Courant-sharp eigenvalues

Based on joint work with Ram Band (Technion) and David Fajman (University of Vienna)

Michael Bersudsky (Technion)

August 1, 2017

Setting - definitions and some notations

- $\Omega \subseteq \mathbb{R}^d$, a bounded connected domain.
- Laplace eigenvalue problem: $-\triangle f = \lambda f$ in Ω, $\frac{\partial f}{\partial n} |_{\partial\Omega} = 0$.
- Set of eigenvalues $\sigma(\Omega)$: discrete, increasing to infinity

$$0 = \lambda_1 < \lambda_2 \leq ... \nearrow \infty.$$

Spectral position and nodal domains

- Counting function (spectral position):
  $N(\lambda) = \#\{i \in \mathbb{N} \mid \lambda_i < \lambda\} + 1,\ \lambda \in \sigma(\Omega).$

Spectral position and nodal domains

- Counting function (spectral position):
  $N(\lambda) = \#\{i \in \mathbb{N} \mid \lambda_i < \lambda\} + 1,\ \lambda \in \sigma(\Omega).$

- Eigenfunction's zero set: $Z_f = \{x \in \Omega \mid f(x) = 0\}$.

Spectral position and nodal domains

- Counting function (spectral position):
  $N(\lambda) = \#\{i \in \mathbb{N} \mid \lambda_i < \lambda\} + 1,\ \lambda \in \sigma(\Omega)$.

- Eigenfunction's zero set: $Z_f = \{x \in \Omega \mid f(x) = 0\}$.

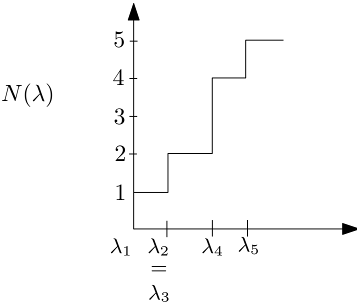
- Nodal count: $\nu(f) = \#\left\{\text{connected components of } \Omega \backslash Z_f\right\}$.

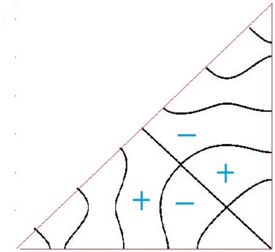
Spectral position and nodal domains

- Counting function (spectral position):
  $N(\lambda) = \#\{i \in \mathbb{N} \mid \lambda_i < \lambda\} + 1, \ \lambda \in \sigma(\Omega).$

- Eigenfunction's zero set: $Z_f = \{x \in \Omega \mid f(x) = 0\}$.

- Nodal count: $\nu(f) = \#\left\{\text{connected components of } \Omega \backslash Z_f\right\}$.



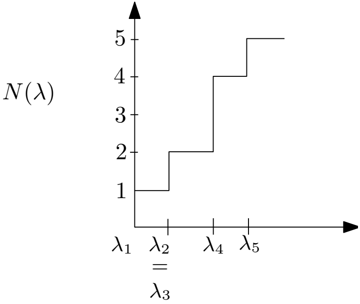


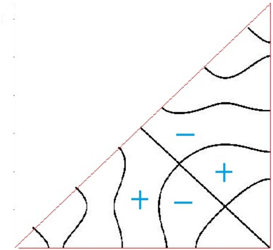
Spectral position and nodal domains

- Counting function (spectral position):
  $N(\lambda) = \#\{i \in \mathbb{N} \mid \lambda_i < \lambda\} + 1, \;\lambda \in \sigma(\Omega).$

- Eigenfunction's zero set: $Z_f = \{x \in \Omega \mid f(x) = 0\}$.

- Nodal count: $\nu(f) = \#\left\{\text{connected components of } \Omega \backslash Z_f\right\}$.





Is the counting function and nodal count related?

