau \varphi_\tau\|^2 d\tau \\
&= 2\int_{t-\delta}^t \sum_{x \in X} (u_\tau\varphi_\tau \partial_\tau(u_\tau\varphi_\tau))(x) m(x) d\tau \\
&= 2\int_{t-\delta}^t \sum_{x \in X} \left(u_\tau^2 \varphi_\tau \partial_\tau \varphi_\tau + u_\tau \varphi_\tau^2 \partial_\tau u_\tau\right)(x) m(x) d\tau \\
&= \int_{t-\delta}^t \sum_{x \in X} \left(u_\tau^2 \partial_\tau \varphi_\tau^2 + 2u_\tau\varphi_\tau^2 \partial_\tau u_\tau\right)(x) m(x) d\tau,
\end{aligned}$$

where we can interchange differentiation and the sum since φ_τ is supported on a fixed finite set. Since u is a solution of the heat equation we infer

$$\ldots = \int_{t-\delta}^{t} \sum_{x\in X} \left(u_\tau^2 \partial_\tau \varphi_\tau^2 - 2u_\tau \varphi_\tau^2 \mathcal{L} u_\tau\right)(x) m(x) d\tau.$$

By a Caccioppoli-type inequality, Theorem 12.4, we obtain

$$\ldots \leq \int_{t-\delta}^{t} \sum_{x\in X} \left(u_\tau^2 \partial_\tau \varphi_\tau^2 + u_\tau^2 |\nabla \varphi_\tau|^2\right)(x) m(x) d\tau,$$

which finishes the proof. □

Next, we take a closer look at the integral error term on the right-hand side of Lemma 14.3. The ultimate goal is to estimate this term by $Ge^{-f(Er)}/r^2$ with constants $E, G > 1$ for large r. We will then iterate the estimate above until we reach time $t = 0$, at which point the initial condition $u_0 = 0$ cancels the second term on the left-hand side. Applying this estimate with an appropriate φ, which is based on the intrinsic metric and will be supported on balls, will the give the necessary growth condition to yield uniqueness.

The required estimate will be achieved below in Grigor′yan's inequality, Lemma 14.5, which is deduced from the following main technical estimate.

Lemma 14.4 (Main technical estimate). *Let b be a graph over (X, m). Let ϱ be an intrinsic metric with finite distance balls* (B) *and finite jump size* (J) *denoted by s. Let $f\colon (0,\infty) \longrightarrow (0,\infty)$ be a monotonically increasing function, let $0 < T \leq \infty$ and let $u\colon [0,T) \times X \longrightarrow \mathbb{R}$ be a solution of the heat equation such that*

$$\int_0^T \|u_t 1_{B_r}\|^2 dt \leq e^{f(r)}$$

for all $r \geq 0$. Let s_r denote the jump size at distance greater than or equal to r with $s_0 = s$. Let $0 < r < R$, $0 < \lambda < 1$, $0 < \delta \leq t < T$, and let C and ε be chosen such that

$$C \geq 8 \exp\left(\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}\right) \qquad \textit{and} \qquad \varepsilon \geq \frac{2s_{r-2s}^2}{C}.$$

Then,

$$\|u_t 1_{B_r}\|^2 \leq e^{\frac{\varepsilon}{C\delta}} \|u_{t-\delta} 1_{B_R}\|^2 + e^{\frac{2\varepsilon}{C\delta}} \frac{2}{(1-\lambda)^2 (R-r)^2} e^{-\frac{((\lambda(R-r)-s)_+)^2}{C\delta} + f(R+s)}.$$

Remark. Note that C can indeed be chosen as asserted in the lemma since the function $a \mapsto (a/8)^a$ is strictly monotonically increasing on $[a_0, \infty)$ for a_0 large enough.

Proof. We define

$$r_\lambda = (1-\lambda)r + \lambda R$$

to interpolate between r and R via λ and fix $\varepsilon > 0$ as assumed in the lemma. We apply Lemma 14.3 to the function $\varphi\colon [0,t]\times X \longrightarrow [0,\infty)$ defined by

$$\varphi_\tau(x) = \eta(x)e^{\xi(x,\tau)},$$

where

$$\eta(x) = \left(1 - \frac{\varrho(x, B_{r_\lambda})}{R - r_\lambda}\right)_+ \qquad \text{and} \qquad \xi(x,\tau) = -\left(\frac{\varrho_o^2(x)+\varepsilon}{C(t+\delta-\tau)}\right)$$

with $\varrho_o(x) = (\varrho(x,o)-r)_+$ for $x\in X$ and $\tau\in[0,t]$, with the convention that $e^{\xi(x,t+\delta)} = 0$.

As $B_r \subseteq B_{r_\lambda}$ we have $\eta = 1$ on B_r. Furthermore, clearly $\varrho_0(x) = 0$ on B_r so that

$$\varphi_\tau = e^{\frac{-\varepsilon}{C(t+\delta-\tau)}}$$

on B_r. In particular, $\varphi_t = e^{-\varepsilon/(C\delta)}$ and $\varphi_{t-\delta} = e^{-\varepsilon/(2C\delta)}$ on B_r. Moreover, as $\varrho(x,B_{r_\lambda}) \geq R - r_\lambda$ for $x \in X\setminus B_R$, we have $\eta = 0$ and thus $\varphi_\tau = 0$ outside of B_R. By the assumption that balls are finite, we infer that $\operatorname{supp}\varphi_\tau$ is included in a fixed finite set for all τ. Therefore, Lemma 14.3 yields

$$\begin{aligned}
e^{-\frac{2\varepsilon}{C\delta}}\|u_t 1_{B_r}\|^2 &\leq \|u_t\varphi_t\|^2 \\
&\leq \|u_{t-\delta}\varphi_{t-\delta}\|^2 + \int_{t-\delta}^t \sum_{x\in X} u_\tau^2(x)\left(\partial_\tau\varphi_\tau^2 + |\nabla\varphi_\tau|^2\right)(x)m(x)d\tau \\
&\leq e^{-\frac{\varepsilon}{C\delta}}\|u_{t-\delta}1_{B_R}\|^2 + \int_{t-\delta}^t \sum_{x\in X} u_\tau^2(x)\left(\partial_\tau\varphi_\tau^2 + |\nabla\varphi_\tau|^2\right)(x)m(x)d\tau,
\end{aligned}$$

where we use the estimate $\xi(x,t-\delta) \leq -\varepsilon/(2c\delta)$ to get $\varphi_{t-\delta} \leq e^{-\varepsilon/(2c\delta)}$ in the last line.

We are left to show

$$\int_{t-\delta}^t \sum_{x\in X} u_\tau^2(x)\left(\partial_\tau\varphi_\tau^2 + |\nabla\varphi_\tau|^2\right)(x)m(x)d\tau \leq \frac{2e^{-\frac{((\lambda(R-r)-s)_+)^2}{C\delta}+f(R+s)}}{(1-\lambda)^2(R-r)^2},$$

which combined with the estimate above will complete the proof. As η is finitely supported, by Young's inequality and the definition of φ we then have

$$\begin{aligned}
|\nabla\varphi_\tau|^2(x)m(x) &= \sum_{y\in X} b(x,y)\left(\eta(x)e^{\xi(x,\tau)} - \eta(y)e^{\xi(y,\tau)}\right)^2 \\
&= \sum_{y\in X} b(x,y)\left(\eta(x)\left(e^{\xi(x,\tau)} - e^{\xi(y,\tau)}\right) + e^{\xi(y,\tau)}(\nabla_{x,y}\eta)\right)^2 \\
&\leq 2\sum_{y\in X} b(x,y)\left(\eta^2(x)\left(e^{\xi(x,\tau)} - e^{\xi(y,\tau)}\right)^2 + e^{2\xi(y,\tau)}(\nabla_{x,y}\eta)^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\varphi_\tau^2(x) \sum_{y \in X} b(x,y) \left(1 - e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2 \\
&\quad + 2 \sum_{y \in X} b(x,y) e^{2\xi(y,\tau)} (\nabla_{x,y}\eta)^2.
\end{aligned}
$$

We furthermore note that

$$
\partial_\tau \varphi_\tau^2(x) = 2\eta^2(x) e^{2\xi(x,\tau)} \partial_\tau \xi(x,\tau) = 2\varphi_\tau^2(x) \partial_\tau \xi(x,\tau).
$$

Hence, putting these two calculations together yields

$$
\begin{aligned}
&(\partial_\tau \varphi_\tau^2 + |\nabla \varphi_\tau|^2)(x) m(x) \\
&\quad \le 2 \sum_{y \in X} b(x,y) e^{2\xi(y,\tau)} (\nabla_{x,y}\eta)^2 \\
&\quad + 2\varphi_\tau^2(x) \left(\partial_\tau \xi(x,\tau) m(x) + \sum_{y \in X} b(x,y) \left(1 - e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2 \right).
\end{aligned}
$$

To continue, we denote the terms in the previous inequality by

$$
I_1(x,\tau) = \sum_{y \in X} b(x,y) e^{2\xi(y,\tau)} (\nabla_{x,y}\eta)^2
$$

and

$$
I_2(x,\tau) = \partial_\tau \xi(x,\tau) m(x) + \sum_{y \in X} b(x,y) \left(1 - e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2.
$$

It remains to show that

$$
\int_{t-\delta}^t \sum_{x \in X} u_\tau^2(x) \left(I_1(x,\tau) + \varphi_\tau^2(x) I_2(x,\tau) \right) d\tau \le \frac{e^{-\frac{((\lambda(R-r)-s)_+)^2}{C\delta} + f(R+s)}}{(1-\lambda)^2 (R-r)^2}.
$$

We estimate the terms involving I_1 and I_2 separately to conclude the proof. In fact, we will see that the upper bound is for the term involving I_1 and the term involving I_2 is actually less than or equal to 0.

Estimating the term involving I_1: Recall that the function η is equal to 1 on B_{r_λ} and is equal to 0 on $X \setminus B_R$. Thus, $I_1(\cdot,\tau)$ vanishes on $B_{r_\lambda - s} \cup X \setminus B_{R+s}$, where s is the jump size of the metric.

Let $x \in B_{R+s} \setminus B_{r_\lambda - s}$ and $y \in X$ with $y \sim x$. We note by the triangle inequality that the term $e^{\xi(x,\tau)}(\eta(x) - \eta(y))$ vanishes whenever $\varrho(o,y) < r_\lambda - s$. This follows since $\varrho(o,x) \le \varrho(o,y) + \varrho(y,x) < r_\lambda - s + s = r_\lambda$, so that $\eta(x) = \eta(y) = 1$ in this case.

On the other hand, for $y \in X \setminus B_{r_\lambda - s}$, the function $\varrho_o(\cdot) = (\varrho(\cdot,o) - r)_+$ satisfies

$$
-\varrho_o^2(y) \le -((r_\lambda - s - r)_+)^2 = -((\lambda(R-r) - s)_+)^2
$$

since $r_\lambda = (1-\lambda)r + \lambda R$. If, additionally, $t-\delta < \tau \le t$, then

$$e^{2\xi(y,\tau)} = \exp\left(-\frac{2(\varrho_o^2(y)+\varepsilon)}{C(t+\delta-\tau)}\right) \le \exp\left(-\frac{2((\lambda(R-r)-s)_+)^2}{C\delta}\right).$$

Finally, observe that as η is $1/(R-r_\lambda)$-Lipschitz with respect to the intrinsic metric ϱ, Proposition 11.29 and $R-r_\lambda = R-(1-\lambda)r+\lambda R = (1-\lambda)(R-r)$ yield

$$\sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2 \le \frac{1}{(1-\lambda)^2(R-r)^2} m(x).$$

Taking these observations into account and integrating over space and time we obtain

$$\begin{aligned}
&\int_{t-\delta}^t \sum_{x\in X} u_\tau^2(x) I_1(x,\tau)d\tau \\
&= \int_{t-\delta}^t \sum_{x\in B_{R+s}\setminus B_{r_\lambda - s}} u_\tau^2(x) \sum_{y\in X\setminus B_{r_\lambda-s}} b(x,y)e^{2\xi(y,\tau)}(\nabla_{x,y}\eta)^2 d\tau \\
&\le e^{-\frac{2((\lambda(R-r)-s)_+)^2}{C\delta}} \int_{t-\delta}^t \sum_{x\in B_{R+s}\setminus B_{r_\lambda-s}} u_\tau^2(x) \sum_{y\in X} b(x,y)(\nabla_{x,y}\eta)^2 d\tau \\
&\le \frac{1}{(1-\lambda)^2(R-r)^2} e^{-\frac{2((\lambda(R-r)-s)_+)^2}{C\delta}} \int_{t-\delta}^t \sum_{x\in B_{R+s}\setminus B_{r_\lambda-s}} u_\tau^2(x)m(x)d\tau \\
&\le \frac{1}{(1-\lambda)^2(R-r)^2} e^{-\frac{2((\lambda(R-r)-s)_+)^2}{C\delta}+f(R+s)},
\end{aligned}$$

where we used the assumption $\int_0^T \|u1_{B_{R+s}}\|^2 d\tau \le e^{f(R+s)}$ in the last estimate.

Estimating the term involving I_2: Specifically, we have to control the term $u^2\varphi^2 I_2$ integrated over space and a time interval. Since φ vanishes on $X\setminus B_R$ it suffices to control $u^2\varphi^2 I_2$ on $B_R\times[t-\delta,t]$. Furthermore,

$$\partial_\tau \xi(x,\tau) = -\partial_\tau\left(\frac{\varrho_o^2(x)+\varepsilon}{C(t+\delta-\tau)}\right) = -\frac{\varrho_o^2(x)+\varepsilon}{C(t+\delta-\tau)^2} \le 0$$

and for $t-\delta\le\tau\le t$, $x\in B_{r-s}$ and $y\sim x$ we have by the definitions of $\varrho_o(\cdot) = (\varrho(\cdot,o)-r)_+$ and $\xi(\cdot,\tau) = -(\varrho_o^2(\cdot)+\varepsilon)/(C(t+\delta-\tau))$ that $\varrho_o(x) = 0 = \varrho_o(y)$ and thus $\xi(x,\tau) = \xi(y,\tau)$. Therefore, we infer for $t-\delta\le\tau\le t$ and $x\in B_{r-s}$ that

$$I_2(x,\tau) = \partial_\tau\xi(x,\tau)m(x) + \sum_{y\in X} b(x,y)\left(1-e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2 \le 0,$$

so that $u^2\varphi^2 I_2 \le 0$ on $B_{r-s}\times[t-\delta,t]$.

Therefore, it suffices to estimate $u^2\varphi^2 I_2$ on $B_R \setminus B_{r-s} \times [t-\delta, t]$. We will ultimately show that $I_2 \le 0$ on this set as well, which will conclude the proof.

Let $t-\delta \le \tau \le t$ and $x \in B_R \setminus B_{r-s}$. Then, $\varrho_o(x) \le R-r$ and $t+\delta-\tau \ge \delta$. Moreover, for $y \sim x$, we have $y \in X \setminus B_{r-2s}$ and, therefore, $\varrho(x,y) \le s_{r-2s} \le s$. Furthermore, the triangle inequality implies $\varrho_o(y) \le \varrho_o(x)+\varrho(x,y)$ and $|\varrho_o(x)-\varrho_o(y)| \le \varrho(x,y)$. Combining these observations we obtain

$$\begin{aligned}
|\xi(x,\tau)-\xi(y,\tau)| &= \frac{|\varrho_o(x)-\varrho_o(y)||\varrho_o(x)+\varrho_o(y)|}{C(t+\delta-\tau)} \\
&\le \frac{\varrho(x,y)(\varrho_o(x)+\varrho_o(y))}{C(t+\delta-\tau)} \\
&\le \frac{\varrho(x,y)(2\varrho_o(x)+\varrho(x,y))}{C(t+\delta-\tau)} \\
&\le \frac{s_{r-2s}(2(R-r)+s)}{C\delta}.
\end{aligned}$$

We use this chain of inequalities repeatedly as well as $(1-e^a)^2 \le a^2 e^{2a\vee 0}$ for $a \in \mathbb{R}$ and Young's inequality to estimate, for all $t-\delta \le \tau \le t$ and $x \in B_R \setminus B_{r-s}$, $y \sim x$ with $\varrho_o(y) \le \varrho_o(x)$,

$$\begin{aligned}
\left(1-e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2 &\le (\xi(y,\tau)-\xi(x,\tau))^2\, e^{2(\xi(y,\tau)-\xi(x,\tau))} \\
&\le \frac{\varrho^2(x,y)(\varrho_o(x)+\varrho_o(y))^2}{C^2(t+\delta-\tau)^2} e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}} \\
&\le \frac{4\varrho_o^2(x)}{C^2(t+\delta-\tau)^2} e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}}\, \varrho^2(x,y)
\end{aligned}$$

and, with $\varrho_o(y) > \varrho_o(x)$,

$$\begin{aligned}
\left(1-e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2 &\le (\xi(y,\tau)-\xi(x,\tau))^2 \\
&\le \frac{\varrho^2(x,y)(2\varrho_o(x)+\varrho(x,y))^2}{C^2(t+\delta-\tau)^2} \\
&\le \frac{4\varrho_o^2(x)+2s_{r-2s}^2}{C^2(t+\delta-\tau)^2}\varrho^2(x,y),
\end{aligned}$$

where we use in the last inequality that $\varrho(x,y) \le s_{r-2s}$ since $x \in X \setminus B_{r-s}$. Putting these two estimates together by separating the sum over $y \in X$ into sums over $\varrho_o(y) \le \varrho_o(x)$ and $\varrho_o(y) > \varrho_o(x)$ and using that ϱ is intrinsic, we obtain

$$\begin{aligned}
&\sum_{y\in X} b(x,y)\left(1-e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2\\
&\qquad\leq \frac{4\varrho_o^2(x)e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}}+4\varrho_o^2(x)+2s_{r-2s}^2}{C^2(t+\delta-\tau)^2}\sum_{y\in X} b(x,y)\varrho^2(x,y)\\
&\qquad\leq \frac{8\varrho_o^2(x)e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}}+2s_{r-2s}^2}{C^2(t+\delta-\tau)^2}m(x).
\end{aligned}$$

We end the proof by putting the estimates above together. Recall that we are interested in estimating $I_2(x,\tau)$ for $x\in B_R\setminus B_{r-s}$ and $\tau\in[t-\delta,t]$. We calculated above that $\partial_\tau\xi(x,\tau)=-(\varrho_o^2(x)+\varepsilon)/(C(t+\delta-\tau)^2)$ for all $x\in X$ and $\tau\in[0,t]$. Thus, we obtain for $x\in B_R\setminus B_{r-s}$ and $t-\delta\leq\tau\leq t$ that

$$\begin{aligned}
I_2(x,\tau)&=\partial_\tau\xi(x,\tau)m(x)+\sum_{y\in X} b(x,y)\left(1-e^{\xi(y,\tau)-\xi(x,\tau)}\right)^2\\
&\leq\left(-\frac{\varrho_o^2(x)+\varepsilon}{C(t+\delta-\tau)^2}+\frac{8\varrho_o^2(x)e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}}+2s_{r-2s}^2}{C^2(t+\delta-\tau)^2}\right)m(x)\\
&=\left(\varrho_o^2(x)\left(\frac{8}{C}e^{\frac{s_{r-2s}(4(R-r)+2s)}{C\delta}}-1\right)+\left(\frac{2s_{r-2s}^2}{C}-\varepsilon\right)\right)\frac{m(x)}{C(t+\delta-\tau)^2}.
\end{aligned}$$

Thus, $I_2(x,\tau)\leq 0$ since the assumption on C and ε read as

$$C\geq 8\exp\left(\frac{s_{r-2s}((4(R-r)+2s)}{C\delta}\right)\qquad\text{and}\qquad \varepsilon\geq\frac{2s_{r-2s}^2}{C}.$$

This finishes the proof. □

We now apply the technical estimate established above to prove Grigor′yan's inequality for solutions of the heat equation on globally local graphs. This allows us to estimate the norm of a solution of the heat equation on a ball by the norm on a larger ball but at a previous time plus an error term.

LEMMA 14.5 (Grigor′yan's inequality). *Let b be a graph over (X,m). Let ϱ be an intrinsic metric with finite distance balls* (B) *and finite jump size* (J). *Let $f\colon(0,\infty)\longrightarrow(0,\infty)$ be a monotonically increasing function such that ϱ is globally local with respect to f. Let $u\colon[0,T]\times X\longrightarrow\mathbb{R}$ be a solution of the heat equation which satisfies*

$$\int_0^T\|u_t1_{B_r}\|^2dt\leq e^{f(r)}$$

for $r \geq 0$. Then, there exist constants $r_1 > 0$ and $D, E, F, G > 1$ such that, for all $r \geq r_1$ and $0 < \delta < t < T$ with

$$\delta \leq \frac{r^2}{Df(Er)},$$

we have

$$\|u_t 1_{B_r}\|^2 \leq F\|u_{t-\delta} 1_{B_{Er}}\|^2 + \frac{Ge^{-f(Er)}}{r^2}.$$

Proof. We prove the statement by applying the main technical estimate, Lemma 14.4, above with a proper choice of constants. From the assumption that the graph is globally local there exist constants $A > 1$, $B > 0$ and $r_0 > 0$ such that $s_r \leq Br/f(Ar)$ for all $r \geq r_0$. Furthermore, $s_0 = s < \infty$, where s is the jump size.

Let $1 < E' < E < A$ and let $R = E'r$. Various subsequent estimates are only true for r large enough and this will eventually determine the constant r_1 in the statement of the lemma. However, since the actual choice of r_1 is as cumbersome as it is irrelevant, we will keep this choice implicit and only refer to choosing r large enough. On the other hand, this choice will only depend on the constants A, E', E and s.

Consider parameters $\alpha, C > 0$, which will be specified later, and let

$$\delta(r, \alpha, C) = \frac{r^2}{\alpha C f(Er)} \qquad \text{and} \qquad \varepsilon(r, C) = \frac{2s_{r-2s}^2}{C}.$$

In order to apply the main technical estimate we have to establish a lower bound on C, which we ultimately do below. For $r \geq 0$ consider

$$a(r) = \frac{\alpha s_{r-2s}(4(E'-1)r + 2s)f(Er)}{r^2} = \frac{s_{r-2s}(4(R-r) + 2s)}{C\delta(r, \alpha, C)},$$

where the second equality is an immediate consequence of the definition of δ and $R = E'r$. We next give an upper bound for $a(r)$. To this end, we observe that for large r we have $f(Er) \leq f(A(r-2s))$, since f is monotonically increasing and $E < A$. Moreover, we use the assumption $s_r \leq Br/f(Ar)$ to estimate

$$\begin{aligned} a(r) &\leq \alpha \frac{B(r-2s)}{f(A(r-2s))} \frac{(4(E'-1)r + 2s)f(Er)}{r^2} \\ &\leq 4\alpha BE' \frac{(r-2s)(r+2s)}{r^2} \frac{f(Er)}{f(A(r-2s))} \\ &\leq 4\alpha BE' \leq 4\alpha AB \end{aligned}$$

for large enough r. So, whenever we choose $C \geq 8e^{4\alpha AB}$ we have, for large enough r,

$$C \geq 8e^{4\alpha AB} \geq 8e^{a(r)} = 8\exp\left(\frac{s_{r-2s}(4(R-r) + 2s)}{C\delta(r, \alpha, C)}\right),$$

which is the required bound on C in the main technical estimate, Lemma 14.4 above.

Noting that with C, $\varepsilon(r,C)$, $\delta(r,\alpha,C)$, $R = E'r$ as above, we have

$$\frac{1}{C\delta(r,\alpha,C)} = \frac{\alpha f(Er)}{r^2} \quad \text{and} \quad R - r = (E'-1)r$$

we apply Lemma 14.4 with r large enough, $T > t > \delta(r,\alpha,C) > 0$ and $\lambda = 1/2$ to obtain

$$\begin{aligned}\|u_t 1_{B_r}\|^2 \leq\; & e^{\frac{\varepsilon(r,C)}{C\delta(r,\alpha,C)}} \|u_{t-\delta(r,\alpha,C)} 1_{B_{E'r}}\|^2 \\ & + e^{\frac{2\varepsilon(r,C)}{C\delta(r,\alpha,C)}} \frac{8}{(E'-1)^2 r^2} e^{-\alpha \frac{((E'-1)r/2-s)_+^2}{r^2} f(Er) + f(E'r+s)}.\end{aligned}$$

We proceed to estimate the remaining terms which still depend on α, r and C to get the asserted inequality. We start with the exponential factor on the right-hand side, which we estimate by $e^{-f(Er)}$ by choosing α appropriately. Thus, choose $\alpha > 1$ such that

$$\alpha \frac{((E'-1)r/2 - s)_+^2}{r^2} \geq 2$$

for r large enough. We note that $\alpha > 1$ can be chosen to only depend on E' and s. Recalling that $f(Er) \geq f(E'r + s)$ for r large enough since $E > E'$ and f is monotonically increasing, we immediately get with this choice of α

$$e^{-\alpha \frac{((E'-1)r/2-s)_+^2}{r^2} f(Er) + f(E'r+s)} \leq e^{-2f(Er) + f(E'r+s)} \leq e^{-f(Er)}$$

for r large enough.

Next, we bound the term $e^{\varepsilon/(C\delta)}$. We use the definitions of $\varepsilon(r,C)$ and $\delta(r,\alpha,C)$, along with the estimates $s_r \leq Br/f(Ar)$, $f(Er) \leq f(A(r-2s))$ for r large enough since $A > E$ and $C \geq 8e^{4\alpha AB}$, to estimate

$$\frac{\varepsilon(r,C)}{C\delta(r,\alpha,C)} = \frac{2\alpha s_{r-2s}^2 f(Er)}{Cr^2} \leq \frac{2\alpha B^2 (r-2s)^2 f(Er)}{Cr^2 f^2(A(r-2s))} \leq \frac{\alpha B^2 e^{-4\alpha AB}}{4f(A(r-2s))}$$

for r large enough. Hence, as f is monotonically increasing, there exists a constant $F > 1$ such that

$$\exp\left(\frac{\varepsilon(r,C)}{C\delta(r,\alpha,C)}\right) \leq F$$

for all $C \geq 8e^{4\alpha AB}$ and r large enough.

Finally, we let

$$G = \frac{8F^2}{(E'-1)^2}.$$

We observe that all of the constants α, E, E', F, G were chosen independently of C. This allows us to further increase $C \geq 8e^{4\alpha AB}$ to get the estimate

$$\|u_t 1_{B_r}\|^2 \leq F\|u_{t-\delta} 1_{B_{E'r}}\|^2 + \frac{Ge^{-f(Er)}}{r^2}$$

for all $0 < \delta < t < T$ with

$$\delta = \delta(r, \alpha, C) = \frac{r^2}{\alpha C f(Er)} \leq \frac{r^2}{Df(Er)},$$

where

$$D = 8\alpha e^{4\alpha AB}.$$

Finally notice that $B_{E'r} \subseteq B_{Er} = B_R$ as $E' < E$, which finishes the proof. □

We now recall the statement of the main result of this section, the uniqueness class assertion of Theorem 14.2. Specifically, we have a graph b with an intrinsic metric ϱ with finite balls and such that b is globally local with respect to a monotone increasing function $f\colon (0,\infty) \longrightarrow (0,\infty)$ with $\int_1^\infty r/f(r)dr = \infty$. Hence, there exist constants $A, B > 1$ such that $s_r \leq Br/f(Ar)$ for all large r, where s_r is the jump size of the graph outside of the ball of radius r. Finally, we assume that u is a solution of the heat equation with initial condition $u_0 = 0$ such that $\int_0^T \|u_t 1_{B_r}\|^2 dt \leq e^{f(r)}$. Our aim is to show that $u = 0$.

As a final step in preparation for the proof we show that the function f appearing in the statement of Theorem 14.2 can be chosen to grow at least linearly. This will be convenient for the proof.

Lemma 14.6. *Let b, ϱ and f be as in the statement of Theorem 14.2. Without loss of generality we can assume that $f(r) \geq r$ for all r large.*

Proof. If f is not greater than or equal to r, then replace f with $g(r) = f(r) \vee r$, which obviously satisfies $g(r) \geq r$ and $e^{f(r)} \leq e^{g(r)}$. Moreover, if $M = \{r \geq 1 \mid f(r) < r\}$, then

$$\int_1^\infty \frac{r}{g(r)}dr = |M| + \int_{[1,\infty)\setminus M} \frac{r}{f(r)}dr,$$

where $|M|$ is the Lebesgue measure of M.

If $|M| = \infty$, then $\int_1^\infty r/g(r)dr = \infty$.

Now, if $|M| < \infty$ and $\int_{[1,\infty)\setminus M} r/f(r)dr = \infty$, then $\int_1^\infty r/g(r)dr = \infty$ as well. Now, assume that $|M| < \infty$ and $\int_{[1,\infty)\setminus M} r/f(r)dr < \infty$. Then, as $\int_1^\infty r/f(r)dr = \infty$, it follows that the function $r \mapsto r/f(r)$ is unbounded on M. Hence, for each $C > 1$, there exists an $r_0 \in M$ such that $r_0 > Cf(r_0)$. Now, using the monotonicity of f we may assume that $f \geq 1$. Then, for all $h < C - 1 \leq (C-1)f(r_0 - h)$, by using the monotonicity of f we get

$$r_0 - h > Cf(r_0) - h \geq Cf(r_0 - h) - h \geq f(r_0 - h),$$

so that $r_0 - h \in M$. But this would imply $|M| = \infty$. Therefore, $\int_1^\infty r/g(r)dr = \infty$.

Finally, using $s_r \leq s_0 = s < \infty$ the globally local assumption $s_r \leq B'r/g(Ar)$ for large r is satisfied with the slight modification of replacing the original B by $B' = B \vee As$. $\square$

We now put all of these results together to prove the uniqueness class result for the heat equation on globally local graphs.

Proof of Theorem 14.2. Let $r_1 > 0$ and $D, E, F, G > 1$ be constants as in Grigor′yan's inequality, Lemma 14.5. Let $r \geq r_1$ and $R_k = E^{k-1}r$ for $k \in \mathbb{N}$. As Lemma 14.6 allows us to assume $f(r) \geq r$, there exists a constant H such that

$$F^{k-1}e^{-f(R_{k+1})} = F^{k-1}e^{-f(E^k r)} \leq F^{k-1}e^{-E^k r} \leq H$$

for all $r \geq r_1$ and $k \in \mathbb{N}$. Since $\int_1^\infty r/f(r)dr = \infty$ and f is monotonically increasing, it follows that

$$\sum_{k=1}^{\infty} \frac{R_k^2}{f(ER_k)} = \infty.$$

Hence, for every $0 < t < T$ there exists a natural number N and $\delta_k \geq 0$ with

$$\delta_k \leq \frac{R_k^2}{Df(ER_k)}$$

for $k = 1, \ldots, N$ such that $\sum_{k=1}^N \delta_k = t$. An iterative application of Grigor′yan's inequality, Lemma 14.5, with radii $R_k = E^{k-1}r$ and time differences δ_k yields

$$\begin{aligned}\sum_{x\in B_r} u_t^2(x)m(x) &\leq F \sum_{x \in B_{R_2}} u_{t-\delta_1}^2(x)m(x) + \frac{Ge^{-f(R_2)}}{r^2} \\ &\leq F^N \sum_{x\in B_{R_{N+1}}} u^2_{t-\sum_{k=1}^N \delta_k}(x)m(x) + G\sum_{k=1}^{N} \frac{F^{k-1}e^{-f(R_{k+1})}}{R_k^2} \\ &\leq \frac{GHE^2}{E^2-1}\frac{1}{r^2},\end{aligned}$$

where for the last inequality we used $F^{k-1}e^{-f(R_{k+1})} \leq H$, $R_k = E^{k-1}r$, $\sum_{k=1}^N \delta_k = t$ and $u_0 = 0$. Noting that the finiteness of the values of the intrinsic metric gives $X = \bigcup_r B_r$, and letting $r \to \infty$ implies that $u_t = 0$ which completes the proof. $\square$

2. Refinements

In this section we study refinements of graphs. More specifically, we introduce a procedure for adding vertices within an edge to make vertices closer together with respect to an intrinsic metric. We then show that if a refinement of a graph is stochastically complete, then the original graph must be stochastically complete.

We recall that our ultimate aim in this chapter is to deduce the volume growth criterion

$$\int_0^\infty \frac{r}{\log^\#(m(B_r))}dr = \infty$$

for graphs with finite distance balls. We will do so from the uniqueness class criterion, Theorem 14.2, which only holds for globally local graphs. The technique of refinements is used to turn a graph which satisfies the volume growth criterion into a globally local graph and then to apply the uniqueness class theorem to the resulting graph.

To do so we have to ensure various things. First of all, we show in Theorem 14.8 that stochastic completeness of a refinement implies stochastic completeness of the original graph. We do this by applying the Omori–Yau maximum principle. Secondly, we show that the metric and volume of a refinement of a graph are comparable to the original graph in Lemma 14.9. Finally, in Lemma 14.10 we prove that for every given control function on the jump size outside of balls, there is a refinement which satisfies this control.

The actual definition of a refinement is rather technical so we explain the basic idea first. Although we work with intrinsic metrics in this section, we note that the construction of the refinement works for a general pseudo metric.

Let b be a graph over (X, m) and let ϱ be an intrinsic metric. Let n be a function on the edges with values in the non-negative integers. The construction of the refinement of b is as follows:

- If (x, y) is an edge, then we replace $\{x, y\}$ by $\{x = x_0, x_1, \ldots, x_{n(x,y)+1} = y\}$. In other words, we "insert" $n(x, y)$ vertices "into" the edge (x, y).
- The new edge weights are set to be $(n + 1)b$. The intuitive understanding of this choice is that the magnitude of the edge weight is the strength of the bond between vertices and by inserting vertices these vertices are now "closer" together and therefore the strength of the bond increases.
- To keep the distance between the original vertices the same, we set the distance for each new edge to be $\varrho/(n + 1)$. We denote the resulting metric by ϱ'.
- Finally, the measure m is left unchanged on the original vertices and m is defined on the new vertices so that the metric ϱ' will be intrinsic.

After this intuitive explanation, we present the formal definition of a refinement.

Definition 14.7 (Refinement of a graph). Let b be a locally finite graph over (X, m) and let ϱ be an intrinsic metric for b. Let $n \colon X \times X \longrightarrow \mathbb{N}_0$ be a symmetric function such that $n(x, y) \geq 1$ only if $x \sim y$

and $\varrho(x,y) > 0$. The *refinement* of b with respect to n is the graph b' over (X', m') with metric ϱ' as follows:

- The vertex set X' is the union
$$X' = X \dot\cup \bigcup_{x,y\in X} X_{x,y},$$
where $X_{x,y} = X_{y,x}$ are pairwise disjoint sets, i.e., $X_{x,y} \cap X_{w,z} = \emptyset$ if $\{x,y\} \neq \{w,z\}$ such that $\# X_{x,y} = n(x,y)$.
- The measure m' is given by
$$m'|_X = m \qquad \text{and} \qquad m'|_{X_{x,y}} = \frac{2b(x,y)\varrho^2(x,y)}{n(x,y)+1}$$
for $x \sim y$.
- The edge weight b' is defined as follows: For $x \sim y$, we fix an enumeration of $X_{x,y} = \{x_1, \ldots, x_{n(x,y)}\}$, set $x_0 = x$, $x_{n(x,y)+1} = y$ and let
$$b'(x_0, x_1) = \ldots = b'(x_{n(x,y)}, x_{n(x,y)+1}) = (n(x,y)+1)b(x,y).$$
Otherwise, we set $b' = 0$ and write $w \sim' z$ if $b'(w,z) > 0$ for $w, z \in X'$.
- The distance ϱ' is defined in three steps:
If $w, z \in \{x,y\} \cup X_{x,y}$ for some $x \sim y$, then we set
$$\varrho'(w,z) = \min\left\{ \frac{k\varrho(x,y)}{n(x,y)+1} \mid w = x_0 \sim' \ldots \sim' x_k = z \text{ in } \{x,y\} \cup X_{x,y} \right\}.$$
If $w \in X$ and $z \in \{x,y\} \cup X_{x,y}$ for some $x \sim y$ and $z \neq x, y$, then we set
$$\varrho'(w,z) = \min_{v\in\{x,y\}} (\varrho(w,v) + \varrho'(v,z)).$$
If $w \in \{x,y\} \cup X_{x,y}$ and $z \in \{x',y'\} \cup X_{x',y'}$ for some $x \sim y$ and $x' \sim y'$ with $\{x,y\} \neq \{x',y'\}$, then we set
$$\varrho'(w,z) = \min_{v\in\{x,y\}, v'\in\{x',y'\}} (\varrho'(w,v) + \varrho(v,v') + \varrho(v',z)).$$

REMARK. We note that we only define refinements for locally finite graphs since we cannot assure that b' is summable about vertices for a non-locally finite b.

We now prove the main theorem of this section. It states that if a refinement of a graph is stochastically complete, then the original graph is stochastically complete. The proof involves the Omori–Yau maximum principle, see Section 7. We note that in Theorem 7.2 we assume connectedness of the graph for the various notions of stochastic completeness to be equivalent. However, for the result below, we do not assume connectedness and rather use the fact that stochastic incompleteness of the entire graph is equivalent to the existence of a connected component which is stochastically incomplete (Exercise 14.3).

