Isotropic Markov processes on ultrametric spaces

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DOCTORAL PROGRAM DISCRETE MATHEMATICS

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joint work with A. Bendikov, W. Cygan, A. Grigor'yan

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Ultrametric spaces

- (X, d) proper ($\equiv$ closed balls are compact) metric space is called ultrametric if instead of the triangle inequality it satisfies the stronger ultrametric inquality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all} \quad x,y,z \in X\,.$$

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- every ball

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is open and compact, and

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- $\Lambda_d(x) = \{d(x,y) : y \in X\,,\; y \neq x\}$ is countable, discrete in $(0\,,\,\infty)$. $r \in \Lambda_d(x) \Rightarrow \mathit{diam}\, B(x,r) = r$

Examples

- $X = G = \bigcup G_n$ direct limit of finite groups

  $$G_n \subset G_{n+1}\,,\; G_0 = \{id\}$$

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- $X = \mathbb{Q}_p$ field of p-adic numbers (ring, when p is not prime).

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- $X =$ space of all rooted graphs (Γ, o) with $\deg(.) \le M$

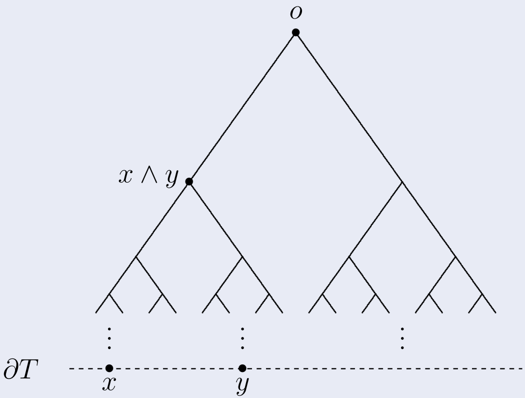
  $d(\Gamma,\Gamma') = 1\Big/\max\Big\{n : \Gamma(o,n) \simeq \Gamma'(o',n)\Big\}$

  ($\Gamma(o,n) = n$-ball in graph metric) non-discrete, compact

  Subspace of rooted trees with $\deg(.) \le M$

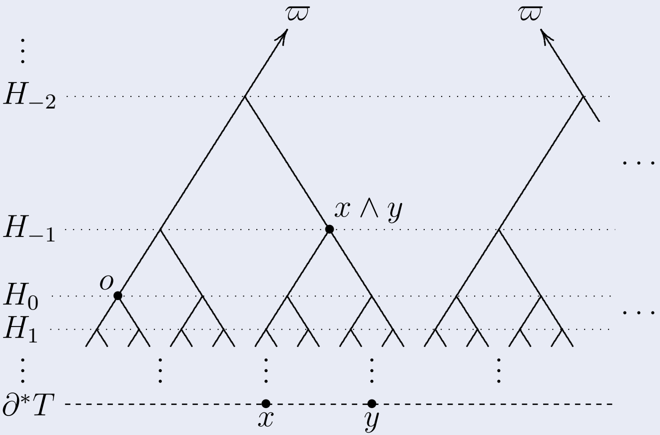
Example 1: tree

Every proper ultrametric space is the boundary of a tree !



$X = \partial T; \quad d(x,y) = 2^{-|x \wedge y|}$ compact, no isolated points.

Example 2: tree



$X = \partial^* T$ non-compact, no isolated points

Selection of previous work

Previous constructions of “Laplacians” and processes on ultrametric spaces:

TAIBLESON (1975) “Taibleson operator”: spectral multiplier on $\mathbb{Q}_p^n$.

VLADIMIROV (1988) “Vladimirov Laplacian” on $\mathbb{Q}_p^n$: sum of p-adic fractional derivatives (spectral multipliers) on factors $\mathbb{Q}_p$.

KOCHUBEI (2001 book) analysis of Vladimirov Laplacian.

FIGÀ-TALAMANCA (1994) and DEL MUTO AND FIGÀ-TALAMANCA (2004, 2006), also BALDI, CASADIO-TARABUSI AND FIGÀ-TALAMANCA (2001) use harmonic analysis to construct processes on homogeneous ultra-metric spaces.

ALBEVERIO AND KARWOWSKI (1994, 2008) construct processes via Chapman-Kolmogorov equations.

KIGAMI (2010, 2013) uses duality between trees and ultra-metric spaces.

PEARSON AND BELLISSARD (2009) via spectral triples.

Isotropic jump processes

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- Averaging operator on $L^1(X, \mathsf{m})$: for $r > 0$

$$Q_r f(x) = \frac{1}{\mathsf{m}\big(B_d(x,r)\big)} \int_{B_d(x,r)} f\,d\mathsf{m}$$

Note: for $r \in \Lambda_d(x)$ we have $Q_s f(x) = Q_r f(x)\,,\ s \in [r\,,\,r')$.