Courant's theorem and the Courant-sharp problem

Theorem

(*Courant's nodal bound, 1923*) *Let* f_λ *be an eigenfunction corresponding to* $\lambda \in \sigma(\Omega)$, *then*

$$\nu(f_\lambda) \leq N(\lambda).$$

Courant's theorem and the Courant-sharp problem

Theorem

(*Courant's nodal bound, 1923*) *Let* f_λ *be an eigenfunction corresponding to* $\lambda \in \sigma(\Omega)$*, then*

$$\nu(f_\lambda) \leq N(\lambda).$$

Question. *Given* Ω*, which eigenfunctions attain Courant's bound?*

Courant's theorem and the Courant-sharp problem

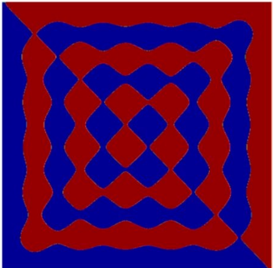
Theorem

(*Courant's nodal bound, 1923*) *Let f_λ be an eigenfunction corresponding to $\lambda \in \sigma(\Omega)$, then*

$$\nu(f_\lambda) \leq N(\lambda).$$

Question. *Given Ω, which eigenfunctions attain Courant's bound?*

A. Stern, 25' - On the dirichlet square $\nu(f_\lambda) = 2$, for infinitely many eigenvalues.



Definition and trivial cases

Definition

- Eigenfunction f_λ is Courant-sharp if $\nu(f_\lambda) = N(\lambda)$.

Definition and trivial cases

Definition

- Eigenfunction f_λ is Courant-sharp if $\nu(f_\lambda) = N(\lambda)$.
- Eigenvalue $\lambda \in \sigma(\Omega)$ is Courant-sharp if there exists a corresponding Courant-sharp eigenfunction.

Definition and trivial cases

Definition

- Eigenfunction f_λ is Courant-sharp if $\nu(f_\lambda) = N(\lambda)$.
- Eigenvalue $\lambda \in \sigma(\Omega)$ is Courant-sharp if there exists a corresponding Courant-sharp eigenfunction.

Trivial cases. The first is Courant-sharp by Courant's theorem and the second is Courant-sharp due to Orthogonality.

Small number of nodal domains - intuition

- Pleijel 56' - Finitely many Courant-sharp.

Small number of nodal domains - intuition

- Pleijel 56' - Finitely many Courant-sharp.
- Main argument - Faber-Krahn inequality

$$|\Omega_i| \geq \frac{C_d}{\lambda_n^{d/2}(\Omega)},$$

where Ω_i is a nodal domain of f_{λ_n} with $\partial\Omega_i \subseteq Z_{f_n}$ and C_d is a constant depending on d.

Small number of nodal domains - intuition

- Pleijel 56' - Finitely many Courant-sharp.
- Main argument - Faber-Krahn inequality

$$|\Omega_i| \geq \frac{C_d}{\lambda_n^{d/2}(\Omega)},$$

where Ω_i is a nodal domain of f_{λ_n} with $\partial\Omega_i \subseteq Z_{f_n}$ and C_d is a constant depending on d.

Remark. Many (almost all) of the present results use Pleijel's argument as first step.

Previous results

General results -

Previous results

General results -

- Finitely many Courant-sharp eigenvalues -
  - A. Pleijel 56' (Dirichlet), I. Polterovich 08' (Neumann, $\mathbb{R}^2$), C. Lena 16' (Robin, $\mathbb{R}^d$)

Previous results

General results -

- Finitely many Courant-sharp eigenvalues -
  - A. Pleijel 56' (Dirichlet), I. Polterovich 08' (Neumann, $\mathbb{R}^2$), C. Lena 16' (Robin, $\mathbb{R}^d$)
- Bounds on the number of Courant-sharp eigenvalues -
  - P. Berard and B. Helffer 15' (Dirichlet, $\mathbb{R}^2$), M. van den Berg, K. Gittins 16' (Dirichlet, $\mathbb{R}^d$).