THEOREM 14.8 (Stability of stochastic completeness under refinements). *Let b be a locally finite graph over (X, m) with an intrinsic metric ϱ. If a refinement of b is stochastically complete, then b is stochastically complete.*

PROOF. Let $n\colon X \times X \longrightarrow \mathbb{N}_0$ be such that $n(x, y) \geq 1$ only if $x \sim y$ and $\varrho(x, y) > 0$. Let b' over (X', m') be the refinement of b with respect to n. Assume that b over (X, m) is stochastically incomplete. Then, b must have at least one stochastically incomplete connected component. The Omori–Yau maximum principle, Theorem 7.2 (iv), yields a function $u \in \mathcal{F}$ on this component with $\sup u \in (0, \infty)$ and $\beta > 0$ such that $\mathcal{L}u < -1$ on the set $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$.

We define a function u' on X' next. First, we let

$$u'|_X = u.$$

For $x, y \in X$ with $n(x, y) \geq 1$ (which occurs only for $x \sim y$ and $\varrho(x, y) > 0$), let $\eta_{x,y}\colon [0, \varrho(x, y)] \longrightarrow \mathbb{R}$ be defined by

$$\eta_{x,y}(t) = \frac{1}{2}t^2 + \left(\frac{u(y) - u(x)}{\varrho(x, y)} - \frac{\varrho(x, y)}{2}\right) t + u(x).$$

We note that $\eta_{x,y}(0) = u(x)$ and $\eta_{x,y}(\varrho(x, y)) = u(y)$. We write $\{x, y\} \cup X_{x,y} = \{x_0, \ldots, x_{n(x,y)+1}\}$ with $x = x_0 \sim' \ldots \sim' x_{n(x,y)+1} = y$. Moreover, we define

$$u'(x_k) = \eta_{x,y}\left(k\frac{\varrho(x, y)}{n(x, y) + 1}\right)$$

for $k = 1, \ldots, n(x, y)$. Furthermore, the enumeration of $X_{x,y}$ is unique up to reversing the order of the vertices. As $\eta_{x,y}(t) = \eta_{y,x}(\varrho(x, y) - t)$, the function u' is well-defined, i.e., u' is independent of the enumeration of $X_{x,y}$.

We will now prove that u' is bounded from above and satisfies $\mathcal{L}'u' < -1/2$ on

$$X'_\beta = \{x' \in X' \mid u'(x') > \sup u' - \beta\},$$

where $\mathcal{L}'$ denotes the formal Laplacian of b' on (X', m'). This will show that the refinement is stochastically incomplete by the Omori-Yau maximum principle.

We note that the second derivative of $\eta_{x,y}$ is equal to 1. Therefore, $\eta_{x,y}$ is convex and, hence, $\eta_{x,y} \leq u(x) \vee u(y)$. So, from $u' = u$ on X we infer

$$\sup_X u = \sup_{X'} u'$$

and thus u' is bounded. Moreover, we deduce

$$X'_\beta \subseteq X_\beta \cup \bigcup_{x \in X_\beta} \bigcup_{y \sim x} X_{x,y}.$$

Thus, it suffices to check $\mathcal{L}'u' < -1/2$ for all $x \in X_\beta$ and all $x' \in X_{x,y}$ for $x \in X_\beta$ with $y \sim x$.

Let $x \in X_\beta$ and $y' \in X'$ with $x \sim' y'$. Then, there exists a unique $y \in X$ with $y' \in X_{x,y}$. We let $n = n(x,y)$. Then,

$$b'(x,y') = (n+1)b(x,y), \qquad m'(x) = m(x), \qquad u'(x) = u(x)$$

and

$$u'(y') = \eta_{x,y}\left(\frac{\varrho(x,y)}{n+1}\right) = \frac{\varrho^2(x,y)}{2(n+1)^2} - \frac{u(x)-u(y)}{n+1} - \frac{\varrho^2(x,y)}{2(n+1)} + u(x).$$

Therefore, we get

$$\begin{aligned}
&b'(x,y')(u'(x) - u'(y')) \\
&\qquad = b(x,y)(n+1)\left(-\frac{\varrho^2(x,y)}{2(n+1)^2} + \frac{u(x)-u(y)}{n+1} + \frac{\varrho^2(x,y)}{2(n+1)}\right) \\
&\qquad \le b(x,y)(u(x)-u(y)) + \frac{b(x,y)\varrho^2(x,y)}{2}.
\end{aligned}$$

We sum up these inequalities over $y' \in X'$, use that ϱ is intrinsic and employ $\mathcal{L}u < -1$ on X_β to obtain

$$\begin{aligned}
\mathcal{L}'u'(x) &= \frac{1}{m'(x)} \sum_{y' \in X'} b'(x,y')(u(x) - u(y')) \\
&\le \mathcal{L}u(x) + \frac{1}{2m(x)} \sum_{y \in X} b(x,y)\varrho^2(x,y) \\
&< -1 + \frac{1}{2} = -\frac{1}{2}.
\end{aligned}$$

This shows $\mathcal{L}'u' < -1/2$ on X_β.

Let $x \in X_\beta$, $y \in X$ with $y \sim x$ and $n = n(x,y) \geq 1$. We enumerate $\{x,y\} \cup X_{x,y} = \{x_0, \ldots, x_n\}$ with $x = x_0 \sim' \ldots \sim' x_{n+1} = y$ and set $r = \varrho(x,y)$. Then, $u'(x_k) = \eta_{x,y}(kr/(n+1))$ for all $k = 1, \ldots, n$ and for $i \in \{\pm 1\}$ we have

$$\begin{aligned}
b'(x_k, x_{k+i}) &= (n+1)b(x,y) \\
m'(x_k) &= \frac{2b(x,y)r^2}{n+1} \\
u'(x_k) &= \frac{k^2r^2}{2(n+1)^2} - \frac{k(u(x)-u(y))}{n+1} - \frac{kr^2}{2(n+1)} + u(x) \\
u'(x_{k+i}) &= \frac{(k+i)^2r^2}{2(n+1)^2} - \frac{(k+i)(u(x)-u(y))}{n+1} - \frac{(k+i)r^2}{2(n+1)} + u(x).
\end{aligned}$$

Therefore, we get

$$\mathcal{L}'u'(x_k) = \frac{1}{m'(x_k)} \sum_{i \in \{\pm 1\}} b'(x_k, x_{k+i})(u(x_k) - u(x_{k+i}))$$

$$= \frac{n+1}{2r^2} \sum_{i\in\{\pm 1\}} \left(-\frac{(2ki+i^2)r^2}{2(n+1)} + i(u(x)-u(y)) + \frac{ir^2}{2} \right)$$

$$= -\frac{1}{4} \sum_{i\in\{\pm 1\}} (2ki+i^2) = -\frac{1}{2}.$$

Hence, there exists a function u' on X' such that $\sup u' \in (0,\infty)$ and $\mathcal{L}'u' < -1/2$ on X'_β. Therefore, b' over (X',m') is stochastically incomplete by Theorem 7.2 (iv). □

Next, we show that the metric defined for a refinement is an intrinsic metric if the original metric is intrinsic. Moreover, we show that the measure of balls of the refinement is comparable to the measure of the original balls. Finally, we show that balls in the refinement are finite whenever they are finite in the original graph.

For two pseudo metrics ϱ on X and ϱ' on X' we denote the distance balls about a vertex $o \in X \cap X'$ for $r \geq 0$ by B_r and B'_r, respectively.

LEMMA 14.9 (Refinements and volume growth). *Let b be a locally finite graph over (X,m) and let ϱ be an intrinsic metric for b. For a refinement b' over (X',m') and metric ϱ' with respect to a function n the following statements hold:*

(a) *ϱ' is an intrinsic metric for b' such that $\varrho = \varrho'$ on $X \times X$.*
(b) *$m(B_r) \leq m'(B_{r'}) \leq 3m(B_r)$ for all $r \geq 0$.*
(c) *The ball B'_r is finite if and only if B_r is finite for all $r \geq 0$.*

PROOF. (a): It follows readily from the definition that ϱ' is a pseudo metric with $\varrho = \varrho'$ on $X \times X$. To see that ϱ' is intrinsic let $x \in X$ and calculate

$$\begin{aligned}\sum_{y'\in X'} b'(x,y')\varrho'(x,y')^2 &= \sum_{y\in X}\sum_{y'\in X_{x,y}} b'(x,y')\varrho'(x,y')^2 \\ &= \sum_{y\in X} b(x,y)(n(x,y)+1)\frac{\varrho^2(x,y)}{(n(x,y)+1)^2} \\ &\leq m(x) = m'(x).\end{aligned}$$

Furthermore, for $x \sim y$ every $x' \in X_{x,y}$ has exactly two neighbors which we denote by y' and y''. For these neighbors, it follows by definition that

$$b'(x',y') = b'(x',y'') = b(x,y)(n(x,y)+1),$$

$$\varrho'(x',y') = \varrho'(x',y'') = \frac{\varrho(x,y)}{n(x,y)+1} \qquad \text{and}$$

$$m'(x') = \frac{2b(x,y)\varrho^2(x,y)}{n(x,y)+1}.$$

Therefore,

$$\sum_{z' \in X'} b'(x', z')\varrho'(x', z')^2 = \frac{2b(x,y)\varrho^2(x,y)}{n(x,y)+1} = m'(x').$$

This proves (a).

(b): By (a) we have $B_r = B'_r \cap X$. Thus,

$$B_r \subseteq B'_r \subseteq B_r \cup \bigcup_{x \in B_r, y \sim x} X_{x,y}.$$

Now, $m'|_X = m$, therefore,

$$m(B_r) \leq m'(B'_r).$$

Moreover, for $x \in X$ we get by the definition of m' and since ϱ is intrinsic

$$m'\left(\bigcup_{y \sim x} X_{x,y}\right) = \sum_{y \sim x} n(x,y)\frac{2b(x,y)\varrho^2(x,y)}{n(x,y)+1} \leq 2m(x).$$

Therefore, as $m = m'$ on X, we conclude

$$m'(B'_r) \leq m(B_r) + \sum_{x \in B_r} m'\left(\bigcup_{y \sim x} X_{x,y}\right) \leq 3m(B_r).$$

This proves (b).

(c): This follows directly from the inclusion

$$B_r \subseteq B'_r \subseteq B_r \cup \bigcup_{x \in B_r, y \sim x} X_{x,y}$$

established at the beginning of the proof of (b), local finiteness of the graph and the finiteness of all $X_{x,y}$ for $x \sim y$. $\square$

Next, we show that given a graph and a function g which is uniformly positive on compact sets we find a function n such that we can control the jump size outside of balls of a refinement by g. To make this precise, we say that a function $g\colon (0,\infty) \longrightarrow (0,\infty)$ is uniformly positive on a set $M \subseteq (0,\infty)$ if there exists a $C_M > 0$ such that $g \geq C_M$ on M.

Lemma 14.10 (Refinements and jump size). *Let b be a locally finite graph over (X,m) and let ϱ be a pseudo metric on X. For every function $g\colon (0,\infty) \longrightarrow (0,\infty)$ which is uniformly positive on every compact set, there exists a symmetric function $n\colon X \times X \longrightarrow \mathbb{N}_0$ with $n(x,y) \geq 1$ only if $x \sim y$ and $\varrho(x,y) > 0$ such that the refinement b' over (X', m') and ϱ' with respect to n satisfies*

$$s'_r \leq g(r),$$

where $s'_r = \sup\{\varrho'(x,y) \mid x,y \in X',\ x \sim' y,\ \varrho(x,o) \wedge \varrho'(y,o) \geq r\}$ for all $r \geq 0$ and some fixed $o \in X$.

Proof. We set $n(x,y) = 0$ if $b(x,y) = 0$ or $\varrho(x,y) = 0$. For $x, y \in X$ with $x \sim y$ and $\varrho(x,y) > 0$, let

$$r_{x,y} = ((\varrho(x,o) \vee \varrho(y,o)) + \varrho(x,y)) \vee 2.$$

Then, choose $n(x,y) \in \mathbb{N}_0$ so large that

$$\frac{\varrho(x,y)}{\inf_{r\in[1,r_{x,y}]} g(r)} \leq n(x,y) + 1,$$

which is possible since g is uniformly positive on $[1, r_{x,y}]$.

We now show that the refinement of b with respect to n satisfies $s'_r \leq g(r)$. Let $r' \geq 1$ and $x', y' \in X' \setminus B'_{r'}$ with $x' \sim' y'$, where $B'_{r'}$ denotes the ball of radius r' about o with respect to ϱ'. Let $x, y \in X$, $x \sim y$, be the unique vertices such that $x', y' \in \{x, y\} \cup X_{x,y}$. Then, by the definition of ϱ' and the fact that ϱ' extends ϱ, we get

$$r' < \varrho'(o,x') \leq \varrho'(o,x) + \varrho'(x,x') \leq \varrho(o,x) + \varrho(x,y) \leq r_{x,y}.$$

By the definitions of ϱ' and n we have, using $1 \leq r' < r_{x,y}$,

$$\varrho'(x',y') = \frac{\varrho(x,y)}{n(x,y)+1} \leq \inf_{r\in[1,r_{x,y}]} g(r) \leq g(r').$$

This completes the proof. $\square$

3. Volume growth criterion for stochastic completeness

In this section we prove the volume growth criterion for stochastic completeness of graphs. We do so by combining the uniqueness class statement for the heat equation with the technique of refinements.

Our goal in this section is to prove the the following volume growth criterion for stochastic completeness of graphs. We recall that we have defined $\log^{\#} = \log \vee 1$. This allows us to cover the case when the measure of all distance balls is small in the result below.

Theorem 14.11 (Grigor′yan's theorem). *Let b be a graph over (X,m) and let ϱ be an intrinsic metric with finite balls* (B). *If*

$$\int_0^\infty \frac{r}{\log^{\#}(m(B_r))} dr = \infty,$$

then the graph is stochastically complete.

Remark. We can also give a related volume growth criterion (Exercise 14.4).

As already discussed in the beginning of this chapter, we prove the theorem by refining the graph to match a given jump rate, which is possible by Lemma 14.10. Moreover, Lemma 14.9 shows that the metric and volume of a graph and a refinement of the graph are comparable. Therefore, by choosing an appropriate function with which to make the graph globally local, we can apply the uniqueness class criterion, Theorem 14.2, to the refinement. Finally, we have shown that stochastic completeness of the refinement implies stochastic completeness of the original graph in Theorem 14.8.

However, refinements are only defined for locally finite graphs and Theorem 14.11 only assumes finite balls. The lemma below takes care of this issue. Specifically, the lemma says that edges corresponding to large jumps can be removed without changing stochastic completeness. This allows us to circumvent the assumption of local finiteness.

Let b be a graph over (X, m) with a pseudo metric ϱ and let $0 < s < \infty$. We define the *truncated edge weight*

$$b_s = b1_{\{\varrho \leq s\}}.$$

Then, b_s is a graph over (X, m) and the pseudo metric ϱ has finite jump size s and thus satisfies (J). Moreover, if ϱ is intrinsic for b over (X, m), then ϱ is obviously intrinsic for b_s over (X, m).

LEMMA 14.12 (Truncation and stability of stochastic completeness). *Let b be a graph over (X, m) with an intrinsic metric ϱ and let $s > 0$. If b_s over (X, m) is stochastically complete, then b over (X, m) is stochastically complete.*

PROOF. As before, we may reduce to the case of connected graphs by working on connected components. We will show that stochastic incompleteness of b implies stochastic incompleteness of b_s. So, assume that b is stochastically incomplete. By the Omori–Yau maximum principle, Theorem 7.2 (iv), there exists a function $u \in \mathcal{F}$ with $\sup u \in (0, \infty)$ and $\beta > 0$ such that $\mathcal{L}u < -1$ on $X_\beta = \{x \in X \mid u(x) > \sup u - \beta\}$. We can assume that

$$\sup u \leq \frac{s^2}{2}$$

as, otherwise, we subtract the constant $\sup u - s^2/2$ from u. Furthermore, we can choose $\alpha < \beta$ such that $\sup u > \alpha$ and since $X_\alpha \subseteq X_\beta$ for $\alpha < \beta$, we infer

$$\mathcal{L}u < -1 \quad \text{and} \quad u > 0 \quad \text{on} \quad X_\alpha.$$

We now denote the formal Laplacian of b_s over (X, m) by $\mathcal{L}_s$. For $x \in X_\alpha$, we estimate, using $u(x) > 0$ and the fact that ϱ is intrinsic,

$$\begin{aligned}
\mathcal{L}_s u(x) &= \mathcal{L}u(x) - \frac{1}{m(x)} \sum_{y \in X, \varrho(x,y) > s} b(x, y)(u(x) - u(y)) \\
&< -1 + \frac{1}{m(x)} \sum_{y \in X, \varrho(x,y) > s} b(x, y)u(y) \frac{\varrho^2(x, y)}{s^2} \\
&\leq -1 + \frac{\sup u}{s^2} \leq -\frac{1}{2}.
\end{aligned}$$

This proves stochastic incompleteness of b_s by the Omori–Yau maximum principle, Theorem 7.2 (iv). $\square$

We now combine all of the pieces to prove the main result of this chapter.

Proof of Theorem 14.11. Let b be a graph over (X, m) and ϱ be an intrinsic metric with finite balls (B) such that

$$\int_0^\infty \frac{r}{\log^\#(m(B_r))} dr = \infty.$$

By Lemma 14.12 we can assume that ϱ has finite jump size (J). Now, finite jump size (J) and finite balls (B) imply that b is locally finite by Lemma 11.28.

Consider now the function $f\colon (0,\infty) \longrightarrow [1,\infty)$ given by

$$f(r) = \log^\#(m(B_r)).$$

Since f is monotonically increasing, the function $g(r) = r/f(Ar)$ for $r \geq 0$ and $A > 1$ is uniformly positive on compact sets. Hence, we can apply Lemma 14.10 to find a function n such that the corresponding refinement b' over (X', m') with the intrinsic metric ϱ' is globally local with respect to f. By Theorem 14.8, it now suffices to show that b' over (X', m') is stochastically complete. By Theorem 7.2 (vi.a) this is equivalent to showing that every bounded solution u of the heat equation with $u_0 = 0$ is trivial as we may assume connectedness by restricting to connected components.

We denote the distance ball about a vertex $o \in X \subseteq X'$ with radius r with respect to ϱ' by B'_r. By Lemma 14.9 we have that ϱ' is an intrinsic metric such that

$$\log(m'(B'_r)) \leq \log(m(B_r)) + \log 3.$$

Let $u\colon (0,\infty) \times X' \longrightarrow \mathbb{R}$ be a bounded solution of the heat equation with $u_0 = 0$. For $T > 0$, u satisfies

$$\int_0^T \|u 1_{B'_r}\|^2 dt \leq T(\sup u)^2 m'(B'_r) = T(\sup u)^2 e^{\log(m'(B'_r))} \leq e^{f(r)+C},$$

where $C > 0$ is a constant. Since b' is globally local with respect to f and, therefore, also with respect to $f + C$ and satisfies the assumption

$$\int_1^\infty \frac{r}{f(r)+C} dr \geq D \int_1^\infty \frac{r}{\log^\#(m(B_r))} dr = \infty,$$

where D is a constant, the uniqueness class criterion, Theorem 14.2 yields that $u = 0$ on $(0, T) \times X'$. As T was chosen arbitrarily, we get $u = 0$. This completes the proof. $\square$

Exercises

Example exercise.

Exercise 14.1 (Non-uniqueness of solutions of the heat equation). Show that there exists a graph with standard weights and bounded vertex degree for which there exists a non-trivial solution of the heat equation u with $u_0 = 0$ such that, for some $C > 0$,

$$\int_0^T \|u_t 1_{B_r}\|^2 dt \leq e^{Cr \log r}$$

for all $r \geq 0$.

(First hint: Consider the graph b over $X = \mathbb{Z}$ and $b(k,n) = 1$ if and only if $|k-n| = 1$ and $m = 1$.)

(Second hint: For $k \in \mathbb{N}$ consider the function $p_k \colon \mathbb{Z} \longrightarrow \mathbb{Z}$

$$p_k(n) = \begin{cases} (n+k)\cdots(n+1)\cdot n \cdots (n-k+1) & \text{for } n \geq 0, \\ (-n-1+k)\cdots(-n)\cdot(-n+1)\cdots(-n-k) & \text{for } n < 0. \end{cases}$$

Note that this function can be thought of as a discrete analogue to the function $x \mapsto x^{2k}$ in the continuum setting with the major difference that $p_k(n)$ vanishes for $|n| < k$. Moreover, for $\beta > 0$, let $\eta \colon [0,\infty] \longrightarrow [0,\infty)$ be defined by

$$\eta(t) = \exp\left(-t^{-\beta}\right)$$

and $\eta(0) = 0$. Consider $u \colon (0,\infty) \times X \longrightarrow \mathbb{R}$ given by

$$u_t(n) = \eta(t) + \sum_{k=1}^{\infty} \frac{\eta^{(k)}(t)}{2k!} p_k(n),$$

where $\eta^{(k)}$ is the k-th derivative of η. Show that u is a solution of the heat equation with $u_0 = 0$ and satisfies the estimate above.)

Extension exercises.

Exercise 14.2 (Uniqueness class for (B) and (J)). Let b be a graph over (X,m) with an intrinsic metric that has finite balls (B) and finite jump size (J). Show that every solution of the heat equation u with $u_0 = 0$ such that there exist $\alpha > 0$ and $C > 0$ with

$$\int_0^T \|u_t 1_{B_r}\|^2 dt \leq C e^{\alpha r}$$

for $r > 0$ satisfies $u = 0$.

Exercise 14.3 (Stochastic completeness and connected components). Let b be a graph over (X,m). Show that the graph is stochastically complete if and only if all connected components of the graph are stochastically complete.

EXERCISE 14.4 (Karp–Li Theorem). Let b be a graph over (X, m) with an intrinsic metric that has finite balls (B). Show that if

$$m(B_r) \leq e^{r^2}$$

for $r \geq 0$, then the graph is stochastically complete.

Notes

The results of this chapter are inspired by the corresponding criteria for stochastic completeness of Riemannian manifolds going back to work of Gaffney [**Gaf59**], Karp/Li [**KL**] and culminating in Grigor′yan's volume growth result in [**Gri86**], see also the survey [**Gri99**]. Other approaches to volume growth criteria for manifolds can be found in [**Dav92, Tak89**]. An extension of Grigor′yan's result to strongly local Dirichlet forms is established in [**Stu94**].

For graphs, the volume growth criterion of Grigor′yan fails badly when using the standard graph metric, see [**Woj08, Woj11**]. This inspired the application of intrinsic metrics to the graph setting to find an analogue for the result of Grigor′yan. The use of intrinsic metrics was first successfully applied by Grigor′yan/Huang/Masamune for jump processes in [**GHM12**]; however, their result did not yield the optimal volume growth bound on graphs. This optimal result was later achieved by Folz [**Fol14b**] using probabilistic techniques, so-called quantum or metric graphs and Sturm's extension of Grigor′yan's result. Huang [**Hua14b**] achieved similar results using quantum graphs and analytic techniques while Huang/Shiozawa [**HS14**] gave a probabilistic proof using refinements instead of quantum graphs. However, the main estimate used in Grigor′yan's proof on manifolds, yielding a much stronger uniqueness class statement for the heat equation, is wrong for graphs, even when using intrinsic metrics, see [**Hua11a, Hua12**].

The approach presented in this chapter follows the recent work of Huang/Keller/Schmidt [**HKS20**], which recovers Grigor′yan's original inequality and uniqueness class statement for globally local graphs. With this result and the technique of refinements from [**HS14**], the paper [**HKS20**] gives a purely analytic proof of Grigor′yan's volume growth criterion for graphs as presented here. Furthermore, [**HKS20**] removes all restrictions on the graph structure found in previous results on graphs in [**Fol14b, Hua14b, HS14**]. For further background and discussion, see [**Woj21**].

Appendix

Notice

The appendices deal with background material. Although there is a natural progression, and thus dependence, between the material presented here, we attempt to make each appendix more self-contained by recalling basic relevant notions when they are needed.

APPENDIX A

The Spectral Theorem

'Cause I bake the cake, then take the cake and eat it, too.
Inspectah Deck.

In this appendix we discuss basics concerning self-adjoint operators on infinite-dimensional Hilbert spaces. In particular, we prove the spectral theorem, which states that every such operator is unitarily equivalent to a multiplication operator on an L^2 space. We then show how this allows us to apply functions to an operator.

We assume that the reader is familiar with basic notions of functional analysis; however, we do recall some standard definitions. For a thorough exposition of the concepts discussed here we recommend the textbooks [**RS80, Tes14, Wei80**]. For general background on measure theory see, e.g., [**Rud87**].

We let H denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We consider complex Hilbert spaces and assume that the inner product is linear in the second argument. Real spaces can be complexified. Thus, all of our results below apply to the real case as well.

An *operator* on H is a linear map

$$A \colon D(A) \longrightarrow H,$$

where $D(A)$ is a subspace of H which we call the *domain* of A. We say that A is *densely defined* if $D(A)$ is dense in H. We call an operator A *closed* if $f_n \to f$ for $f_n \in D(A)$ along with $Af_n \to g$ imply $f \in D(A)$ and $Af = g$. We say that an operator A is *bounded* if there exists a constant $C \geq 0$ such that $\|Af\| \leq C\|f\|$ for all $f \in D(A)$. In this case, $\|A\|$, the *norm* of A, is the smallest such constant C.

If A is densely defined and bounded, then A can be uniquely extended to a bounded operator on the entire Hilbert space H and we denote this extension by A as well. We note that a bounded operator defined on the entire space is always closed. We denote the space of bounded operators defined on the entire Hilbert space H by $B(H)$.

For operators A and B on H we define the sum $A + B$ to be the linear map with domain

$$D(A + B) = D(A) \cap D(B)$$

by $(A + B)f = Af + Bf$.

We will also consider operators between different Hilbert spaces H_1 and H_2. In this case, the above definitions hold with the obvious modifications. In particular, an operator A from H_1 to H_2 is a linear map from a subspace $D(A)$ of H_1 into H_2. A most relevant instance is the product AB of operators B from H_1 into H_2 and A from H_2 into H_3. This product is defined on

$$D(AB) = \{f \in D(B) \mid Bf \in D(A)\}$$

and acts by $ABf = A(Bf)$. Finally, if an operator A from H_1 to H_2 is injective, we define A^{-1} to be the unique inverse of A with domain $D(A^{-1}) = AD(A) \subseteq H_2$.

Whenever A is an operator on H and $z \in \mathbb{C}$, we write $(A - z)$ for the operator $A - zI$ on $D(A)$, where I denotes the identity operator on H. We define the *resolvent set of* A to be

$$\varrho(A) = \{z \in \mathbb{C} \mid (A - z) \text{ is bijective and } (A - z)^{-1} \text{ is bounded}\}$$

and the *spectrum* of A as

$$\sigma(A) = \mathbb{C} \setminus \varrho(A).$$

We recall the standard fact that $\sigma(A)$ is always a closed set. For $z \in \varrho(A)$, we call the operator $(A - z)^{-1}$ the *resolvent of* A *at* z.

For an operator A that is not closed, we have

$$\varrho(A) = \emptyset.$$

This follows since if $(A-z)^{-1}$ is bijective and bounded, then $(A-z)^{-1}$ is bounded on the entire Hilbert space and, therefore, closed. But then $(A-z)$ and, hence, A would also be closed. This shows that the notion of a resolvent set is only relevant for closed operators.

On the other hand, for a closed operator A the definition of the resolvent set can be simplified to

$$\varrho(A) = \{z \in \mathbb{C} \mid (A - z) \text{ is bijective}\}.$$

This follows since if A is closed and $A - z$ is bijective, then $(A - z)^{-1}$ is bounded by the closed graph theorem.

An operator A is called *invertible* if $A\colon D(A) \longrightarrow H$ is bijective. If A and B are invertible operators and $D(B) \subseteq D(A)$, then

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1},$$

as follows by a direct calculation. In particular, if $z_1, z_2 \in \varrho(A)$, then

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}.$$

We refer to these formulae as *resolvent identities*. As a particular consequence, we note that the second formula implies that resolvents commute. Moreover, the second formula and boundedness properties, see Corollary A.12 below, imply that the resolvent map

$$\varrho(A) \longrightarrow B(H), \quad z \mapsto (A - z)^{-1}$$

is continuous, i.e., for a sequence (z_n) in $\varrho(A)$ with $z_n \to z \in \varrho(A)$ we have

$$\lim_{n\to\infty} \|(A - z_n)^{-1} - (A - z)^{-1}\| = 0.$$

If A is densely defined, then we define the *adjoint* A^* of A to be the operator with domain

$$D(A^*) = \left\{ f \in H \;\middle|\; \begin{array}{c} \text{there exists a } g \in H \text{ with } \langle f, Ah\rangle = \langle g, h\rangle \\ \text{for all } h \in D(A) \end{array} \right\}$$

acting as

$$A^* f = g.$$

In particular,

$$\langle Af, g\rangle = \langle f, A^* g\rangle$$

for all $f \in D(A)$ and $g \in D(A^*)$ and A^* has the maximal domain among all operators with this property. The operator A^* is always closed.

We note that $D((A - \lambda)^*) = D(A^*)$ and

$$(A - \lambda)^* = A^* - \overline{\lambda}$$

for all $\lambda \in \mathbb{C}$. Furthermore, for $z \in \varrho(A)$,

$$((A - z)^{-1})^* = (A^* - \overline{z})^{-1}.$$

If A is densely defined, we say that A is *symmetric* if A^* is an extension of A, that is, $D(A) \subseteq D(A^*)$ and $Af = A^* f$ for all $f \in D(A)$. Equivalently, A is symmetric if and only if A is densely defined and

$$\langle Af, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in D(A)$. With these preparations, we now define the class of operators of primary interest.

Definition A.1 (Self-adjoint operators). We call a densely defined operator A *self-adjoint* if $A = A^*$.

By definition, a self-adjoint operator is symmetric. Moreover, as the adjoint is always a closed operator, all self-adjoint operators are closed.

Self-adjoint operators are the topic of this appendix. Before we develop the general theory further, we pause for a moment to present a key example which should be kept in mind, namely, that of multiplication operators. The main result of this section is that any self-adjoint operator is equivalent, in a sense that will be made precise, to such an operator where the multiplying function is real-valued.

Example A.2 (Multiplication operators). Let (X, μ) be a measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. The operator M_u of *multiplication by u* has domain

$$D(M_u) = \{f \in L^2(X, \mu) \mid uf \in L^2(X, \mu)\}$$

and acts as

$$M_u f = uf$$

for all $f \in D(M_u)$. As discussed below, the operator M_u is densely defined. Furthermore, it can readily be seen that M_u is closed. If $u \neq 0$ almost everywhere, then M_u is injective and $M_u^{-1} = M_{1/u}$. The adjoint of M_u is given by $(M_u)^* = M_{\overline{u}}$. In particular, M_u is self-adjoint if u is real-valued. Finally, M_u is bounded if u is bounded.

The proofs of these statements are rather straightforward. We only sketch how to show that the domain of M_u is dense. For $n \in \mathbb{N}$, we define

$$X_n = \{x \in X \mid |u(x)| \leq n\}.$$

Then, the characteristic functions 1_{X_n} for $n \in \mathbb{N}$ tend pointwise increasingly towards the constant function with value 1. In particular, we have $1_{X_n} f \to f$ for any $f \in L^2(X, \mu)$ by Lebesgue's dominated convergence theorem. On the other hand, by the definition of X_n, the function $1_{X_n} f$ belongs to $D(M_u)$ for any $n \in \mathbb{N}$.

As the discussion above shows, we can infer features of M_u from those of u. For our subsequent theory, it will also be important that the converse holds, i.e., that u is actually determined by M_u. To conclude this, we need the measure space to satisfy an additional weak condition which we now introduce.

LEMMA A.3. *Let (X, μ) be a measure space and let $u\colon X \longrightarrow \mathbb{C}$ be measurable. Then, the following statements are equivalent:*

(i) *Any measurable set $B \subseteq X$ with $\mu(B) > 0$ contains a measurable subset $B' \subseteq B$ with $0 < \mu(B') < \infty$.*
(ii) *If $u_1, u_2\colon X \longrightarrow \mathbb{C}$ satisfy $M_{u_1} = M_{u_2}$, then $u_1 = u_2$ almost everywhere.*

PROOF. (i) $\Longrightarrow$ (ii): Let u_1, u_2 be as in (ii). By restricting to

$$X_n = \{x \in X \mid |u_1(x)| \leq n \text{ and } |u_2(x)| \leq n\}$$

for $n \in \mathbb{N}$ and noting that $X = \bigcup_n X_n$ we can assume without loss of generality that both u_1 and u_2 are bounded functions. Hence, M_{u_1} and M_{u_2} are defined on $L^2(X, \mu)$.

Now, consider the set

$$B = \{x \in X \mid u_1(x) \neq u_2(x)\}.$$

As $M_{u_1} = M_{u_2}$, we have $u_1 1_{B'} = u_2 1_{B'}$ for any measurable set $B' \subseteq B$ with $1_{B'} \in L^2(X, \mu)$. Hence, for any measurable set $B' \subseteq B$ with $\mu(B') < \infty$ we infer that u_1 equals u_2 almost everywhere on B'. On the other hand, by $B' \subseteq B$, we know that $u_1 \neq u_2$ almost everywhere on B'. This gives that B' has measure zero. Hence, any measurable set $B' \subseteq B$ with finite measure has measure zero and we conclude from (i) that B has measure zero. This gives $u_1 = u_2$ almost everywhere.

(ii) $\Longrightarrow$ (i): Assume that a measurable set $B \subseteq X$ has the property that any measurable subset $B' \subseteq B$ satisfies either $\mu(B') = 0$ or $\mu(B') = \infty$. We have to show that $\mu(B) = 0$.

Now, for any $n \in \mathbb{N}$ and $f \in L^2(X, m)$, the set

$$B'_n = \{x \in B \mid |f(x)| \geq \frac{1}{n}\}$$

must have measure zero, as otherwise we have $\mu(B'_n) = \infty$, yielding a contradiction to $f \in L^2(X, m)$. As this holds for each n, we infer that any $f \in L^2(X, \mu)$ must vanish almost everywhere on B. Thus, the operator of multiplication by $u_1 = 1_B$ agrees with the operator of multiplication by $u_2 = 0$. Hence, by (ii) we infer that B must have measure zero. $\square$

Definition A.4 (No atoms of infinite mass). We say that (X, μ) has *no atoms of infinite mass* if any measurable set $B \subseteq X$ with $\mu(B) > 0$ contains a measurable subset $B' \subseteq B$ with $0 < \mu(B') < \infty$.

Part (ii) of the preceding lemma shows that this condition is exactly the appropriate condition for our aim of concluding properties of u from those of M_u. However, this condition does not seem to be commonly considered in the literature. Thus, we would like to emphasize that if (X, μ) is σ-finite, i.e., if X can be expressed as a countable union of measurable sets each of which has finite measure, then (X, μ) has no atoms of infinite mass.

In our dealing with measure spaces, sets of measure zero will not play a role. For this reason it will not be the range of u but rather a modified version of the range which takes the measure into account, which will be relevant for us when we derive features of M_u from those of u.

Definition A.5 (Essential range). Let (X, μ) be a measure space and $u\colon X \longrightarrow \mathbb{C}$. The *essential range of* u is

$$\{\lambda \in \mathbb{C} \mid \mu(u^{-1}(B_\varepsilon(\lambda))) > 0 \text{ for all } \varepsilon > 0\}.$$

It is not hard to see that the essential range of a function is always a closed subset of $\mathbb{C}$ and that no measurability assumption on u is needed for this. Examples show that, in general, neither the range of a function is contained in the essential range nor must the essential range be contained in the range.

Remark (When the essential range is the closure of the range). The essential range of a function is clearly contained in the closure of the range of the function. A converse holds in a suitable topological situation for continuous functions. More specifically, if X carries a topology generating the σ-algebra and the measure μ on X has full support, i.e., any open non-empty subset of X has positive measure, then a short argument shows that the range of any continuous function is contained in the essential range of the function. As the essential range is closed, we infer that in this case the essential range agrees with the closure of the range of the function. We will make use of this when presenting the spectral mapping theorem.

We now the give the connection between the spectrum and the essential range.

LEMMA A.6 (Spectrum of multiplication operators). *Let (X, μ) be a measure space which has no atoms of infinite mass. Let $u \colon X \longrightarrow \mathbb{C}$ be measurable and M_u be the operator of multiplication by u. Then, $\sigma(M_u)$ equals the essential range of u, i.e.,*

$$\sigma(M_u) = \{\lambda \in \mathbb{C} \mid \mu(u^{-1}(B_\varepsilon(\lambda))) > 0 \text{ for all } \varepsilon > 0\}.$$

PROOF. For λ not in the essential range, the operator $M_{1/(u-\lambda)}$ is obviously a bounded inverse for $M_u - \lambda = M_{u-\lambda}$. Conversely, consider λ belonging to the essential range of u. Using the assumption that there are no atoms of infinite mass, for any $\varepsilon > 0$, we can construct $f \in L^2(X, \mu)$ with $\|f\| = 1$ and $\|(M_u - \lambda)f\| < \varepsilon$. This contradicts the existence of a bounded inverse to $M_u - \lambda$. □

From the definitions it is not hard to derive the following additional properties of multiplication operators. They will be used repeatedly in what follows.

