Tunnels of Positive Scalar Curvature

(joint with J. Basilio and C. Sormani, arXiv:1703.00984)

Józef Dodziuk

Queens College and the Graduate Center
of The City University of New York

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Scalar curvature at a point p of a Riemannian manifold M^n is defined as the trace of the Ricci tensor or as the average of sectional curvatures. It appears as a coefficient in the Taylor series expansion in r of the volume $\mathrm{vol}_M(B(p,r)$

$$Sc(p) = \lim_{r \to 0^+} c(n) \frac{\mathrm{vol}_{\mathbb{E}^n}(B(0,r)) - \mathrm{vol}_M(B(p,r))}{r^2 \, \mathrm{vol}_{\mathbb{E}^n}(B(0,r))}.$$

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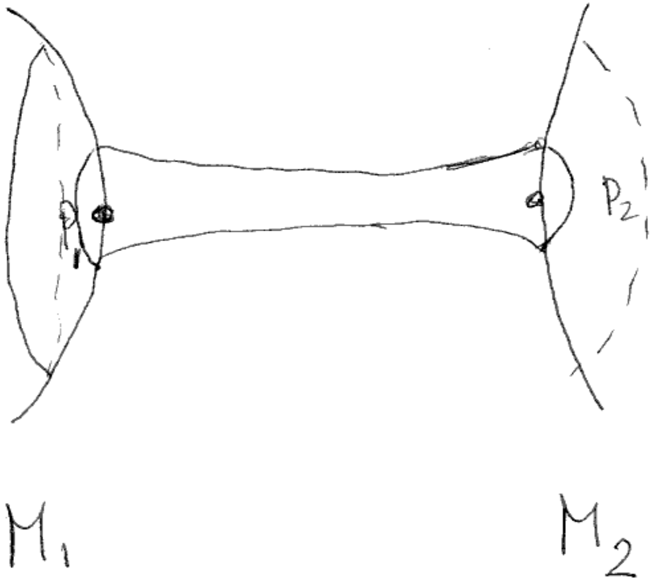
Around 1979, Gromov and Lawson and independently Schoen and Yau (for dimensions less than or equal to 7) proved that connected sums of manifolds with $Sc > 0$ admit metrics with $Sc > 0$. They did it by constructing tunnels, i.e. removing small disks centered at $p_1 \in M_1$ and $p_2 \in M_2$, gluing in $S^{n-1} \times$ interval and showing that the resulting manifold carries a positive scalar curvature metric.



P1
P2
M1
M2

As a preliminary they showed that the original metrics on on M_i can be deformed so that on $B(p_i, \delta)$ have constant and equal sectional curvatures. The old constructions, including one by Rosenberg and Stolz (2001) correcting an error in Gromov-Lawson paper, did not give any estimates of the length of the tunnel. We now give a precise statement of our result. Let $B(p, \delta)$ be a ball of small radius δ in the sphere of constant sectional curvature $K \in (0, 1]$.

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Theorem

For every $\delta_0 \in (0, \delta/2])$ there exists a metric of positive scalar curvature on $U = (B(p, \delta) \setminus B(p, \delta_0)) \sqcup S_{\delta_0} \times [0, l]$ such that the new metric agrees with the old one on $B(p, \delta) \setminus B(p, \delta_0)$; $\operatorname{diam} S_{\delta_0} \times [0, l]) = O(\delta_0)$ *and* $\operatorname{vol}(S_{\delta_0} \times [0, l]) = O(\delta_0^3)$. *Morever, the new metric is a product of the round sphere with an interval $(l - \epsilon, l]$ near the end of the tube. Finally, U has the same rotational symmetry as the ball $B(p, \delta)$.*

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The motivation for the construction is to use the tunnels to create examples of sequences of compact manifolds of positive scalar curvature that converge, say, in Gromov-Hausdorff sense to metric measured spaces whose generalized scalar curvature is not positive. As a matter of fact, the singular space in the limit will have points where the scalar curvature is $-\infty$.