Previous results

General results -

- Finitely many Courant-sharp eigenvalues -
  - A. Pleijel 56' (Dirichlet), I. Polterovich 08' (Neumann, $\mathbb{R}^2$), C. Lena 16' (Robin, $\mathbb{R}^d$)
- Bounds on the number of Courant-sharp eigenvalues -
  - P. Berard and B. Helffer 15' (Dirichlet, $\mathbb{R}^2$), M. van den Berg, K. Gittins 16' (Dirichlet, $\mathbb{R}^d$).
- Minimal Partitions, B. Helffer, T. Hoffmann-Ostenhof, and S.Terracini, 09'.

Specific cases- determination of the Courant-sharp eigenfunctions and eigenvalues -

Previous results

General results -

- Finitely many Courant-sharp eigenvalues -
  - A. Pleijel 56' (Dirichlet), I. Polterovich 08' (Neumann, $\mathbb{R}^2$), C. Lena 16' (Robin, $\mathbb{R}^d$)
- Bounds on the number of Courant-sharp eigenvalues -
  - P. Berard and B. Helffer 15' (Dirichlet, $\mathbb{R}^2$), M. van den Berg, K. Gittins 16' (Dirichlet, $\mathbb{R}^d$).
- Minimal Partitions, B. Helffer, T. Hoffmann-Ostenhof, and S.Terracini, 09'.

Specific cases- determination of the Courant-sharp eigenfunctions and eigenvalues -

Contributors (07'-16') - Helffer, Hoffman-Ostenhof, Terracini,Berard, Sundqvist, Lena.

- Planar domains - the square, the disc, the annulus, irrational rectangles, the torus and some triangles.

Previous results

General results -

- Finitely many Courant-sharp eigenvalues -
  - A. Pleijel 56' (Dirichlet), I. Polterovich 08' (Neumann, $\mathbb{R}^2$), C. Lena 16' (Robin, $\mathbb{R}^d$)
- Bounds on the number of Courant-sharp eigenvalues -
  - P. Berard and B. Helffer 15' (Dirichlet, $\mathbb{R}^2$), M. van den Berg, K. Gittins 16' (Dirichlet, $\mathbb{R}^d$).
- Minimal Partitions, B. Helffer, T. Hoffmann-Ostenhof, and S.Terracini, 09'.

Specific cases- determination of the Courant-sharp eigenfunctions and eigenvalues -

Contributors (07'-16') - Helffer, Hoffman-Ostenhof, Terracini,Berard, Sundqvist, Lena.

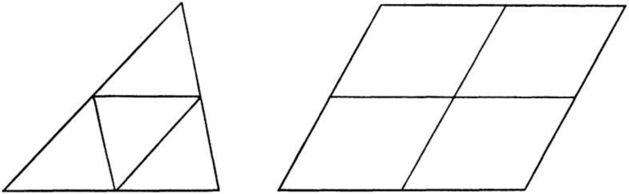
- Planar domains - the square, the disc, the annulus, irrational rectangles, the torus and some triangles.
- Domains in $\mathbb{R}^d$ - $d > 2$ - the cube $\mathbb{R}^3$, the torus in $\mathbb{R}^3$, Balls in $\mathbb{R}^d$, $d \geq 2$.

Reptiles

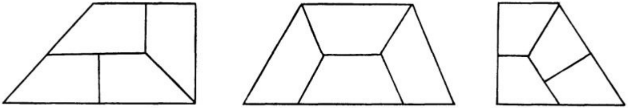
Reptiles

Rep-tiles (mathematical)

W. Golomb 64' - a rep-tile or reptile is a shape that can be dissected into smaller copies of the same shape. Shape is a **n-rep-tile** if disected into n shapes.



