Some recent results for Ollivier Ricci curvature on graphs

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Overview

- Introduction into Ollivier's coarse Ricci curvature
- Ollivier Ricci curvature on graphs and idleness
- Some classical results: *Bonnet-Myers and Lichnerowicz*
- Our Main Result: *"$p \mapsto \kappa_p$ has three linear pieces"*

Abstract: *Ollivier proposed in 2009 a curavture notion of Markov chains on metric spaces, based on optimal transport of probability measures associated to a random walk. In the special setting of graphs, this concept provides a curvature on the edges and depends on an idleness parameter of the random walk. Lin, Lu, and Yau modified this notion in 2011. In this talk, I will recall this curvature notion and present some specific results, which are based on joint work with D. Bourne, D. Cushing, R. Kangaslampi, Sh. Liu, and F. Muench.*

Introduction into Ollivier's coarse Ricci curvature

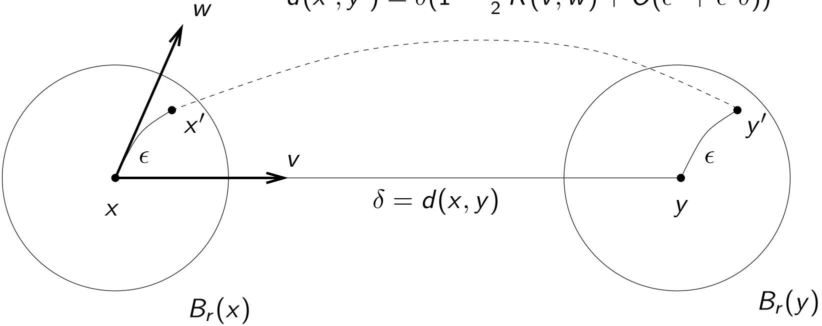
Based on moving "dirt" from here to there...

Motivation from Riemannian Geometry

(M, g) a complete, connected Riemannian manifold, $n = \dim(M)$.

Ollivier: *If $\mathrm{Ric}_x > 0$, the average distance of corresponding points in nearby balls of small radius $r > 0$ is smaller than the distance between their centers:*

$$d(x', y') = \delta(1 - \tfrac{\epsilon^2}{2} K(v, w) + O(\epsilon^3 + \epsilon^2 \delta))$$



where $v, w \in S_x M$, $K(v, w) = \langle R(v, w)w, v \rangle$, and taking average over the ball $B_r(x)$. $(R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$

Transport of balls and Ricci curvature

$A \subset M$ gives rise to a (probability) distribution

$$\mu_A(x) = \frac{1}{\mathrm{vol_g}(A)} \chi_A(x) d\mathrm{vol}_g(x).$$

Integrating $d(x', y') = \delta(1 - \frac{\epsilon^2}{2} K(v, w) + O(\epsilon^3 + \epsilon^2 \delta))$ over $B_r(x)$ yields:
Minimal cost $W_1(\mu_{B_r(x)}, \mu_{B_r(y)})$ to transport the distribution $\mu_{B_r(x)}$ to $\mu_{B_r y}$ is given by

$$W_1(\mu_{B_r(x)}, \mu_{B_r(y)}) \approx d(x, y) \left(1 - \frac{r^2}{2n(n+2)} \mathrm{Ric}(v)\right).$$

($W_1(\mu, \nu)$ called 1-*Wasserstein distance* between distributions μ, ν.)

Definition (Ollivier's coarse Ricci curvature, JFA 2009)

$x, y \in M$ nearby, $r > 0$ small. Then $\kappa(x, y)$ is defined as

$$\kappa(x, y) = 1 - \frac{W_1(\mu_{B_r(x)}, \mu_{B_r(y)})}{d(x, y)} \approx \frac{r^2}{2n(n+2)} \mathrm{Ric}(v).$$

Bringing in general context: Ricci curvature generalizations

$$\textbf{Ollivier(JFA2009)}: \quad \kappa(x,y) = 1 - \frac{W_1(\mu_{B_r(x)}, \mu_{B_r(y)})}{d(x,y)}$$

Advantage: Can be defined on arbitrary complete metric space (X, d).

Comparison with Sturm/Lott-Villani's definition: They define lower Ricci curvature bounds via convexity properties of certain entropy functions along Wasserstein geodesics (displacement convexity) in the associated 2-Wasserstein space.