PROPOSITION A.7 (Further features of multiplication operators). *Let (X, μ) be a measure space which has no atoms of infinite mass and let $u \colon X \longrightarrow \mathbb{C}$ be measurable. Then, the following statements hold:*

(a) *The operator M_u is self-adjoint if and only if the essential range of u is contained in $\mathbb{R}$, which, in turn, holds if and only if u is real-valued almost everywhere.*
(b) *The operator M_u is bounded if and only if the essential range of u is bounded, which, in turn, holds if and only if $u \in L^\infty(X, \mu)$. In this case,*

$$\|M_u\| = \|u\|_\infty = \sup\{|\lambda| \mid \lambda \text{ is in the essential range of } u\}.$$

We now highlight an example of a multiplication operator which will arise throughout.

EXAMPLE A.8 (Multiplication by the identity). We let $X = \mathbb{R}$, μ be a finite measure on $\mathbb{R}$ and let $u = \mathrm{id} \colon \mathbb{R} \longrightarrow \mathbb{R}$ be the identity mapping, that is,

$$\mathrm{id}(x) = x$$

for all $x \in \mathbb{R}$. We denote the corresponding operator by M_{id} and refer to M_{id} as the operator of multiplication by the identity. As id maps into $\mathbb{R}$, the operator M_{id} is self-adjoint. As μ is finite, μ does not have atoms of infinite mass. In particular, the spectrum of M_{id} is the essential range of the identity mapping. This essential range can easily be seen to be equal to the *support of* μ, i.e.,

$$\sigma(M_{\mathrm{id}}) = \operatorname{supp}(\mu) = \{\lambda \in \mathbb{R} \mid \mu((\lambda - \varepsilon, \lambda + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

By the previous proposition, M_{id} is bounded if and only if $\operatorname{supp}(\mu)$ is a bounded set.

After this discussion of multiplication operators, we now work towards introducing a natural notion of equivalence between operators. We consider the case of two Hilbert spaces $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$. An operator $U\colon H_1 \longrightarrow H_2$ is called *unitary* if $D(U) = H_1$, U is onto and

$$\langle f, g\rangle_1 = \langle Uf, Ug\rangle_2$$

for all $f, g \in H_1$. In particular, U is a bounded operator which is bijective with $U^* = U^{-1}$.

Given an operator A_1 with domain $D(A_1) \subseteq H_1$ and an operator A_2 with domain $D(A_2) \subseteq H_2$ we say that A_1 is *unitarily equivalent* to A_2 if there exists a unitary operator $U\colon H_2 \longrightarrow H_1$ such that $D(A_1) = UD(A_2)$ and

$$A_1 = UA_2U^{-1}.$$

As a special case, if A is an operator on H with domain $D(A)$, then we say that A is *unitarily equivalent to a multiplication operator* M_u if there exists a measure space (X, μ), a measurable function $u\colon X \longrightarrow \mathbb{C}$ and a unitary operator $U\colon L^2(X, \mu) \longrightarrow H$ such that $D(A) = UD(M_u)$ and

$$A = UM_uU^{-1}.$$

By definition, unitary operators are structure preserving maps between Hilbert spaces. So, unitarily equivalent operators can naturally be considered indistinguishable in terms of structural properties. In particular, we note that the spectrum of an operator is preserved by unitary equivalence. Therefore, if A is unitarily equivalent to M_u, then

$$\sigma(A) = \sigma(M_u),$$

which is equal to the essential range of u by Lemma A.6 whenever (X, μ) has no atoms of infinite mass.

Having introduced these notions we can now precisely announce the main result that we prove in this appendix: That every self-adjoint operator is unitarily equivalent to a multiplication operator and even a direct sum of operators which are multiplication by the identity. We refer to this result as the spectral theorem.

We now start to work towards the proof of the spectral theorem. We start with a basic formula which will be used in several places in what follows.

LEMMA A.9. *Let A be a self-adjoint operator on H with domain $D(A)$ and let $z = a + ib \in \mathbb{C}$. Then,*

$$\|(A - z)f\|^2 = \|(A - a)f\|^2 + b^2\|f\|^2$$

for all $f \in D(A)$.

PROOF. We expand the left-hand side via the inner product and use the fact that if A is self-adjoint, then $(A - a)$ is also self-adjoint, which implies

$$\langle (A - a)f, ibf\rangle + \langle ibf, (A - a)f\rangle = 0$$

for all $f \in D(A)$. This allows us to cancel the mixed terms and prove the equality. $\square$

We now use the lemma above to characterize when $\lambda \in \mathbb{C}$ belongs to the resolvent set.

PROPOSITION A.10. *Let A be a self-adjoint operator on H with domain $D(A)$ and let $\lambda \in \mathbb{C}$. Then, $\lambda \in \varrho(A)$ if and only if there exists a constant $C > 0$ with*

$$\|(A - \lambda)f\| \geq C\|f\|$$

for all $f \in D(A)$. If such a $C > 0$ exists, then $\|(A - \lambda)^{-1}\| \leq 1/C$.

PROOF. Let $\lambda \in \varrho(A)$. Then, there exists a bounded linear operator B with $B(A-\lambda) = I_{D(A)}$, where $I_{D(A)}$ denotes the identity operator on $D(A)$. This gives

$$\|f\| = \|B(A - \lambda)f\| \leq \|B\|\|(A - \lambda)f\|$$

for all $f \in D(A)$. The desired statement follows with $C = 1/\|B\|$.

Conversely, suppose that there exists a constant $C > 0$ as assumed. Then, clearly, $(A - \lambda)$ is injective. We next show that $(A - \lambda)$ is surjective as well. To do so, it suffices to show that the image of $(A - \lambda)$ is both dense and closed in H.

Claim. $(A - \lambda)D(A)$ is closed.

Proof of the claim. Let g be in the closure of $(A - \lambda)D(A)$. Then, there exists a sequence $f_n \in D(A)$ with $(A - \lambda)f_n \to g$ as $n \to \infty$. Therefore, $((A - \lambda)f_n)$ is a Cauchy sequence and, by the assumption on A, it follows that (f_n) must be a Cauchy sequence as well. As H is a Hilbert space, there exists an $f \in H$ such that $f_n \to f$ as $n \to \infty$. It remains to show that $f \in D(A)$ and $(A - \lambda)f = g$.

For $h \in D(A)$, we obtain from the self-adjointness of A

$$\langle f, (A - \overline{\lambda})h\rangle = \lim_{n\to\infty}\langle f_n, (A - \overline{\lambda})h\rangle = \lim_{n\to\infty}\langle (A - \lambda)f_n, h\rangle = \langle g, h\rangle.$$

This yields $f \in D((A - \overline{\lambda})^*) = D(A^*) = D(A)$ and

$$(A - \lambda)f = (A - \overline{\lambda})^*f = g.$$

Hence, $(A - \lambda)D(A)$ is closed as claimed.

Claim. $(A-\lambda)D(A)$ is dense.

Proof of the claim. Let $g \perp (A-\lambda)D(A)$ so that

$$0 = \langle g, (A-\lambda)f\rangle$$

for all $f \in D(A)$. This implies $g \in D(A^*) = D(A)$ with

$$(A-\overline{\lambda})g = 0.$$

Now, Lemma A.9 implies $\|(A-\overline{\lambda})g\| = \|(A-\lambda)g\|$. Therefore, $0 = \|(A-\lambda)g\| \geq C\|g\|$ with $C>0$ so that $g=0$ and we infer the desired statement.

Finally, we note that $(A-\lambda)^{-1}\colon H \longrightarrow D(A)$ is bounded by $1/C$ as, for every $f \in H$,

$$\|f\| = \|(A-\lambda)(A-\lambda)^{-1}f\| \geq C\|(A-\lambda)^{-1}f\|,$$

which directly yields $\|(A-\lambda)^{-1}\| \leq 1/C$. This completes the proof. □

Passing to complements in Proposition A.10 gives a characterization of when $\lambda \in \mathbb{C}$ belongs to the spectrum. We refer to this characterization as Weyl's criterion for the spectrum. We will see variations on this criterion in later appendices, see Section 2.

Corollary A.11 (Weyl's criterion – spectrum). *Let A be a self-adjoint operator on H with domain $D(A)$ and let $\lambda \in \mathbb{C}$. Then, $\lambda \in \sigma(A)$ if and only if there exists a normalized sequence $f_n \in D(A)$ with*

$$\lim_{n\to\infty} \|(A-\lambda)f_n\| = 0.$$

We also obtain the following fundamental results on the spectrum of self-adjoint operators as well as a bound on the norm of the resolvent.

Corollary A.12 (Spectrum is real). *If A is a self-adjoint operator, then $\sigma(A) \subseteq \mathbb{R}$. Moreover,*

$$\|(A-z)^{-1}\| \leq \frac{1}{|\mathrm{Im}\, z|}$$

for $z \in \mathbb{C}\setminus\mathbb{R}$ and the resolvent map $z \mapsto (A-z)^{-1}$ is continuous.

Proof. Consider $z = a+ib$ with $b \neq 0$. Then, Lemma A.9 implies $\|(A-z)f\| \geq |b|\|f\|$ for all $f \in D(A)$. So $z \in \varrho(A)$ and the estimate follows from Proposition A.10. The continuity of the resolvent map then follows from the resolvent identity and the estimate. □

Key quantities for the spectral theorem are the spectral measures, which we introduce next. We let A be a self-adjoint operator on H. We will show that for every $f \in H$ there exists a unique finite positive regular Borel measure μ_f on $\mathbb{R}$ with

$$\langle f, (A-z)^{-1}f\rangle = \int \frac{1}{x-z} d\mu_f(x)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. This measure satisfies $\mu_f(\mathbb{R}) = \|f\|^2$. We call μ_f the *spectral measure* associated to f. By polarization, for all $f, g \in H$, there then exists a unique finite signed regular Borel measure μ on $\mathbb{R}$ with

$$\langle f, (A-z)^{-1} g\rangle = \int \frac{1}{x-z} d\mu(x)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

We first present some properties of the map $z \mapsto \langle f, (A-z)^{-1} f\rangle$. We let

$$\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}$$

denote the upper and lower half plane.

Proposition A.13. *Let A be a self-adjoint operator on H. Let $f \in H$ with $f \neq 0$ and define $F_f \colon \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}$ by*

$$F_f(z) = \langle f, (A-z)^{-1} f\rangle.$$

Then, the following statements hold:

(a) *F_f is holomorphic.*
(b) *$\pm \operatorname{Im} F_f(z) > 0$ for $z \in \mathbb{C}^{\pm}$.*
(c) *$|F_f(z) \operatorname{Im} z| \leq \|f\|^2$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. (a): This follows from the resolvent formula

$$(A-z)^{-1} - (A-z_0)^{-1} = (z-z_0)(A-z)^{-1}(A-z_0)^{-1}$$

and the continuity of the mapping $z \mapsto (A-z)^{-1}$ given in Corollary A.12.

(b): Since A is self-adjoint, $\langle (A-a)g, g\rangle$ is real for all $g \in D(A)$ and $a \in \mathbb{R}$. We now calculate as follows:

$$\begin{aligned}\operatorname{Im} F_f(z) &= \operatorname{Im} \langle f, (A-z)^{-1} f\rangle \\ &= \operatorname{Im} \langle (A-z)(A-z)^{-1} f, (A-z)^{-1} f\rangle \\ &= \operatorname{Im} z \langle (A-z)^{-1} f, (A-z)^{-1} f\rangle \\ &= \operatorname{Im} z \|(A-z)^{-1} f\|^2.\end{aligned}$$

This gives the statement.

(c): This follows by the estimate for $\|(A-z)^{-1}\|$ given in Corollary A.12 and the Cauchy-Schwarz inequality as

$$\begin{aligned}|F_f(z) \operatorname{Im} z| &= |\langle f, (A-z)^{-1} f\rangle \operatorname{Im} z| \\ &\leq \|f\| \|(A-z)^{-1} f\| |\operatorname{Im} z| \\ &\leq \|f\|^2.\end{aligned}$$

This finishes the proof. $\square$

Given the preceding proposition, the existence and uniqueness of the spectral measure is a direct consequence of the following famous result concerning Herglotz–Nevanlinna functions. We refrain from giving a proof but rather refer to the already mentioned literature [**Tes14, Wei80**].

THEOREM A.14 (Borel transform). *Let $F\colon \mathbb{C}\setminus\mathbb{R} \longrightarrow \mathbb{C}$ satisfy the following properties:*

(a) *F is holomorphic*
(b) *$\pm \operatorname{Im} F(z) > 0$ for all $z \in \mathbb{C}^{\pm}$*
(c) *$|F(z)\operatorname{Im} z| \leq M$ for all $z \in \mathbb{C}\setminus\mathbb{R}$ and some M.*

Then, there exists a unique positive finite regular Borel measure μ on $\mathbb{R}$ with

$$F(z) = \int \frac{1}{x-z} d\mu(x)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$ and $\mu(\mathbb{R}) \leq M$.

The preceding theorem gives the existence of the spectral measures as well as the estimate $\mu_f(\mathbb{R}) \leq \|f\|^2$. It does not give the stronger statement that $\mu_f(\mathbb{R}) = \|f\|^2$. This statement can be inferred from our discussion of the little spectral theorem below. However, we now give a direct proof.

PROPOSITION A.15. *Let A be self-adjoint on H. Then, $\mu_f(\mathbb{R}) = \|f\|^2$ for all $f \in H$.*

PROOF. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. By the characterizing property of μ_f we obtain, after multiplication by $-\lambda i$,

$$\langle f, -\lambda i (A-\lambda i)^{-1} f\rangle = \int \frac{-\lambda i}{x - \lambda i} d\mu_f(x).$$

Now, we consider the limit as $\lambda \to \infty$ on both sides. On the right-hand side we obtain from Lebesgue's dominated convergence theorem

$$\int \frac{-\lambda i}{x-\lambda i} d\mu_f(x) \to \int 1 d\mu_f(x) = \mu_f(\mathbb{R})$$

as $\lambda \to \infty$. On the left-hand side we obtain

$$\langle f, -\lambda i (A-\lambda i)^{-1} f\rangle \to \langle f, f\rangle = \|f\|^2$$

since

$$-\lambda i (A-\lambda i)^{-1} f \to f$$

as $\lambda \to \infty$, as we now show.

To show that $\lim_{\lambda\to\infty} -\lambda i (A - \lambda i)^{-1} f = f$ we first note by Lemma A.9 that

$$\begin{aligned}\|f\|^2 &= \|(A-\lambda i)(A-\lambda i)^{-1} f\|^2 \\ &= \|A(A-\lambda i)^{-1}\, f\|^2 + \lambda^2 \|(A-\lambda i)^{-1} f\|^2,\end{aligned}$$

where the last term is non-negative. This gives

$$\|A(A-\lambda i)^{-1}\| \le 1$$

for all $\lambda > 0$. Moreover, as $A(A-\lambda i)^{-1}f = (A-\lambda i)^{-1}Af$ for all $f \in D(A)$, we have by Corollary A.12

$$\|A(A-\lambda i)^{-1}f\| = \|(A-\lambda i)^{-1}Af\| \le \frac{1}{\lambda}\|Af\|^2 \to 0$$

as $\lambda \to \infty$. Thus, $A(A-\lambda i)^{-1}$ is bounded in norm by 1 uniformly in $\lambda > 0$ and converges pointwise to 0 as $\lambda \to \infty$ on the dense set $D(A)$. Hence, it converges pointwise to 0 on the entire space. Thus, the operators

$$-\lambda i(A-\lambda i)^{-1} = (A-\lambda i)(A-\lambda i)^{-1} - A(A-\lambda i)^{-1} = I - A(A-\lambda i)^{-1}$$

converge pointwise to the identity. This proves the statement. $\square$

As we will see later, the spectral measures are the key object in the presentation of a self-adjoint operator as a multiplication operator. In order to simplify the subsequent discussion we define, for $z \in \mathbb{C} \setminus \mathbb{R}$, the function $\varphi_z \colon \mathbb{R} \longrightarrow \mathbb{C}$ by

$$\varphi_z(x) = \frac{1}{x-z}.$$

Moreover, for $f \in H$, we let

$$H_f = \overline{\mathrm{Lin}\{(A-z)^{-1}f \mid z \in \mathbb{C} \setminus \mathbb{R}\}}.$$

We will show that $(A-z)^{-1}$ acting on H_f is unitarily equivalent to multiplication by φ_z on $L^2(\mathbb{R}, \mu_f)$. Along the way, we will need that the closure of the linear span of the functions φ_z for $z \in \mathbb{C} \setminus \mathbb{R}$ is dense in $L^2(\mathbb{R}, \mu_f)$. This is a rather direct consequence of the Stone–Weierstrass theorem. As this will be used in various places below we state it as a separate lemma.

LEMMA A.16. *Let $\mathcal{A}$ be the closure of the linear span of the functions φ_z for $z \in \mathbb{C} \setminus \mathbb{R}$, i.e.,*

$$\mathcal{A} = \overline{\mathrm{Lin}\{\varphi_z \mid z \in \mathbb{C} \setminus \mathbb{R}\}}.$$

Then, $\mathcal{A}$ is an algebra and is equal to the space of continuous complex-valued functions on $\mathbb{R}$ vanishing at ∞, i.e.,

$$\mathcal{A} = C_0(\mathbb{R}, \mathbb{C}).$$

PROOF. First, we note that for $z_1, z \in \mathbb{C} \setminus \mathbb{R}$ with $z_1 \neq z$ we have

$$\varphi_{z_1}\varphi_z = \frac{\varphi_{z_1} - \varphi_z}{z_1 - z} \in \mathcal{A}.$$

Letting $z_1 \to z$, then gives that $\varphi_z^2 \in \mathcal{A}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. This shows that $\mathcal{A}$ is an algebra. As the functions φ_z clearly vanish nowhere, separate points and $\overline{\varphi_z} = \varphi_{\overline{z}} \in \mathcal{A}$, the rest of the statement follows by the Stone–Weierstrass theorem. $\square$

REMARK. The argument given in the preceding proof allows for some variants. Among them we mention the following two statements. They will not be used in what follows below but may be useful in other contexts.

(a) For $t > 0$ we define the function $\psi_t\colon [0,\infty) \longrightarrow \mathbb{R}$ via

$$\psi_t(x) = \frac{1}{t+x}.$$

Then, the linear span of the functions ψ_t for $t > 0$ is dense in the algebra $C_0([0,\infty),\mathbb{R})$ of continuous real-valued functions on $[0,\infty)$ vanishing at ∞. Indeed, this follows by simply mimicking the argument in the proof above.

(b) Let J be a subset of $\mathbb{C}\setminus\mathbb{R}$. Assume that there exists a $w \in \mathbb{C}\setminus\mathbb{R}$ such that both w and $\overline{w}$ are accumulation points of J. Then, the linear span $\mathrm{Lin}(J)$ of $\{\varphi_z \mid z \in J\}$ is dense in $C_0(\mathbb{R},\mathbb{C})$. Indeed, arguing as in the proof above we can show that products $\varphi_{z_1}\cdots\varphi_{z_n}$ for pairwise different $z_1,\ldots,z_n \in J$ belong to $\mathrm{Lin}(J)$. As both w and $\overline{w}$ are accumulation points of J, this gives that all powers of the form $\varphi_w^m\varphi_{\overline{w}}^k$ for $k,m \in \mathbb{N}$ belong to closure of $\mathrm{Lin}(J)$. Hence, the algebra generated by φ_w and $\varphi_{\overline{w}}$ belongs to the closure of $\mathrm{Lin}(J)$ and this algebra can be seen to agree with $C_0(\mathbb{R},\mathbb{C})$ by the Stone–Weierstrass theorem.

We now gather some basic features of the space H_f and the relationship between H_f and $L^2(\mathbb{R},\mu_f)$.

PROPOSITION A.17. *Let A be a self-adjoint operator on H and let $f \in H$.*
(a) *$(A-\lambda)^{-1}H_f \subseteq H_f$ for any $\lambda \in \varrho(A)$.*
(b) *For all $z_1,z_2 \in \mathbb{C}\setminus\mathbb{R}$,*

$$\langle (A-z_1)^{-1}f,(A-z_2)^{-1}f\rangle = \langle \varphi_{z_1},\varphi_{z_2}\rangle_{L^2(\mathbb{R},\mu_f)}.$$

PROOF. We first note that for $z_1,z_2 \in \mathbb{C}\setminus\mathbb{R}$, with $z_1 \neq z_2$,

$$\varphi_{z_1} - \varphi_{z_2} = (z_1 - z_2)\varphi_{z_1}\varphi_{z_2}.$$

Furthermore, we recall the resolvent identity, which states

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1-z_2)(A-z_1)^{-1}(A-z_2)^{-1}.$$

These identities will be used below.

(a) We first consider $\lambda = z_1 \in \mathbb{C}\setminus\mathbb{R} \subseteq \varrho(A)$. For any $z_2 \in \mathbb{C}\setminus\mathbb{R}$ with $z_2 \neq z_1$, the resolvent identity gives

$$(A-z_1)^{-1}(A-z_2)^{-1}f = \frac{1}{z_1-z_2}((A-z_1)^{-1}f - (A-z_2)^{-1}f) \in H_f.$$

By continuity of the resolvent map $z \mapsto (A-z)^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$ given in Corollary A.12, we infer $(A-z_1)^{-1}(A-z_1)^{-1}f \in H_f$ for all $z_1 \in \mathbb{C}\setminus\mathbb{R}$. This implies that H_f is invariant under $(A-z)^{-1}$ for any $z \in \mathbb{C}\setminus\mathbb{R}$. As any $\lambda \in \varrho(A)$ can be obtained as a limit of a sequence (z_n) in $\mathbb{C}\setminus\mathbb{R}$, we obtain the desired statement by the continuity of the resolvent map.

(b) A direct computation for $z_1 \neq \overline{z_2}$ invoking the formulas discussed at the beginning of the proof gives

$$\begin{aligned}
\langle (A-z_1)^{-1}f, (A-z_2)^{-1}f\rangle &= \langle f, (A-\overline{z}_1)^{-1}(A-z_2)^{-1}f\rangle \\
&= (\overline{z}_1 - z_2)^{-1}\left\langle f, \left((A-\overline{z}_1)^{-1} - (A-z_2)^{-1}\right) f\right\rangle \\
&= (\overline{z}_1 - z_2)^{-1}\int (\varphi_{\overline{z}_1} - \varphi_{z_2})\, d\mu_f \\
&= \int \varphi_{\overline{z}_1}\varphi_{z_2} d\mu_f \\
&= \int \overline{\varphi_{z_1}}\varphi_{z_2} d\mu_f \\
&= \langle \varphi_{z_1}, \varphi_{z_2}\rangle.
\end{aligned}$$

This shows the desired equality for $\overline{z}_1 \neq z_2$. The general case then follows from the continuity of the resolvent map. □

We next prove a first version of the spectral theorem. In particular, we find that $(A-z)^{-1}$ is unitarily equivalent to multiplication by φ_z and the restriction of A to H_f is unitarily equivalent to multiplication by the identity.

LEMMA A.18 (Little spectral theorem). *Let A be a self-adjoint operator on H and let $f \in H$.*

(a) *There is a unique unitary operator $U_f\colon L^2(\mathbb{R}, \mu_f) \longrightarrow H_f$ with*
$$U_f\varphi_z = (A-z)^{-1}f$$
for all $z \in \mathbb{C}\setminus\mathbb{R}$. This operator satisfies $U_f 1 = f$.

(b) *For all $z \in \mathbb{C}\setminus\mathbb{R}$,*
$$(A-z)^{-1} = U_f M_{\varphi_z} U_f^{-1}.$$

(c) *The operator A maps $D(A)\cap H_f$ into H_f and the restriction $A|_{H_f}$ of A to H_f with domain $D(A|_{H_f}) = D(A)\cap H_f$ and $A|_{H_f}g = Ag$ is a self-adjoint operator on H_f satisfying*
$$A|_{H_f} = U_f M_{\mathrm{id}} U_f^{-1}.$$

(d) $\mathrm{supp}(\mu_f) \subseteq \sigma(A)$.

PROOF. (a) and (b): We first show the existence of such a unitary operator U_f. Part (b) of Proposition A.17 easily gives

$$\|\sum_{n=1}^{N} a_n\varphi_{z_n}\| = \|\sum_{n=1}^{N} a_n(A-z_n)^{-1}f\|$$

for all $z_1, \ldots, z_n \in \mathbb{C} \setminus \mathbb{R}$ and $a_n \in \mathbb{C}$. This implies that the map $\mathrm{Lin}\{\varphi_z \mid z \in \mathbb{C} \setminus \mathbb{R}\} \longrightarrow H_f$ given by

$$\sum_{n=1}^{N} a_n \varphi_{z_n} \mapsto \sum_{n=1}^{N} a_n (A - z_n)^{-1} f$$

is well-defined and isometric. We can thus extend this map to an isometric map on $\overline{\mathrm{Lin}\{\varphi_z \mid z \in \mathbb{C} \setminus \mathbb{R}\}}$ which, by Lemma A.16, is the set of continuous functions which vanish at infinity. Hence, the map is densely defined on $L^2(\mathbb{R}, \mu_f)$. Furthermore, the map has dense range by the definition of H_f. As the map is isometric, it can then be extended uniquely to a unitary operator U_f.

Next, we show that $(A - z)^{-1} U_f = U_f M_{\varphi_z}$. First, by definition, for $z_1 \in \mathbb{C} \setminus \mathbb{R}$ we have

$$(A - z)^{-1} U_f \varphi_{z_1} = (A - z)^{-1} (A - z_1)^{-1} f.$$

Furthermore, by using the resolvent identity for $z \neq z_1$, we get

$$\begin{aligned} U_f M_{\varphi_z} \varphi_{z_1} &= U_f (\varphi_z \varphi_{z_1}) \\ &= U_f \left((z - z_1)^{-1} (\varphi_z - \varphi_{z_1}) \right) \\ &= (z - z_1)^{-1} \left((A - z)^{-1} f - (A - z_1)^{-1} f \right) \\ &= (A - z)^{-1} (A - z_1)^{-1} f, \end{aligned}$$

so that $(A - z)^{-1} U_f \varphi_{z_1} = U_f M_{\varphi_z} \varphi_{z_1}$ for all $z \neq z_1$ and, thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$ by continuity. Therefore, as the set $\{\varphi_z \mid z \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $L^2(\mathbb{R}, \mu_f)$, we get

$$(A - z)^{-1} = U_f M_{\varphi_z} U_f^{-1}$$

on $L^2(\mathbb{R}, \mu_f)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Now, we prove $U_f 1 = f$. In particular, this implies $f \in H_f$ as U_f maps into H_f. As $(A - z)^{-1}$ is injective, it suffices to show $(A - z)^{-1} U_f 1 = (A - z)^{-1} f$. By what we have already shown and the definition of U_f we directly compute

$$\begin{aligned} (A - z)^{-1} U_f 1 &= U_f M_{\varphi_z} 1 \\ &= U_f \varphi_z \\ &= (A - z)^{-1} f. \end{aligned}$$

This gives $U_f 1 = f$.

(c) As stated in (a) of Proposition A.17, the subspace H_f is invariant under $(A - z)^{-1}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, i.e., $(A - z)^{-1} H_f \subseteq H_f$. By a short computation, this shows that the orthogonal complement $H_f^{\perp}$ of H_f is invariant as well, i.e., $(A - z)^{-1} H_f^{\perp} \subseteq H_f^{\perp}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

These invariance properties in turn easily give the equality

$$D(A) \cap H_f = (A - z)^{-1} H_f$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$. Indeed, the inclusion $\supseteq$ follows from $(A-z)^{-1}H_f \subseteq H_f$ as $(A-z)^{-1}$ clearly maps into $D(A)$. The inclusion $\subseteq$ follows as each $g \in D(A)$ can be written as $(A-z)^{-1}h$ for some $h \in H$. Decomposing h into $h_1 \in H_f$ and $h_2 \in H_f^{\perp}$, using that both H_f and $H_f^{\perp}$ are invariant under $(A-z)^{-1}$ and that $(A-z)^{-1}$ is injective, we conclude that h_2 must be zero if g belongs to H_f. So, for $g \in D(A) \cap H_f$, we find $g = (A-z)^{-1}h_1$ with $h_1 \in H_f$. This finishes the proof of the equality $D(A) \cap H_f = (A-z)^{-1}H_f$.

This equality has strong consequences for the restriction $A|_{H_f}$ of A to $D(A)\cap H_f$. More specifically, it gives that $(A|_{H_f}-z)$ maps $D(A)\cap H_f$ onto H_f. Hence,

$$(A|_{H_f} - z)\colon D(A) \cap H_f \longrightarrow H_f$$

is bijective as it is a restriction of the injective operator $A - z$. This bijectivity implies

$$\left(A|_{H_f} - z\right)^{-1} g = (A-z)^{-1}g$$

for all $g \in H_f$.

Combining this with what we have shown already in (b), we conclude

$$(A|_{H_f} - z)^{-1}g = (A-z)^{-1}g = U_f M_{\varphi_z} U_f^{-1} g$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $g \in H_f$. Hence, the operators $(A|_{H_f} - z)^{-1}$ and $U_f M_{\varphi_z} U_f^{-1}$ agree on H_f. By taking inverses we get

$$A|_{H_f} - z = U_f M_{\frac{1}{\varphi_z}} U_f^{-1}.$$

As $M_{1/\varphi_z} + z = M_{\mathrm{id}}$, we have

$$A|_{H_f} = U_f M_{\mathrm{id}} U_f^{-1}.$$

By Proposition A.7 (a), M_{id} is a self-adjoint operator. As self-adjointness is preserved by unitary equivalence, we infer that $A|_{H_f}$ is self-adjoint as well.

(d) We have

$$\operatorname{supp}(\mu_f) = \sigma(M_{\mathrm{id}}) = \sigma(A|_{H_f}) \subseteq \sigma(A).$$

Here, the first equality follows from our discussion of multiplication operators, see Example A.8, the second equality follows from the unitary equivalence between M_{id} and $A|_{H_f}$ established in (c) and the last inclusion can be shown as follows: By (a) of Proposition A.17, H_f is invariant under $(A-\lambda)^{-1}$ for $\lambda \in \varrho(A)$. Therefore, the restriction of $(A-\lambda)^{-1}$ to H_f gives a bounded operator, which obviously is an inverse to $(A|_{H_f} - \lambda)$. Thus, any $\lambda \in \varrho(A)$ belongs to $\varrho(A|_{H_f})$ and the desired inclusion follows. $\square$

For later use, we note the following corollary of (d) from the previous result. The corollary will be used repeatedly, often tacitly, in the remaining part of this appendix.

COROLLARY A.19. *Let A be a self-adjoint operator on H, $f \in H$ and μ_f be the spectral measure associated to f. Then,*

$$\int_{\mathbb{R}} \psi d\mu_f = \int_{\sigma(A)} \psi d\mu_f$$

for all measurable $\psi\colon \mathbb{R} \longrightarrow [0,\infty)$, where both sides may take the value ∞. In particular,

$$L^2(\mathbb{R},\mu_f) = L^2(\sigma(A),\mu_f).$$

If $\psi \in L^1(\mathbb{R},\mu_f)$, then both sides are finite.

The little spectral theorem deals with the action of A on the Hilbert space H_f. By decomposing an arbitrary Hilbert space into a direct sum of such subspaces H_f it is possible to show that any self-adjoint operator is unitarily equivalent to a multiplication operator. This is presented next.

THEOREM A.20 (Spectral theorem). *Let A be a self-adjoint operator on H. Then, there exists a measure space (X,μ) without atoms of infinite mass, a measurable function $u\colon X \longrightarrow \mathbb{R}$ and a unitary map $U\colon L^2(X,\mu) \longrightarrow H$ with*

$$A = UM_uU^{-1}.$$

In particular, $\sigma(A)$ is equal to the essential range of u and A is bounded if and only if u is essentially bounded. If H is separable, then the measure space can be chosen to be σ-finite.

PROOF. Using the Kuratowski–Zorn lemma, we can find a set I and $f_\iota \in H$ for $\iota \in I$ decomposing H into a sum of mutually orthogonal subspaces

$$H = \bigoplus_{\iota\in I} H_{f_\iota}.$$

The little spectral theorem, Lemma A.18, then gives for each $\iota \in I$ a unitary map

$$U_\iota\colon L^2(\mathbb{R},\mu_{f_\iota}) \longrightarrow H_{f_\iota}$$

such that μ_{f_ι} is a finite measure and the restriction of A to H_{f_ι} is given by $U_\iota M_{\mathrm{id}} U_\iota^{-1}$. Taking sums, we obtain a unitary map

$$V = \bigoplus_{\iota\in I} U_\iota\colon \bigoplus_{\iota\in I} L^2(\mathbb{R},\mu_{f_\iota}) \longrightarrow \bigoplus_{\iota\in I} H_{f_\iota} = H$$

with

$$A = V \bigoplus_{\iota\in I} M_{\mathrm{id}} V^{-1}.$$

We will now discuss how to express the orthogonal sum of L^2 spaces as a single L^2 space. Consider $X = \mathbb{R} \times I$. Set $\mathbb{R}_\iota = \{(x,\iota) \mid x \in \mathbb{R}\}$ for $\iota \in I$ and equip X with the σ-algebra consisting of all subsets B such that $B \cap \mathbb{R}_\iota$ is measurable for any $\iota \in I$. Extend μ_{f_ι} to $\mathbb{R}_\iota$ via

$\mu_{f_\iota}(B \times \{\iota\}) = \mu_{f_\iota}(B)$ for measurable $B \subseteq \mathbb{R}$. We define a measure μ on X via

$$\mu(B) = \sum_{\iota \in I} \mu_{f_\iota}(B \cap \mathbb{R}_\iota).$$

As each μ_{f_ι} is a finite measure, it is clear that μ has no atoms of infinite mass. Let $\widetilde{\mathrm{id}} \colon X \longrightarrow \mathbb{R}$ denote the projection onto the first coordinate, i.e.,

$$\widetilde{\mathrm{id}}(x, \iota) = x.$$

By construction, $L^2(X, \mu)$ is canonically unitarily equivalent to the direct sum $\oplus_\iota L^2(\mathbb{R}, \mu_{f_\iota})$ and under this unitary equivalence $M_{\widetilde{\mathrm{id}}}$ becomes $\oplus_\iota M_{\mathrm{id}}$. In particular, by combining this unitary equivalence with the map V above, we get a unitary map

$$U \colon L^2(X, \mu) \longrightarrow H$$

with

$$A = U M_{\widetilde{\mathrm{id}}} U^{-1}.$$

This is the desired statement.

We now prove the statement on the spectrum and boundedness of A. By Lemma A.6, the essential range is the spectrum of the multiplication operator M_u. As the spectrum of an operator is preserved by unitary equivalence, this is also the spectrum of A. Furthermore, A is bounded if and only if M_u is bounded if and only if u is essentially bounded by Proposition A.7 (b).

If H is separable, then we can provide a more explicit construction which avoids using the Kuratowski–Zorn lemma and yields a σ-finite measure space. More specifically, in the case that H is separable, there exists a countable dense subset (g_n) in H. By induction, we then construct a sequence (f_n) such that the spaces H_{f_n} are pairwise orthogonal and $\{g_1, \ldots, g_n\} \subseteq \oplus_{k=1}^n H_{f_k}$: Define $f_1 = g_1$. Now, assume that $f_1, \ldots, f_n$ have the desired properties. Let m be the smallest integer larger than n such that g_m does not belong to $\oplus_{k=1}^n H_{f_k}$. Define f_{n+1} to be the orthogonal projection of g_m onto $(\oplus_{k=1}^n H_{f_k})^\perp$. Then $H_{f_{n+1}}$ is orthogonal to H_{f_k} for $k = 1, \ldots, n$ as each H_{f_k} is invariant under $(A - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, and hence so is $(\oplus_{k=1}^n H_{f_k})^\perp$.

By construction, we have

$$\{g_n \mid n \in \mathbb{N}\} \subseteq \bigoplus_{n \in \mathbb{N}} H_{f_n}.$$

As $\{g_n \mid n \in \mathbb{N}\}$ is dense in H by assumption, we infer that $\oplus_n H_{f_n}$ is dense in H and, hence, agrees with H. So, we have a countable decomposition. Now, we can mimic the above considerations to arrive at the desired statement. $\square$

Remark. From the proof above, as X is a disjoint union of copies of $\mathbb{R}$, we can think of X as a topological space with a topology generating the σ-algebra of the measure space (X, μ). Furthermore, by removing those points of X that do not belong to the support of the measure, μ can be taken to have full support on X. Finally, u is clearly a continuous mapping on X. Hence, we are in the case when the essential range of u is the closure of the range of u, see the remark following Definition A.5. In particular, we get that the spectrum of A is equal to $\overline{u(X)}$.

The spectral theorem allows us to define functions of a self-adjoint operator. This is known as the *functional* or *spectral calculus*. We now give a precise definition.

Definition A.21 (Functional calculus). If $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, A is self-adjoint and (X, μ), u and U are as in Theorem A.20, then we define the operator $\varphi(A)$ acting on the domain

$$D(\varphi(A)) = UD(M_{\varphi \circ u})$$

as

$$\varphi(A) = UM_{\varphi \circ u}U^{-1}.$$

Remark (Consistency, uniqueness and domain of φ). We consider the situation of the previous definition.

(a) For $\varphi = \mathrm{id}\colon \mathbb{R} \longrightarrow \mathbb{R}$, we find $\varphi(A) = UM_uU^{-1} = A$. Moreover, as $(M_u - z)^{-1} = M_{1/(u-z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ whenever u maps into $\mathbb{R}$, we obtain

$$\varphi_z(A) = (A - z)^{-1},$$

where $\varphi_z(x) = 1/(x - z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, for the constant function $1\colon \mathbb{R} \longrightarrow \mathbb{R}$ via $1(x) = 1$ for all $x \in \mathbb{R}$, we find $1(A) = I$, where I is the identity operator. In this sense our definition of functions of A is consistent with what is expected.