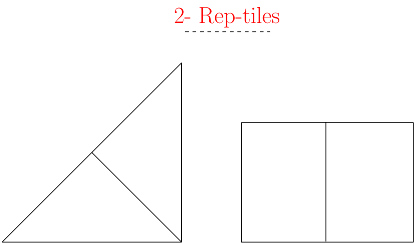
An arbitrary triangle or parallelogram can be dissected into four replicas.



Three trapezoids which can be dissected into four replicas.

Symmetric 2-Rep-tiles

- Disected at a symmetry plane - eigenfunctions are self tiling.



Our result

Theorem

(Band, Bersudsky, Fajman, 16')

1. *The Courant-sharp eigenvalues of the Neumann Laplacian on the right triangle with equal legs are* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6$
2. *Let* $d \in \mathbb{N}$, $d \geq 2$, *and let* $\mathcal{B}^{(d)}$ *be a* d*-dimensional box of measures* $l_1 \times l_2 \times ... \times l_d$, *such that* $\frac{l_j}{l_{j+1}} = 2^{1/d}$ $(1 \leq j \leq d-1)$.

Our result

Theorem

(Band, Bersudsky, Fajman, 16')

1. *The Courant-sharp eigenvalues of the Neumann Laplacian on the right triangle with equal legs are* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6$
2. *Let* $d \in \mathbb{N}$, $d \geq 2$, *and let* $\mathcal{B}^{(d)}$ *be a* d*-dimensional box of measures* $l_1 \times l_2 \times ... \times l_d$, *such that* $\frac{l_j}{l_{j+1}} = 2^{1/d}$ $(1 \leq j \leq d-1)$.

The Courant-sharp eigenvalues of the Neumann Laplacian on $\mathcal{B}^{(d)}$ *are:*

- $\lambda_1, \lambda_2, \lambda_4, \lambda_6$ *for* $d = 2$

Our result

Theorem

(Band, Bersudsky, Fajman, 16')

1. *The Courant-sharp eigenvalues of the Neumann Laplacian on the right triangle with equal legs are* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6$
2. *Let* $d \in \mathbb{N}$, $d \geq 2$, *and let* $\mathcal{B}^{(d)}$ *be a* d*-dimensional box of measures* $l_1 \times l_2 \times ... \times l_d$, *such that* $\frac{l_j}{l_{j+1}} = 2^{1/d}$ $(1 \leq j \leq d-1)$.

The Courant-sharp eigenvalues of the Neumann Laplacian on $\mathcal{B}^{(d)}$ *are:*

- $\lambda_1, \lambda_2, \lambda_4, \lambda_6$ *for* $d = 2$
- λ_1, λ_2 *for* $d \geq 3$.

Our result

Theorem

(Band, Bersudsky, Fajman, 16')

1. *The Courant-sharp eigenvalues of the Neumann Laplacian on the right triangle with equal legs are* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6$
2. *Let* $d \in \mathbb{N}$, $d \geq 2$, *and let* $\mathcal{B}^{(d)}$ *be a* d*-dimensional box of measures* $l_1 \times l_2 \times ... \times l_d$, *such that* $\frac{l_j}{l_{j+1}} = 2^{1/d}$ $(1 \leq j \leq d-1)$.

The Courant-sharp eigenvalues of the Neumann Laplacian on $\mathcal{B}^{(d)}$ *are:*

- $\lambda_1, \lambda_2, \lambda_4, \lambda_6$ *for* $d = 2$
- λ_1, λ_2 *for* $d \geq 3$.

Remark. It is impossible to use Pleijel's argument (Faber-Krahn) to solve the problem for the boxes in every dimension.

Some ideas in the proof

Lemma

For the domains we consider, if an eignevalue is Courant-sharp, then it is a simple eigenvalue.

Some ideas in the proof

Lemma

For the domains we consider, if an eignevalue is Courant-sharp, then it is a simple eigenvalue.

Remark. This is not always the case - the second eigenvalue of the square has multiplicity 2.

Some ideas in the proof

Lemma

For the domains we consider, if an eignevalue is Courant-sharp, then it is a simple eigenvalue.