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- Transition operator (self-adjoint, bounded Markov operator)

$$Pf(x) = \int_{\mathbb{R}^+} Q_r f(x)\,d\sigma(r)\,.$$

Isotropic jump processes

- Crucial identity based on ultrametric inequality

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- Therefore: powers P^t, $t > 0$, are

$$P^t f(x) = \int_{\mathbb{R}^+} Q_r f(x)\, d\sigma^t(r),$$

where σ^t is the probability on $\mathbb{R}^+$ with distribution function $\sigma^t(r) = \sigma\big([0, r)\big)^t$.

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- $(P^t)_{t>0}$ is a strongly continuous Markov semigroup, gives rise to a Markov process $(\mathcal{X}_t)_{t\geq 0}$ on our ultrametric space: the (d, m, σ)-process.

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- THEOREM. P is self-adjoint on $\mathcal{L}^2(X,\mathrm{m})$ and has pure point spectrum

$$\big(\{0\}\cup\big)\quad \Big\{\sigma\big(\mathrm{diam}(B)\big) : B \in \mathcal{B}'\Big\} \cup \{1\}.$$

with complete system of compactly supported eigenfunctions.

Hierarchical Laplacian

- The infinitesimal generator of the Markov process $(\mathcal{X}_t)_{t\geq 0}$ is

$$Lf(x) = \sum_{r\in\Lambda_d(x)} \left(\log\frac{1}{\sigma(r)} - \log\frac{1}{\sigma(r')}\right)\Big(f(x) - Q_r f(x)\Big)$$

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- Considered in different contexts by more complicated methods by various authors , e.g. TAIBLESON (1975),

  VLADIMIROV (1988), ALBEVERIO AND KARWOWSKI (1994, 2008),

  KRITCHEVSKI (2007), ZUNIGA-GALINDO (2008,...)

Oscillations

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- Transition density (heat kernel)

$$P^t f(x) = \int_{\mathbb{R}^+} Q_r f(x)\, d\sigma^t(r) = \int_X p_t(x,y)\, f(y)\, dm(y)\,,$$

$$p_t(x,y) = \sum_{r \in \Lambda_d(x)\,:\, r > d(x,y)} \frac{\sigma^t(r) - \sigma^t(r_-)}{m\big(B(x,r_-)\big)}$$

r_- next smaller element than r in $\Lambda_d(x)$.

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- Step length distribution $\sigma_\alpha(r) = \exp\big(-(p/r)^\alpha\big)$, $\alpha > 0$.

  $\leftrightarrow$ p-adic fractional derivative [Vladimirov, 1988],

  resp. spectral multiplier [Taibleson, 1975] of order α.

Oscillations

$$p_t(x,x) = (p-1)\sum_{k\in\mathbb{Z}} p^{-k}\,\exp\Big(-t\,p^{-\alpha k}\Big) = t^{-1/\alpha}\,A\left(\frac{1}{\alpha}\log_p t\right),$$

with substitution $t = p^{\alpha u}$

$$A(u) = (p-1)\sum_{k\in\mathbb{Z}} p^{u+k}\,\exp\Big(-p^{\alpha(u+k)}\Big).$$

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- Amplitude: as $p \to \infty$,

$$\min_u A(u) \sim \frac{(\log p)^{1/\alpha}}{p} \quad \text{and} \quad \max_u A(u) \to (e\,\alpha)^{-1/\alpha}.$$

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  One-step operator $Pf = \mu * f$
- Our process is the random walk on S_∞ with law $\mu\,.$

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- Suppose $\pi(n) = \Lambda(n!)$,

  where $\Lambda(v) = v^{-\alpha}\,\phi(\log v)$, and ϕ is regularly varying at ∞ ;

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- Define $\rho(t) = \dfrac{\log t}{\alpha \log\log t}$ and

$$\psi(t) = t^{-1/\alpha} \cdot \phi\big(\rho(t)\log\rho(t)\big)^{-1/\alpha} \cdot \big(\log\rho(t)\big)^{1/\alpha} \quad \text{and}$$

$$\Psi(t) = t^{-1/\alpha} \cdot \phi\big(\rho(t)\log\rho(t)\big)^{-1/\alpha} \cdot (e\,\alpha)^{-1/\alpha}\rho(t)$$

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- Then

$$\limsup_{t\to\infty} \frac{p_t(x,x)}{\Psi(t)} = 1 \qquad \text{and} \qquad \liminf \limsup_{t\to\infty} \frac{p_t(x,x)}{\psi(t)} = 1$$

Remarks

- For random walks on certain finitely generated groups (lamplighter groups and generalisations), pure point spectrum with finitely supported eigenfunctions occurs, but no oscillations of return probabilities.

  E.g. Revelle (2003), Bartholdi and Woess (2005), Bartholdi, Neuhauser and Woess (2008).

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- For random walks on certain fractal graphs, one has both pure point spectrum with finitely supported eigenfunctions and oscillations of return probabilities.

  E.g. Grabner and Woess (1997), Teufl and Krön (2003).