(b) The consistency discussed in (a) implies that the definition of $\varphi(A)$ does not depend on the actual choice of U and (X, μ). Indeed, (a) shows the independence of U and (X, μ) for $\varphi = \varphi_z$ for $z \in \mathbb{C} \setminus \mathbb{R}$. By Lemma A.16 this easily gives independence for any continuous $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ with compact support. By taking suitable limits, compare Lemma A.27 below, this gives the desired independence for all measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$.

(c) We do not need φ to be defined on all of $\mathbb{R}$. In fact, it suffices for φ to be defined on the essential range of u, which is equal to $\sigma(A)$. Then any extension of φ to $\mathbb{R}$ will give the same operator when applied to A.

By (c) of the preceding remark, to establish properties of $\varphi(A)$ it suffices to consider the properties of the restriction of φ to $\sigma(A)$. We denote this restriction by $\varphi|_{\sigma(A)}$, i.e., $\varphi|_{\sigma(A)}\colon \sigma(A) \longrightarrow \mathbb{C}$ is given by $\varphi|_{\sigma(A)}(x) = \varphi(x)$. With this notation, we now give some basic properties of the functional calculus.

Proposition A.22 (Basic properties of $\varphi(A)$). *Let A be a self-adjoint operator on H with spectrum $\sigma(A)$ and let $\varphi, \psi \colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable on $\sigma(A)$. Then, the following statements hold:*
(a) $\varphi(A)^* = \overline{\varphi}(A)$.
(b) *The operator $\varphi(A)$ is self-adjoint if and only if the essential range of $\varphi|_{\sigma(A)}$ is contained in $\mathbb{R}$.*
(c) *The operator $\varphi(A)$ is bounded if and only if $\varphi|_{\sigma(A)}$ is essentially bounded, in which case*

$$\|\varphi(A)\| = \|\varphi|_{\sigma(A)}\|_\infty.$$

(d) $D(\varphi(A)\psi(A)) \subseteq D((\varphi\psi)(A))$ *and on* $D(\varphi(A)\psi(A))$ *we have*

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

(e) $D(\varphi(A)+\psi(A)) \subseteq D((\varphi+\psi)(A))$ *and on* $D(\varphi(A)+\psi(A))$ *we have*

$$(\varphi + \psi)(A) = \varphi(A) + \psi(A).$$

Proof. This follows from the definition of $\varphi(A)$ and the corresponding properties of multiplication operators, in particular, see Proposition A.7 for properties (b) and (c). □

Remark. Note that in (d) and (e) the domains of $(\varphi\psi)(A)$ and $(\varphi+\psi)(A)$ can indeed be strictly bigger than the domains of $\varphi(A)\psi(A)$ and $\varphi(A) + \psi(A)$, respectively. For example, it is possible that φ and ψ are unbounded functions which result in operators not defined on the entire Hilbert space whereas $\varphi + \psi$ or $\varphi\psi$ are zero, resulting in an operator defined on the entire Hilbert space.

Remark (Spectral mapping theorem). The spectral theorem also gives a relationship between the spectra of A and $\varphi(A)$ whenever $\varphi :$ $\sigma(A) \longrightarrow \mathbb{R}$ is continuous. More specifically, we get

$$\sigma(\varphi(A)) = \overline{\varphi(\sigma(A))}.$$

This can be seen as follows: Let A be unitarily equivalent to M_u. As unitary equivalence preserves the spectrum, the spectrum of A is equivalent to the essential range of u which, by the remark following the spectral theorem, is equivalent to the closure of the range of u. By the same argument, the spectrum of $\varphi(A)$ is equivalent to the closure of the range of $\varphi \circ u$ which, by the continuity of φ, is equivalent to the closure of $\varphi(\sigma(A))$. This completes the proof.

The spectral theorem makes it possible to exhibit the spectral measure in the following way.

Proposition A.23 (Spectral measures via the spectral theorem). *Let A be a self-adjoint operator on H and let (X, μ), u and U be as in Theorem A.20. Let $f \in H$ and let $\psi = U^{-1}f$. Then,*

$$\int_{\mathbb{R}} \varphi d\mu_f = \int_X (\varphi \circ u)|\psi|^2 d\mu$$

for all measurable $\varphi\colon \mathbb{R} \longrightarrow [0,\infty)$*, where both sides may take the value* ∞*. If* $\varphi \in L^1(\mathbb{R},\mu_f)$*, then both sides are finite.*

Proof. As $U\colon L^2(X,\mu) \longrightarrow H$ is unitary, we find

$$\begin{aligned}\int_{\mathbb{R}} \varphi_z d\mu_f &= \langle f, (A-z)^{-1} f\rangle \\ &= \langle f, U(M_u - z)^{-1}U^{-1} f\rangle \\ &= \langle \psi, (M_u - z)^{-1}\psi\rangle \\ &= \int_X (\varphi_z \circ u)|\psi|^2 d\mu\end{aligned}$$

for $f \in H$ and $z \in \mathbb{C}\setminus\mathbb{R}$. By Lemma A.16 this equality extends to all continuous functions $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ with compact support. From monotone convergence we then extend the equality to all measurable functions $\varphi\colon \mathbb{R} \longrightarrow [0,\infty)$. Decomposing $\varphi \in L^1(\mathbb{R},\mu_f)$ as a linear combination of functions in $L^1(\mathbb{R},\mu_f)$ with values in $[0,\infty)$ we then obtain the statement for $\varphi \in L^1(\mathbb{R},\mu_f)$. □

Given the preceding computation of the spectral measure, we now give some connections between the spectral calculus and the spectral measures. The arising formulas will be most useful for our subsequent considerations.

Proposition A.24 (Functional calculus and spectral measures). *Let* A *be a self-adjoint operator on* H*, let* $f \in H$ *and let* $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ *be measurable.*

(a) $f \in D(\varphi(A))$ *if and only if* $\varphi \in L^2(\mathbb{R},\mu_f) = L^2(\sigma(A),\mu_f)$*, in which case*
$$\|\varphi(A)f\|^2 = \int |\varphi|^2 d\mu_f.$$
In particular, $f \in D(A)$ *if and only if* $\int x^2 d\mu_f < \infty$*.*

(b) *If* $f \in D(\varphi(A))$*, then*
$$\langle f, \varphi(A) f\rangle = \int \varphi d\mu_f \qquad \text{and} \qquad |\varphi|^2\mu_f = \mu_{\varphi(A)f}.$$

(c) *The map* $L^2(\mathbb{R},\mu_f) \longrightarrow H$ *given by* $\varphi \mapsto \varphi(A)f$ *is isometric with range* H_f *and maps* φ_z *to* $(A-z)^{-1}f$ *for any* $z \in \mathbb{C}\setminus\mathbb{R}$*.*

Remark. We note that (c) may be rephrased as saying that the operator U_f from the little spectral theorem, Lemma A.18, is given by $U_f\varphi = \varphi(A)f$.

Proof. Let (X,μ), u and U be as in Theorem A.20, i.e.,

$$A = UM_uU^{-1},$$

where $U : L^2(X,\mu) \longrightarrow H$ is unitary. Then, by definition,

$$\varphi(A) = UM_{\varphi\circ u}U^{-1}$$

holds for all measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$. We set

$$\psi = U^{-1} f.$$

(a) We first show the characterization of the domain of $\varphi(A)$. By Proposition A.23, we have $\varphi \in L^2(\mathbb{R}, \mu_f)$ if and only if

$$\int |\varphi \circ u|^2 |\psi|^2 d\mu < \infty$$

which, by the definition of the domain of a multiplication operator, is equivalent to

$$\psi \in D(M_{\varphi \circ u}).$$

As $\psi = U^{-1} f$, this holds if and only if $f \in D(\varphi(A))$ from the definition of the domain of $\varphi(A)$.

Now, if φ belongs to $L^2(\mathbb{R}, \mu_f)$, then, as U is unitary, Proposition A.23 gives

$$\|\varphi(A) f\|^2 = \|M_{\varphi \circ u} \psi\|^2 = \int |\varphi \circ u|^2 |\psi|^2 d\mu = \int |\varphi|^2 d\mu_f,$$

which proves the formula given in (a). The last statement of (a) is immediate by taking $\varphi = \mathrm{id}$.

(b) As U is unitary and $U^{-1}\varphi(A) = M_{\varphi \circ u} U^{-1}$, we obtain

$$\langle f, \varphi(A) f\rangle = \langle U^{-1} f, U^{-1}\varphi(A) f\rangle = \langle \psi, M_{\varphi \circ u}\psi\rangle = \int (\varphi \circ u)|\psi|^2 d\mu.$$

Since we assume $f \in D(\varphi(A))$, part (a) gives $\varphi \in L^2(\mathbb{R}, \mu_f)$. As μ_f is finite, $\varphi \in L^2(\mathbb{R}, \mu_f)$ implies $\varphi \in L^1(\mathbb{R}, \mu_f)$ and Proposition A.23 yields

$$\int (\varphi \circ u)|\psi|^2 d\mu = \int \varphi d\mu_f.$$

Putting these equations together gives the first formula claimed in (b).

We now show

$$|\varphi|^2 \mu_f = \mu_{\varphi(A) f}.$$

It suffices to show

$$\int \chi |\varphi|^2 d\mu_f = \int \chi d\mu_{\varphi(A) f}$$

for all bounded measurable functions $\chi\colon \mathbb{R} \longrightarrow \mathbb{C}$. As χ is bounded and μ_f is finite for all $f \in H$, by part (a) the operator $\chi(A)$ is defined on the entire Hilbert space H. Hence, from the already established first formula of (b), the fact that U is unitary and the definitions of $\varphi(A)$

and $\chi(A)$, we find

$$\begin{aligned}\int \chi d\mu_{\varphi(A)f} &= \langle \varphi(A)f, \chi(A)\varphi(A)f\rangle \\ &= \langle U^{-1}\varphi(A)f, U^{-1}\chi(A)UU^{-1}\varphi(A)f\rangle \\ &= \langle M_{\varphi\circ u}\psi, M_{\chi\circ u}M_{\varphi\circ u}\psi\rangle \\ &= \int (\chi\circ u)|\varphi\circ u|^2|\psi|^2 d\mu \\ &= \int \chi|\varphi|^2 d\mu_f,\end{aligned}$$

where we used Proposition A.23 in the last equality. This is the desired statement.

(c) From the formula $\|\varphi(A)f\|^2 = \int|\varphi|^2 d\mu_f$ for $\varphi \in L^2(X,\mu)$ proven in (a) we see that the map in question is isometric. For $\varphi = \varphi_z$ with $z \in \mathbb{C}\setminus\mathbb{R}$ we have $\varphi_z(A)f = (A-z)^{-1}f$, as can be seen directly. Hence, we see that the map in question maps $\mathrm{Lin}\{\varphi_z \mid z \in \mathbb{C}\setminus\mathbb{R}\}$ into H_f. As this linear hull is a dense subspace of $L^2(\mathbb{R},\mu_f)$ by Lemma A.16 and the map is isometric and thus continuous, it maps $L^2(\mathbb{R},\mu_f)$ into H_f. As the set $\{(A-z)^{-1}f \mid z \in \mathbb{C}\setminus\mathbb{R}\}$ is dense in H_f and belongs to the range of the map, we see that the map has dense range in H_f. As the map is isometric, we infer that the range of the map is actually equal to H_f. □

A direct consequence of the preceding proposition is the following statement concerning bounded functions.

Corollary A.25 (Bounded functional calculus). *Let A be a self-adjoint operator on H and $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ be measurable and bounded on $\sigma(A)$. Then, $D(\varphi(A)) = H$ and, for every $f \in H$,*

$$\|\varphi(A)f\|^2 = \int |\varphi|^2 d\mu_f, \qquad \langle f, \varphi(A)f\rangle = \int \varphi d\mu_f$$

and

$$|\varphi|^2\mu_f = \mu_{\varphi(A)f}.$$

Proof. As φ is bounded on $\sigma(A)$ and μ_f is a finite measure supported on $\sigma(A)$, we have $D(\varphi(A)) = H$ from (a) of Proposition A.24, which also gives the first equality. The remaining equalities then follow from (b) of Proposition A.24. □

We also note the following consequence which concerns signed spectral measures corresponding to two elements of the Hilbert space.

Proposition A.26. *Let A be a self-adjoint operator on H and let $f, g \in H$. Then, there exists a unique finite signed regular Borel measure μ on $\mathbb{R}$ with*

$$\langle f, (A-z)^{-1}g\rangle = \int \frac{1}{x-z} d\mu(x)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$*. If* $\varphi \colon \mathbb{R} \longrightarrow \mathbb{C}$ *is measurable and* $f, g \in D(\varphi(A))$*, then*

$$\langle f, \varphi(A) g \rangle = \int \varphi d\mu.$$

In particular, this holds for all $f, g \in H$ *when* φ *is bounded and measurable.*

Proof. The existence of such a signed measure is given by the existence of μ_f and μ_g and polarization. The remaining statements follow by Proposition A.24 (b) and polarization as well. □

We also note the following continuity features of the functional calculus.

Lemma A.27 (Continuity of the functional calculus). *Let* A *be a self-adjoint operator on* H*. Let* $\varphi_n \colon \mathbb{R} \longrightarrow \mathbb{C}$ *for* $n \in \mathbb{N}$ *and* $\varphi \colon \mathbb{R} \longrightarrow \mathbb{C}$ *be measurable.*

(a) *If* φ*,* φ_n *are bounded on* $\sigma(A)$ *with* $\|(\varphi_n - \varphi)|_{\sigma(A)}\|_\infty \to 0$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} \|\varphi_n(A) - \varphi(A)\| = 0.$$

(b) *If* $|\varphi_n(x)| \leq |\varphi(x)|$ *for* $n \in \mathbb{N}$ *and* $\varphi_n(x) \to \varphi(x)$ *as* $n \to \infty$ *for all* $x \in \sigma(A)$*, then for all* $f \in D(\varphi(A))$*,*

$$\lim_{n\to\infty} \|(\varphi_n(A) - \varphi(A)) f\| = 0.$$

Proof. (a) By Proposition A.22 we have

$$\|\varphi_n(A) - \varphi(A)\| = \|(\varphi_n - \varphi)(A)\| \leq \|(\varphi_n - \varphi)|_{\sigma(A)}\|_\infty$$

and (a) follows.

(b) We use part (a) of Proposition A.24 repeatedly. First, as $f \in D(\varphi(A))$, it follows that $\varphi \in L^2(\mathbb{R}, \mu_f)$. Furthermore, as $|\varphi_n(x)| \leq |\varphi(x)|$ for all $n \in \mathbb{N}$ and $x \in \sigma(A)$, we have $\varphi_n \in L^2(\mathbb{R}, \mu_f)$ and thus $f \in D(\varphi_n(L))$ for all $n \in \mathbb{N}$. Finally,

$$\|\varphi(A) f - \varphi_n(A) f\|^2 = \int |\varphi - \varphi_n|^2 d\mu_f \to 0$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem since the integrand converges to 0 pointwise, the φ_n are bounded by φ and $\varphi \in L^2(\mathbb{R}, \mu_f)$. □

Of particular relevance for applications of the functional calculus are characteristic functions of measurable sets. They are discussed next. Recall that $B(H)$ denotes the space of bounded operators on H. Given a self-adjoint operator A we define

$$E \colon \{\text{measurable subsets of } \mathbb{R}\} \longrightarrow B(H)$$

via

$$E(B) = 1_B(A).$$

We call the operator $E(B)$ the *spectral projection* associated to B.

We now highlight some properties of E which follow directly from Proposition A.22. By definition, for every measurable set B, the operator $E(B)$ is unitarily equivalent to multiplication by the characteristic function $v = 1_B \circ u = 1_{u^{-1}(B)}$, where $u^{-1}(B)$ is the preimage of B. Clearly, $v = \overline{v} = v^2$, so that $E(B)$ is an orthogonal projection, i.e., satisfies

$$E(B) = E(B)^* = E(B)E(B).$$

Similarly, we infer from $1_{B_1}1_{B_2} = 1_{B_1 \cap B_2}$ that

$$E(B_1)E(B_2) = E(B_1 \cap B_2) = E(B_2)E(B_1)$$

whenever B_1, B_2 are measurable subsets of $\mathbb{R}$. Moreover, we obviously have $E(\emptyset) = 0$ as $1_\emptyset = 0$. These consideration give, in particular,

$$E(B_1)E(B_2) = E(\emptyset) = 0$$

whenever $B_1 \cap B_2 = \emptyset$. Moreover, as $1_{\cup_n B_n}$ is the monotone pointwise limit of $\sum_{n=1}^N 1_{B_n}$ whenever the sets B_n are mutually disjoint, we infer $E(\bigcup_n B_n) = \oplus_n E(B_n)$. Furthermore, $E(\mathbb{R}) = I$ is the identity operator.

To summarize, we note that E satisfies the following properties:

- $E(B)$ is an orthogonal projection for each measurable $B \subseteq \mathbb{R}$.
- $E(\bigcup_n B_n) = \oplus_n E(B_n)$ for mutually disjoint measurable sets.
- $E(\emptyset) = 0$.

In this sense, the map E resembles a measure. We refer to E as the *projection valued measure associated to* A or the *spectral family*.

The map E is intimately linked to the spectral measures. This is discussed in the subsequent two propositions.

Proposition A.28 (Spectral measure via projection valued measures). *Let A be a self-adjoint operator on H with associated projection valued measure E. Then, for any $f \in H$, we have*

$$\mu_f(B) = \langle f, E(B)f \rangle = \|E(B)f\|^2$$

for any measurable set $B \subseteq \mathbb{R}$.

Proof. By the definition of $E(B) = 1_B(A)$ and (b) of Proposition A.24 we have

$$\mu_f(B) = \int 1_B d\mu_f = \langle f, E(B)f \rangle$$

for any measurable set B in $\mathbb{R}$ and any $f \in H$. Moreover, as $E(B)$ satisfies $E(B)^2 = E(B) = E(B)^*$, we also find

$$\langle f, E(B)f \rangle = \langle f, E(B)^*E(B)f \rangle = \langle E(B)f, E(B)f \rangle = \|E(B)f\|^2.$$

This finishes the proof. $\square$

Remark. We note that for $B = \mathbb{R}$, the proposition above gives $\mu_f(\mathbb{R}) = \|f\|^2$, which was already shown in Proposition A.15.

Proposition A.29. *Let A be a self-adjoint operator on H with associated projection valued measure E. Let $B \subseteq \mathbb{R}$ be measurable. Then, for all $f \in H$,*

$$\mu_{E(B)f} = 1_B \mu_f.$$

In particular, for any $g \in E(B)H$, we have $\mu_g = 1_B \mu_g$,

$$\operatorname{supp}(\mu_g) \subseteq \overline{B},$$

where $\overline{B}$ denotes the closure of B, and $g \in D(\varphi(A))$ if and only if $\varphi \in L^2(\sigma(A), \mu_g)$ for any measurable $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$.

Proof. The first statement follows from (b) of Proposition A.24. Now, for $g \in E(B)H$, we have $g = E(B)g$ as $E(B)$ is an orthogonal projection and we find $\mu_g = 1_B \mu_g$. Clearly, $1_B \mu_g$ is supported on $\overline{B}$. Now, if $\varphi\colon \mathbb{R} \longrightarrow \mathbb{C}$ is measurable, then the statement on the domain of $\varphi(A)$ follows from Proposition A.24 (a). This finishes the proof. $\square$

It is possible to characterize the spectrum of A via E. To do so we define the *support of* E as

$$\operatorname{supp}(E) = \{\lambda \in \mathbb{R} \mid E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \text{ for all } \varepsilon > 0\}.$$

With this definition, we can show that the spectrum of A is equal to the support of E.

Theorem A.30. *Let A be a self-adjoint operator with associated projection-valued measure E. Then,*

$$\sigma(A) = \operatorname{supp}(E).$$

Proof. As the spectrum is preserved by unitary equivalence, we may assume that $A = M_u$, where (X, μ) is a measure space without atoms of infinite mass and $u\colon X \longrightarrow \mathbb{R}$ is measurable by Theorem A.20. In particular, the spectrum of A is given by the essential range of u. Hence, it remains to show that $\operatorname{supp}(E)$ equals the essential range of u. Now, for a measurable set $B \subseteq \mathbb{R}$, we have

$$E(B) = M_{1_B \circ u}$$

and, hence, $E(B)$ is not trivial if and only if

$$0 \neq 1_B \circ u = 1_{u^{-1}(B)}$$

if and only if $\mu(u^{-1}(B)) > 0$. This easily shows that $\operatorname{supp}(E)$ is equal to the essential range of u. $\square$

From the considerations above, we can also express the spectrum in terms of the spectral measures associated to elements $f \in H$. In fact, we have already seen that the support of the spectral measure for each element in the Hilbert space is contained in the spectrum. It turns out that the union of all such supports covers the entire spectrum. So, for any point in the spectrum, there will be an element in the Hilbert space that will see this point via its spectral measure.

COROLLARY A.31. *Let A be a self-adjoint operator on H. Then,*

$$\sigma(A) = \bigcup_{f \in H} \operatorname{supp}(\mu_f).$$

PROOF. We have to show two inclusions. From the little spectral theorem, Lemma A.18 (d), we already know that $\operatorname{supp}(\mu_f) \subseteq \sigma(A)$ for all $f \in H$. This gives the inclusion $\supseteq$.

It remains to show $\subseteq$. Let $\lambda \in \sigma(A)$. By the spectral theorem, Theorem A.20, we can assume without loss of generality that $A = M_u$ for some real-valued measurable function u on a measure space (X, μ) without atoms of infinite mass. Then, λ belongs to the essential range of u. For $n \in \mathbb{N}$, we define

$$C_n = \{x \in X \mid \frac{1}{n+1} \leq |u(x) - \lambda| < \frac{1}{n}\}.$$

Clearly, the sets C_n are pairwise mutually disjoint. Moreover, as λ belongs to the essential range of u, we infer that $\mu(C_n) > 0$ for infinitely many n. Without loss of generality, we assume $\mu(C_n) > 0$ for all $n \in \mathbb{N}$. As (X, μ) does not have atoms of infinite mass, any C_n admits a measurable subset B_n with $0 < \mu(B_n) < \infty$. Clearly, the sets B_n are pairwise mutually disjoint.

Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{1}{\mu(B_n)^{1/2}} \right) 1_{B_n}.$$

Then, f belongs to $L^2(X, \mu)$. Moreover, from Proposition A.23, we find

$$\mu_f(S) = \int (1_S \circ u) |f|^2 d\mu$$

for all measurable $S \subseteq \mathbb{R}$. Combining this with the obvious inequality

$$1_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})} \circ u \geq 1_{C_n} \geq 1_{B_n}$$

we then obtain

$$\mu_f \left((\lambda - \frac{1}{n}, \lambda + \frac{1}{n}) \right) \geq \int 1_{B_n} |f|^2 d\mu = \frac{1}{2^{2n}} \left(\frac{1}{\mu(B_n)^{1/2}} \right)^2 > 0$$

for all $n \in \mathbb{N}$. Hence, λ belongs to the support of μ_f. $\square$

REMARK. If H is separable, then it is even possible to find a single $f \in H$ with

$$\sigma(A) = \operatorname{supp}(\mu_f).$$

To see this we consider the decomposition $H = \oplus_n H_{f_n}$ provided in the proof of Theorem A.20. Let A_{f_n} be the self-adjoint restriction of A to H_{f_n}. Then, A is the orthogonal sum of the operators A_{f_n}. This easily gives

$$\sigma(A) = \overline{\bigcup_n \sigma(A_{f_n})}.$$

Consider now

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|f_n\|)} f_n.$$

As each H_{f_n} is invariant under $(A - z)^{-1}$ by Proposition A.17 and the spaces H_{f_n} are pairwise orthogonal for all $n \in \mathbb{N}$, we infer

$$\langle f, (A - z)^{-1} f \rangle = \sum_{n=1}^{\infty} \frac{1}{(2^n(1 + \|f_n\|))^2} \langle f_n, (A - z)^{-1} f_n \rangle$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. By the defining feature of the spectral measures this gives

$$\mu_f = \sum_{n=1}^{\infty} \frac{1}{(2^n(1 + \|f_n\|))^2} \mu_{f_n}.$$

This implies

$$\operatorname{supp}(\mu_f) = \overline{\bigcup_n \operatorname{supp}(\mu_{f_n})}.$$

As A_{f_n} is unitarily equivalent to multiplication by the identity on $L^2(X, \mu_{f_n})$, it follows that the spectrum of A_{f_n} equals the support of μ_{f_n}. Thus, by putting everything together we have

$$\sigma(A) = \overline{\bigcup_n \sigma(A_{f_n})} = \overline{\bigcup_n \operatorname{supp}(\mu_{f_n})} = \operatorname{supp}(\mu_f).$$

This shows the desired statement.

We now use the functional calculus developed above to establish some basic properties of semigroups and resolvents. In particular, we will discuss how the semigroups and resolvents generate solutions for certain equations. We will further develop this theory for general Banach spaces in Appendix D.

For our discussion, we will be interested in self-adjoint operators A such that $\sigma(A) \subseteq [0, \infty)$. We call such an operator A *positive* and write $A \geq 0$. We give some equivalent formulations and further discussion of this condition in Appendix B.

We start with the semigroup. Consider $A \geq 0$. Then, for any $t \geq 0$, the map $\varphi_t \colon \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\varphi_t(x) = e^{-tx}$$

is a bounded real-valued function on $[0, \infty)$. Hence, due to $\sigma(A) \subseteq [0, \infty)$, the operator $\varphi_t(A)$ is bounded, self-adjoint and defined on the entire Hilbert space H for each $t \geq 0$ by Proposition A.22. We denote this operator by e^{-tA}. The family e^{-tA} for $t \geq 0$ is called the *semigroup associated to A*. We now consider some properties of this family of operators. These properties play a significant role in our considerations.

Proposition A.32 (Basic properties of the semigroup). *Let A be a positive operator on H. Then,*

(a) *For all $s, t \geq 0$,*

$$e^{-(s+t)A} = e^{-sA}e^{-tA}.$$

(b) *For all $f \in H$,*

$$\lim_{t \to 0^+} e^{-tA} f = f.$$

(c) *For all $t \geq 0$,*

$$\|e^{-tA}\| \leq 1.$$

Proof. (a) This follows immediately from Proposition A.22 (d).

(b) By Corollary A.25, for $f \in H$, we have

$$\|e^{-tA} f - f\|^2 = \int_0^\infty (e^{-tx} - 1)^2 d\mu_f(x) \to 0$$

as $t \to 0^+$ by Lebesgue's dominated convergence theorem. This follows as the integral is bounded above by 1, converges pointwise to 0 and each spectral measure is finite.

(c) By Corollary A.25 and Proposition A.15, for $f \in H$ we have

$$\|e^{-tA} f\|^2 = \int_0^\infty e^{-2tx} d\mu_f(x) \leq \mu_f([0,\infty)) = \|f\|^2.$$

This gives the desired conclusion. □

Remark. We call property (a) in the proposition above the *semigroup property*, property (b) *strong continuity* and property (c) *contraction*. Thus, we summarize the proposition above by saying that e^{-tA} is a strongly continuous contraction semigroup.

We will now show that the semigroup generates solutions of the parabolic equation involving A. In order to make this precise, we recall that a function $u\colon (0,\infty) \longrightarrow H$ is called *differentiable* if for any $t > 0$ the limit

$$\lim_{h \to 0} \frac{1}{h}(u(t+h) - u(t))$$

exists. In this case, we denote this limit as $\partial_t u$ and call it the *derivative* of u. We call the equation $(A + \partial_t)u = 0$ the *parabolic equation* associated to A. Any differentiable function $u\colon (0,\infty) \longrightarrow H$ which satisfies $(A + \partial_t)u = 0$ is called a *solution of the parabolic equation.*

When dealing with parabolic equations, we are usually interested in solutions $u\colon (0,\infty) \longrightarrow H$ satisfying some additional properties as $t \to 0^+$. For example, a common requirement is that $u(t) \to f \in H$ as $t \to 0^+$. In this case we refer to u as a *solution of the parabolic equation with initial condition f*. Finally, we note that when dealing with functions $u\colon [0,\infty) \longrightarrow H$ we often write u_t instead of $u(t)$ for $t \geq 0$. We now show that the semigroup generates a solution of the parabolic equation with a prescribed initial condition.

Theorem A.33 (Solution of the parabolic equation). *Let A be a positive operator on H and let $f \in H$. Then, $u\colon [0,\infty) \longrightarrow H$ given by*

$$u_t = e^{-tA} f$$

is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, satisfies $u_t \in D(A)$ and

$$\partial_t u_t = -Au_t$$

for all $t > 0$ as well as $u(t) \to f$ for $t \to 0^+$.

Proof. We prove the theorem through a series of claims.

Claim. The function u is continuous on $[0,\infty)$.

Proof of the claim. Let $t \geq 0$. Then, for all $h \in \mathbb{R}$ with $t + h \geq 0$, the operator $e^{-(t+h)A}$ is bounded and Corollary A.25 gives

$$\left\| e^{-(t+h)A} f - e^{-tA} f \right\|^2 = \int_0^\infty \left| e^{-(t+h)x} - e^{-tx} \right|^2 d\mu_f(x).$$

Now, $|e^{-(t+h)x} - e^{-tx}|^2$ is bounded by 4 and converges to 0 pointwise as $h \to 0$ for $x \geq 0$. Thus, we obtain from Lebesgue's dominated convergence theorem

$$\lim_{h\to 0} \int_0^\infty \left| e^{-(t+h)x} - e^{-tx} \right|^2 d\mu_f(x) = 0.$$

This proves the continuity of u at t.

Claim. For any $t > 0$, $u_t \in D(A)$.

Proof of the claim. By Proposition A.24 (a), we have to show $\int x^2 d\mu_{u_t}(x) < \infty$ for $t > 0$. Now, by Corollary A.25, as $f \in H = D(e^{-tA})$ we have $\mu_{u_t} = \mu_{e^{-tA}f} = |e^{-t\cdot}|^2 \mu_f$. This easily gives

$$\int x^2 d\mu_{u_t}(x) = \int_{[0,\infty)} x^2 e^{-2tx} d\mu_f(x) < \infty,$$

where we used that μ_f is supported on $\sigma(A) \subseteq [0,\infty)$ and $x \mapsto x^2 e^{-2tx}$ is bounded on $[0,\infty)$.

Claim. For any $t > 0$, the function u is differentiable in t and satisfies

$$\partial_t u_t = -Au_t.$$

Proof of the claim. For $h \in \mathbb{R}$ with $|h| \leq t$, we define the function $\psi_h\colon [0,\infty) \longrightarrow \mathbb{R}$ by

$$\psi_h(x) = \frac{e^{-(t+h)x} - e^{-tx}}{h} - xe^{-tx}.$$

Then, (d) and (e) of Proposition A.22, give

$$\frac{1}{h}(e^{-(t+h)A} f - e^{-tA} f) - Ae^{-tA} f = \psi_h(A) f,$$

where we use $e^{-tA}f \in D(A)$ for $t > 0$, which was established in the preceding claim to write down the expression on the left-hand side. Hence, Proposition A.24 (a) yields

$$\left\| \frac{1}{h}(e^{-(t+h)A}f - e^{-tA}f) - Ae^{-tA}f \right\|^2 = \int_0^\infty |\psi_h(x)|^2 d\mu_f(x).$$

Now, ψ_h can easily be seen to converge pointwise to 0 as $h \to 0$ and to be bounded by $x \mapsto 2xe^{-tx}$, which is bounded on $[0, \infty)$. Hence, by Lebesgue's dominated convergence theorem, we see that $\int |\psi_h|^2 d\mu_f \to 0$ as $h \to 0$ and this gives the desired claim.

Claim. $u_t \to f$ as $t \to 0^+$.

Proof of the claim. This is immediate from the already established continuity of u on $[0, \infty)$ and $u_0 = f$. $\square$

Remark. (a) The theorem above can be phrased as saying that u with $u_t = e^{-tA}f$ is a solution of the parabolic equation with initial condition f.

(b) In the above theorem, if $f \in D(A)$, then we obtain the differentiability of u at 0 and that the derivative of u at $t = 0$ is given by $-Af$. The proof follows by the obvious modification of the proof of differentiability given in the theorem.

(c) We will later show in Appendix D that $e^{-tA}f$ is the unique solution to the equation $\partial_t u_t = -Au_t$ with $u_0 = f$. As a particular consequence of this uniqueness, if $e^{-tA} = e^{-tB}$ for positive operators A and B, then $A = B$.

We now turn to resolvents. For a positive operator A, for every $\lambda < 0$, the resolvent $(A - \lambda)^{-1}$ exists and is a bounded operator on H for $\lambda < 0$. We now switch notation and write

$$(A + \alpha)^{-1}$$

for $\alpha > 0$ instead. In particular, the resolvent $(A+\alpha)^{-1}$ is obtained by applying the bounded function $\varphi\colon [0, \infty) \longrightarrow \mathbb{R}$ given by

$$\varphi(x) = \frac{1}{x + \alpha}$$

for $\alpha > 0$ to A. We now gather some basic properties of resolvents.

Proposition A.34 (Basic properties of resolvents). *Let A be a positive operator on H. Then,*

(a) *For all $\alpha, \beta > 0$,*

$$(A + \alpha)^{-1} - (A + \beta)^{-1} = -(\alpha - \beta)(A + \alpha)^{-1}(A + \beta)^{-1}.$$

(b) *For all $f \in H$,*

$$\lim_{\alpha \to \infty} \alpha(A + \alpha)^{-1} f = f.$$

(c) *For all $\alpha > 0$,*

$$\|\alpha(A + \alpha)^{-1}\| \leq 1.$$

Proof. (a) This follows directly from the identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

for invertible operators A and B with $D(B) \subseteq D(A)$.

(b) For every $f \in H$ we have by Corollary A.25

$$\|\alpha(A+\alpha)^{-1}f - f\|^2 = \int_0^\infty \left|\frac{\alpha}{x+\alpha} - 1\right|^2 d\mu_f(x) \to 0$$

as $\alpha \to \infty$ by Lebesgue's dominated convergence theorem.

(c) By Corollary A.25 and Proposition A.15, for every $f \in H$ we have

$$\|\alpha(A+\alpha)^{-1}f\|^2 = \int_0^\infty \left|\frac{\alpha}{x+\alpha}\right|^2 d\mu_f(x) \le \mu_f([0,\infty)) = \|f\|^2.$$

This completes the proof. □

Remark. We call property (a) in Proposition A.34 the *resolvent identity*, property (b) *strong continuity* and property (c) *contraction*. Thus, we summarize the proposition above by saying that $\alpha(A+\alpha)^{-1}$ is a strongly continuous contraction resolvent.

From our discussion above, the semigroup e^{-tA} generates a solution of the parabolic equation $\partial_t u_t = -Au_t$. Similarly, the resolvent also generates a solution of a naturally arising equation. More specifically, it is immediate that $u = (A+\alpha)^{-1}f$ for $f \in H$ gives a solution of

$$(A+\alpha)u = f.$$

We refer to this equation as the *Poisson equation*.

We conclude this appendix by presenting the connection between the semigroup and the resolvent associated to an operator. In order to state this connection, we need to integrate Hilbert space-valued functions. Hence, if $g\colon [a,b] \longrightarrow H$ is continuous, we define the integral

$$\int_a^b g(t)dt$$

as a Riemann integral via approximation by Riemann sums of step functions. We will use this to integrate the semigroup directly below.

Theorem A.35 (Semigroups and resolvents). *Let A be a positive operator on H.*

(a) *For every $\alpha > 0$,*

$$(A+\alpha)^{-1} = \int_0^\infty e^{-t\alpha}e^{-tA}dt.$$

("Laplace transform")

(b) *For every $t > 0$,*

$$e^{-tA} = \lim_{n\to\infty} \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^n .$$

PROOF. (a) From the formula

$$(x+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tx} dt,$$

which holds for all $x \geq 0$ and $\alpha > 0$, we obtain by applying the functional calculus

$$(A+\alpha)^{-1} = \int_0^\infty e^{-t\alpha} e^{-tA} dt.$$

This gives the conclusion.

(b) We note that

$$\varphi_n(x) = \left(\frac{n}{t} \left(x + \frac{n}{t} \right)^{-1} \right)^n = \left(1 + \frac{tx}{n} \right)^{-n} \to e^{-tx}$$

as $n \to \infty$ for $x, t \geq 0$. Hence, by Corollary A.25 and Lebesgue's dominated convergence theorem, we obtain, for every $f \in H$,

$$\left\| e^{-tA} f - \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^n f \right\|^2 = \int_0^\infty \left| e^{-tx} - \varphi_n(x) \right|^2 d\mu_f(x) \to 0$$

as $n \to \infty$. This completes the proof. $\square$

APPENDIX B

Closed Forms on Hilbert spaces

Upside downside inside and outside, hittin you from every angle
there's no doubt ...
Method Man.

This appendix deals with forms. In particular, we will show that a positive closed form gives rise to a unique positive operator. We will also characterize the domain of this operator.

We first discuss the operators which will be of interest in this appendix. These are self-adjoint operators whose spectrum is contained in the non-negative real numbers.

We start by recalling the relevant notions. We let H denote a complex Hilbert space. We call an operator A with dense domain $D(A) \subseteq H$ self-adjoint if $A = A^*$, where A^* denotes the adjoint of A. We denote the spectrum of A by $\sigma(A)$. If A is a self-adjoint operator, then $\sigma(A) \subseteq \mathbb{R}$, as was shown in Corollary A.12. We will now restrict our attention further to those operators whose spectrum is contained in the non-negative real numbers.