Our Aim for rest of the talk: Investigate Ollivier's coarse Ricci curvature in the discrete setting of graphs. Other interesting curvature notions for discrete spaces and graphs: Bakry-Émery's CD-condition, CDE, CDE' (S.T. Yau and co-authors), $CD\psi$ (F. Münch), Erbar-Maas curvature.

Ollivier Ricci curvature on graphs and idleness

The importance of being idle...

Random walk on graph with idleness

Given: $G = (V, E)$ locally finite, connected, simple (= w/o loops and multiple edges) graph, d_x = degree of vertex $x \in V$, idleness parameter $p \in [0, 1]$.

Let $d : V \times V \to \mathbb{N} \cup \{0\}$ be the *combinatorial distance function*.

We replace the distributions $\mu_{B_r(x)}$ in the smooth setting by the probability measures

$$\mu_x^p(z) = \begin{cases} p, & \text{if } z = x, \\ \frac{1-p}{d_x}, & \text{if } z \sim x, \\ 0, & \text{otherwise}, \end{cases}$$

for each $x \in V$. They represent a (lazy) simple random walk on G with idleness p to stay at a vertex.

Next: Define 1-Wasserstein distance $W_1(\mu_x^p, \mu_y^p)$ and $\kappa_p(x, y)$ properly in this setting.

1-Wasserstein distance in the graph case

$\pi : V \times V \to [0, \infty)$ is a *transport plan* for probability measures $\mu_1 \to \mu_2$ if

$$\sum_{w \in V} \pi(z, w) = \mu_1(z) \quad \text{and} \quad \sum_{z \in V} \pi(z, w) = \mu_2(z),$$

where $\pi(z, w) =$ mass transported from z to w. The cost to do this is $\pi(z, w) d(z, w)$.

Set of all transport plans: $\Pi(\mu_1, \mu_2)$.

Then

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_x, \mu_y)} \sum_{z, w \in V} \pi(z, w) d(z, w),$$

where any π realising the infimum is called an *optimal transport plan*.

Ollivier's Ricci curvature for graphs

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1,\mu_2)} \sum_{z,w \in V} \pi(z,w) d(z,w),$$

is a distance function on the set of probability measures and

$$\kappa_p(x,y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x,y)}.$$

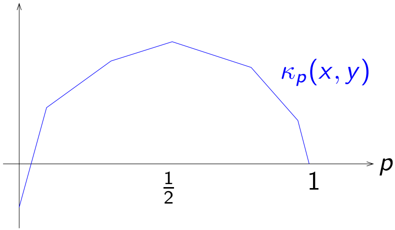
for any pair $x, y \in V$. If $x \sim y$, $\kappa(x,y) = 1 - W_1(\mu_x^p, \mu_y^p)$ can also considered as curvature of the edge $\{x,y\} \in E$.

Example (Lin/Lu/Yau, Tohoku MJ 2011): We have for the n-dimensional hypercube $Q^n = (V, E)$ and $\{x,y\} \in E$:

$$\kappa_p(x,y) = \begin{cases} 2p, & \text{if } p \in [0, \frac{1}{n+1}], \\ \frac{2}{n}(1-p), & \text{if } p \in [\frac{1}{n+1}, 1]. \end{cases}$$

Lin/Lu/Yau's modification of Ollivier's curvature

- $\kappa_1(x, y) = 0$,
- $p \mapsto \kappa_p(x, y)$ is concave,
- $\frac{\kappa_p(x,y)}{1-p} \leq \frac{2}{d(x,y)}$.



Definition (Lin/Lu/Yau, Tohoku MJ 2011):

$$\kappa_{LLY}(x, y) = \lim_{p \to 1} \frac{\kappa_p(x, y)}{1 - p}.$$

Then $\kappa_p(x, y) \leq \kappa_{LLY}(x, y)$ for all $p \in [0, 1]$. For the hypercube Q^n:

$$\kappa_{LLY}(x, y) = \frac{2}{n}.$$

Curvature signs of κ_0 and κ_{LLY}

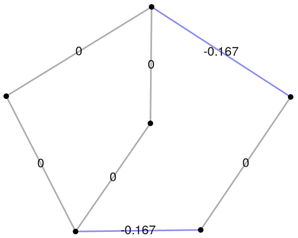
A graph $G = (V, E)$ is *regular* if there exists D such that $d_x = D$ for all $x \in V$.