Remark. This is not always the case - the second eigenvalue of the square has multiplicity 2.

This Lemma and the following counting principle prove the theorem for the boxes. For the triangle it is more involved.

Ideas in the proof for $\mathcal{B}^{(2)}$

We consider the scaling - $\mathcal{B}^{(2)} = \left[0, \sqrt{2}\pi\right] \times [0, \pi]$.

Ideas in the proof for $\mathcal{B}^{(2)}$

We consider the scaling - $\mathcal{B}^{(2)} = \left[0, \sqrt{2}\pi\right] \times [0, \pi]$.

- The eigenvalues are given by
  $\lambda_{m,n} = \frac{m}{2}^2 + n^2, \; (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Ideas in the proof for $\mathcal{B}^{(2)}$

We consider the scaling - $\mathcal{B}^{(2)} = \left[0, \sqrt{2}\pi\right] \times [0, \pi]$.

- The eigenvalues are given by
  $\lambda_{m,n} = \frac{m}{2}^2 + n^2, \ (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.
- A basis of eigenfunctions is given by
  $\varphi_{m,n}(x, y) = \cos \frac{mx}{\sqrt{2}} \cdot \cos nx, \ (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Ideas in the proof for $\mathcal{B}^{(2)}$

We consider the scaling - $\mathcal{B}^{(2)} = \left[0, \sqrt{2}\pi\right] \times [0, \pi]$.

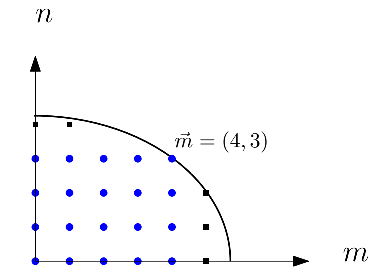
- The eigenvalues are given by
  $\lambda_{m,n} = \frac{m}{2}^2 + n^2, \ (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.
- A basis of eigenfunctions is given by
  $\varphi_{m,n}(x, y) = \cos \frac{mx}{\sqrt{2}} \cdot \cos nx, \ (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$.
- Note that $\nu(\varphi_{m,n}) = (m + 1)(n + 1)$.

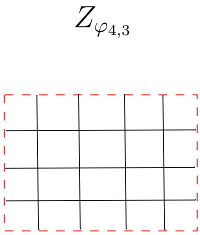
Ideas in the proof for $\mathcal{B}^{(2)}$

We consider the scaling - $\mathcal{B}^{(2)} = \left[0, \sqrt{2}\pi\right] \times [0, \pi]$.

- The eigenvalues are given by
  $\lambda_{m,n} = \frac{m}{2}^2 + n^2, \ (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$.
- A basis of eigenfunctions is given by
  $\varphi_{m,n}(x,y) = \cos \frac{mx}{\sqrt{2}} \cdot \cos nx, \ (m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$.
- Note that $\nu(\varphi_{m,n}) = (m+1)(n+1)$.

By the lemma it remains to examine the simple eigenvalues.





$N(\lambda_{m,n}) =$ ● + ▪

$\nu(\varphi_{m,n}) =$ ●

Concluding remarks

- The proof for $\mathcal{B}^{(d)}$, $d \geq 3$ is the same.

Concluding remarks

- The proof for $\mathcal{B}^{(d)}$, $d \geq 3$ is the same.
- The proof for the triangle - more involved.

Concluding remarks

- The proof for $\mathcal{B}^{(d)}$, $d \geq 3$ is the same.
- The proof for the triangle - more involved.
- General remark - degenerate eigenvalues are hard to treat - rectangles with degenerate eigenspace still unsolved.

Concluding remarks

- The proof for $\mathcal{B}^{(d)}$, $d \geq 3$ is the same.
- The proof for the triangle - more involved.
- General remark - degenerate eigenvalues are hard to treat - rectangles with degenerate eigenspace still unsolved.

Thanks for listening!