Lemma B.1. *Let A be a self-adjoint operator on H with domain $D(A)$. Then, the following statements are equivalent:*

(i) $\sigma(A) \subseteq [0, \infty)$.
(ii) *A is unitarily equivalent to multiplication by an almost everywhere positive function.*
(iii) $\langle f, Af\rangle \geq 0$ *for all* $f \in D(A)$.
(iv) *There exists a self-adjoint operator S with $A = S^2$.*

Proof. According to the spectral theorem, Theorem A.20, we can assume without loss of generality that A is the operator M_u of multiplication by a measurable function $u\colon X \longrightarrow \mathbb{R}$, where (X, μ) is a measure space without atoms of infinite mass and $D(M_u) = \{f \in L^2(X, \mu) \mid uf \in L^2(X, \mu)\}$. The spectrum of A is then the essential range of u by Lemma A.6. Now, the essential range is contained in $[0, \infty)$ if and only if $u \geq 0$ almost everywhere and this in turn holds if and only if $\int u|f|^2 d\mu = \langle f, M_u f\rangle \geq 0$ for all $f \in D(M_u)$. This shows the equivalence between (i), (ii) and (iii).

Now, if $u \geq 0$ almost everywhere, then $M_u = M_v^2$ with $v = \sqrt{u}$ and, thus, (ii) implies (iv). Finally, (iv) implies (iii) via

$$\langle f, Af\rangle = \langle f, S^2 f\rangle = \langle Sf, Sf\rangle = \|Sf\|^2 \geq 0.$$

This finishes the proof. □

We highlight the class of operators appearing in the previous statement by giving a definition.

Definition B.2 (Positive operator). We say that a self-adjoint operator A is *positive* if A satisfies one of the equivalent conditions of Lemma B.1. We write $A \geq 0$ in this case.

Remark. We caution that some communities refer to operators on function spaces as positive if they map positive functions to positive functions. In this book we refer to such operators as positivity preserving, see, for example, Section 3 or Appendix C.

When A is positive the function $\varphi\colon [0,\infty) \longrightarrow \mathbb{R}$ given by $\varphi(x) = \sqrt{x}$ is defined on the spectrum of A. We can thus use the functional calculus, see Definition A.21 and the subsequent remark, to define

$$A^{1/2} = \sqrt{A} = \varphi(A)$$

and call this operator the *square root* of A. We now collect some basic properties of the square root.

Lemma B.3 (Square root). *Let A be a positive operator on H. Then, the following statements hold:*

(a) *$\sqrt{A}$ is self-adjoint and positive.*

(b) *$f \in D(A)$ if and only if $f \in D(\sqrt{A})$ and $\sqrt{A}f \in D(\sqrt{A})$. In particular, $D(A) \subseteq D(\sqrt{A})$.*

(c) *$(\sqrt{A})^2 = A$ on $D(A)$.*

Proof. By Lemma B.1 we may assume that A is unitarily equivalent to multiplication by a function u which is positive almost everywhere on a measure space (X,μ) without atoms of infinite mass. It follows by the definition of the spectral calculus that $\sqrt{A}$ is then unitarily equivalent to multiplication by $\sqrt{u}$. As $\sqrt{u}$ is real-valued almost everywhere, the self-adjointness of $\sqrt{A}$ follows from Proposition A.7. Positivity of $\sqrt{A}$ then follows by Lemma B.1. This proves (a). Property (b) follows by a short argument involving the definition of the domain of a multiplication operator, see Example A.2. Property (c) is then obvious from the discussion above. □

We will study positive operators by means of forms. Roughly speaking, a form is like an inner product except that it may be degenerate and is not defined on the entire Hilbert space. We make this notion precise in the following definition.

Definition B.4 (Symmetric positive form). A *symmetric positive form Q* on H consists of a dense subspace $D(Q) \subseteq H$ called the *domain of Q* together with a map

$$Q\colon D(Q) \times D(Q) \longrightarrow \mathbb{C}$$

satisfying

- $Q(f,g) = \overline{Q(g,f)}$ (*"Symmetry"*)
- $Q(f,\alpha g + \beta h) = \alpha Q(f,g) + \beta Q(f,h)$ (*"Linearity"*)
- $Q(f,f) \geq 0$ (*"Positivity"*)

for all $f,g,h \in D(Q)$ and $\alpha,\beta \in \mathbb{C}$.

In this appendix we will deal exclusively with forms that are symmetric and positive. We refer to such forms simply as positive forms. We note that the theory developed below can easily be adapted to a slightly more general class of forms, namely those which are bounded below. We do not treat this case here, as we do not meet such forms in the context of graphs discussed in the bulk of the book.

For $f \in H$, we define $Q(f)$ as

$$Q(f) = \begin{cases} Q(f,f) & \text{if } f \in D(Q) \\ \infty & \text{otherwise.} \end{cases}$$

We note that we can recover the form Q from the values $Q(f)$ for $f \in H$ as the domain of Q is given by

$$D(Q) = \{f \in H \mid Q(f) < \infty\}$$

and $Q(f,g)$ can be obtained by using the polarization identity, i.e.,

$$Q(f,g) = \frac{1}{4}\sum_{k=0}^{3} i^k Q(g + i^k f)$$

for $f,g \in D(Q)$.

Every form Q induces an inner product on the subspace $D(Q)$ via

$$\langle f,g\rangle_Q = Q(f,g) + \langle f,g\rangle.$$

The associated norm is given by

$$\|f\|_Q = \langle f,f\rangle_Q^{1/2} = \left(Q(f) + \|f\|^2\right)^{1/2}.$$

In the next example we begin to establish the connection between forms and positive operators. In particular, we show how to define a form from a positive operator.

Example B.5 (Form associated to a positive operator). Let A be a positive operator on H. We define the form Q_A by letting $D(Q_A) = D(\sqrt{A})$ and

$$Q_A(f,g) = \langle \sqrt{A}f, \sqrt{A}g\rangle$$

for all $f,g \in D(\sqrt{A})$. We call Q_A the *form associated to A*.

In particular, if we let (X,μ) be a measure space, $u\colon X \longrightarrow [0,\infty)$ be measurable and M_u be the operator of multiplication by u, then Q_{M_u} has domain

$$D(Q_{M_u}) = D(M_{\sqrt{u}}) = \{f \in L^2(X,\mu) \mid \int u|f|^2 d\mu < \infty\}$$

and acts by

$$Q_{M_u}(f,g) = \int u\overline{f}g d\mu$$

for $f, g \in D(Q_{M_u})$. We note that the integral defining $Q_{M_u}(f,g)$ exists as $u|fg| \leq u|f|^2 + u|g|^2$.

We will show that the converse of the preceding example holds under some additional assumptions. For forms with suitable boundedness properties, this is not hard to see by using the Riesz representation theorem. This is the content of the next proposition.

Proposition B.6 (Bounded forms and operators). *Let Q be a positive form with $D(Q) = H$ such that there exists a constant $C \geq 0$ with*

$$Q(f,g) \leq C\|f\|\|g\|$$

for all $f, g \in H$. Then, there exists a unique positive operator A with $D(A) = H$, $\|A\| \leq C$ and

$$Q(f,g) = \langle f, Ag\rangle = \langle Af, g\rangle$$

for all $f, g \in H$.

Proof. For a fixed $f \in H$, we consider the map from H to $\mathbb{C}$ given by

$$g \mapsto Q(f,g).$$

This map is linear and bounded by the assumptions on Q. Hence, by the Riesz representation theorem, there exists a unique $f' \in H$ with

$$Q(f,g) = \langle f', g\rangle$$

for all $g \in H$. We define $A\colon H \longrightarrow H$ by

$$Af = f'.$$

It follows that A is linear and

$$Q(f,g) = \langle Af, g\rangle$$

for all $f, g \in H$. In particular, we infer

$$\|Af\| = \sup\{\langle Af, g\rangle \mid \|g\| \leq 1\} \leq C\|f\|$$

and, thus, $\|A\| \leq C$ follows. Moreover, by using the symmetry of Q, we have

$$\langle Af, g\rangle = Q(f,g) = \overline{Q(g,f)} = \overline{\langle Ag, f\rangle} = \langle f, Ag\rangle,$$

so that A is symmetric. As A is bounded, it follows that A is self-adjoint.

Finally, using the positivity of Q, we obtain

$$\langle f, Af\rangle = Q(f,f) \geq 0$$

and, thus, A is positive. The uniqueness of A is clear. □

Any form Q with $D(Q) = H$ which satisfies $Q(f,g) \le C\|f\|\|g\|$ for all $f, g \in H$ and some constant $C \ge 0$ is called *bounded*. Hence, we see from the proposition above that any positive bounded form gives rise to a unique positive bounded operator. Conversely, if A is a bounded positive operator, then $\sqrt{A}$ is bounded and, thus, Q_A as defined in Example B.5 is a bounded form. Hence, from the considerations above, we see that there is a one-to-one correspondence between bounded positive operators and bounded positive forms. We will extend this result to a larger class of forms in what follows.

We first show that we can weaken the boundedness assumption on the form to a completeness assumption and still obtain the existence of an operator. This is the content of the next lemma.

Lemma B.7 (Associated operator). *Let Q be a positive form on H. If $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is complete, then there exists a positive operator A with $D(Q) = D(\sqrt{A})$ and*

$$Q(f,g) = \langle \sqrt{A} f, \sqrt{A} g\rangle$$

for all $f, g \in D(Q)$, i.e., $Q = Q_A$ is the form associated to A.

Proof. By assumption, $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space, which we denote by H_Q. Consider

$$\langle\cdot,\cdot\rangle \colon H_Q \times H_Q \longrightarrow \mathbb{C},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on H. Then, as Q is positive,

$$|\langle f, g\rangle| \le \|f\|\|g\| \le \|f\|_Q \|g\|_Q,$$

so that $\langle\cdot,\cdot\rangle$ is a bounded form on H_Q. Hence, by Proposition B.6, there exists a unique positive operator T with $D(T) = H_Q$ and

$$\langle f, g\rangle = \langle f, Tg\rangle_Q = Q(f, Tg) + \langle f, Tg\rangle$$

for all $f, g \in H_Q$.

We will ultimately show that

$$A = T^{-1} - I$$

has the desired properties. Indeed, assuming the definition of A as $T^{-1} - I$ makes sense, letting $g' = Tg$ and noting that $g - g' = Ag'$ we see from the above that

$$\langle f, Ag'\rangle = Q(f, g')$$

for all $f, g \in H_Q$. Using $A = \sqrt{A}\sqrt{A} = (\sqrt{A})^*(\sqrt{A})$ then gives

$$Q(f, g') = \langle \sqrt{A} f, \sqrt{A} g'\rangle$$

for all $f, g \in H_Q$.

To turn this into a rigorous argument, we have to show that T is injective and that $T^{-1} - I$ can be seen as a positive operator on H. One obstacle to overcome is that T and A are only defined on H_Q and we have to extend them to subspaces of H.

After this sketch, we now proceed to give the proof. As

$$\langle f, Tf\rangle_Q = \langle f, f\rangle = \|f\|^2$$

for all $f \in H_Q$, the operator T is positive and bounded on H_Q with $\|T\| \leq 1$. By the spectral theorem, Theorem A.20, applied to T on H_Q, there exists a measure space (X, μ_Q) without atoms of infinite mass, a measurable function $u\colon X \longrightarrow [0,1]$ and a unitary map $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ such that

$$T = V^{-1} M_u V.$$

Here, $0 \leq u \leq 1$ follows from the fact that T is positive and bounded with $\|T\| \leq 1$. Furthermore, as $\langle f, Tf\rangle_Q = \|f\|^2$, the operator T is injective and thus $u > 0$ almost everywhere so that $0 < u \leq 1$ almost everywhere.

We now define $a\colon X \longrightarrow [0,\infty)$ by

$$a = \frac{1}{u} - 1.$$

For all $f, g \in H_Q$, from

$$\langle f, g\rangle = \langle f, Tg\rangle_Q = \int u(\overline{Vf})(Vg)d\mu_Q$$

we infer

$$\begin{aligned}
Q(f,g) &= \langle f, g\rangle_Q - \langle f, g\rangle \\
&= \int (\overline{Vf})(Vg)d\mu_Q - \int u(\overline{Vf})(Vg)d\mu_Q \\
&= \int (1-u)(\overline{Vf})(Vg)d\mu_Q \\
&= \int a(\overline{Vf})(Vg)u d\mu_Q \\
&= \int a(\overline{Vf})(Vg)d\mu,
\end{aligned}$$

where we define the measure $\mu = u\mu_Q$ and use that $1 - u = au$.

This is almost the desired formula for Q. It just remains to show that we can use $V\colon H_Q \longrightarrow L^2(X, \mu_Q)$ to define a unitary map

$$U\colon H \longrightarrow L^2(X, \mu)$$

which satisfies $Q(f,g) = \int a(\overline{Uf})(Ug)d\mu$ for all $f, g \in D(Q)$ and

$$UD(Q) = \{f \in L^2(X,\mu) \mid \int a|f|^2 d\mu < \infty\}.$$

If so, then we can define $M_{\sqrt{a}}$ on $D(M_{\sqrt{a}}) = UD(Q)$ and

$$\begin{aligned}
Q(f,g) &= \int a(\overline{Uf})(Ug)d\mu \\
&= \langle M_{\sqrt{a}}Uf, M_{\sqrt{a}}Ug\rangle_{L^2(X,\mu)} \\
&= \langle U^{-1}M_{\sqrt{a}}Uf, U^{-1}M_{\sqrt{a}}Ug\rangle
\end{aligned}$$

for all $f, g \in D(Q)$. We then let

$$\sqrt{A} = U^{-1}M_{\sqrt{a}}U \quad \text{with} \quad D(\sqrt{A}) = U^{-1}D(M_{\sqrt{a}}) = D(Q),$$

which will complete the proof.

To this end, we note that V is isometric as a map from $D(Q) \subseteq H$ to $L^2(X,\mu)$ as

$$\begin{aligned}
\langle f, g\rangle = \langle f, Tg\rangle_Q &= \int (\overline{Vf})(Vg)ud\mu_Q = \int (\overline{Vf})(Vg)d\mu \\
&= \langle Vf, Vg\rangle_{L^2(X,\mu)}.
\end{aligned}$$

Furthermore, as $L^2(X,\mu_Q)$ is dense in $L^2(X,\mu)$, the image of V is dense. As $D(Q)$ is dense in H, we can extend V to an isometric operator $U\colon H \longrightarrow L^2(X,\mu)$ which is onto. As U is also one-to-one, U is unitary.

Moreover, the images of H_Q under U and V are equal. This image, by definition, is $L^2(X,\mu_Q)$ and clearly agrees with

$$\{f \in L^2(X,\mu) \mid \int a|f|^2 d\mu < \infty\}.$$

Hence, we obtain the asserted formula for $UD(Q)$, which completes the proof. □

We now give the operator constructed above a name.

Definition B.8 (Associated operator). Let Q be a positive form on H such that $(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space. The positive operator A such that

$$D(\sqrt{A}) = D(Q) \quad \text{and} \quad Q(f,g) = \langle\sqrt{A}f, \sqrt{A}g\rangle$$

is called the operator *associated* to Q.

From the preceding we see that every form which induces a Hilbert space structure on its domain gives rise to an associated operator. We will now show that all such forms come from positive operators. Along the way, we also characterize the completeness assumption in terms of lower semi-continuity.

Theorem B.9 (Characterization of closed forms). *Let Q be a positive form on H. Then, the following statements are equivalent:*

(i) *There exists a positive operator A with $Q = Q_A$, i.e., $D(Q) = D(\sqrt{A})$ and*

$$Q(f,g) = \langle\sqrt{A}f, \sqrt{A}g\rangle$$

for all $f, g \in D(Q)$.

(ii) *Q is lower semi-continuous, i.e.,*

$$Q(f) \leq \liminf_{n\to\infty} Q(f_n)$$

whenever $f_n \to f$ as $n \to \infty$ in H.

(iii) *$(D(Q), \langle\cdot,\cdot\rangle_Q)$ is a Hilbert space.*

Proof of Theorem B.9. (i) $\Longrightarrow$ (ii): We will show that Q is the supremum of continuous functions Q_n, from which (ii) follows easily.

Since $A \geq 0$, the operator $(A+n)^{-1}$ exists and is bounded on H for all $n \in \mathbb{N}$. For $f \in H$, we denote by μ_f the spectral measure associated to f and note by Lemma A.18 (d) that $\operatorname{supp}(\mu_f) \subseteq [0,\infty)$. We let $\varphi_n : [0,\infty) \longrightarrow \mathbb{R}$ be given by $\varphi_n(x) = nx/(x+n)$ and note that φ_n is bounded for every $n \in \mathbb{N}$. Thus, by the bounded functional calculus, Corollary A.25, we may define a continuous map $Q_n : H \longrightarrow [0,\infty)$ via

$$Q_n(f) = \int_0^\infty \frac{nx}{x+n} d\mu_f(x) = \langle f, nA(A+n)^{-1} f\rangle.$$

We now claim that

$$Q_n(f) \nearrow \int_0^\infty x d\mu_f = Q(f)$$

as $n \to \infty$ for every $f \in H$. Here, the convergence follows easily by the monotone convergence theorem as $\varphi_n(x) \nearrow x$ as $n \to \infty$. The equality follows from Proposition A.24 (a), which gives $f \in D(\sqrt{A}) = D(Q)$ if and only if $\int x d\mu_f < \infty$, in which case

$$Q(f) = \|\sqrt{A} f\|^2 = \int_0^\infty x d\mu_f.$$

This completes the proof.

(ii) $\Longrightarrow$ (iii): Let (f_n) be a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$. Then, (f_n) is a Cauchy sequence in H. In particular, there exists an $f \in H$ with $f_n \to f$ with respect to $\|\cdot\|$.

Let $\varepsilon > 0$. As (f_n) is a Cauchy sequence in $(D(Q), \langle\cdot,\cdot\rangle_Q)$, there exists an $N \in \mathbb{N}$ with

$$\|f_n - f_m\|_Q < \varepsilon$$

for all $n, m \geq N$. Consider now $m \geq N$. Then, using (ii), we get

$$Q(f - f_m) \leq \liminf_{n\to\infty} Q(f_n - f_m) \leq \varepsilon.$$

This implies $f \in D(Q)$ and $Q(f - f_m) \leq \varepsilon$ for all $m \geq N$. Therefore, $f_n \to f$ with respect to $\|\cdot\|_Q$.

(iii) $\Longrightarrow$ (i): This is shown in Lemma B.7. $\square$

We highlight the class of forms appearing in the previous statement by giving a definition.

Definition B.10 (Closed form). We say that a positive form Q on H is *closed* if Q satisfies one of the equivalent conditions of Theorem B.9.

The preceding considerations show that all positive closed forms come from positive operators. We now discuss how to further describe the domain of the operator associated to such a form.

Theorem B.11 (Domain and action of the operator). *Let Q be a positive closed form on H. Then, the associated operator A has domain*

$$D(A) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \textit{there exists a } g \in H \textit{ with } Q(h,f) = \langle h, g\rangle \\ \textit{for all } h \in D(Q) \end{array} \right\}$$

and acts on $D(A)$ via

$$Af = g.$$

Proof. This follows from the definitions of the associated operator and the adjoint of the square root, the fact that $\sqrt{A}$ is self-adjoint, so that $D(\sqrt{A}) = D(\sqrt{A}^*)$, and the fact that $f \in D(A)$ if and only if $\sqrt{A}f \in D(\sqrt{A}^*) = D(\sqrt{A})$, see Lemma B.3.

More specifically, for $f \in D(\sqrt{A}) = D(Q)$ we have $\sqrt{A}f \in D(\sqrt{A}^*)$ if and only if there exists an element $g \in H$ such that

$$\langle h, \sqrt{A}\sqrt{A}f\rangle = \langle h, g\rangle$$

for all $h \in D(\sqrt{A}) = D(Q)$, which is equivalent to

$$\langle \sqrt{A}h, \sqrt{A}f\rangle = Q(h,f) = \langle h, g\rangle$$

for all $h \in D(Q)$. This completes the proof. □

The following consequence of the previous theorem is a convenient way to think about the operator associated to a closed form. As a further fact, we also show that the operator domain is dense in the form domain with respect to the inner product arising from the form.

Corollary B.12. *Let Q be a positive closed form on H. Then, there exists a unique self-adjoint operator L with*

$$Q(f,g) = \langle f, Lg\rangle$$

for all $f \in D(Q)$ and $g \in D(L)$. The operator L is positive and the form Q satisfies

$$D(Q) = D(\sqrt{L}) \quad \textit{and} \quad Q(f,g) = \langle \sqrt{L}f, \sqrt{L}g\rangle$$

for all $f, g \in D(Q)$. Furthermore, $D(L) \subseteq D(Q)$ is dense with respect to $\|\cdot\|_Q$.

Proof. We first show uniqueness. Let L be such an operator. Then, L is a restriction of A, the operator associated to Q, by Theorem B.11. As both L and A are self-adjoint, they must agree.

The existence of such an operator as well as the connection to the form follow from Theorem B.9 and Lemma B.3. Finally, to show that $D(L)$ is dense in $D(Q)$ with respect to $\|\cdot\|_Q$ we suppose not. Then there exists an $f \in D(Q)$, $f \neq 0$, which is in the orthogonal complement of $D(L)$ with respect to $\langle \cdot, \cdot \rangle_Q$, that is,

$$\langle f, g \rangle_Q = \langle f, g \rangle + Q(f, g) = 0$$

for all $g \in D(L)$. By the connection between the operator and form we then obtain

$$\langle f, Lg \rangle = -\langle f, g \rangle$$

for all $g \in D(L)$ so that $f \in D(L^*)$. As L is self-adjoint, it follows that $f \in D(L)$ so that $f = 0$. This contradiction yields the claim. $\square$

We now briefly discuss several concepts of independent interest in the context of forms. These concepts provide a self-adjoint extension to a symmetric operator that is bounded below.

Given forms Q and Q' with domains $D(Q)$ and $D(Q')$ in H, we call Q' an *extension* of Q if

$$D(Q) \subseteq D(Q') \quad \text{and} \quad Q'(f, g) = Q(f, g)$$

for all $f, g \in D(Q)$. We call a form Q *closable* if there exists a closed extension of Q. Equivalently, Q is closable if and only if for every sequence (f_n) in $D(Q)$ such that $f_n \to 0$ as $n \to \infty$ and $Q(f_n - f_m) \to 0$ as $n, m \to \infty$, it follows that $Q(f_n) \to 0$ as $n \to \infty$.

If Q is closable, then we call the smallest closed extension of Q the *closure* of Q and denote the closure by $\overline{Q}$. The form $\overline{Q}$ can be constructed by letting $D(\overline{Q})$ be the closure of $D(Q)$ in H with respect to the form norm $\| \cdot \|_Q$, that is,

$$D(\overline{Q}) = \overline{D(Q)}^{\|\cdot\|_Q}$$

and letting

$$\overline{Q}(f) = \lim_{n \to \infty} Q(f_n)$$

if $f_n \in D(Q)$ satisfies $f_n \to f$ in H and $Q(f_n - f_m) \to 0$ as $n, m \to \infty$. The closability assumption on Q is then required to show that this procedure is well-defined and that we can embed $D(\overline{Q})$ into H.

After these preparation we now briefly discuss how we can use forms to obtain a self-adjoint extension of a symmetric operator.

Example B.13 (Friedrichs extension). We recall that a densely defined operator A_0 on H is called symmetric if A_0^* is an extension of A_0, that is, $D(A_0) \subseteq D(A_0^*)$ and $A_0^* f = A_0 f$ for all $f \in D(A_0)$. If A_0 is a symmetric operator and $\langle A_0 f, f \rangle \geq 0$ for all $f \in D(A_0)$, then we define a positive form Q_0 by letting $D(Q_0) = D(A_0)$ and

$$Q_0(f, g) = \langle A_0 f, g \rangle$$

for all $f, g \in D(A_0)$. It follows by a short argument that the form Q_0 is closable. Let Q be the closure of Q_0. Then, by Theorem B.9, there exists a positive self-adjoint operator A which is associated to Q.

We now show that A is an extension of A_0. To do so, let $f \in D(A_0) = D(Q_0) \subseteq D(Q)$. By the definition of the closure of Q_0 we can find for any $h \in D(Q)$ a sequence (h_n) in $D(Q_0) = D(A_0)$ with $h_n \to h$ in H and $Q(h - h_n) \to 0$ as $n \to \infty$. Thus, we obtain for all $h \in D(Q)$

$$Q(h, f) = \lim_{n\to\infty} Q(h_n, f) = \lim_{n\to\infty} Q_0(h_n, f) = \lim_{n\to\infty} \langle h_n, A_0 f\rangle = \langle h, A_0 f\rangle.$$

By Theorem B.11, we infer $f \in D(A)$ and $Af = A_0 f$. As $f \in D(A_0)$ was arbitrary this implies that A is an extension of A_0.

We refer to the extension A constructed above as the *Friedrichs extension* of A_0. This construction will be used in Section 3 of Appendix E.

We finish this appendix with two ways to approximate the value on the diagonal of a closed form. More specifically, given a closed form we define two quadratic forms using the resolvent and semigroup arising from the associated operator and show that in the limit they agree with the diagonal of the form.

We call a map $q\colon H \longrightarrow (-\infty, \infty]$ a *quadratic form* if $q(zf) = |z|^2 q(f)$ for all $z \in \mathbb{C}$ and $f \in H$ and if

$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$

for all $f, g \in H$. The domain of q is given by

$$D(q) = \{f \in H \mid q(f) < \infty\}.$$

We can then extend q to a sesquilinear map on $D(q) \times D(q)$ via polarization, that is, we let

$$q(f,g) = \frac{1}{4}\sum_{k=0}^{3} i^k q(g + i^k f)$$

for $f, g \in D(q)$.

Now, given a positive closed form Q on H, as the associated operator L is positive we can use the functional calculus to define both the semigroup e^{-tL} for $t \geq 0$ and the resolvent $(L+\alpha)^{-1}$ for $\alpha > 0$. We then define the quadratic forms $Q^\alpha\colon H \longrightarrow \mathbb{R}$ associated to the resolvent by

$$Q^\alpha(f) = \alpha\langle (I - \alpha(L+\alpha)^{-1})f, f\rangle$$

for $\alpha > 0$ and the quadratic forms $Q_t\colon H \longrightarrow \mathbb{R}$ associated to the semigroup by

$$Q_t(f) = \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for $t \geq 0$. As both the resolvent and semigroup are bounded self-adjoint operators, see Propositions A.32 and A.34, we have $D(Q^\alpha) = H = D(Q_t)$ as well as

$$Q^\alpha(f,g) = \alpha\langle (I - \alpha(L+\alpha)^{-1})f, g\rangle$$

and

$$Q_t(f,g) = \frac{1}{t}\langle (I - e^{-tL})f, g\rangle$$

for all $f, g \in H$, as follows from polarization.

We now show that the value of a closed form on the diagonal is the limit of the value of the quadratic forms associated to the resolvent and the semigroup.

COROLLARY B.14. *Let Q be a positive closed form on H with associated operator L. Then, for all $f \in H$,*

$$Q(f) = \lim_{\alpha\to\infty} Q^\alpha(f) = \lim_{t\to 0^+} Q_t(f),$$

where the value is finite if and only if $f \in D(Q)$.

PROOF. The statement follows directly from the connection between the operator and form, properties of the functional calculus given in Proposition A.24, and the monotone convergence theorem as

$$\alpha\left(1 - \alpha(x+\alpha)^{-1}\right) \nearrow x \qquad \text{and} \qquad \frac{1}{t}(1 - e^{-tx}) \nearrow x$$

as $\alpha \to \infty$ and $t \to 0^+$, respectively, for all $x \geq 0$. $\square$

APPENDIX C

Dirichlet Forms and Beurling–Deny Criteria

Looks like the work of a master, evidence indicates that's it's stature.
Masta Killa.

In this appendix we discuss some general theory of Dirichlet forms, including the Beurling–Deny criteria. While this material provides useful background, it is not necessary to follow the discussion of regular Dirichlet forms on discrete spaces which forms a substantial part of the book. However, we use this material when dealing with non-regular Dirichlet forms.

The material presented here can be found in one form or another in the monographs [**BH91, RS78**] or in the special case of locally compact spaces in [**Dav89, FŌT11**].

We first define the concept of a Dirichlet form. Let (X,μ) be a σ-finite measure space. Let H be the Hilbert space of square integrable real-valued functions on X, i.e., $H = L^2(X,\mu)$. We let Q be a positive closed form with domain $D(Q) \subseteq H$, as discussed in Appendix B. In particular, Q is a positive symmetric form and $D(Q)$ is complete with respect to the form norm $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$ for all $f \in D(Q)$, where $\|\cdot\|$ denotes the norm arising from the inner product on H. We recall that Q is extended on the diagonal to all of H via $Q(f) = \infty$ for $f \in H \setminus D(Q)$.

We call a map $C\colon \mathbb{R} \longrightarrow \mathbb{R}$ a *normal contraction* if $C(0) = 0$ and $|C(s) - C(t)| \leq |s - t|$ for all $s,t \in \mathbb{R}$. We now state the additional requirement for a closed form to be a Dirichlet form.

Definition C.1 (Dirichlet form). A positive closed form Q with domain $D(Q)$ in $H = L^2(X,\mu)$ is called a *Dirichlet form* if $C \circ f \in D(Q)$ and

$$Q(C \circ f) \leq Q(f)$$

for all $f \in D(Q)$ and all normal contractions C.

We say that the form is compatible with all normal contractions in this case. This condition has a number of surprising consequences which we will discuss. We note that while the definition requires compatibility with all normal contractions, it actually suffices to check the condition for the normal contraction given by

$$C(s) = 0 \vee s \wedge 1,$$

that is, cutting below by 0 and above by 1. This follows directly from the proof of Theorem C.4 given below.

We recall that whenever Q is a positive closed form, the associated operator L is positive, that is, L is self-adjoint and $\sigma(L) \subseteq [0,\infty)$, see Theorem B.9 for more details and Lemma B.7 for the construction of L. As $\sigma(L) \subseteq [0,\infty)$, it follows that we can use the functional calculus to define both the resolvent $(L+\alpha)^{-1}$ for $\alpha > 0$ and the semigroup e^{-tL} for $t \geq 0$, which are bounded operators on H, see Propositions A.32 and A.34 for basic properties and Theorem A.35 for the connection between the two. In particular, we recall that the semigroup is a strongly continuous contraction semigroup and the resolvent is a strongly continuous contraction resolvent.

We now give some consequences for both semigroups and resolvents when the associated operator comes from a Dirichlet form. We say that an operator A with domain $D(A) \subseteq L^2(X,\mu)$ is *positivity preserving* if $Af \geq 0$ whenever $f \in D(A)$ satisfies $f \geq 0$. We say that A is *contracting* if $Af \leq 1$ whenever $f \in D(A)$ satisfies $f \leq 1$. When A is both positivity preserving and contracting, i.e., $0 \leq Af \leq 1$ for all $f \in D(A)$ with $0 \leq f \leq 1$, we say that A is *Markov*.

We start with a lemma which will be applied to the semigroup and resolvent in what follows.

Lemma C.2. *Let A be a bounded self-adjoint positivity preserving operator on $H = L^2(X,\mu)$. Then, the quadratic form Q_{I-A} defined by*

$$Q_{I-A}(f,g) = \langle (I-A)f, g\rangle$$

satisfies

$$Q_{I-A}(|f|) \leq Q_{I-A}(f)$$

for all $f \in L^2(X,\mu)$. If, furthermore, A is Markov, then Q_{I-A} is a Dirichlet form and for $f, g \in L^2(X,\mu) \cap L^\infty(X,\mu)$ we have

$$Q_{I-A}(fg) \leq 2\|g\|_\infty^2 Q_{I-A}(f) + 2\|f\|_\infty^2 Q_{I-A}(g).$$

Proof. We show the first statement for simple functions. The statement for functions in $L^2(X,\mu)$ then follows by approximation. Let $f = \sum_{k=1}^n f_k 1_{U_k}$ for $f_1,\ldots,f_n \in \mathbb{R}$ and $U_1,\ldots,U_n \subseteq X$ which are measurable disjoint sets of finite measure. Then, by a direct calculation we find the following explicit formula for $Q_{I-A}(f)$

$$Q_{I-A}(f) = \frac{1}{2}\sum_{k,l=1}^n b_{k,l}(f_k - f_l)^2 + \sum_{k=1}^n c_k f_k^2,$$

where $b_{k,l} = \langle 1_{U_k}, A1_{U_l}\rangle$ and $c_k = \mu(U_k) - \sum_{l=1}^n b_{k,l}$.

If A is positivity preserving, then $b_{k,l} \geq 0$ and the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(|f|) \leq Q_{I-A}(f)$.

If A is Markov, then for $U = \bigcup_{l=1}^n U_l$ we have $0 \leq A1_U \leq 1$ and thus

$$\sum_{l=1}^n b_{k,l} = \langle 1_{U_k}, A1_U \rangle \leq \mu(U_k).$$

Therefore, $c_k \geq 0$ by definition. Then, the explicit formula for $Q_{I-A}(f)$ above easily gives $Q_{I-A}(C \circ f) \leq Q_{I-A}(f)$ for any normal contraction C. As Q_{I-A} is clearly symmetric positive and closed, this shows that Q_{I-A} is a Dirichlet form.

For the last statement, we let $g = \sum_{k=1}^n g_k 1_{U_k}$, where we alter the sets $U_1, \ldots, U_n$ appearing in the definition of f if necessary. Then, using Young's inequality we get

$$(f_k g_k - f_l g_l)^2 = (g_k(f_k - f_l)) + f_l(g_k - g_l))^2 \leq 2g_k^2(f_k - f_l)^2 + 2f_l^2(g_k - g_l)^2,$$

which, along with the estimate

$$\sum_{k=1}^n c_k f_k^2 g_k^2 \leq \|f\|_\infty^2 \sum_{k=1}^n c_k g_k^2,$$

yields

$$\begin{aligned} Q_{I-A}(fg) &= \frac{1}{2} \sum_{k,l=1}^n b_{k,l}(f_k g_k - f_l g_l)^2 + \sum_{k=1}^n c_k f_k g_k \\ &\leq 2\|g\|_\infty^2 Q(f) + 2\|f\|_\infty^2 Q(g). \end{aligned}$$

This concludes the proof. □

We now state and prove the Beurling–Deny criteria for positive closed forms. The first criterion shows that a form being compatible with the absolute value is equivalent to the fact that the heat semigroup and the resolvent are positivity preserving.

Theorem C.3 (First Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X, \mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) *$Q(|f|) \leq Q(f)$ for all $f \in H$.*
(ii) *$\alpha(L + \alpha)^{-1}$ is positivity preserving for every $\alpha > 0$.*
(iii) *e^{-tL} is positivity preserving for every $t \geq 0$.*

Proof. (i) $\Longrightarrow$ (ii): We note that the norm $\|\cdot\|_\alpha$ arising from the scalar product

$$\langle f, g \rangle_\alpha = Q(f, g) + \alpha \langle f, g \rangle$$

for $\alpha > 0$ and $f, g \in D(Q)$ is equivalent to $\|\cdot\|_Q$. We denote the Hilbert space $D(Q)$ equipped with $\langle \cdot, \cdot \rangle_\alpha$ by H_α.

Let

$$g = (L + \alpha)^{-1} f$$

for $f \in H$ with $f \geq 0$. In order to prove $|g| = g$, we will show the following two inequalities:

$$\|g\|_\alpha^2 \leq \langle |g|, g\rangle_\alpha$$

and

$$\||g|\|_\alpha \leq \|g\|_\alpha.$$

Combining these two inequalities with the Cauchy–Schwarz inequality and the assumption on Q yields

$$\|g\|_\alpha^2 \leq \langle |g|, g\rangle_\alpha \leq \||g|\|_\alpha \|g\|_\alpha \leq \|g\|_\alpha^2,$$

so that we get equalities in the above and thus $|g|$ is a multiple of g. Then, $\|g\|_\alpha^2 \leq \langle |g|, g\rangle_\alpha$ implies that $|g| = g$ and thus $g \geq 0$.

We are left to show the two inequalities above. To this end consider the injection $J\colon H_\alpha \longrightarrow H$, which is bounded, and note that

$$J^* f = (L + \alpha)^{-1} f$$

for $f \in H$. Since $g = J^* f$ and $(L + \alpha)^{-1} H \subseteq D(Q)$, we conclude $g \in H_\alpha$ and by (i) we have $|g| \in H_\alpha$. Therefore, using $f \geq 0$, we have

$$\begin{aligned}\|g\|_\alpha^2 &= |\langle g, J^* f\rangle_\alpha| = |\langle Jg, f\rangle| = |\langle g, f\rangle| \\ &\leq \langle |g|, f\rangle = \langle J|g|, f\rangle = \langle |g|, J^* f\rangle_\alpha = \langle |g|, g\rangle_\alpha.\end{aligned}$$

This shows the first inequality. For the second inequality, we use the assumption on Q to obtain

$$\||g|\|_\alpha^2 = Q(|g|) + \alpha\||g|\|^2 \leq Q(g) + \alpha\|g\|^2 = \|g\|_\alpha^2.$$

This concludes the proof of (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem A.35 (b).