Theorem (Kangaslampi, 2017)

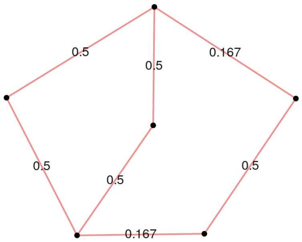
Assume $G = (V, E)$ is regular. Then we have, for every edge $\{x, y\} \in E$:

$$\kappa_{LLY}(x, y) > 0 \Rightarrow \kappa_0(x, y) \geq 0.$$

This is no longer true for non-regular graphs:

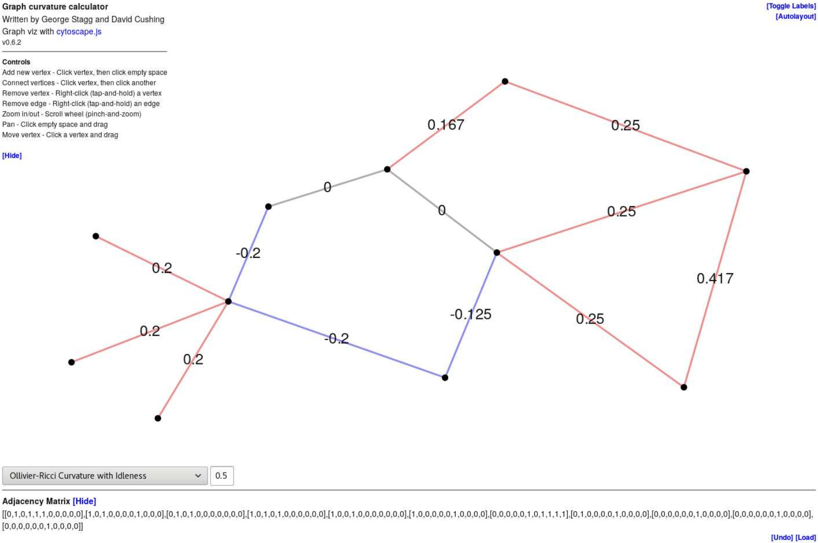


κ_0



κ_{LLY}

Curvature calculator tool by D.Cushing/G. Stagg



Web Link

This tool is freely available at

http://teggers.eu/graph/

Easy to use and very helpful to make lots of discoveries!!

Alternatively, it can also be installed locally on your computer. For installation details, see

https://mas-gitlab.ncl.ac.uk/graph-curvature

Some articles with various idleness assumptions...

- Ollivier (JFA 2009) considered κ_0 (Examples 5, 15) and $\kappa_{\frac{1}{2}}$ (Example 8).
- Lin/Lu/Yau (Tohoku MJ 2011) considered κ_{LLY}.
- For D-regular graphs, Ollivier-Villani (SIAM J. Discr. M. 2012, Q^n) considered $\kappa_{\frac{1}{D+1}}$.
- Jost/Liu (Disc Comp Geom 2014) considered κ_0 (lower curvature estimate in terms of triangles).

Some classical results: *Bonnet-Myers and Lichnerowicz*

Not 2000 years old (like this Greek mosaic) but important!

Discrete Bonnet-Myers

Theorem (Discrete Bonnet-Myers, Ollivier, JFA 2009)

For any $z, w \in V$:

$$d(z,w) \leq \frac{W_1(\delta_z, \mu_z^p) + W_1(\mu_w^p, \delta_w)}{\kappa_p(z,w)} = \frac{2(1-p)}{\kappa_p(z,w)}.$$

Moreover, if, for all edges $\{x,y\} \in E$, $\kappa_p(x,y) \geq K > 0$, then

$$\mathrm{diam}(G) \leq \frac{2(1-p)}{K}. \tag{1}$$

Finally, if, for all edges $\{x,y\} \in E$, $\kappa_{LLY}(x,y) \geq K > 0$, then

$$\mathrm{diam}(G) \leq \frac{2}{K}. \tag{2}$$

For hypercube Q^n: (2) is sharp and (1) is sharp for idleness $p \in [\frac{1}{n+1}, 1]$.
At idleness $p = 0$, Q^n has zero curvature, so Theorem not applicable!

Graphs with positive curvature

Bonnet-Myers is sharp for Q^n for idleness $p \in [\frac{1}{n+1}, 1]$ and, in the smooth setting, Bonnet-Myers is sharp for round spheres S^n.

General philosophy: Hypercubes can be viewed as discrete analogues of round spheres.