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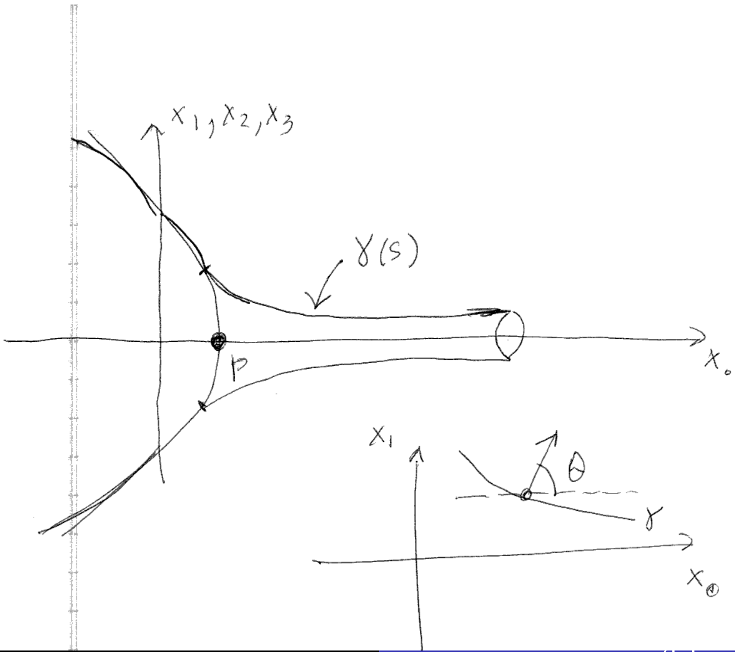
To describe the construction we restrict our attention to the case $n = 3$ and consider $B(p, \delta)$ as the geodesic ball on the round sphere in the Euclidean space $\mathbb{E}^4$ with coordinates x_0, x_1, x_2, x_3. Let p be the point where x_0-axis intersects the sphere. We will construct the tunnel as the manifold of revolution of a curve $\gamma(s) = (x_0(s), x_1(s))$, i.e. as the set $\{(x_0, x_1, x_2, x_3) \mid x_0 = x_0(s), x_1^2 + x_2^2 + x_3^2 = x_1(s)^2\}$ with the metric induced from $\mathbb{E}^3$.


x1, x2, x3
γ(s)
p
x0
x1
θ
γ
x0

Here s is the arclength parameter along γ and θ denotes the angle between the normal to γ and the horizontal direction. γ will be constructed by defining its geodesic curvature function $k(s)$ and recovering γ from k. The scalar curvature depends only on s and is given by the formula

$$Sc(s) = \frac{2 \sin \theta(s)}{x_1(s)} \left[\frac{\sin \theta(s)}{x_1(s)} - 2k(s) \right].$$

In the construction $x_1(s)$ will be decreasing, and $\theta(s)$ will be increasing to eventually reach $\pi/2$. Suppose $k(s)$ has been defined on $[0, s_0]$ and we wish to extend it further. The ratio $\sin(\theta(s))/x_1(s)$ is increasing. Thus if $k(s)$ is constant for $s \geq s_0$, the scalar curvature will stay positive. But $k(s) = k = \text{const}$ means that the segment of γ is a piece of a circle of radius $1/k$. Another consideration is that the curve $\gamma(s)$ does not cross the x_0 axis. Thus we define $\Delta s_0 = x_1(s_0)/2$ and $k(s)$ on $[s_0, s_0 + \Delta s_0]$ to be $k = \sin \theta(s_0)/4x_1(s_0)$.