(iii) $\Longrightarrow$ (i): By Lemma C.2 we have

$$\frac{1}{t}\langle (I - e^{-tL})|f|, |f|\rangle \leq \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

for all $t \geq 0$ and $f \in H$. Letting $Q_t(f) = \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$, Corollary B.14 gives $\lim_{t\to 0^+} Q_t(f) = Q(f)$ for all $f \in H$. Thus, we conclude

$$Q(|f|) \leq Q(f).$$

This finishes the proof. □

Remark. One can check that (i) in Theorem C.3 is equivalent to:

(i.a) $Q(f_+) \leq Q(f)$ for all $f \in H$,

where $f_+ = f \vee 0$ denotes the positive part of f.

Indeed, as $f_+ = (f + |f|)/2$, it is clear that (i) implies (i.a). On the other hand, (i.a) implies $Q(f_-) \leq Q(f)$ and, by considering $f_s = f_+ - sf_-$ for $s > 0$, so that $(f_s)_+ = f_+$ and using bilinearity, $Q(f_+, f_-) \leq 0$, where $f_- = -f \vee 0$ is the negative part of f. Now, using the bilinearity of the form once more implies (i).

The second Beurling–Deny criterion deals with Dirichlet forms. In particular, being a Dirichlet form turns out to be equivalent to the Markov property for both the heat semigroup and the resolvent.

THEOREM C.4 (Second Beurling–Deny criterion). *Let Q be a positive closed form on $H = L^2(X, \mu)$ and let L be the associated positive operator. Then, the following statements are equivalent:*

(i) *Q is a Dirichlet form.*
(ii) *$\alpha(L + \alpha)^{-1}$ is Markov for every $\alpha > 0$.*
(iii) *e^{-tL} is Markov for every $t \geq 0$.*

PROOF. (i) $\Longrightarrow$ (ii): As in the proof of Theorem C.3, we write $\|f\|_\alpha^2 = Q(f) + \alpha\|f\|^2$ for $\alpha > 0$ and $f \in D(Q)$ and note that $\|\cdot\|_\alpha$ is equivalent as a norm on $D(Q)$ to $\|\cdot\|_Q$. Let $f \in H$ satisfy $0 \leq f \leq 1$ and let $\alpha > 0$. We let

$$g = \alpha(L + \alpha)^{-1} f \qquad \text{and} \qquad h = 0 \vee g \wedge 1$$

and show that $g = h$.

We first note that $\langle g, k\rangle_\alpha = \alpha\langle f, k\rangle$ for all $k \in D(Q)$. We now use the definition of $\|\cdot\|_\alpha$, the facts that $\langle g, h\rangle_\alpha = \alpha\langle f, h\rangle$ and $\|g\|_\alpha^2 = \alpha\langle f, g\rangle$, basic algebraic manipulations, $0 \leq f \leq 1$ and that Q is a Dirichlet form to obtain

$$\begin{aligned}
\|g - h\|_\alpha^2 &= \|g\|_\alpha^2 - 2\langle g, h\rangle_\alpha + \|h\|_\alpha^2 \\
&= \|g\|_\alpha^2 - 2\alpha\langle f, h\rangle + \alpha\|h\|^2 + Q(h) \\
&= \|g\|_\alpha^2 - \alpha\|f\|^2 + \alpha\|f - h\|^2 + Q(h) \\
&\leq \|g\|_\alpha^2 - \alpha\|f\|^2 + \alpha\|f - g\|^2 + Q(g) \\
&= 2\|g\|_\alpha^2 - \alpha\left(\|f\|^2 - \|f - g\|^2 + \|g\|^2\right) \\
&= 2\|g\|_\alpha^2 - 2\alpha\langle f, g\rangle = 0.
\end{aligned}$$

Since $\|\cdot\|_\alpha$ is a norm, we conclude $g = h$ and, therefore, we have shown (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): This follows directly from Theorem A.35 (b).

(iii) $\Longrightarrow$ (i): As e^{-tL} is Markov for every $t \geq 0$, the form

$$Q_t(f) = \frac{1}{t}\langle (I - e^{-tL})f, f\rangle$$

is a Dirichlet form by Lemma C.2. Since $Q(f) = \lim_{t\to 0^+} Q_t(f)$ by Corollary B.14, the statement follows. □

REMARK. From the proof of Theorem C.4 we see that (i) is equivalent to

(i.a) $Q(0 \vee f \wedge 1) \leq Q(f)$ for all $f \in L^2(X, \mu)$.

Furthermore, (i) clearly implies

(i.b) $Q(f \wedge 1) \leq Q(f)$ for all $f \in L^2(X, \mu)$.

Using $t \vee (-\varepsilon) = -\varepsilon(-\varepsilon^{-1}t \wedge 1)$ for $t \in \mathbb{R}$ and $\varepsilon > 0$, one easily sees that (i.b) implies $Q(0 \vee f) \leq Q(f)$ for all $f \in L^2(X, \mu)$. Therefore, (i.b) is equivalent to both (i.a) and to (i).

REMARK. By monotone convergence we see that (iii) in Theorem C.4 is equivalent to

(iii.a) $0 \leq e^{-tL} f \leq 1$ for all $f \in L^\infty(X, \mu)$ with $0 \leq f \leq 1$.

By duality and the Riesz–Thorin interpolation theorem, see Appendix 4, one sees that (iii.a) is equivalent to

(iii.b) $0 \leq e^{-tL} f \leq 1$ for all $f \in L^p(X, \mu)$ with $0 \leq f \leq 1$ and $1 \leq p \leq \infty$.

We end this section by discussing approximating forms for a Dirichlet form. Recall that for a positive closed form Q with associated operator L the quadratic forms Q^α for $\alpha > 0$ and Q_t for $t \geq 0$ on $L^2(X, \mu)$ are defined by

$$Q^\alpha(f, g) = \alpha \langle (I - \alpha(L + \alpha)^{-1}) f, g \rangle$$

and

$$Q_t(f, g) = \frac{1}{t} \langle (I - e^{-tL}) f, g \rangle.$$

By the theory above we see that these forms are Dirichlet forms whenever Q is a Dirichlet form.

COROLLARY C.5 (Approximating forms are Dirichlet). *Let Q be a Dirichlet form on $H = L^2(X, \mu)$. Then, Q^α and Q_t are Dirichlet forms.*

PROOF. By the second Beurling–Deny criterion, Theorem C.4, the resolvent and the semigroup of a Dirichlet form are Markov. Hence, the statement follows from Lemma C.2. □

By using the approximating forms, we now derive another consequence of Lemma C.2. More specifically, we show that the bounded functions in the form domain form an algebra.

COROLLARY C.6 (Bounded functions form an algebra). *Let Q be a Dirichlet form on $H = L^2(X, \mu)$. Let $D(Q)$ denote the domain of Q. Then, $D(Q) \cap L^\infty(X, \mu)$ is an algebra.*

PROOF. By Corollary C.5, the approximating forms are Dirichlet forms. The conclusion then follows from the definition of either form, the last assertion of Lemma C.2 and the fact that the value of either approximating form converges to the value of the form as shown in Corollary B.14. □

APPENDIX D

Semigroups, Resolvents and their Generators

Punks in the back, come on and attract to what ...
GZA.

In this appendix we discuss general background from the theory of semigroups and resolvents on Banach spaces. In particular, we will show that every strongly continuous contraction semigroup gives rise to both an operator and a strongly continuous contraction resolvent. This operator is called the generator of both the semigroup and the resolvent. In the case of Hilbert spaces, we discuss how the spectral theorem allows us to reverse these constructions if the operator is positive. This gives a one-to-one correspondence between semigroups, resolvents and positive operators on Hilbert spaces.

For a general introduction to Banach spaces, see [**RS80**]. For the spectral theorem and positive operators, see Appendices A and B.

We let E denote a Banach space. An *operator* on E consists of a subspace $D(A) \subseteq E$ called the *domain* of the operator and a linear map $A\colon D(A) \longrightarrow E$. We say that an operator A is *bounded* if there exists a constant C such that $\|Af\| \leq C\|f\|$ for all $f \in D(A)$. If A is a bounded operator on E, we let $\|A\|$ denote the norm of A, which is the smallest such constant C. Furthermore, we assume that $D(A) = E$ whenever A is bounded in what follows. We let $B(E)$ denote the set of all bounded operators on E and note that $B(E)$ is a Banach space with respect to the operator norm.

We now define the first central notion of this appendix, namely, that of a semigroup on a Banach space.

Definition D.1 (Semigroup). Let E be a Banach space and let $B(E)$ denote the Banach space of bounded operators on E.

(a) We call a map $S\colon [0,\infty) \longrightarrow B(E)$ a *semigroup* if S satisfies $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$.
(b) We call a semigroup S *strongly continuous* if S additionally satisfies $S(t)f \to f$ as $t \to 0^+$ for all $f \in E$.
(c) We call a semigroup S with $\|S(t)\| \leq 1$ for every $t \geq 0$ a *contraction semigroup*.

We first present a central example which should be kept in mind.

Example D.2 (Bounded operators and semigroups). Let A be a bounded operator on E. Then, for any $t \geq 0$, the series $\sum_{n=0}^{\infty}(-tA)^n/n!$

converges absolutely and the mapping $S_A\colon [0,\infty) \longrightarrow B(E)$ given by

$$S_A(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A^n$$

defines a strongly continuous semigroup.

We are mainly interested in strongly continuous contraction semigroups as these give rise to the operators of interest. However, we will first establish various properties of general strongly continuous semigroups and observe that by rescaling we can obtain a strongly continuous contraction semigroup.

PROPOSITION D.3. *Let S be a strongly continuous semigroup on E. Then, the following statements hold:*

(a) $S(0) = I$.
(b) *There exist constants $M \geq 0$ and $\beta > 0$ with $\|S(t)\| \leq Me^{\beta t}$ for all $t \geq 0$.*
(c) *For every $f \in E$, the map $[0,\infty) \longrightarrow E$ given by $t \mapsto S(t)f$ is continuous.*

PROOF. (a) For any $f \in E$, we infer by the strong continuity of the semigroup that

$$S(0)f = \lim_{t\to 0^+} S(t)S(0)f = \lim_{t\to 0^+} S(t)f = f.$$

This shows (a).

(b) By the uniform boundedness principle we claim that there exists a constant $\delta > 0$ with

$$M = \sup\{\|S(t)\| \mid 0 \leq t \leq \delta\} < \infty.$$

Indeed, assuming that this is not the case, we find a sequence $t_n \to 0^+$ with $\|S(t_n)\| \to \infty$ as $n \to \infty$. However, by strong continuity we have $S(t_n)f \to f$ for all $f \in E$ as $n \to \infty$ and the uniform boundedness principle implies $\sup_n \|S(t_n)\| < \infty$, which is a contradiction.

Now, by the semigroup property and induction, we then easily infer $\|S(n\delta)\| \leq M^n$ for all $n \in \mathbb{N}$. The desired statement then follows by taking any $\beta > \log M/\delta$ and applying the semigroup property.

(c) Continuity at $t = 0$ is clear by strong continuity and part (a). So, consider now $t > 0$. Then, continuity from the right follows easily from the semigroup property and strong continuity as

$$S(s+t)f = S(s)S(t)f \to S(t)f$$

as $s \to 0^+$. To show continuity from the left we calculate for $0 < s < t$

$$(S(t) - S(t-s))f = S(t-s)(S(s)f - f) \to 0$$

as $s \to 0^+$, where we use (b) to get the uniform boundedness of $\|S(t-s)\|$ and strong continuity to conclude that $S(s)f - f \to 0$ as $s \to 0^+$. $\square$

Remark. Part (b) of the previous proposition shows that there is not much of a difference between a strongly continuous semigroup and a strongly continuous contraction semigroup. More specifically, whenever $\alpha > \beta$, then $\widetilde{S}(t) = e^{-t\alpha} S(t)$ is a semigroup with essentially the same properties as S and $\|\widetilde{S}(t)\| \leq 1$ for all sufficiently large t. We refer to this as rescaling the semigroup and will return to this idea later.

From the previous proposition we also infer the following continuity feature of strongly continuous semigroups.

Lemma D.4 (Continuity of compositions). *Let S be a strongly continuous semigroup on E. Then,*

$$S(t_n)f_n \to S(t)f$$

as $t_n \to t$ in $[0, \infty)$ and $f_n \to f$ in E.

Proof. By $t_n \to t$ there exists a constant $c > 0$ such that $t_n \in [0, c]$ for all $n \in \mathbb{N}$. By (b) of Proposition D.3 there then exists a constant $C > 0$ with $\|S(t)\| \leq C$ and $\|S(t_n)\| \leq C$ for all $n \in \mathbb{N}$. Now, the desired statement follows from

$$\begin{aligned}\|S(t_n)f_n - S(t)f\| &\leq \|S(t_n)f_n - S(t_n)f\| + \|S(t_n)f - S(t)f\| \\ &\leq C\|f_n - f\| + \|S(t_n)f - S(t)f\| \to 0\end{aligned}$$

as $n \to \infty$, where the convergence to 0 follows from $f_n \to f$ and part (c) of Proposition D.3. $\square$

Strongly continuous semigroups arise naturally in the context of certain initial value problems. We can already see this connection in the study of ordinary differential equations. More specifically, if A is an $n \times n$ matrix and $f \in \mathbb{R}^n$, the solution to the linear equation

$$\partial_t \varphi = -A\varphi,$$

with $\varphi(0) = f$ in the space of differentiable functions $\varphi\colon [0, \infty) \longrightarrow \mathbb{R}^n$, is given by

$$\varphi(t) = e^{-tA} f.$$

Here, $S_A(t) = e^{-tA}$ is a strongly continuous semigroup as in Example D.2.

A very similar statement is true when $\mathbb{R}^n$ is replaced by an arbitrary Banach space E. To make sense of the corresponding terms, we recall that a function $w\colon [0, \infty) \longrightarrow E$ is called *differentiable* if for any $t \geq 0$ the limit

$$\lim_{h \to 0} \frac{1}{h}(w(t + h) - w(t)) = \lim_{s \to t} \frac{1}{t - s}(w(t) - w(s))$$

exists. We note that for $t = 0$ we only take the right limit. In this case, we denote this limit as $\partial_t w(t)$ and call it the *derivative* of w at t. The equation leading to the semigroup is

$$\partial_t w = -Aw$$

with $w(0) = u$, where $u \in E$, A is an operator on E and w is the desired differentiable function. We call such equations *parabolic equations with initial condition u*. We call a differentiable function w which satisfies the equation a *solution* of the parabolic equation.

A most prominent example is the heat equation. For this reason we will also refer to such equations as generalized heat equations. If A is a bounded operator, then the solution of the generalized heat equation can be directly seen to be $w_t = S_A(t)u$ with $S_A(t) = e^{-tA}$, as in Example D.2. Below we will treat the general case of operators A which are not necessarily bounded. This possible unboundedness will mean that we have to be very careful in various places.

The basic underlying concept is that of a generator of a semigroup. As we will see, the generator is the operator appearing in the parabolic equation for which the semigroup gives a unique solution. We next discuss this circle of ideas connecting generators, semigroups and the parabolic equation.

Definition D.5 (Generator of a semigroup). Let S be a strongly continuous semigroup on E. We call the operator A on E with domain

$$D(A) = \{f \in E \mid g = \lim_{t \to 0^+} \frac{1}{t}(f - S(t)f) \text{ exists}\}$$

acting as

$$Af = g$$

for $f \in D(A)$ the *generator* of the semigroup S.

Indeed, it is not hard to see that $D(A)$ is a subspace of E and that A is linear on $D(A)$.

Remark. Sometimes the generator is defined with the reverse sign. However, in view of our applications in the main text, the sign convention chosen above is more convenient.

We now show that a strongly continuous semigroup gives the unique solution of the parabolic equation for the generator. We will later use this to show that a strongly continuous contraction semigroup is uniquely determined by the generator.

Theorem D.6 (Semigroups and the parabolic equation). *Let S be a strongly continuous semigroup on E with generator A. Then, for every $u \in D(A)$ and every $t \geq 0$,*

$$\partial_t S(t)u = \lim_{h \to 0} \frac{1}{h}(S(t+h)u - S(t)u)$$

exists, $S(t)u \in D(A)$ and

$$AS(t)u = S(t)Au = -\partial_t S(t)u.$$

In fact, the function $w\colon [0,\infty) \longrightarrow E$ *given by* $w_t = S(t)u$ *is the unique solution of the equation*

$$\partial_t w_t = -Aw_t$$

in $D(A)$ *with* $w_0 = u$.

Proof. A direct computation using the semigroup property gives

$$\lim_{h\to 0^+} \frac{1}{h}(S(t+h)u - S(t)u) = \lim_{h\to 0^+} \frac{1}{h} S(t)(S(h)u - u) = -S(t)Au,$$

where we use that $u \in D(A)$ and the definition of the generator. This shows the existence of the right limit needed for the differentiability of $S(t)u$ as well as the differentiability at $t = 0$. For the existence of the left limit for $t > 0$, we have, for all h with $0 < h < t$,

$$\frac{1}{-h}(S(t-h)u - S(t)u) = \frac{1}{-h}(S(t-h)(u - S(h)u)) \to -S(t)Au$$

as $h \to 0^+$, where we use Lemma D.4 and $u \in D(A)$. Hence, $S(t)u$ is differentiable for all $t \geq 0$ and

$$\partial_t S(t)u = -S(t)Au.$$

Therefore,

$$-S(t)Au = \lim_{h\to 0} \frac{1}{h}((S(t+h) - S(t))u) = \lim_{h\to 0^+} \frac{1}{h}((S(h) - I)(S(t)u)),$$

which implies $S(t)u \in D(A)$ and $AS(t)u = S(t)Au$. Combining these equalities proves the first statement of the theorem.

We now turn to proving the second statement. By the already proven first statement, the function $w_t = S(t)u$ solves the parabolic equation. Furthermore, $w_0 = S(0)u = u$ by Proposition D.3 (a). It remains to show the uniqueness of the solution. Let φ be another solution of the equation with initial condition u. For $t > 0$, let $v\colon [0,t] \longrightarrow E$ be given by

$$v(s) = S(t-s)\varphi(s).$$

Lemma D.4 and the continuity of φ easily give that v is continuous on $[0,t]$. Moreover, a short computation shows that v is differentiable on $(0,t)$ with

$$\begin{aligned}
\partial_s v(s) &= \lim_{r\to s} \frac{1}{s-r}(v(s) - v(r)) \\
&= \lim_{r\to s} \frac{1}{s-r}\big(S(t-s)\varphi(s) - S(t-r)\varphi(r)\big) \\
&= \lim_{r\to s} \frac{1}{s-r}\big(S(t-s) - S(t-r)\big)\varphi(s) \\
&\quad + \lim_{r\to s} \frac{1}{s-r} S(t-r)(\varphi(s) - \varphi(r))
\end{aligned}$$

$$\begin{aligned}
&= S(t-s)\lim_{r\to s}\frac{1}{s-r}\big(I - S(s-r)\big)\varphi(s)\\
&\quad + \lim_{r\to s}\frac{1}{s-r}S(t-r)(\varphi(s)-\varphi(r))\\
&= S(t-s)A\varphi(s) + S(t-s)\partial_s\varphi(s)\\
&= S(t-s)(A\varphi(s) + \partial_s\varphi(s))\\
&= 0,
\end{aligned}$$

where we used the already proven first statement of the theorem, Lemma D.4 and the fact that φ is a solution.

From these considerations we see that for any differentiable function $\gamma\colon E \longrightarrow \mathbb{R}$, the function $\gamma\circ v\colon [0,t] \longrightarrow \mathbb{R}$ is continuous on $[0,t]$ and differentiable on $(0,t)$ with derivative equal to 0. Thus, $\gamma\circ v$ is constant. As γ is arbitrary, we infer that v is constant on $[0,t]$. Hence, $v(t) = v(0)$ and we obtain

$$\varphi(t) = S(0)\varphi(t) = v(t) = v(0) = S(t)\varphi(0) = S(t)u,$$

so that φ agrees with $S(t)u$ for $t \geq 0$. This completes the proof. □

For our further investigations we need an integral version of the heat equation. This statement can be thought of as a version of the fundamental theorem of calculus in our setting. To formulate it, we need to integrate Banach space-valued functions. Thus, whenever $g\colon [a,b] \longrightarrow E$ is continuous, we define the integral

$$\int_a^b g(s)ds$$

as a Riemann integral via approximation by Riemann sums of step functions. It follows that the fundamental theorem of calculus holds for continuous functions with basically the same proof. We will need this in what follows.

Lemma D.7 (Integral version of derivative). *Let S be a strongly continuous semigroup on E with generator A. Then, for all $f \in E$ and $\delta \geq 0$, we have*

$$\int_0^\delta S(s)fds \in D(A)$$

and

$$S(\delta)f - f = -A\int_0^\delta S(s)fds.$$

Proof. We clearly have

$$\begin{aligned}
S(t)\int_0^\delta S(s)fds - \int_0^\delta S(s)fds &= \int_t^{t+\delta} S(s)fds - \int_0^\delta S(s)fds\\
&= \int_\delta^{t+\delta} S(s)fds - \int_0^t S(s)fds
\end{aligned}$$

$$= (S(\delta) - I) \int_0^t S(s) f ds$$

for all $t \geq 0$. By the fundamental theorem of calculus we have

$$\frac{1}{t} \int_0^t g(s) ds \to g(0)$$

as $t \to 0^+$ whenever $g\colon \mathbb{R} \longrightarrow E$ is continuous. Therefore, as $t \mapsto S(t)f$ is continuous by Proposition D.3 (c), we get

$$\lim_{t \to 0^+} \frac{1}{t} \left(S(t) \int_0^\delta S(s) f ds - \int_0^\delta S(s) f ds \right) = S(\delta) f - f,$$

so that $\int_0^\delta S(s) f ds \in D(A)$ and

$$-A \int_0^\delta S(s) f ds = S(\delta) f - f.$$

This completes the proof. $\square$

We now return to the notion of rescaling a semigroup by an exponential function. This will be used later when we discuss the inverse of the operator $A + \alpha$ for $\alpha > 0$.

Lemma D.8 (Rescaling the semigroup). *Let S be a strongly continuous contraction semigroup on E with generator A. Then, for $\alpha > 0$, the function $\widetilde{S}\colon [0, \infty) \longrightarrow B(E)$ given by*

$$\widetilde{S}(t) = e^{-t\alpha} S(t)$$

is a strongly continuous contraction semigroup with generator $A + \alpha$.

Proof. Clearly $\widetilde{S}$ is a semigroup. It is strongly continuous as S is strongly continuous and $t \mapsto e^{-t\alpha}$ is a continuous function. The fact that $\widetilde{S}$ is a contraction semigroup is obvious from the fact that S is a contraction semigroup and $\alpha > 0$.

We next turn to proving the statement about the generator. Denote the generator of $\widetilde{S}$ by $\widetilde{A}$. Let $f \in D(A)$. Then,

$$\begin{aligned}
\frac{1}{t}(\widetilde{S}(t) f - f) &= \frac{1}{t}(e^{-t\alpha} S(t) f - f) \\
&= \frac{1}{t}(e^{-t\alpha} S(t) f - e^{-t\alpha} f) + \frac{1}{t}(e^{-t\alpha} f - f) \\
&= e^{-t\alpha} \frac{1}{t}(S(t) f - f) + \frac{1}{t}(e^{-t\alpha} f - f) \\
&\to -Af - \alpha f
\end{aligned}$$

as $t \to 0^+$, where we used in the last step that $f \in D(A)$ and the formula for the derivative of the exponential function. This shows $D(A) \subseteq D(\widetilde{A})$ and $\widetilde{A} f = (A + \alpha) f$ for $f \in D(A)$.

It remains to show that $D(\widetilde{A}) \subseteq D(A)$. Consider $f \in D(\widetilde{A})$. Then,

$$\frac{1}{t}(S(t)f - f) = e^{t\alpha}\frac{1}{t}(\widetilde{S}(t)f - f) + \frac{1}{t}(e^{t\alpha}f - f) \to -\widetilde{A}f + \alpha f$$

as $t \to 0^+$. This convergence shows $f \in D(A)$. □

We recall that an operator A is called *closed* if $f_n \to f$ for $f_n \in D(A)$ together with $Af_n \to g$ implies that $f \in D(A)$ and $Af = g$. We will now show that the generator of a strongly continuous semigroup is closed. Furthermore, we will show that adding any strictly positive constant to the generator produces an invertible operator which has a bounded inverse. For this property, we need the additional assumption that the semigroup is a contraction semigroup.

THEOREM D.9 (Basic spectral theory of generators). *Let S be a strongly continuous contraction semigroup on E with generator A. Then, A is closed and $D(A)$ is dense in E. For any $\alpha > 0$, the operator $A+\alpha$ is bijective with bounded inverse given by*

$$(A+\alpha)^{-1} = \int_0^\infty e^{-t\alpha}S(t)dt$$

with $\|(A+\alpha)^{-1}\| \leq 1/\alpha$.

PROOF. We first show that $D(A)$ is dense in E. Let $f \in E$. By Lemma D.7, we have, for all $\delta \geq 0$,

$$\int_0^\delta S(s)fds \in D(A).$$

By part (c) of Proposition D.3, the function $t \mapsto S(t)f$ is continuous so, by the fundamental theorem of calculus, we have

$$\lim_{\delta \to 0^+} \frac{1}{\delta}\int_0^\delta S(s)fds = f.$$

The denseness of $D(A)$ follows.

We next show that A is closed. Consider a sequence (f_n) in $D(A)$ with $f_n \to f$ and $Af_n \to g$ for $f, g \in E$. Then, by Lemma D.7 and Theorem D.6, we have

$$S(t)f_n - f_n = -A\int_0^t S(s)f_nds = -\int_0^t S(s)Af_nds$$

for any $n \in \mathbb{N}$ and any $t > 0$. Taking the limit as $n \to \infty$ and using continuity gives

$$S(t)f - f = -\int_0^t S(s)gds$$

for any $t > 0$. Dividing both sides by t and using the fundamental theorem of calculus to take the limit as $t \to 0^+$ we infer $f \in D(A)$ and $Af = g$.

We now turn to the statement concerning the inverse of $A+\alpha$. For $\alpha > 0$, we define $G(\alpha)$ by

$$G(\alpha) = \int_0^\infty e^{-t\alpha} S(t) dt.$$

Then, for every $\alpha > 0$, the operator $G(\alpha)$ is bounded with

$$\|G(\alpha)\| \le \frac{1}{\alpha}.$$

Indeed, clearly $G(\alpha)$ is linear and, as S is a contraction semigroup, we find

$$\|G(\alpha)f\| \le \int_0^\infty e^{-t\alpha}\|S(t)f\|dt \le \|f\| \int_0^\infty e^{-t\alpha} dt \le \frac{1}{\alpha}\|f\|$$

for any $f \in E$. Therefore, $G(\alpha)$ is bounded with $\|G(\alpha)\| \le 1/\alpha$ as claimed.

We finish the proof by showing that $G(\alpha)$ is the inverse of $A+\alpha$ for $\alpha > 0$. We first show that for $f \in E$ and $\alpha > 0$, $G(\alpha)f \in D(A)$ and $(A+\alpha)G(\alpha)f = f$.

Let $\widetilde{S}(t) = e^{-t\alpha}S(t)$. Then, $\widetilde{S}$ is a strongly continuous semigroup with generator $A+\alpha$ by Lemma D.8. Now, Lemma D.7 applied to $\widetilde{S}$ gives

$$\widetilde{S}(\delta)f - f = (-A-\alpha)\int_0^\delta \widetilde{S}(s) f ds$$

for any $\delta > 0$. We take the limit as $\delta \to \infty$ on both sides and use that $A+\alpha$ is closed to obtain $G(\alpha)f \in D(A)$ and

$$(A+\alpha)G(\alpha)f = f.$$

Similarly, from the definition of $G(\alpha)$ and Theorem D.6 we infer, for any $f \in D(A)$,

$$\begin{aligned} G(\alpha)(A+\alpha)f &= \int_0^\infty e^{-t\alpha} S(t)(A+\alpha) f dt \\ &= (A+\alpha)\int_0^\infty e^{-t\alpha} S(t) f dt \\ &= (A+\alpha)G(\alpha)f \\ &= f, \end{aligned}$$

from what we have already shown. These equalities show that $(A+\alpha)$ is the inverse of the bounded operator $G(\alpha)$. Hence, $(A+\alpha)$ is a bijective operator and

$$(A+\alpha)^{-1} = G(\alpha).$$

This completes the proof. $\square$

We now use the preceding considerations to show that if two strongly continuous contraction semigroups have the same generator, then they must be equal.

Corollary D.10 (Uniqueness of generators). *If S and $\widetilde{S}$ are strongly continuous contraction semigroups on E with generator A, then $S = \widetilde{S}$.*

Proof. As $D(A)$ is dense in E by Theorem D.9, it suffices to show that $S(t)u = \widetilde{S}(t)u$ for all $u \in D(A)$ and all $t \geq 0$. As both S and $\widetilde{S}$ are strongly continuous contraction semigroups, it follows that $t \mapsto S(t)u$ and $t \mapsto \widetilde{S}(t)u$ are both solutions of

$$\partial_t w = -Aw$$

with $w(0) = u$ by Theorem D.6. Hence, by the uniqueness statement in Theorem D.6, it follows that $S(t)u = \widetilde{S}(t)u$ for all $t \geq 0$. This completes the proof. □

We now apply the theory developed above to the case when the Banach space is a Hilbert space. We let A be a densely defined operator on a Hilbert space H with adjoint A^*. We recall that the operator A is self-adjoint if $A = A^*$. Furthermore, for a self-adjoint operator A on H, the spectral theorem allows us to define functions of an operator. More specifically, if A is unitarily equivalent to multiplication by u and φ is a measurable function on the essential range of u, then $\varphi(A)$ is unitarily equivalent to multiplication by $\varphi \circ u$. For more details, see the definition and discussion following Theorem A.20.

We let $\sigma(A)$ denote the spectrum of A and recall that $\sigma(A) \subseteq \mathbb{R}$ for a self-adjoint operator A. Furthermore, a self-adjoint operator A is called positive if $\sigma(A) \subseteq [0, \infty)$, equivalently, if A is unitarily equivalent to multiplication by a positive function u, see Lemma B.1.

For general Banach spaces, we have shown that the generator of a strongly continuous contraction semigroup is closed and densely defined. We now show that on a Hilbert space, if the semigroup additionally takes values in the self-adjoint operators, then the generator is positive.

Lemma D.11. *Let H be a Hilbert space. If S is a strongly continuous contraction semigroup on H taking values in the self-adjoint operators, then the generator A of S is positive.*

Proof. We first note that A is densely defined by Theorem D.9. Furthermore, A is symmetric, as follows from the self-adjointness of $S(t)$ for $t \geq 0$. Hence, A^* is an extension of A. Thus, in order to establish self-adjointness it suffices to show that $D(A^*) \subseteq D(A)$.

Let $f \in D(A^*)$ and let $\alpha > 0$. By Theorem D.9, $A + \alpha$ is bijective with bounded inverse. Hence, there exists a $g \in D(A)$ with

$$(A + \alpha)g = (A^* + \alpha)f.$$

As A is symmetric, this gives $(A^* + \alpha)(f - g) = 0$, so that $f - g$ belongs to the kernel of $(A^* + \alpha)$. This kernel agrees with the orthogonal complement of the range of $A+\alpha$ and thus is trivial. This shows $f = g \in D(A)$. Therefore, A is self-adjoint.

Finally, as $A+\alpha$ is bijective with bounded inverse for any $\alpha > 0$, it follows that $(-\infty, 0)$ belongs to the resolvent set of A. Thus, A is positive by Lemma B.1. □

We now establish a one-to-one correspondence between positive operators and strongly continuous contraction semigroups taking values in the space of bounded self-adjoint operators.

COROLLARY D.12 (Hilbert space semigroups and generators). *Let H be a Hilbert space.*

(a) *If A is a positive operator on H, then $S\colon [0,\infty) \longrightarrow B(H)$ given by*

$$S(t) = e^{-tA}$$

is a strongly continuous contraction semigroup with self-adjoint generator A. Furthermore, if $\widetilde{A}$ is a positive operator and $e^{-tA} = e^{-t\widetilde{A}}$, then $A = \widetilde{A}$.

(b) *If S is a self-adjoint strongly continuous contraction semigroup on H, then $S(t) = e^{-tA}$, where A is the generator of S.*

PROOF. (a) The fact that $S(t) = e^{-tA}$ is a strongly continuous contraction semigroup was already shown in Proposition A.32 by using the functional calculus and Lebesgue's dominated convergence theorem.

We now show that A is the generator of e^{-tA}. Let $f \in D(A)$. By Proposition A.24 we get

$$\left\| \frac{1}{t}(e^{-tA}f - f) + Af \right\|^2 = \int_0^\infty \left| \frac{e^{-tx}-1}{t} + x \right|^2 d\mu_f(x) \to 0$$

as $t \to 0^+$. The convergence follows by Lebesgue's dominated convergence theorem. This follows since we have pointwise convergence by the differentiability of the exponential function and the integrand is bounded above by $2x$, which is in $L^2(\mathbb{R}, \mu_f)$ by Proposition A.24 as f belongs to $D(A)$.

This convergence shows that f is in the domain of the generator of S and that A agrees with the generator applied to f. As A is self-adjoint by assumption and the generator of S is self-adjoint by Lemma D.11, it follows that A is the generator of S.

Finally, let $e^{-tA} = e^{-t\widetilde{A}}$ for $\widetilde{A}$ positive. Then both e^{-tA} and $e^{-t\widetilde{A}}$ are strongly continuous contraction semigroups from what we have already shown. By definition of the generator we have, for all $f \in D(A)$,

$$Af = \lim_{t\to 0^+} \frac{1}{t}(f - e^{-tA}f) = \lim_{t\to 0^+} \frac{1}{t}(f - e^{-t\widetilde{A}}f).$$

This shows $f \in D(\widetilde{A})$ and, therefore, the right-hand side is equal to $\widetilde{A}f$. Therefore, $Af = \widetilde{A}f$ for $f \in D(A)$. As both A and $\widetilde{A}$ are self-adjoint, it follows that $A = \widetilde{A}$.

(b) By Lemma D.11, the generator A of S is positive. Hence, by part (a), we have that $t \mapsto e^{-tA}$ is a strongly continuous contraction semigroup with generator A. Therefore,

$$S(t) = e^{-tA}$$

by Corollary D.10. This completes the proof. □

After this discussion of semigroups on Banach and Hilbert spaces we now turn to resolvents. As semigroups, resolvents are families of operators determined by a generator. Resolvents and semigroups are strongly related. Indeed, they can be seen as two different perspectives on the same object which is the generator. Unlike the case of semigroups, for resolvents the generator is already determined from any single element of the family. This is a structural advantage of working with resolvents.

Definition D.13 (Resolvents). Let E be a Banach space.

(a) We call a map $G\colon (0,\infty) \longrightarrow B(E)$ a *resolvent* if G satisfies the *resolvent identity*, that is,

$$G(\alpha) - G(\beta) = -(\alpha - \beta)G(\alpha)G(\beta)$$

for all $\alpha, \beta > 0$.

(b) We call a resolvent G *strongly continuous* if G additionally satisfies $\alpha G(\alpha)f \to f$ as $\alpha \to \infty$ for all $f \in E$.

(c) We call a resolvent G with $\|\alpha G(\alpha)\| \leq 1$ for every $\alpha > 0$ a *contraction resolvent*.

Remark. The definitions above show that in some situations it makes sense to think of $\alpha G(\alpha)$ as the primary object rather than $G(\alpha)$.

We now gather some simple properties of resolvents. In particular, we show that resolvents commute.

Proposition D.14. *Let G be a resolvent on E.*

(a) *For all $\alpha, \beta > 0$,*

$$G(\alpha)G(\beta) = G(\beta)G(\alpha)$$

and

$$\mathrm{Range}(G(\alpha)) = \mathrm{Range}(G(\beta)).$$

(b) *If G is additionally a contraction resolvent, then $G(\alpha_n)f \to G(\alpha)f$ as $\alpha_n \to \alpha$ for $\alpha_n, \alpha > 0$.*

PROOF. (a) The first part of the claim is a direct consequence of the resolvent identity since

$$\begin{aligned} -(\alpha-\beta)G(\alpha)G(\beta) &= G(\alpha) - G(\beta) \\ &= -(G(\beta)-G(\alpha)) = (\beta-\alpha)G(\beta)G(\alpha). \end{aligned}$$

From the resolvent identity, we also easily obtain

$$\mathrm{Range}(G(\beta)) \subseteq \mathrm{Range}(G(\alpha))$$

for all $\alpha, \beta > 0$ since $G(\beta) = G(\alpha)(I + (\alpha-\beta)G(\beta))$. By symmetry, this gives the desired statement.

(b) This follows directly from the contraction and resolvent identity properties as

$$\|(G(\alpha)-G(\alpha_n))f\| = \|(\alpha-\alpha_n)G(\alpha)G(\alpha_n)f\| \leq \frac{|\alpha-\alpha_n|}{\alpha\alpha_n}\|f\|.$$

This completes the proof. □

We now establish some additional properties of strongly continuous resolvents which will lead to the definition of the generator.

PROPOSITION D.15. *Let G be a strongly continuous resolvent on E. Then, for each $\alpha > 0$, the operator $G(\alpha)\colon E \longrightarrow \mathrm{Range}(G(\alpha))$ is bijective and the operator with domain $D(A) = \mathrm{Range}(G(\alpha))$ acting as*

$$Af = G(\alpha)^{-1}f - \alpha f$$

for $f \in D(A)$ does not depend on $\alpha > 0$.