Question: Are there *infinite graphs* with $\kappa_0(x, y) > 0$ along all edges?

Graphs with positive curvature

Bonnet-Myers is sharp for Q^n for idleness $p \in [\frac{1}{n+1}, 1]$ and, in the smooth setting, Bonnet-Myers is sharp for round spheres S^n.

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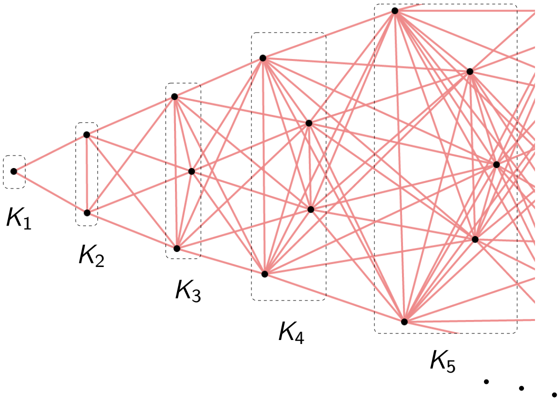
Question: Are there *infinite graphs* with $\kappa_0(x,y) > 0$ along all edges?

Answer: YES! Anti-tree $\mathcal{AT}((j))$:



Curvature of anti-trees

Theorem (Cushing, Liu, Münch, Peyerimhoff, 2017)

Let $(a_j)_{j\in\mathbb{N}}$ be monotone increasing, $a_1 = 1$. Then, for the anti-tree $\mathcal{AT}((a_j))$, we have the following curvature results:

- *For radial root edges:* $\kappa_0 = \frac{a_2-1}{a_2+a_3} > 0$, $\kappa_{LLY} = \frac{a_2+1}{a_2+a_3} > 0$,
- *For radial inner edges from K_{a_i} to $K_{a_{i+1}}$ $(i \geq 2)$:*

$$\kappa_p = \left(\frac{2a_i + a_{i+1} - 1}{a_i + a_{i+1} + a_{i+2} - 1} - \frac{2a_{i-1} + a_i - 1}{a_{i-1} + a_i + a_{i+1} - 1}\right)(1-p),$$

- *For spherical edges in K_{a_i} $(i \geq 2)$:* $\kappa_0 = \frac{a_{i-1}+a_i+a_{i+1}-2}{a_{i-1}+a_i+a_{i+1}-1} > 0$, $\kappa_{LLY} = \frac{a_{i-1}+a_i+a_{i+1}}{a_{i-1}+a_i+a_{i+1}-1} > 0$.

Corollary

Anti-trees have strictly positive curvature κ_p, $p \in [0,1)$, for arithmetic and geometric progressions (e.g., $\mathcal{AT}((j))$ or $\mathcal{AT}((2^{j-1}))$).

Discrete Lichnerowicz

Normalized Laplacian is defined as $\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} (f(x) - f(y))$.
Self-adjoint operator with eigenvalues (respecting multiplicities)

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{|V|-1} \leq 2,$$

provided $G = (V, E)$ is finite and connected.

Theorem (Discrete Lichnerowicz, Lin/Lu/Yau, Tohoku MJ 2011)

Let $G = (V, E)$ to be finite and connected. Assume for all edges $\{x, y\} \in E$, $\kappa_{LLY}(x, y) \geq K > 0$. Then

$$\lambda_1 \geq K.$$

For hypercube Q^n: Eigenvalues $\frac{2k}{n}$ with multiplicity $\binom{n}{k}$, $k \in \{0, \ldots, n\}$.
Therefore, $\lambda_1 = 2/n$ and $\kappa_{LLY}(x, y) = 2/n$ for all edges. So Discrete Lichnerowicz is sharp. Lichnerowicz also sharp for complete graphs K_n ($\lambda_1 = \frac{n}{n-1}$).

Our Main Result: *“$p \mapsto \kappa_p$ has three linear pieces”*

Based on detailed and thorough investigations...

Our main result

Theorem (Bourne, Cushing, Liu, Münch, Peyerimhoff, 2017)

Let $G = (V, E)$ be a simple graph and $\{x, y\} \in E$ an edge. Then the function $p \mapsto \kappa_p(x, y)$ is concave and piecewise linear over $[0, 1]$ with at most 3 linear pieces. Furthermore, $\kappa_p(x, y)$ is linear on the intervals

$$\left[0, \frac{1}{\mathrm{lcm}(d_x, d_y) + 1}\right] \quad \textit{and} \quad \left[\frac{1}{\max(d_x, d_y) + 1}, 1\right].$$

Thus, if we have $d_x = d_y$, then $p \mapsto \kappa_p(x, y)$ has at most two linear pieces with only possible change of slope at $p = \frac{1}{d_x+1}$.