Then the scalar curvature remains positive and the curve stays above the horizontal axis in the interval $[s_0, s_0 + \Delta s_0] = [s_0, s_1]$. Along the newly constructed circular arc $\theta(s)$ increases by

$$\Delta\theta_0 = \int k\, ds = \frac{\sin\theta(s_0)}{4x_1(s_0)} \times \frac{x_1(s_0)}{2} = \frac{\sin\theta(s_0)}{8} \geq \frac{\sin\theta(0)}{8}.$$

Now we can repeat the construction starting with the endpoint $(x_0(s_1), x_1(s_1))$ of newly defined segment of the curve and continue inductively to add circular arcs and gain at least a fixed amount of angle at every stage to arrive at $\theta = \pi/2$ in finitely many steps. We remark that if, at any stage of the construction, θ exceeds $\pi/2$ then we stop at the value of s for which $\theta = \pi/2$.
We also need to control the total length of the tunnel. Note that by definition

$$\frac{\Delta s_1}{\Delta s_0} = \frac{x_1(s_1)}{x_1(s_0)}.$$

It is intuitively clear that if $\theta(s_n)$ is very close to zero, then the ratio

$$\frac{x_1(s_{n+1})}{x_1(s_n)} \approx 1.$$

On the other hand we prove that there exists a constant $C \in (0, 1)$ such that for the angle $\theta_n \in (0, \pi/4)$

$$\frac{\Delta s_{n+1}}{\Delta s_n} = \frac{x_1(s_{n+1)})}{x_1(s_n)} \leq C < 1.$$

Thus, provided $\theta_n = \theta_0 + \Delta\theta_0 + \ldots + \Delta\theta_n << \leq \pi/4$, we get a comparison with a geometric progression that yields the length of the tunnel

$$\Delta s_0 + \ldots + \Delta s_n = O(\Delta s_0) = O(x_1(s_0)) = O(\delta_0).$$

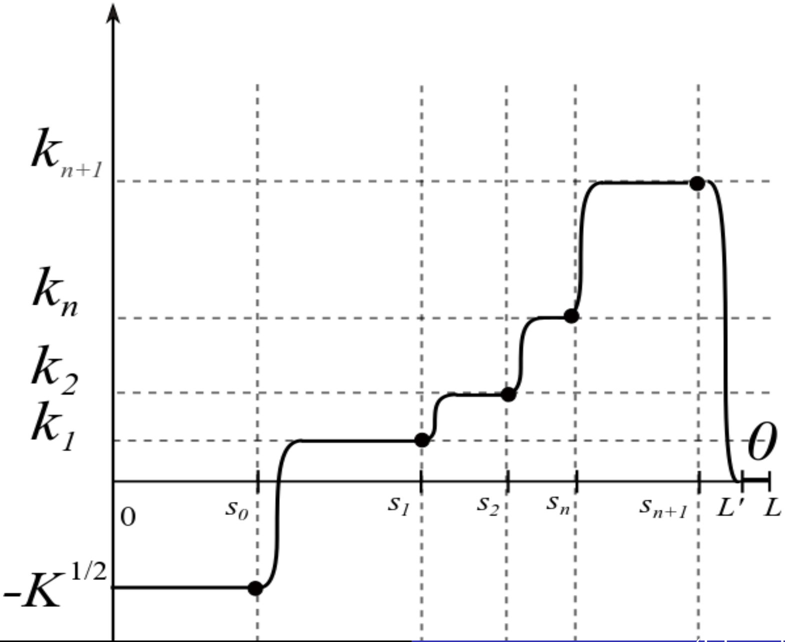
Finally, once we reach the angle $\theta_n = \pi/4$, we can finish the construction with *one eighth of the circle* of radius

$$R = 1/k$$

with k chosen so that

$$0.29289322 \approx \frac{2-\sqrt{2}}{2} < kx_1(s_n) < \frac{\sqrt{2}}{4} \approx 0.35355339.$$

There is one more technical complication as $k(s)$ is piecewise constant so that $\gamma(s)$ is only $\mathcal{C}^1$ at points where two successive circles meet. One needs an additional argument to smooth it without changing the total increment of the angle angle and the length of γ.


k_{n+1}
k_n
k_2
k_1
θ
0
s_0
s_1
s_2
s_n
s_{n+1}
L'
L
$-K^{1/2}$