PROOF. We first show that $G(\alpha)$ injective. Let f satisfy $G(\alpha)f = 0$ for some $\alpha > 0$. Then, the resolvent identity gives $G(\beta)f = 0$ for all $\beta > 0$. By strong continuity we conclude $0 = \alpha G(\alpha)f \to f$ as $\alpha \to \infty$, so that $f = 0$. Hence, $G(\alpha)\colon E \longrightarrow \mathrm{Range}(G(\alpha))$ is bijective as claimed.

By Proposition D.14 (a), the range of $G(\alpha)$ does not depend on $\alpha > 0$. Therefore, $D(A)$ does not depend on α. We now show that $G(\alpha)^{-1}f - \alpha f$ also does not depend on α. Applying the resolvent identity twice we infer

$$\begin{aligned} &G(\alpha)\big((G(\alpha)^{-1}f - \alpha f) - (G(\beta)^{-1}f - \beta f)\big) \\ &\qquad = f - \alpha G(\alpha)f - \big(G(\beta) - (\alpha-\beta)G(\alpha)G(\beta)\big)(G(\beta)^{-1}f - \beta f) \\ &\qquad = \beta\big(G(\beta)f - G(\alpha)f - (\alpha-\beta)G(\alpha)G(\beta)f\big) = 0. \end{aligned}$$

By the already proven injectivity of $G(\alpha)$ this shows

$$G(\alpha)^{-1}f - \alpha f = G(\beta)^{-1}f - \beta f,$$

so that Af does not depend on $\alpha > 0$. Clearly, A is linear. This finishes the proof. □

Given the preceding result, we now define the generator of a resolvent.

DEFINITION D.16 (Generator of a resolvent). Let G be a strongly continuous resolvent on E. The operator with $D(A) = \mathrm{Range}(G(\alpha))$ and acting as

$$Af = G(\alpha)^{-1}f - \alpha f$$

for $f \in D(A)$ is called the *generator of the resolvent.*

As with the generator of a strongly continuous semigroup, we now highlight some properties of the generator of a strongly continuous resolvent. In particular, we show that if two resolvents have the same generator, then they must be equal.

COROLLARY D.17 (Basic properties of resolvents). *Let G be a strongly continuous resolvent on E. Then, the generator A of G is closed, $D(A)$ is dense in E and for any $\alpha > 0$ the operator $A + \alpha$ is bijective with*

$$G(\alpha) = (A + \alpha)^{-1}.$$

In particular, if G and $\widetilde{G}$ are both strongly continuous contraction resolvents with generator A, then $G = \widetilde{G}$.

PROOF. That $G(\alpha) = (A + \alpha)^{-1}$ is a direct consequence of the definitions. As $G(\alpha)$ is a bounded operator for every $\alpha > 0$, $G(\alpha)$ is closed. Hence, as the inverse of a closed operator is closed, it follows that A is closed.

We now show that $D(A)$ is dense in E. Let $f \in E$. Then, $\alpha G(\alpha)f \in D(A) = \mathrm{Range}(G(\alpha))$ and $\alpha G(\alpha)f \to f$ as $\alpha \to \infty$ by strong continuity. Hence, $D(A)$ is dense in E. The uniqueness statement follows directly. □

Given the preceding concepts we can reformulate Theorem D.9 as describing the close relationship between resolvents and semigroups. In particular, we show that any strongly continuous contraction semigroup defines a strongly continuous contraction resolvent.

THEOREM D.18 (Relationship between resolvents and semigroups). *Let S be a strongly continuous contraction semigroup on E. Then, the map $G\colon (0, \infty) \longrightarrow B(E)$ given by*

$$G(\alpha) = \int_0^\infty e^{-t\alpha} S(t)dt$$

defines a strongly continuous contraction resolvent and the generator of G and S agree. In particular, if A is the generator of S, then

$$G(\alpha) = (A + \alpha)^{-1}$$

for all $\alpha > 0$, so that

$$(A + \alpha)^{-1} = \int_0^\infty e^{-t\alpha} S(t)dt.$$

(*"Laplace transform"*)

PROOF. If A is the generator of S, then Theorem D.9 gives

$$G(\alpha) = \int_0^\infty e^{-t\alpha} S(t)dt = (A+\alpha)^{-1}.$$

The resolvent identity then follows from

$$\begin{aligned} G(\alpha) - G(\beta) &= (A+\alpha)^{-1} - (A+\beta)^{-1} \\ &= (A+\alpha)^{-1}((A+\beta) - (A+\alpha))(A+\beta)^{-1} \\ &= (\beta - \alpha)(A+\alpha)^{-1}(A+\beta)^{-1} \\ &= -(\alpha - \beta)G(\alpha)G(\beta). \end{aligned}$$

The fact that G is a contraction resolvent also follows directly from Theorem D.9, which gives $\|(A+\alpha)^{-1}\| \le 1/\alpha$.

To show that G is strongly continuous, let $f \in E$ and $\varepsilon > 0$. By the strong continuity of S, there exists a $\delta > 0$ such that $\|S(t)f - f\| < \varepsilon$ for all $t \in [0, \delta)$. Then, using the fact that S is a contraction semigroup, we get

$$\begin{aligned} \|\alpha G(\alpha)f - f\| &= \left\| \int_0^\infty \alpha e^{-t\alpha}(S(t)f - f)dt \right\| \\ &\le \int_0^\delta \alpha e^{-t\alpha}\|S(t)f - f\|dt + \int_\delta^\infty \alpha e^{-t\alpha}\|S(t)f - f\|dt \\ &< \varepsilon \int_0^\delta \alpha e^{-t\alpha}dt + 2\|f\| \int_\delta^\infty \alpha e^{-t\alpha}dt \\ &= \varepsilon(1 - e^{-\delta\alpha}) + 2\|f\|e^{-\delta\alpha} \\ &\to \varepsilon \end{aligned}$$

as $\alpha \to \infty$. Hence, G is strongly continuous.

The fact that A is the generator of G follows from $G(\alpha) = (A+\alpha)^{-1}$ and Corollary D.17. This completes the proof. □

We now have a closer look at resolvents and their generators in the case of Hilbert spaces. In particular, we use the functional calculus to show that there is a one-to-one correspondence between positive operators and strongly continuous contraction resolvents which take values in the self-adjoint operators.

COROLLARY D.19 (Hilbert space resolvents and generators). *Let H be a Hilbert space.*

(a) *If A is a positive operator on H, then $G\colon (0,\infty) \longrightarrow B(H)$ given by*

$$G(\alpha) = (A+\alpha)^{-1}$$

defines the unique strongly continuous contraction resolvent with generator A taking values in the self-adjoint operators.

(b) *If G is a strongly continuous contraction resolvent on H taking values in the self-adjoint operators, then the generator A of G is the positive operator with $G(\alpha) = (A + \alpha)^{-1}$ for $\alpha > 0$.*

PROOF. (a) That $G(\alpha) = (A + \alpha)^{-1}$ is a strongly continuous contraction resolvent on H was already shown in Proposition A.34 by using the functional calculus and Lebesgue's dominated convergence theorem. Alternatively, by using the material in this appendix, we note that as A is positive, $S(t) = e^{-tA}$ is the unique strongly continuous contraction semigroup with generator A which takes values in the self-adjoint operators by Corollary D.12 (a). Hence, $G(\alpha) = (A+\alpha)^{-1}$ for $\alpha > 0$ defines a strongly continuous contraction resolvent by Theorem D.18. As A is self-adjoint, it is clear that $(A + \alpha)^{-1}$ is self-adjoint for every $\alpha > 0$. Uniqueness follows from Corollary D.17.

(b) That the generator A of G satisfies $G(\alpha) = (A + \alpha)^{-1}$ follows from Corollary D.17. As $G(\alpha)$ is self-adjoint for every $\alpha > 0$, it follows that A is self-adjoint, so $\sigma(A) \subseteq \mathbb{R}$ by Corollary A.12. As $(A + \alpha)^{-1}$ exists and is a bounded operator for all $\alpha > 0$, it follows that $\sigma(A) \subseteq [0, \infty)$. Thus, A is a positive operator by definition, see Lemma B.1. □

We note that Theorem D.18 allows us to pass from a strongly continuous contraction semigroup to a strongly continuous contraction resolvent on Banach spaces via the Laplace transform formula. For Hilbert spaces, we have also proven this via the functional calculus in Theorem A.35 (a). We now show that in the case of Hilbert spaces, we may pass from a strongly continuous contraction resolvent to a strongly continuous contraction semigroup. This extends Theorem A.35 (b) and completes the cycle connecting operators, semigroups and resolvents in the case of Hilbert spaces.

PROPOSITION D.20. *Let H be a Hilbert space. Let G be a strongly continuous contraction resolvent on H taking values in the self-adjoint operators and let A be the generator of G. Then,*

$$S(t) = \lim_{n\to\infty} \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^{n}$$

gives the unique strongly continuous contraction semigroup on H with generator A, i.e., $S(t) = e^{-tA}$ for all $t \geq 0$.

PROOF. By Corollary D.19 (b) it follows that the generator A of G is a positive operator. Therefore, by Corollary D.12 (b), $S(t) = e^{-tA}$ gives the strongly continuous contraction semigroup on H with generator A. The fact that

$$e^{-tA} = \lim_{n\to\infty} \left(\frac{n}{t} \left(A + \frac{n}{t} \right)^{-1} \right)^{n}$$

was shown in Theorem A.35 (b) by using the functional calculus and Lebesgue's dominated convergence theorem. □

We finish this appendix by briefly discussing adjoint operators on Banach spaces and their resolvents. We recall that whenever E is a Banach space, E^*, the dual space of E consisting of all linear continuous maps from E to the underlying field, is a Banach space as well with respect to the norm

$$\|\phi\|_{E^*} = \sup\{|\phi(f)| \mid \|f\| \leq 1\}.$$

For any bounded operator $A\colon E \longrightarrow E$, we then define the *adjoint operator* $A^*\colon E^* \longrightarrow E^*$ by $(A^*\phi) = \phi \circ A$, that is,

$$(A^*\phi)f = \phi(Af)$$

for every $f \in E$. Clearly, A^* is a bounded operator as well since

$$\|A^*\phi\|_{E^*} = \|\phi \circ A\|_{E^*} \leq \|\phi\|_{E^*}\|A\|$$

for all $\phi \in E^*$.

For an arbitrary densely defined operator A on E, we define the adjoint A^* on E^* by

$$D(A^*) = \{\phi \in E^* \mid \text{ there exists a } \psi \in E^* \text{ extending } \phi \circ A\}$$

via

$$A^*\phi = \psi$$

for all $\phi \in D(A^*)$. We note that ψ is unique as A is densely defined.

It turns out that strongly continuous contraction resolvents on Banach spaces can be extended to their dual spaces.

Proposition D.21. *Let E be a Banach space. Let S be a strongly continuous contraction semigroup on E with generator A. Denote the associated resolvent by G, i.e., $G\colon (0,\infty) \longrightarrow B(E)$ via*

$$G(\alpha) = (A + \alpha)^{-1}.$$

Then, for any $\alpha > 0$, the operator $A^* + \alpha$ is bijective and

$$(A^* + \alpha)^{-1} = G(\alpha)^*.$$

Proof. As $(A + \alpha)^{-1} = G(\alpha)$ is a bounded operator for $\alpha > 0$, so is $T = G(\alpha)^*$. Clearly,

$$T\phi = \phi \circ G(\alpha) = \phi \circ (A + \alpha)^{-1}.$$

We prove a series of claims:

Claim 1. For any $\phi \in E^*$ the element $T\phi$ belongs to $D(A^*)$.

Proof of Claim 1. We have to show that $(T\phi) \circ A$ can be extended to an element of E^*. This, however, is clear as on $D(A)$ we find

$$\begin{aligned}(T\phi) \circ A &= \phi \circ (A + \alpha)^{-1}A \\ &= \phi \circ (A + \alpha)^{-1}(A + \alpha) - \alpha\phi \circ (A + \alpha)^{-1} \\ &= \phi - \alpha\phi \circ G(\alpha)\end{aligned}$$

and the last term can obviously be extended to an element of E^*.

Claim 2. For any $\phi \in E^*$, we have $(A^* + \alpha)T\phi = \phi$.

Proof of Claim 2. By Claim 1, the operator $(A^* + \alpha)T$ is defined on E^*. Moreover, for any $\phi \in E^*$, we have

$$(A^* + \alpha)(T\phi) = \phi \circ (A + \alpha)^{-1} \circ (A + \alpha) = \phi$$

on $D(A)$. As $D(A)$ is dense in E by Corollary D.17, $(A^* + \alpha)T\phi = \phi$ for any $\phi \in E^*$. This proves the claim.

Claim 3. For any $\phi \in D(A^*)$, we have $T(A^* + \alpha)\phi = \phi$.

Proof of Claim 3. A direct computation shows

$$T(A^* + \alpha)\phi = \phi \circ (A + \alpha)G(\alpha) = \phi.$$

Combining Claim 2 and Claim 3 we obtain that

$$G(\alpha)^* = T = (A^* + \alpha)^{-1}$$

and this is the desired statement. □

APPENDIX E

Aspects of Operator Theory

... you see without a trace, a whole bunch of people gathered in around the place.
Mellow Dee.

This final appendix collects various pieces of operator theory. First, we prove a characterization of the resolvent of an operator as the unique minimizer of an equation involving the form. Then, we give criteria for a number to be in the spectrum and essential spectrum of an operator in terms of the spectral family and Weyl sequences. This has direct consequences for compact perturbations of an operator which we discuss. We next consider bounds on the bottom of the spectrum and prove the min-max principle. We then turn to upper bounds for the bottom of the essential spectrum via a Persson theorem. Next, we study the notion of commuting operators. In particular, we show that a bounded operator commutes with a symmetric operator if and only if the resolvent or semigroup of the Friedrichs extension of the symmetric operator commute with the bounded operator. Finally, we recall the Riesz–Thorin interpolation theorem.

1. A characterization of the resolvent

In this section we prove a characterization of the resolvent of an operator. More specifically, given a closed form, we show that the resolvent of the associated operator gives the unique minimizer of an equation involving the form.

We recall that for a positive symmetric form Q on a complex Hilbert space H with domain $D(Q)$ we define the inner product $\langle \cdot, \cdot \rangle_Q$ as

$$\langle f, g \rangle_Q = Q(f, g) + \langle f, g \rangle$$

for $f, g \in D(Q)$. We denote the corresponding norm by $\| \cdot \|_Q$. If Q is closed, then $D(Q)$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle_Q$, see Theorem B.9 for various formulations of this notion. In particular, a positive closed form gives rise to a unique self-adjoint operator L which satisfies $D(Q) = D(\sqrt{L})$ and

$$Q(f, g) = \langle \sqrt{L} f, \sqrt{L} g \rangle$$

for all $f, g \in D(Q)$.

We say that the operator L is associated to Q, see Lemma B.7 for a construction of L. Furthermore, we note that the domain and action of L can be characterized as

$$D(L) = \left\{ f \in D(Q) \;\middle|\; \begin{array}{c} \text{there exists a } g \in H \text{ with } Q(h,f) = \langle h, g \rangle \\ \text{for all } h \in D(Q) \end{array} \right\}$$

with $Lf = g$, see Theorem B.11. In particular,

$$Q(f,g) = \langle Lf, g \rangle$$

for all $f \in D(L) \subseteq D(Q)$ and $g \in D(Q)$, see Corollary B.12.

We note that L is positive in this case, that is, L is self-adjoint and $(L+\alpha)^{-1}$ exists and is a bounded self-adjoint operator which maps into $D(L)$ for every $\alpha > 0$. This operator is called the resolvent associated to L.

Given these preparations, we now state a characterization of the resolvent in terms of the form.

Theorem E.1 (Characterization of the resolvent as a minimizer). *Let Q be a positive closed form on H with associated operator L. For $f \in H$ and $\alpha > 0$, define $j \colon D(Q) \longrightarrow [0,\infty)$ by*

$$j(v) = Q(v) + \alpha \left\| v - \frac{1}{\alpha} f \right\|^2.$$

Then, j satisfies the formula

$$j(v) = j\left((L+\alpha)^{-1} f\right) + Q\left((L+\alpha)^{-1} f - v\right) + \alpha \left\|(L+\alpha)^{-1} f - v\right\|^2.$$

In particular, $(L+\alpha)^{-1} f$ is the unique minimizer of j on $D(Q)$.

Proof. It suffices to show the formula for j. The statement on the minimizer is then immediate. For ease of notation we set

$$G_\alpha = (L+\alpha)^{-1}$$

and

$$Q_\alpha(u,v) = Q(u,v) + \alpha \langle u, v \rangle$$

for $\alpha > 0$. Given this, the right-hand side of the formula for j can be written as

$$\text{RHS} = j(G_\alpha f) + Q_\alpha(G_\alpha f - v).$$

We will compute the two terms appearing in RHS. In order to do so, we need a little bit of preparation. We obviously have

$$Q_\alpha(G_\alpha f, v) = \langle f, v \rangle$$

for all $f \in H$ and $v \in D(Q)$, which directly yields

$$Q_\alpha(G_\alpha f) = \langle f, G_\alpha f \rangle = \langle G_\alpha f, f \rangle,$$

where we use the self-adjointness of G_α. Furthermore, a direct computation gives

$$j(v) = Q_\alpha(v) - \langle v, f\rangle - \langle f, v\rangle + \frac{1}{\alpha}\|f\|^2.$$

Now, we turn to computing the two terms in RHS: By the last two equalities we obtain for the first term

$$\begin{aligned} j(G_\alpha f) &= Q_\alpha(G_\alpha f) - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= \langle f, G_\alpha f\rangle - \langle G_\alpha f, f\rangle - \langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2 \\ &= -\langle f, G_\alpha f\rangle + \frac{1}{\alpha}\|f\|^2. \end{aligned}$$

For the second term, using $Q_\alpha(G_\alpha f, v) = \langle f, v\rangle$ repeatedly we obtain

$$\begin{aligned} Q_\alpha(G_\alpha f - v) &= Q_\alpha(G_\alpha f) - Q_\alpha(G_\alpha f, v) - Q_\alpha(v, G_\alpha f) + Q_\alpha(v) \\ &= \langle f, G_\alpha f\rangle - \langle f, v\rangle - \langle v, f\rangle + Q_\alpha(v). \end{aligned}$$

Putting the two terms together we can now compute

$$\begin{aligned} \mathrm{RHS} &= j(G_\alpha f) + Q_\alpha(G_\alpha f - v) \\ &= Q_\alpha(v) - \langle f, v\rangle - \langle v, f\rangle + \frac{1}{\alpha}\|f\|^2 = j(v), \end{aligned}$$

which finishes the proof. $\square$

Remark (Geometric interpretation). It is possible to interpret the previous result in terms of Hilbert space geometry on a suitably chosen Hilbert space. First, $v = G_\alpha f$ is equivalent to $(L+\alpha)v = f$, which in turn is equivalent to the fact that $v \in D(Q)$ with $Q_\alpha(v, w) = \langle f, w\rangle$ for all $w \in D(Q)$. We can write this as

$$Q(v, w) + \alpha\langle v - \frac{1}{\alpha}f, w\rangle = 0$$

for $w \in D(Q)$. Rewriting this with the (semi-)inner product

$$\langle (a, b), (c, d)\rangle_* = Q(a, c) + \alpha\langle b, d\rangle$$

on $D(Q) \times D(Q)$ we infer that $v = G_\alpha f$ if and only if $(v, v - \alpha^{-1}f)$ is perpendicular to the diagonal, i.e.,

$$(v, v - \frac{1}{\alpha}f) = (v, v) - (0, \frac{1}{\alpha}f) \perp U,$$

where U is the subspace

$$U = \{(w, w) \mid w \in D(Q)\}.$$

So, if $x = -(0, \alpha^{-1}f)$, then we want to find an element $\widetilde{v} \in U$ such that $x + \widetilde{v}$ is perpendicular to U.

By standard theory this problem has a unique solution, which is given by the minimizer of $\|\cdot\|_*$ on $x + U$ whenever $\langle\cdot,\cdot\rangle_*$ is an inner product inducing a Hilbert space structure. Now, in general, $\langle\cdot,\cdot\rangle_*$ is not an inner product and completeness may fail on $D(Q) \times D(Q)$. So, the basic theory does not apply directly. However, it is not necessary for $\langle \cdot, \cdot \rangle_*$ to be an inner product giving a Hilbert space structure on the entire space, it suffices that $\langle \cdot, \cdot \rangle_*$ is an inner product on U making U into a Hilbert space. This is indeed the case in our situation and we infer that $v = G_\alpha f$ holds if and only if v minimizes $\| \cdot \|_*$ on $x + U$. As, $j(\cdot) = \| \cdot \|_*^2$ on $x + U$ we obtain the statement of the theorem.

2. The discrete and essential spectrum

In this section we discuss basic spectral features of self-adjoint operators. More specifically, we give criteria for a number to be in the spectrum and the essential spectrum of an operator. This is first done in terms of the spectral family and then in terms of Weyl sequences. This has direct consequences for compact perturbations of an operator. Afterwards, we consider bounds on the bottom of the spectrum and prove the min-max principle. Finally, we turn to upper bounds for the bottom of the essential spectrum via a Persson theorem.

We let L be a self-adjoint operator on a Hilbert space H with domain $D(L)$ and spectrum $\sigma(L)$. We note that $\sigma(L) \subseteq \mathbb{R}$ by Corollary A.12. In particular, if there exist $\lambda \in \mathbb{R}$ and $f \in D(L)$ with $f \neq 0$ such that

$$Lf = \lambda f$$

then $(L - \lambda)$ is not one-to-one and, thus, $\lambda \in \sigma(L)$. In this case, we call λ an *eigenvalue* for L corresponding to the *eigenfunction* f. We call the space spanned by all eigenfunctions corresponding to an eigenvalue λ the *eigenspace* of λ. Clearly, the eigenspace is a subspace. It is closed, as can easily be seen from the fact that L is a closed operator. If λ is an eigenvalue for L, then the *multiplicity* of λ is the dimension of the eigenspace of λ.

We decompose $\sigma(L)$ into two subsets. Specifically, the *discrete spectrum* $\sigma_{\mathrm{disc}}(L)$ consists of isolated eigenvalues of finite multiplicity. The *essential spectrum* $\sigma_{\mathrm{ess}}(L)$ is the complement in $\sigma(L)$ of the discrete spectrum, that is, $\sigma_{\mathrm{ess}}(L) = \sigma(L) \setminus \sigma_{\mathrm{disc}}(L)$.

2.1. Characterization via the spectral family. We first characterize when a number belongs to the spectrum, the discrete or the essential spectrum as well as when the number is an eigenvalue in terms of spectral projections.

Recall that we denote by $E(B)$ the spectral projection associated to a measurable set $B \subseteq \mathbb{R}$ defined via the spectral theorem as

$$E(B) = 1_B(L),$$

where 1_B is the characteristic function of the set B.

Let $f \in H$. For the spectral measure μ_f associated f we have

$$\mu_f(B) = \langle f, E(B)f \rangle = \|E(B)f\|^2,$$

as shown in Proposition A.28. Furthermore, as $E(B)$ is an orthogonal projection for every measurable set B, it follows that if B_1 and B_2 are disjoint sets, then $E(B_1)H \perp E(B_2)H$, i.e., $E(B_1)g$ and $E(B_2)g$ are orthogonal for every $g \in H$.

We will denote the support of the spectral measure μ_f by

$$\operatorname{supp}(\mu_f) = \{\lambda \in \mathbb{R} \mid \mu_f((\lambda - \varepsilon, \lambda + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\}.$$

We note by Corollary A.31 that $\operatorname{supp}(\mu_f) \subseteq \sigma(L)$. Furthermore, by Proposition A.29, if $f \in E(B)H$, then the support of μ_f is contained in the closure of B, that is,

$$\operatorname{supp}(\mu_f) \subseteq \overline{B}.$$

The connection between the spectral family and the spectrum is discussed next.

Proposition E.2 (Spectral parts and spectral family). *Let L be a self-adjoint operator on H and let E be the associated spectral family. Let $\lambda \in \mathbb{R}$.*

(a) *$\lambda \in \sigma(L)$ if and only if $\lambda \in \operatorname{supp}(E)$, i.e.,*

$$E((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$$

for all $\varepsilon > 0$.

(b) *λ is an eigenvalue of L if and only if $E(\{\lambda\}) \neq 0$, in which case the range of $E(\{\lambda\})$ is the eigenspace of λ. Furthermore, $f \in H$ is an eigenfunction corresponding to λ if and only if μ_f is supported on $\{\lambda\}$.*

(c) *$\lambda \in \sigma_{\mathrm{disc}}(L)$ if and only if $\lambda \in \sigma(L)$ and there exists an $\varepsilon > 0$ such that the range of $E((\lambda - \varepsilon, \lambda + \varepsilon))$ is finite-dimensional.*

(d) *$\lambda \in \sigma_{\mathrm{ess}}(L)$ if and only if the range of $E((\lambda - \varepsilon, \lambda + \varepsilon))$ is infinite-dimensional for all $\varepsilon > 0$.*

Proof. (a) This has already been shown in Theorem A.30.

(b) From Proposition A.24, as $(x - \lambda)^2 = 0$ for $x = \lambda$, we get

$$\|(L - \lambda)f\|^2 = \int (x - \lambda)^2 d\mu_f = \int_{\mathbb{R}\setminus\{\lambda\}} (x - \lambda)^2 d\mu_f$$

for any $f \in D(L)$. As $(x - \lambda)^2 > 0$ for all $x \neq \lambda$, we infer that $f \in D(L)$ with $f \neq 0$ is an eigenfunction corresponding to λ if and only if $\mu_f = 1_{\{\lambda\}}\mu_f$, i.e., if and only if $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$. Now, $1_{\mathbb{R}\setminus\{\lambda\}}\mu_f = 0$ if and only if $\mu_f(\mathbb{R} \setminus \{\lambda\}) = 0$ and Proposition A.28 gives

$$\|E(\mathbb{R} \setminus \{\lambda\})f\|^2 = \mu_f(\mathbb{R} \setminus \{\lambda\}) = 0.$$

Thus, we infer that $f \in D(L)$ is an eigenfunction corresponding to λ if and only if $E((\mathbb{R} \setminus \{\lambda\}))f = 0$. This, in turn, is equivalent to

$f = E(\{\lambda\})f$ as $f = E(\{\lambda\})f + E(\mathbb{R} \setminus \{\lambda\})f$, where the summands are orthogonal. This shows the first statement of (b). The other statement has been shown along the way.

(c) If λ is an isolated eigenvalue of finite multiplicity, then there exists an $\varepsilon > 0$ such that $E((\lambda - \varepsilon, \lambda + \varepsilon)) = E(\{\lambda\})$ is finite-dimensional, as follows from (a) and (b). Conversely, if $E((\lambda - \varepsilon, \lambda + \varepsilon))$ is finite-dimensional for some $\varepsilon > 0$ and λ belongs to the spectrum, i.e., $E((\lambda - \varepsilon', \lambda + \varepsilon') \neq 0$ for all $\varepsilon' > 0$ by (a), we infer from (b) that λ must be an eigenvalue of finite multiplicity.

(d) This follows immediately from (c) and the definition of the essential spectrum as the complement of the discrete spectrum within the spectrum. □

2.2. Weyl sequences and compact perturbations. We give characterizations for a value to be in the spectrum and essential spectrum of an operator via sequences of functions. We then give consequences for compact perturbations of an operator.

A famous criterion for a number λ to be in the spectrum of L states that $\lambda \in \sigma(L)$ if and only if there exists a normalized sequence (f_n) in $D(L)$ such that

$$\lim_{n\to\infty} \|(L - \lambda)f_n\| = 0,$$

see Corollary A.11. This criterion goes back to work of Weyl [**Wey10**]. Consequently, the sequence (f_n) is called a *Weyl sequence* and the criterion for the existence of such a sequence is called *Weyl's criterion*. We now introduce a variant of the Weyl sequence criterion which is adapted from [**Sto01**]. In contrast to the above, the formulation we give here is in terms of the form and, therefore, (f_n) only has to be in the form domain.

To this end, we assume additionally that L is positive and denote the form associated to L by Q. We recall that in this case $D(Q) = D(\sqrt{L})$ with

$$Q(f, g) = \langle Lf, g \rangle$$

for all $f \in D(L)$ and $g \in D(Q)$. Furthermore, as usual we denote the form norm arising from Q by

$$\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$$

for $f \in D(Q)$.

Theorem E.3 (Weyl's criterion – spectrum). *Let Q be a positive closed form on H, let L be the associated operator and let $\lambda \in \mathbb{R}$. Then, $\lambda \in \sigma(L)$ if and only if there exists a normalized sequence (f_n) in $D(Q)$ such that*

$$\lim_{n\to 0} \sup_{g \in D(Q), \|g\|_Q = 1} (Q - \lambda)(f_n, g) = 0.$$

Proof. If $\lambda \in \sigma(L)$, then by Corollary A.11 there exists a normalized sequence (f_n) in $D(L) \subseteq D(Q)$ such that $\|(L-\lambda)f_n\| \to 0$ as $n \to \infty$. Then, for all $g \in D(Q)$ with $\|g\|_Q = 1$, we have

$$|(Q-\lambda)(f_n, g)| = |\langle (L-\lambda)f_n, g\rangle| \leq \|(L-\lambda)f_n\| \to 0$$

as $n \to \infty$.

Conversely, suppose that (f_n) is a normalized sequence in $D(Q)$ which satisfies $\sup_{g\in D(Q),\|g\|_Q=1}(Q-\lambda)(f_n,g) \to 0$ as $n \to \infty$ and suppose that $\lambda \not\in \sigma(L)$. Then, $(L-\lambda)^{-1}$ is a bounded operator and, thus, $h_n = (L-\lambda)^{-1}f_n$ for $n \in \mathbb{N}$ is a bounded sequence in $D(L)$. Furthermore, (h_n) is also bounded with respect to $\|\cdot\|_Q$ as

$$\|h_n\|_Q^2 = \langle Lh_n, h_n\rangle + \|h_n\|^2 = \langle f_n, h_n\rangle + (1+\lambda)\|h_n\|^2 \leq C$$

for some constant C and all $n \in \mathbb{N}$. Thus, we obtain the contradiction

$$\begin{aligned} 1 = \|f_n\|^2 &= (Q-\lambda)\langle f_n, (L-\lambda)^{-1}f_n\rangle = (Q-\lambda)\langle f_n, h_n\rangle \\ &\leq C^{1/2} \sup_{g\in D(Q),\|g\|_Q=1} (Q-\lambda)(f_n, g) \to 0 \end{aligned}$$

as $n \to \infty$. Therefore, $\lambda \in \sigma(L)$, which completes the proof. $\square$

Next, we present a Weyl's criterion for $\lambda \in \sigma(L)$ to be in the essential spectrum. To this end, we additionally need that the Weyl sequence converges weakly to zero. We recall that a sequence (f_n) is said to *converge weakly* to f in H if $\langle g, f_n\rangle \to \langle g, f\rangle$ for all $g \in H$. In particular, any orthonormal sequence converges weakly to 0.

A crucial feature of a weak convergence is the following: Whenever f_n converges weakly to f and P is the projection onto a finite-dimensional subspace, then

$$\lim_{n\to\infty} Pf_n = Pf.$$

Indeed, P can be written as

$$P = \sum_{k=1}^{N} \langle g_k, \cdot\rangle g_k$$

for $N \in \mathbb{N}$ and normalized pairwise orthogonal g_k for $k = 1, \ldots, N$. This easily gives the claim.

Theorem E.4 (Weyl's criterion – essential spectrum). *Let Q be a positive closed form on H, let L be the associated operator and let $\lambda \in \mathbb{R}$. Then, $\lambda \in \sigma_{\mathrm{ess}}(L)$ if and only if there exists a normalized sequence (f_n) in $D(Q)$ converging weakly to zero such that*

$$\lim_{n\to\infty} \sup_{g\in D(Q),\|g\|_Q=1} (Q-\lambda)(f_n, g) = 0.$$

Proof. Let $\lambda \in \sigma_{\mathrm{ess}}(L)$. Then, for any sequence $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ as $n \to \infty$ we have that the range of E_nH is infinite-dimensional by Proposition E.2, where $E_n = E((\lambda-\varepsilon_n, \lambda+\varepsilon_n))$. Letting

$f_n \in E_n H$ for $n \in \mathbb{N}$ be orthonormal we obtain a sequence which is weakly convergent to zero. By Proposition A.29, the spectral measure μ_{f_n} is supported on $[\lambda - \varepsilon_n, \lambda + \varepsilon_n]$ and $f_n \in D(L) \subseteq D(Q)$. Moreover, by Proposition A.28 and the fact that (f_n) is orthonormal we get

$$\mu_{f_n}([\lambda - \varepsilon_n, \lambda + \varepsilon_n]) \leq \|f_n\|^2 = 1.$$

Combining these facts and using Proposition A.24 (a) we obtain

$$\|(L - \lambda) f_n\|^2 = \int_{\lambda - \varepsilon_n}^{\lambda + \varepsilon_n} (x - \lambda)^2 d\mu_{f_n} \leq \varepsilon_n^2 \to 0$$

as $n \to \infty$. As in the proof of Theorem E.3, for all $g \in D(Q)$ with $\|g\|_Q = 1$ we thus obtain

$$|(Q - \lambda)(f_n, g)| = |\langle (L - \lambda) f_n, g \rangle| \leq \|(L - \lambda) f_n\| \to 0$$

as $n \to \infty$. Thus, (f_n) has the desired properties.

Conversely, let (f_n) in $D(Q)$ be a normalized sequence converging weakly to zero such that $(Q - \lambda)(f_n, g) \to 0$ as $n \to \infty$ uniformly with respect to $\|g\|_Q = 1$. Suppose that λ is not in the essential spectrum and let $\varepsilon > 0$ be such that the space $H_\varepsilon = E_\varepsilon H = E((\lambda - \varepsilon, \lambda + \varepsilon))H$ is finite-dimensional. By construction, $(L - \lambda)$ has a bounded inverse on $H_\varepsilon^\perp = (E_\varepsilon H)^\perp$. We argue next that (f_n) can be chosen to be in $H_\varepsilon^\perp$.

By the discussion before the statement of the theorem, we have $E_\varepsilon f_n \to 0$ as $f_n \to 0$ weakly for $n \to \infty$. As $(L - \lambda)$ is a bounded operator on the finite-dimensional space H_ε, this implies

$$|(Q - \lambda)(E_\varepsilon f_n, g)| = |\langle (L - \lambda) E_\varepsilon f_n, g \rangle| \leq C \|E_\varepsilon f_n\| \|g\| \to 0$$

as $n \to \infty$. Thus, by considering $f_n = E_\varepsilon f_n + (E_\varepsilon f_n)^\perp$, we can assume without loss of generality that $E_\varepsilon f_n = 0$, i.e., $f_n \in H_\varepsilon^\perp$. Therefore,

$$h_n = ((L - \lambda)|_{H_\varepsilon^\perp})^{-1} f_n$$

exists, is in $D(L)$ and is bounded with respect to $\| \cdot \|_Q$ as in the proof of Theorem E.3, i.e.,

$$\|h_n\|_Q^2 = \langle f_n, h_n \rangle + (1 + \lambda) \|h_n\|^2 \leq C$$

for some $C \geq 0$. Hence, with $g_n = h_n / \|h_n\|_Q$, we obtain the contradiction

$$1 = \|f_n\|^2 = |(Q - \lambda)(f_n, h_n)| \leq C^{1/2} |(Q - \lambda)(f_n, g_n)| \to 0$$

as $n \to \infty$. This proves the statement. $\square$

We refer to the sequences (f_n) appearing in Theorems E.3 and E.4 also as *Weyl sequences*.

From the proof of the result above, we get the following criterion for a number to be in the essential spectrum in terms of Weyl sequences. We also refer to this as Weyl's criterion and note that we do not require positivity of the operator as we do not work with forms in this formulation.

Corollary E.5 (Weyl's criterion – essential spectrum). *Let L be a self-adjoint operator on H and let $\lambda \in \mathbb{R}$. Then, $\lambda \in \sigma_{\mathrm{ess}}(L)$ if and only if there exists a normalized sequence (f_n) in $D(L)$ converging weakly to zero such that*

$$\lim_{n\to\infty} \|(L-\lambda)f_n\| = 0.$$

As a direct consequence, we now show that the essential spectrum does not change under compact perturbations. We recall that an operator $A\colon H \longrightarrow H$ with $D(A) = H$ is called *compact* if A maps bounded sets to relatively compact sets, i.e., to sets whose closure is compact. We now give a consequence of this definition.

Lemma E.6. *Let A be a compact operator on H. Then, A maps weakly convergent sequences to convergent sequences.*

Proof. Let (f_n) be a weakly convergent sequence with weak limit f. Then, the set $\{f_n\}$ is bounded by the uniform boundedness principle. Since A is compact, the set $\overline{\{Af_n \mid n \in \mathbb{N}\}}$ is compact. Moreover, the set is clearly separable. Hence, any sequence in this set has a convergent subsequence. In particular, (Af_n) and each of its subsequences has a convergent subsequence. Weak convergence of (f_n) then implies that any of these convergent subsequences must have the same limit: Indeed, let g be the limit of the subsequence Af_{n_k}. Then, we find

$$\langle g, h\rangle = \lim_{k\to\infty} \langle Af_{n_k}, h\rangle = \lim_{k\to\infty} \langle f_{n_k}, A^*h\rangle = \langle f, A^*h\rangle$$

for any $h \in H$. So, g only depends on f and is independent of the subsequence. This gives convergence of the sequence itself. □

Remark. On a separable Hilbert space one can also show the converse, i.e., that an operator is compact if and only if it maps weakly convergent sequences to convergent sequences. On arbitrary Hilbert spaces, an analogous characterization holds if sequences are replaced by nets. We refrain from giving details.