Important consequence: This result allows us to relate curvatures of egdes for different values of idleness: for example, $\kappa_{\frac{1}{2}}(x, y)$, $\kappa_{LYY}(x, y)$, $\kappa_{\frac{1}{D+1}}(x, y)$ (for D-regular graphs):

$$\kappa_{LLY}(x, y) = 2\kappa_{\frac{1}{2}}(x, y) = \frac{D+1}{D}\kappa_{\frac{1}{D+1}}(x, y) \quad \text{for } \{x, y\} \in E.$$

Very few words about the proof...

Fundamental tool is "Duality": Let $\{x, y\} \in E$. Then

$$\underbrace{\inf_{\pi \in \Pi(\mu_x^p, \mu_y^p)} \sum_{z,w \in V} \pi(z,w) d(z,w)}_{=W_1(\mu_1,\mu_2)} = \sup_{\phi \in 1-\mathrm{Lip}} \underbrace{\sum_{x \in V} \phi(x)(\mu_1(x) - \mu_2(x))}_{(*)}.$$

Since $d : V \times V \to \mathbb{N} \cup \{0\}$ is integer-valued, it suffices to choose integer-values 1-Lipschitz functions ϕ on the RHS. Moreover, expression $(*)$ does not change by replacing ϕ by $\phi + \mathrm{constant}$. The considered 1-Lipschitz functions ϕ can therefore be divided into three classes:

- $\phi(x) = 1$ and $\phi(y) = 0$,
- $\phi(x) = 0$ and $\phi(y) = 0$,
- $\phi(x) = -1$ and $\phi(y) = 0$.

This indicates that we will have at most 3 linear pieces of $p \mapsto \kappa_p(x, y)$. The estimates for the lengths of the first and last linear piece require further detailed investigations...

Some applications

Corollary (of Main Result and Lin/Lu/Yau, Tohoku MJ 2011)

G and H two regular graphs, $\{x_1, x_2\} \in E_G$, $y \in V_H$. *Then*

$$\kappa_p^{G\times H}((x_1,y),(x_2,y)) = \frac{d_G}{d_G + d_H} \times$$

$$\begin{cases} \kappa_p(x_1,x_2) + d_H(\kappa_{LLY}(x_1,x_2) - \kappa_0(x_1,x_2))p, & \text{if } p \in [0, \frac{1}{d_G+d_H+1}], \\ \kappa_{LLY}(x_1,x_2)(1-p), & \text{if } p \in [\frac{1}{d_G+d_H+1}, 1]. \end{cases}$$

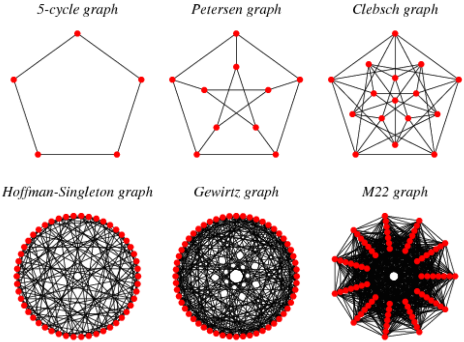
Theorem (Cushing, Kangaslampi, Liu, Peyerimhoff 2017)

$G = (V, E)$ *strongly regular. Let* $\{x, y\} \in E$.

- *If girth is* 4, *we have* $\kappa_0(x,y) = 0$ *and* $\kappa_{LLY}(x,y) = \frac{2}{d}$ *(same as* Q^d*).*
- *If girth is* 5, *we have* $\kappa_0(x,y) = \frac{2}{d} - 1$ *and* $\kappa_{LLY}(x,y) = \frac{3}{d} - 1$.

Main Result implies explicit curvature for all idleness (since 2 *linear pieces).*

A final conjecture



Conjecture (Cushing, Kangaslampi, Liu, Peyerimhoff)

All strongly regular graphs of girth 3 have non-negative Ollivier Ricci curvature κ_0.

We checked many known examples, including those given in

http://mathworld.wolfram.com/StronglyRegularGraph.html

Thank you for your attention!