The magnetic bottle on graphs

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S. Golénia and F. Truc : The magnetic Laplacian acting on discrete cusps, arxiv :150702638v2

Plan

1. Preliminaries and notation
   - Definitions and notation
   - Linear like graph
   - Operator

2. magnetic bottle

Definitions and notation

- A graph G is a couple (V, E) where :
  - V is a countable elements are called vertices.
  - $E \subset V \times V$ is the set of edges.
- Tow vertices x and y are called neighbors if they are connected with an edge and we write $x \sim y$.
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- Let $\gamma = \{x_0, x_1, ..., x_n\}$ be a part of V, we say γ is a path if

$$x_k \sim x_{k+1}, \ \forall 0 \leq k \leq n-1.$$

- A graph is connected if for all x, y $\in$ V, there exists a path γ joining x and y :

$$\forall x \neq y \in V, \ \exists \text{ a path } \gamma = (x_0, x_1, ..., x_n); x_0 = x, \ x_n = y.$$

- A cycle is a path whose end and origin are identical
- **A finite linear graph** is a couple (V, E) where $V = \{1, ..., n\}$ and $E = \{\{i, i+1\}, \ 1 \leqslant i \leqslant n-1\}$

Linear like graph

- Let $(N_n)_n$ be a strictly increasing sequence of $\mathbb{N}$ with $N_0 = 0$. We suppose for $n \in \mathbb{N}$:

$$I_n = [N_n + 1, N_{n+1}]$$

it's a finite linear graph.

- Let $G_n = (V_n, E_n)$ be a finite graphs where E_n is the set of edges, and V_n is the set of vertices. we suppose that there exist $v_n^1, v_n^2 \in V_n$ with :

$$e_n^+ = \{N_n, v_n^1\}$$

$$e_n^- = \{N_n + 1, v_n^2\}.$$

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We consider the graph $G = (V, E)$ where :

- $V = \bigcup_{n=1}^{\infty} V_n \cup I_n$.
- $E = \bigcup_{n=0}^{\infty} (E_n \cup \{e_n^+, e_n^-\} \cup E_{I_n})$

$G = (V, E)$ is called linear like graph.

$m : V \to (0, \infty)$ with : $m(v) = \begin{cases} m_{|V_n^-} & = m_{V_n^-} \\ m_{|L_n} & = m_{L_n} \end{cases}$ Where :

$V_n^- = V_n \backslash \{v_n^1, v_n^2\}$, $I_n^+ = \{v_n^1, v_n^2\}$ and $L_n = I_n \cup I_n^+$

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We consider on G the separable Hilbert space :

$$\ell^2(V,m)=\{f\in C(V)\quad|\quad \|f\|<\infty\},$$

endowed with the inner product :

$$\langle f,g\rangle_V=\sum_{x\in V} m(x)f(x)g(x)$$

and the norm

$$||f||_V=\sqrt{\langle f,f\rangle_V}.$$

we called magnetic potential the function :

$$\begin{array}{cccc}\theta & : & V_n\times V_n & \longrightarrow & \mathbb{R}\\ & & (x,y) & \longmapsto & \theta_{x,y}\end{array}$$

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which checks :

1. $\theta_{x,y} = -\theta_{y,x}$
2. $\theta_{x,y} = 0,$ if x and y are not neighber

We define the quadratic form q_θ :

$$q_\theta(f) = \sum_{n\geqslant 0} \sum_{x,y\in V_n} |f(x) - \exp(i\theta)f(y)|^2 + \sum_{n\geqslant 0} \sum_{i=N_n+1}^{N_{n+1}} |f(x_i) - f(x_{i+1})|^2$$

$$+ \sum_{n\geqslant 0} \sum_{\substack{x,y\in I_n^+ \\ x\backsim y}} |f(x) - f(y)|^2$$

For all $f \in C_c(V)$, there is a unique non-negative symmetric operator Δ_θ such that :

$$< f, \Delta_\theta f >= q_\theta(f), \ \forall \ f \ \in \ C_c(V)$$

moreover, the magnetic Laplacian acts as follows :

$$\Delta_\theta f(x) = \frac{1}{m_n(x)} \sum_{y \sim x} (f(x) - \exp(i\theta) f(y))$$

We denote by Δ_θ the Friedrichs extension of $\Delta_\theta|_{C_{c(V)}}$.

We assume that :

- $q_{V_n^-}(u) = \sum_{x,y \in V_n} |u(x) - \exp(i\theta)u(y)|^2$
- $q_{L_n} = \sum_{x,y \in L_n} |u(x) - u(y)|^2$
- $\tilde{a}(v) = \begin{cases} a_n \text{ if } v \in V_n^- \\ b_n \text{ if } v \in L_n \end{cases}$

where :

- a_n the bottom of spectrum of q_{V_n} on $\ell^2(V_n, 1)$
- b_n the bottom of spectrum of q_{L_n} on $\ell^2(L_n, 1)$

Plan

Theorem 2.1

Let G a linear like graph such that $\sum_{x\in V} m(x) = \infty$ *then :*

$$\sigma_{ess}(q_0) \neq \varnothing$$

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$$\sigma_{ess}(q_0) \neq \emptyset$$

Lemma {S. Golénia}

Let A, B tow self-adjoint operators with form-domains $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$, respectively. if A and B have the same form-domian and

$$0 \leq < \psi, A\psi > \leq < \psi, B\psi >$$

for all $\psi \in \mathcal{D}(\mathcal{B})$ then $\sigma_{ess}(B) = \varnothing$ if and only if $\sigma_{ess}(A) = \varnothing$

Theorem 2.2

Let G a linear like graph and q_θ the quadratic form on G, such that :

$$\lim_{n \to +\infty} \frac{\tilde{a}(v)}{m(v)} = +\infty$$

then the spectrum of q_θ is discreet.

Proposition 2.3

Let G a linear like graph and q_θ the quadratic form on G if $|L_n|$ fixed and a_n, a_{n+1} close to 0, then there exist D such that :

$$b_n \geq \frac{D}{2} a_n$$

Proposition 2.4

Let G be a linear like graph and q_θ the quadratic form on G if a_n, a_{n+1} fixed and $|L_n|$ tends to infinity then :

$$b_n \backsim (\frac{\pi}{(l_n + 1)})^2$$

Y. Colin de Verdière : Asymptotique de Weyl pour les bouteilles magnétiques, Commun. Math. Phys. 105 (1986), 327–335 .

Y.Colin de Verdière, N.Torki-Hamza and F.Truc : Essential self-adjointness for combinatorial Schrödinger operators III-Magnetic fields, Ann. Fac. Sci. Toulouse Math. Sér, 6, **20** (2011), no. 3, 599–611.

S. Golénia and F. Truc : The magnetic Laplacian acting on discrete cusps, arxiv :150702638v2
Analysis of operators Academic Press.

M. Reed and B. Simon : Methods of Modern Mathematical Physics, Tome I (1980).

Thank for your attention