Given an operator L and a bounded operator A on H, we may define the operator $L + A$ with domain $D(L + A) = D(L)$ and $(L + A)f = Lf + Af$ for all $f \in D(L + A)$. Given these notions, we easily obtain the following statement.

Theorem E.7 (Stability of essential spectrum under compact perturbation). *Let A and L be self-adjoint operators on H. If A is compact, then*

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L + A).$$

Proof. The proof follows immediately from Corollary E.5 and Lemma E.6. □

Remark (Characterization of essential spectrum by compact perturbations). Theorem E.7 shows that the essential spectrum is stable under compact perturbation. This is not just a feature of the essential spectrum but can be considered as a characterizing property of the essential spectrum. There are various ways to make this precise. Here, we note that

$$\sigma_{\mathrm{ess}}(L) = \bigcap_A \sigma(L + A),$$

where the intersection is taken over all compact self-adjoint operators A.

Indeed, the inclusion $\subseteq$ follows easily from the previous theorem since

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(L + A) \subseteq \sigma(L + A)$$

for all compact operators A.

The inclusion $\supseteq$ follows as $\sigma(L) \supseteq \bigcap_A \sigma(L + A)$ since $A = 0$ is a compact operator and, for any $\lambda \in \sigma_{\mathrm{disc}}(L)$, we have $\lambda \notin \sigma(L - P)$ for the finite-dimensional and, thus, compact projection P onto the eigenspace of λ.

2.3. The min-max principle. We first prove a variational characterization for the bottom of the spectrum of a positive operator. We then state and prove a general min-max principle and discuss an application to non-linear functions of operators.

We start by proving a characterization of the bottom of the spectrum of a positive operator. This is a special case of the general min-max principle which we prove later. However, as it is the most commonly appearing manifestation of the variational principle and the proof is rather straightforward, we establish it first before proceeding to the more general statement. The equality we show here is also sometimes referred to as the Rayleigh–Ritz formula.

THEOREM E.8 (Variational characterization of λ_0). *Let Q be a positive closed form and let L be the associated operator on H. Let $\lambda_0(L)$ denote the bottom of the spectrum of L. Then,*

$$\lambda_0(L) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle = \inf_{f \in D(Q), \|f\|=1} Q(f).$$

Furthermore, if $f \in D(Q)$ is normalized and satisfies $Q(f) = \lambda_0(L)$, then $f \in D(L)$ and $Lf = \lambda_0(L)f$, i.e., f is a normalized eigenfunction corresponding to the eigenvalue $\lambda_0(L)$.

PROOF. The second equality is clear from the connection between the form and the operator and the fact that $D(L)$ is dense with respect to the form norm $\|\cdot\|_Q$ by Corollary B.12. Hence, we focus on proving the first equality. In order to do so, we will show two inequalities.

To this end, we recall that by Proposition A.24 (b) we have

$$\langle f, Lf \rangle = \int x d\mu_f,$$

where the integral is over the support of the spectral measure μ_f for $f \in D(L)$. Furthermore, Proposition A.15 gives $\mu_f(\sigma(L)) = \|f\|^2$.

Now, we let $\lambda_0 = \lambda_0(L)$ and let $f \in D(L)$ be normalized. As $\sigma(L) \subseteq [\lambda_0, \infty)$, we obtain

$$\langle f, Lf\rangle = \int_{\lambda_0}^{\infty} x d\mu_f \geq \lambda_0 \int_{\lambda_0}^{\infty} d\mu_f = \lambda_0 \|f\|^2 = \lambda_0.$$

This shows $\lambda_0 \leq \inf\langle f, Lf\rangle$ for all normalized $f \in D(L)$.

Conversely, since $\lambda_0 \in \sigma(L)$, we have $E([\lambda_0, \lambda_0 + \varepsilon)) \neq 0$ for all $\varepsilon > 0$ by Proposition E.2 (a). Hence, for every $\varepsilon > 0$ there exists a normalized f with $f = E([\lambda_0, \lambda_0 + \varepsilon))f$. By Proposition A.29, f has spectral measure supported on $[\lambda_0, \lambda_0+\varepsilon]$ and $f \in D(L)$. We then find

$$\langle f, Lf\rangle = \int_{\lambda_0}^{\lambda_0+\varepsilon} x d\mu_f \leq (\lambda_0 + \varepsilon)\|f\|^2 = \lambda_0 + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this gives $\lambda_0 \geq \inf\langle f, Lf\rangle$ for all normalized $f \in D(L)$.

For the furthermore statement, suppose that $f \in D(Q) = D(\sqrt{L})$ is normalized and satisfies $Q(f) = \lambda_0$. By Proposition A.24 (a), we now get

$$0 = Q(f) - \lambda_0 = \|\sqrt{L}f\|^2 - \lambda_0\|f\|^2 = \int_{\lambda_0}^{\infty} (x - \lambda_0) d\mu_f.$$

As the integrand is non-negative, μ_f is supported on $\{\lambda_0\}$ which, by Proposition E.2 (b), is equivalent to λ_0 being an eigenvalue with eigenfunction f. □

The characterization above deals with the bottom of the spectrum. We will now present a generalization that will capture all eigenvalues below the bottom of the essential spectrum.

We recall that for $t \in \mathbb{R}$, we let

$$E_t = E((-\infty, t]) = 1_{(-\infty,t]}(L)$$

denote the spectral family associated to L. We will first show how the dimension of the range of the spectral family is related to a generalization of the variational constant appearing in the previous theorem.

Lemma E.9. *Let Q be a positive closed form and let L be the associated operator on H. For $n \in \mathbb{N}$, set*

$$\nu_n(L) = \sup_{\varphi_1,\ldots,\varphi_n \in H} \inf_{\substack{f\in\{\varphi_1,\ldots,\varphi_n\}^{\perp}\cap D(L)\\ \|f\|=1}} \langle f, Lf\rangle$$

and set $\nu_0(L) = \inf_{f\in D(L),\|f\|=1}\langle f, Lf\rangle$.

(a) *If $t < \nu_n$, then* dim Ran $E_t \leq n$.

(b) *If $t > \nu_n$, then* dim Ran $E_t \geq n + 1$.

PROOF. We prove both (a) and (b) by contraposition. Suppose dim Ran $E_t > n$. Then, for any $\varphi_1, \ldots, \varphi_n \in H$, there exists an $f \in$ Ran E_t such that $\|f\| = 1$ and $f \in \{\varphi_1, \ldots, \varphi_n\}^\perp$. Since $f \in$ Ran E_t, it follows by Proposition A.29 that $f \in D(L)$ and by Proposition A.24 we get

$$\langle f, Lf \rangle = \int_{\sigma(L)} x d\mu_f = \int_0^t x d\mu_f \leq t\|f\|^2 = t.$$

This gives $\nu_n \leq t$.

Similarly, for (b) suppose that dim Ran $E_t \leq n$. Choose $\varphi_1, \ldots \varphi_n \in H$ such that the span of $\{\varphi_1, \ldots \varphi_n\}$ is Ran E_t. Then, for every $f \in D(L)$ with $\|f\| = 1$ and $f \in \{\varphi_1, \ldots, \varphi_n\}^\perp$, it follows that $f \in$ Ran $E((t, \infty))$. Therefore, by Proposition A.29 and Proposition A.24

$$\langle f, Lf \rangle = \int_t^\infty x d\mu_f \geq t,$$

which implies $\nu_n \geq t$. This completes the proof. □

Let $n(L) \in \mathbb{N}_0 \cup \{\infty\}$ be the dimension of the range of the spectral projection below the infimum of the essential spectrum, i.e.,

$$n(L) = \dim \text{ Ran } E((-\infty, \lambda_0^{\text{ess}}(L)).$$

We note that $n(L)$ is the number of isolated eigenvalues, counted with multiplicity, below the bottom of the essential spectrum. In particular, if $\lambda_0(L) < \lambda_0^{\text{ess}}(L)$, then we denote the eigenvalues below $\lambda_0^{\text{ess}}(L)$ by $\lambda_n(L)$ for $0 \leq n < n(L)$ in increasing order counted with multiplicity.

The following result gives a way of calculating the eigenvalues below the bottom of the essential spectrum as well as the bottom of the essential spectrum. It contains Theorem E.8 as the case $n = 0$.

THEOREM E.10 (Min-max principle). *Let Q be a positive closed form and let L be the associated operator on H. For $n \in \mathbb{N}$, set*

$$\nu_n(L) = \sup_{\varphi_1, \ldots, \varphi_n \in H} \inf_{\substack{f \in \{\varphi_1, \ldots, \varphi_n\}^\perp \cap D(L) \\ \|f\|=1}} \langle f, Lf \rangle$$

and $\nu_0(L) = \inf_{f \in D(L), \|f\|=1} \langle f, Lf \rangle$. Then, for $0 \leq n < n(L)$, we have

$$\nu_n(L) = \lambda_n(L).$$

Moreover, if $n(L) < \infty$, then

$$\nu_n(L) = \lambda_0^{\text{ess}}(L)$$

for $n \geq n(L)$ and if $n(L) = \infty$, then

$$\nu_n(L) \nearrow \lambda_0^{\text{ess}}(L)$$

as $n \to \infty$.

PROOF. We let ν_n denote $\nu_n(L)$ and λ_0^{ess} denote $\lambda_0^{\text{ess}}(L)$. We show a series of claims.

Claim 1. $\nu_n \leq \lambda_0^{\text{ess}}$ for all $n \in \mathbb{N}_0$.

Proof of Claim 1. Assume the contrary. Then, there exists $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $\lambda_0^{\text{ess}} + \varepsilon < \nu_n$. Lemma E.9 then gives

$$\dim \text{Ran } E_{\lambda_0^{\text{ess}}+\varepsilon} \leq n,$$

which contradicts $\lambda_0^{\text{ess}} \in \sigma_{\text{ess}}(L)$ by Proposition E.2 (d).

Claim 2. For each $n \in \mathbb{N}$, there are at most n eigenvalues strictly less than ν_n.

Proof of Claim 2. Assume that $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_k$ are eigenvalues with $\lambda_k < \nu_n$ and $k \geq n$. Then, for $\varepsilon > 0$ such that $\lambda_k + \varepsilon < \nu_n$, we have $n + 1 \leq k + 1 \leq \dim \text{Ran } E_{\lambda_k+\varepsilon}$, yielding a contradiction to Lemma E.9.

Claim 3. $\nu_n \in \sigma(L)$ for all $n \in \mathbb{N}_0$.

Proof of Claim 3. Combining the two implications in Lemma E.9, we see that $E((\nu_n - \varepsilon, \nu_n + \varepsilon)) \neq 0$ for all $\varepsilon > 0$, that is, $\nu_n \in \text{supp}(E)$. Therefore, by Proposition E.2 (a), we have $\nu_n \in \sigma(L)$ for all $n \in \mathbb{N}_0$.

From Claim 3 and the decomposition of the spectrum into discrete and essential parts, we infer that ν_n either belongs to the discrete spectrum or to the essential spectrum. We analyze these two cases separately in the subsequent two claims.

Claim 4. If $\nu_n \in \sigma_{\text{ess}}(L)$, then for all $k \in \mathbb{N}_0$ we have

$$\nu_{n+k} = \lambda_0^{\text{ess}}.$$

Proof of Claim 4. By $\nu_n \in \sigma_{\text{ess}}(L)$, the definition of ν_k and Claim 1 we find

$$\lambda_0^{\text{ess}} \leq \nu_n \leq \nu_{n+k} \leq \lambda_0^{\text{ess}}$$

for all $k \in \mathbb{N}_0$. This gives the desired statement.

Claim 5. If $\nu_n \in \sigma_{\text{disc}}(L)$, then ν_k is the k-th isolated eigenvalue below the infimum of the essential spectrum counted with multiplicity.

Proof of Claim 5. From Claim 4, we find $\nu_k \in \sigma_{\text{disc}}(L)$ for all $k = 0, 1, \ldots, n$. Now, consider $k \in \{0, \ldots, n\}$. As ν_k is isolated in the spectrum, there exists an $\varepsilon > 0$ such that $(\nu_k - \varepsilon, \nu_k + \varepsilon) \cap \sigma(L) = \{\nu_k\}$ and thus

$$\dim \text{Ran } E_{\nu_k} = \dim \text{Ran } E_{\nu_k+\varepsilon} \geq k + 1$$

by Lemma E.9. Therefore, there are at least $k + 1$ eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_k \leq \nu_k$. On the other hand, by Claim 2, there are at most k eigenvalues strictly below ν_k and the proof is finished.

The statement of the theorem concerning ν_n for $0 \leq n < n(L)$ follows immediately from Claim 5. Given this, the last statement of the theorem then follows from Claim 4 and the fact that, if the monotone increasing ν_n converge, their limit must belong to the essential spectrum as an accumulation point of the spectrum and, hence, be the infimum of the essential spectrum. □

We now apply the min-max principle above to estimate the eigenvalues when applying functions to the value of a form and to the eigenvalues. For a positive closed form Q with domain $D(Q)$ we call $D_0 \subseteq D(Q)$ a *form core* if D_0 is dense in $D(Q)$ with respect to the form norm. In particular, as $D(L)$ is dense in $D(Q)$ in the form norm by Corollary B.12, it follows that we can choose the functions appearing in the definition of $\nu_n(L)$ in the min-max principle to be in the form core. This will be used in the proof below.

THEOREM E.11 (Generalized min-max principle). *Let $(Q_1, D(Q_1))$ and $(Q_2, D(Q_2))$ be positive closed forms with a common form core D_0 and let L_1 and L_2 be the associated operators. Assume that there exist continuous monotonically increasing functions $f_1, f_2 \colon [\lambda_0(L_2), \infty) \longrightarrow \mathbb{R}$ such that for all $\varphi \in D_0$ with $\|\varphi\| = 1$*

$$f_1(Q_2(\varphi)) \leq Q_1(\varphi) \leq f_2(Q_2(\varphi)).$$

Then, for $0 \leq n < \min(n(L_1), n(L_2))$,

$$f_1(\lambda_n(L_2)) \leq \lambda_n(L_1) \leq f_2(\lambda_n(L_2)).$$

Moreover, if $\lim_{r\to\infty} f_1(r) = \lim_{r\to\infty} f_2(r) = \infty$, then $\sigma_{\mathrm{ess}}(L_1) = \emptyset$ if and only if $\sigma_{\mathrm{ess}}(L_2) = \emptyset$.

PROOF. By the min-max principle, Theorem E.10, we know for a self-adjoint operator L and $n \in \mathbb{N}_0$ that $\nu_n(L) = \lambda_n(L)$ if $\nu_n(L) < \lambda_0^{\mathrm{ess}}(L)$ and $\nu_n(L) = \lambda_0^{\mathrm{ess}}(L)$ otherwise. Assume $n < \min\{n(L_1), n(L_2)\}$ and let $\varphi_0^{(j)}, \ldots, \varphi_{n-1}^{(j)}$ be the eigenfunctions of L_j corresponding to $\lambda_0(L_j), \ldots, \lambda_{n-1}(L_j)$ for $j = 1, 2$. Let

$$U_j^{(n)} = \{\varphi_0^{(j)}, \ldots, \varphi_{n-1}^{(j)}\}^\perp \cap \{\psi \in D_0 \mid \|\psi\| = 1\}$$

for $j = 1, 2$.

Now, observe that for a continuous monotonically increasing function
$f \colon [0, \infty) \longrightarrow \mathbb{R}$ and a function $g \colon X \longrightarrow [0, \infty)$ defined on an arbitrary set X we have

$$\inf_{x\in X} f(g(x)) = f\left(\inf_{x\in X} g(x)\right).$$

We apply this observation with $f = f_1$ and $g = g_1 \colon U_2^{(n)} \longrightarrow [0, \infty)$ given by $g(\psi) = Q_1(\psi)$ first and $f = f_2$ and $g = g_2 \colon U_1^{(n)} \longrightarrow [0, \infty)$ given by $g_2(\psi) = Q_2(\psi)$ second to obtain

$$f_1(\lambda_n(L_2)) = f_1\left(\inf_{\psi\in U_2^{(n)}} Q_2(\psi)\right) = \inf_{\psi\in U_2^{(n)}} f_1\left(Q_2(\psi)\right) \leq \inf_{\psi\in U_2^{(n)}} Q_1(\psi)$$

$$
\begin{aligned}
&\le \nu_n(L_1) = \lambda_n(L_1) = \inf_{\psi \in U_1^{(n)}} Q_1(\psi) \\
&\le \inf_{\psi \in U_1^{(n)}} f_2\left(Q_2(\psi)\right) = f_2\left(\inf_{\psi \in U_1^{(n)}} Q_2(\psi)\right) \\
&\le f_2(\nu_n(L_2)) = f_2(\lambda_n(L_2)).
\end{aligned}
$$

This directly implies the first statement.

If $\lambda_0^{\mathrm{ess}}(L_2) = \infty$, then $n(L_2) = \infty$ and $\lim_{n\to\infty} \lambda_n(L_2) = \infty$. Therefore, $\lim_{n\to\infty} f_1(\lambda_n(L_2)) = \infty$ by the assumption on f_1. Hence, by the above we get $\lambda_0^{\mathrm{ess}}(L_1) = \infty$. The other implication follows analogously. $\square$

2.4. The bottom of the essential spectrum. We end this section by giving a bound on the bottom of the essential spectrum of a positive operator.

THEOREM E.12 (Persson theorem). *Let Q be a positive closed form on H and let L be the associated operator. Assume that there exists a normalized sequence (f_n) in $D(Q)$ that converges weakly to zero in H. Then,*

$$\lambda_0^{\mathrm{ess}}(L) \le \liminf_{n\to\infty} Q(f_n).$$

PROOF. The inequality is trivial if $\lambda_0^{\mathrm{ess}}(L) = 0$ as $Q(f) \ge 0$ for all $f \in D(Q)$. Hence, we assume that $\lambda_0^{\mathrm{ess}}(L) > 0$ and let $0 < \lambda < \lambda_0^{\mathrm{ess}}(L)$. We will show that there exists an $N \ge 0$ such that $Q(f_n) > \lambda$ for all $n \ge N$.

Let λ_1 be such that $\lambda < \lambda_1 < \lambda_0^{\mathrm{ess}}(L)$ and let $\varepsilon > 0$ be arbitrary. As $\lambda_1 < \lambda_0^{\mathrm{ess}}(L)$, the range of the spectral projection $E_{\lambda_1} = E((-\infty, \lambda_1])$ is finite-dimensional by Proposition E.2 (d). Hence, E_{λ_1} is a compact operator. Therefore, as (f_n) converges weakly to zero, $\|E_{\lambda_1} f_n\| \to 0$ as $n \to \infty$ by Lemma E.6. Thus, there exists an $N \ge 0$ such that $\|E_{\lambda_1} f_n\|^2 < \varepsilon$ for $n \ge N$. Letting μ_{f_n} be the spectral measure associated to f_n, we estimate by using Proposition A.24 (a) and Proposition A.28 that, for $n \ge N$,

$$
\begin{aligned}
Q(f_n) = \|L^{1/2} f_n\|^2 = \int_{\sigma(L)} x \, d\mu_{f_n} &\ge \int_{\lambda_1}^{\infty} x \, d\mu_{f_n} \\
&\ge \lambda_1 \int_{\lambda_1}^{\infty} d\mu_{f_n} \\
&= \lambda_1 (\|f_n\|^2 - \|E_{\lambda_1} f_n\|^2) \\
&> \lambda_1 (1 - \varepsilon).
\end{aligned}
$$

We conclude the asserted inequality by choosing $\varepsilon = (\lambda_1 - \lambda)/\lambda_1 > 0$. This completes the proof. $\square$

Remark. We note that the theorem above can also be easily deduced from the min-max principle. However, since we use it independently of the min-max principle, we give an independent proof.

3. Reducing subspaces and commuting operators

In this section we study the notion of commuting operators. In particular, we show that a bounded operator commutes with a symmetric operator if and only if the resolvent or the semigroup of the Friedrichs extension of the symmetric operator commute with the bounded operator. Along the way we introduce the notion of a reducing subspace for a symmetric operator.

We let H denote a Hilbert space. We recall that an operator L with dense domain $D(L) \subseteq H$ is called self-adjoint if $L = L^*$, where L^* denotes the adjoint of L. Furthermore, we call a self-adjoint operator L positive if $\sigma(L) \subseteq [0, \infty)$, where $\sigma(L)$ denotes the spectrum of L. In this case, we can use the spectral theorem to apply measurable functions with domain $[0, \infty)$ to the operator. See the discussion following Definition A.21 as well as Proposition A.24 for more details.

Whenever an operator is bounded, we assume that the domain of the operator is the entire Hilbert space H. We note, in particular, that if L is positive and φ is bounded on $[0, \infty)$, then $\varphi(L)$ is bounded and, thus, $D(\varphi(L)) = H$, see Corollary A.25. We recall that an operator L with domain $D(L)$ is called closed if whenever a sequence (f_n) in $D(L)$ converges to $f \in H$ and Lf_n converges to $g \in H$, then $f \in D(L)$ and $Lf = g$. In particular, this is always the case for self-adjoint operators. For example, whenever L is positive and $\varphi\colon [0, \infty) \longrightarrow \mathbb{R}$ is measurable, $\varphi(L)$ is self-adjoint and therefore closed.

In what follows, we will need the fact that the spectral calculus is well-behaved with respect to convergence of functions. A sufficient statement for our purposes is given in the next lemma, which was already shown as part of Lemma A.27.

Lemma E.13. *Let L be a positive operator on H. Suppose that*

$$\varphi_n, \varphi\colon [0, \infty) \longrightarrow \mathbb{R}$$

are measurable functions with $|\varphi_n(x)| \leq |\varphi(x)|$ and $\varphi_n(x) \to \varphi(x)$ as $n \to \infty$ for all $x \in [0, \infty)$. If $f \in D(\varphi(L))$, then

$$\lim_{n \to \infty} \varphi_n(L) f = \varphi(L) f.$$

Let A be a bounded operator on H. We say that a subspace $U \subseteq H$ is *invariant under A* if A maps U into U. We will be interested in the case when the domain of an operator is invariant under A and we can interchange the operator and A. This is characterized in various ways in the following theorem.

THEOREM E.14 (Characterization of commuting operators). *Let L be a positive operator on H with domain $D(L)$ and let A be a bounded operator on H. Then, the following statements are equivalent:*

(i) *$D(L)$ is invariant under A and $LA = AL$ on $D(L)$.*
(ii) *$D(L^{1/2})$ is invariant under A and $L^{1/2}A = AL^{1/2}$ on $D(L^{1/2})$.*
(iii) *$1_{[0,t]}(L)A = A1_{[0,t]}(L)$ for all $t \geq 0$.*
(iv) *$e^{-tL}A = Ae^{-tL}$ for all $t \geq 0$.*
(v) *$(L+\alpha)^{-1}A = A(L+\alpha)^{-1}$ for all $\alpha > 0$.*
(vi) *$\varphi(L)A = A\varphi(L)$ for all bounded measurable functions $\varphi\colon [0,\infty) \longrightarrow \mathbb{C}$.*

PROOF. (i) $\Longrightarrow$ (v): From (i) we get $A(L+\alpha)f = (L+\alpha)Af$ for all $\alpha \in \mathbb{R}$ and $f \in D(L)$. As L is positive, it follows that $(L+\alpha)$ is invertible for all $\alpha > 0$ and thus

$$(L+\alpha)^{-1}A = A(L+\alpha)^{-1}$$

on $D(L)$. As both A and $(L+\alpha)^{-1}$ are bounded, the equality can be extended to H. This is the desired conclusion.

We next show that (iii), (iv), (v) and (vi) are all equivalent:

(iii) $\Longrightarrow$ (iv): This follows by a simple approximation argument using Lemma E.13.

(iv) $\Longrightarrow$ (v): This follows immediately from the Laplace transform formula given in Theorem A.35, i.e.,

$$(L+\alpha)^{-1} = \int_0^\infty e^{-t\alpha}e^{-tL}dt.$$

(v) $\Longrightarrow$ (vi): The assumption (v) together with a Stone–Weierstrass argument shows $\psi(L)A = A\psi(L)$ for all continuous functions $\psi\colon [0,\infty) \longrightarrow \mathbb{C}$ which vanish at infinity, see Lemma A.16. Now, it is not hard to see by using Lemma E.13 that the set

$$\{\varphi\colon [0,\infty) \longrightarrow \mathbb{C} \mid \varphi \text{ measurable and bounded with } \varphi(L)A = A\varphi(L)\}$$

is closed under pointwise convergence of uniformly bounded sequences. This gives the desired statement (vi).

(vi) $\Longrightarrow$ (iii): This is obvious.

We finally show (ii) $\Longrightarrow$ (i) and (vi) $\Longrightarrow$ (ii):

(ii) $\Longrightarrow$ (i): If $f \in D(L)$, then $L^{1/2}f \in D(L^{1/2})$, see Lemma B.3. Therefore, by assumption we have $AL^{1/2}f \in D(L^{1/2})$ and

$$AL^{1/2}f = L^{1/2}Af.$$

Thus, as $L^{1/2}Af \in D(L^{1/2})$, we have $Af \in D(L)$ by Lemma B.3 again so that $D(L)$ is invariant under A. Now, $LAf = ALf$ is clear as $L = L^{1/2}L^{1/2}$.

(vi) $\Longrightarrow$ (ii): We denote the spectral family associated to L by E_n, i.e., $E_n = 1_{[0,n]}(L)$ and note that E_n maps into $D(L^{1/2})$ by Proposition A.29. Furthermore, by Proposition A.22, $L^{1/2}E_n$ is a bounded operator and thus commutes with A by assumption.

Let $f \in D(L^{1/2})$ and $f_n = E_n f$. Since A commutes with E_n we get

$$Af_n = AE_n f = E_n Af,$$

so that $Af_n \in D(L^{1/2})$ for all n. Furthermore,

$$L^{1/2}Af_n = L^{1/2}AE_n f = L^{1/2}E_n Af = AL^{1/2}E_n f.$$

Since $f_n \to f$ and $L^{1/2}E_n f \to L^{1/2}f$ by Lemma E.13, using that A is bounded we now get $Af_n \to Af$ and $L^{1/2}Af_n \to AL^{1/2}f$ as $n \to \infty$. As $L^{1/2}$ is closed, this gives $Af \in D(L^{1/2})$ and $L^{1/2}Af = AL^{1/2}f$ for all $f \in D(L^{1/2})$. This completes the proof. $\square$

The preceding theorem naturally leads to the following definition.

Definition E.15 (Commuting operators). Let L be a positive operator on H and let A be a bounded operator on H. We say that A *commutes* with L if one of the equivalent statements of Theorem E.14 holds.

We also note the following consequence of the preceding considerations.

Corollary E.16. *Let L be a positive operator on H and let A be a bounded operator on H. Then, A commutes with L if and only if A^* commutes with L.*

Proof. Suppose that A commutes with L. Let $f \in D(L^*) = D(L)$. We want to show that $A^*f \in D(L^*) = D(L)$. For this, it suffices to show that the mapping $g \mapsto \langle Lg, A^*f\rangle$ is bounded on $D(L)$. However, this follows from the fact that A is bounded, L is self-adjoint and that A and L commute as

$$\begin{aligned}|\langle Lg, A^*f\rangle| &= |\langle ALg, f\rangle| = |\langle LAg, f\rangle| = |\langle Ag, Lf\rangle| \\ &\leq \|Lf\|\|A\|\|g\|.\end{aligned}$$

Therefore, $D(L)$ is invariant under A^* and for $f \in D(L)$ we have

$$LA^*f = (AL)^*f = (LA)^*f = A^*Lf.$$

This shows that A^* commutes with L.

An analogous argument shows that if L commutes with A^*, then L commutes with A. This completes the proof. $\square$

Theorem E.14 deals with symmetries of a self-adjoint operator L. Often, the self-adjoint operator arises as the Friedrichs extension of a symmetric operator. We now recall this construction. We let L_0 be a symmetric operator on H with domain $D_0 = D(L_0)$. We let Q_0 be the form with domain $D(Q_0) = D_0$ acting via

$$Q_0(f, g) = \langle L_0 f, g\rangle$$

for $f, g \in D_0$. We assume that Q_0 is positive, i.e., $Q_0(f) = Q_0(f, f) \geq 0$ for all $f \in D_0$. Then, Q_0 is closable, i.e., there exists a closed extension of Q_0. We let Q be the closure of Q_0, that is, Q is the smallest closed extension of Q_0 and let $D(Q)$ be the domain of Q. Then the positive self-adjoint operator L which is associated to the closed form Q is called the Friedrichs extension of L_0. For more details on this construction, see Example B.13. In particular, we note that the domain $D(Q)$ is the closure in H of D_0 with respect to the form norm $\|\cdot\|_{Q_0}$, i.e.,

$$D(Q) = \overline{D(Q_0)}^{\|\cdot\|_{Q_0}},$$

where $\|f\|_{Q_0} = (Q_0(f) + \|f\|^2)^{1/2}$ for $f \in D_0$. Furthermore,

$$D(L) = \{f \in D(Q) \mid Q(f, h) = \langle g, h\rangle \text{ for all } h \in D(Q)\}$$

with $Lf = g$, see Theorem B.11, as well as $D(Q) = D(L^{1/2})$ with $Q(f) = \|L^{1/2} f\|^2$, see Lemma B.7.

We will now show that in the case when L is the Friedrichs extension, L commuting with a bounded operator A is equivalent to some compatibility conditions between A and the forms Q and Q_0.

LEMMA E.17. *Let L_0 be a symmetric operator with domain D_0, let Q_0 be the associated form and assume that Q_0 is positive. Let L be the Friedrichs extension of L_0 associated to the form Q. Let A be a bounded operator on H such that D_0 is invariant under both A and A^* with*

$$L_0 A = A L_0 \qquad \text{and} \qquad L_0 A^* = A^* L_0$$

on D_0. Then, the following statements are equivalent:

(i) *A commutes with L.*

(ii) *$D(Q)$ is invariant under both A and A^* and for all $f, g \in D(Q)$*

$$Q(Af, g) = Q(f, A^* g).$$

(iii) *There exists a constant $C \geq 0$ such that for all $f \in D_0$*

$$Q_0(Af) \leq C Q_0(f) \qquad \text{and} \qquad Q_0(A^* f) \leq C Q_0(f).$$

PROOF. (i) $\Longrightarrow$ (iii): From Theorem E.14 and the assumption we infer $L^{1/2} A f = A L^{1/2} f$ for all $f \in D(L^{1/2})$. Now, for $f \in D_0$ we have

$$Q_0(f) = \langle L_0 f, f\rangle = \langle L^{1/2} f, L^{1/2} f\rangle = \|L^{1/2} f\|^2.$$

As $Af \in D_0$ for $f \in D_0$, a direct calculation gives

$$Q_0(Af) = \|L^{1/2} A f\|^2 = \|A L^{1/2} f\|^2 \leq \|A\|^2 \|L^{1/2} f\|^2 = \|A\|^2 Q_0(f).$$

This gives the statement for Q_0 and A.

Now, from Corollary E.16 we get that A^* and L also commute. A similar argument then shows that $Q_0(A^* f) \leq \|A^*\|^2 Q_0(f)$ for $f \in D_0$. This finishes the proof.

(iii) $\Longrightarrow$ (ii): Since $L_0 A f = A L_0 f$ for all $f \in D_0$, we infer $Q_0(Af, g) = Q_0(f, A^* g)$ for all $f, g \in D_0$. As Q is the closure of Q_0, it now suffices to show that both (Af_n) and $(A^* f_n)$ are Cauchy sequences with respect to $\| \cdot \|_Q$ whenever (f_n) is a Cauchy sequence with respect to $\| \cdot \|_Q$ in D_0. This follows directly from (iii).

(ii) $\Longrightarrow$ (i): Let $f \in D(L) \subseteq D(Q)$. By (ii), $Af \in D(Q)$. Thus, we calculate, for all $g \in D(Q)$,

$$Q(Af, g) = Q(f, A^* g) = \langle Lf, A^* g \rangle = \langle ALf, g \rangle.$$

This implies $Af \in D(L)$ and $LAf = ALf$. Hence, we obtain (i). $\square$

We now turn to the special case when A is the projection onto a closed subspace. In this case, some further strengthening of the above result is possible. We first provide an appropriate definition for symmetric operators.

Definition E.18 (Reducing subspace). Let S be a symmetric operator on H with domain $D(S)$. If U is a closed subspace of H and P is the orthogonal projection onto U, then we call U a *reducing subspace for S* if $D(S)$ is invariant under P and

$$SPf = PSPf$$

for all $f \in D(S)$.

The previous definition is a commutation condition, as shown in the next lemma.

Lemma E.19. *Let S be a symmetric operator on H with domain $D(S)$ and let P be the orthogonal projection onto a closed subspace U of H. Then, the following statements are equivalent:*

(i) *U is a reducing subspace for S.*
(ii) *$D(S)$ is invariant under P and $SP = PS$ on $D(S)$.*

Proof. (ii) $\Longrightarrow$ (i): This is obvious as $P^2 = P$.

(i) $\Longrightarrow$ (ii): We first show that $PSf = 0$ for all $f \in D(S)$ with $Pf = 0$, that is, for f which are orthogonal to U. Let $g \in D(S)$. Then, as $Pf \in D(S) \subseteq D(S^*)$ we obtain

$$\begin{aligned}\langle PSf, g \rangle &= \langle Sf, Pg \rangle = \langle f, S^* Pg \rangle = \langle f, SPg \rangle = \langle f, PSPg \rangle \\ &= \langle Pf, SPg \rangle = 0.\end{aligned}$$

As $D(S)$ is dense, we infer $PSf = 0$.

Let now $f \in D(S)$ be arbitrary. Then, $f = Pf + (1 - P)f$ and both Pf and $(1 - P)f$ belong to $D(S)$. Thus, we calculate

$$PSf = PSPf + PS(1 - P)f = PSPf = SPf.$$

This finishes the proof. □

In particular, we note that if L is a positive operator, then U is a reducing subspace for L if and only if P commutes with L.

We now come to the main result of this section which ties symmetric operators, reducing subspaces and commuting operators together. In particular, we see that on a reducing subspace, both the semigroup and the resolvent of the Friedrichs extension of a symmetric operator commute with the projection onto the subspace.

Corollary E.20 (Characterization of reducing subspaces). *Let L_0 be a symmetric operator with domain D_0, let Q_0 be the associated form and assume that Q_0 is positive. Let L be the Friedrichs extension of L_0 associated to the form Q. Let U be a closed subspace of H and let A be the orthogonal projection onto U. Assume that D_0 is invariant under A. Then, the following statements are equivalent:*

(i) *U is a reducing subspace for L_0, i.e., $L_0 A = A L_0$ on D_0.*
(ii) *$Q_0(Af, Ag) = Q_0(Af, g) = Q_0(f, Ag)$ for all $f, g \in D_0$.*
(iii) *$D(Q)$ is invariant under A and*

$$Q(Af, Ag) = Q(Af, g) = Q(f, Ag)$$

for all $f, g \in D(Q)$.
(iv) *U is a reducing subspace for L, i.e., A commutes with L.*
(v) *A commutes with e^{-tL} for all $t \geq 0$.*
(vi) *A commutes with $(L + \alpha)^{-1}$ for all $\alpha > 0$.*

Proof. Obviously, (i) and (ii) are equivalent. The equivalence of (iii) and (iv) follows from the equivalence of (i) and (ii) in Lemma E.17. The equivalence between (iv), (v) and (vi) follows immediately from Theorem E.14. The implication (iii) $\Longrightarrow$ (ii) is clear as $AD_0 \subseteq D_0$ by assumption.

(ii) $\Longrightarrow$ (iii): A direct calculation using (ii) gives for all $f \in D_0$ that

$$\begin{aligned} Q_0(f) &= Q_0((A + (1 - A))f, f) \\ &= Q_0(Af, f) + Q_0((1 - A)f, f) \\ &= Q_0(Af) + Q_0((1 - A)f). \end{aligned}$$

As Q_0 is positive, this shows

$$Q_0(Af) \leq Q_0(f)$$

for all $f \in D_0$. Now, the implication (iii) $\Longrightarrow$ (ii) from Lemma E.17 gives (iii). □

4. The Riesz–Thorin interpolation theorem

The following theorem is used in several places when discussing bounded operators on ℓ^p spaces. We state it here without proof. We note that the theorem implies that any $f \in L^1(X, m) \cap L^\infty(X, m)$ is in $L^p(X, m)$ for all $p \in [1, \infty]$.

This result is the work of Riesz [**Rie27**] and Thorin [**Tho48**]. A proof of a more general result called the Stein interpolation theorem, which implies the Riesz–Thorin theorem, can be found in [**SW71**].

THEOREM E.21 (Riesz–Thorin interpolation theorem). *Let (X, m) be a σ-finite measure space. Let $p_n, q_n \in [1, \infty]$ for $n = 1, 2$ and let A be a linear operator from $L^{p_1}(X, m) \cap L^{p_2}(X, m)$ to $L^{q_1}(X, m) + L^{q_2}(X, m)$ which satisfies*

$$\|Af\|_{q_n} \leq C_n \|f\|_{p_n}$$

for all $f \in L^{p_1}(X, m) \cap L^{p_2}(X, m)$ and some C_n for $n = 1, 2$. Let $0 < t < 1$ and define p and q by

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}.$$

Then,

$$\|Af\|_q \leq C_1^{1-t} C_2^t \|f\|_p.$$

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