Glivenko–Cantelli Theory, Banach-space valued Ergodic Theorems and uniform approximation of the integrated density of states

Ivan Veselić, TU Dortmund

Universität Potsdam, August 2017

joint with
Christoph Schumacher (TU Dortmund)
& Fabian Schwarzenberger (HTW Dresden)

Spectral distribution function of Hamiltonians on $\mathbb{Z}^d$

$$\begin{aligned} \text{Hamiltonian} &= \text{(negative) Laplacian} + \text{multiplication operator} \\ &= \text{kinetic energy} + \text{potential energy (in Quantum Mechanics)} \\ (H\phi)(x) &= \sum_{y \text{ neighbour of } x} \big(\phi(x) - \phi(y)\big) + V(x)\phi(x) \end{aligned}$$

$\Rightarrow H$ selfadjoint and $\sigma(H) \subset \mathbb{R}$.

How to describe '*density of the spectrum*'?

Spectral distribution function of Hamiltonians on $\mathbb{Z}^d$

$$\begin{aligned} \text{Hamiltonian} &= \text{(negative) Laplacian} + \text{multiplication operator} \\ &= \text{kinetic energy} + \text{potential energy (in Quantum Mechanics)} \\ (H\phi)(x) &= \sum_{y \text{ neighbour of } x} \big(\phi(x) - \phi(y)\big) + V(x)\phi(x) \end{aligned}$$

$\Rightarrow H$ selfadjoint and $\sigma(H) \subset \mathbb{R}$.

How to describe '*density of the spectrum*'?

Local procedure, using cubes $\Lambda = \mathbb{Z}^d \cap [-L, L]^d$ and $H_L = \mathbf{1}_\Lambda^* H \mathbf{1}_\Lambda$:

$$\mathbb{R} \ni E \to N_L(E) := \frac{\#\{\text{ eigenvalues of} H_L \leq E\}}{(2L+1)^d}$$

is a probability distribution function.

Spectral distribution function of Hamiltonians on $\mathbb{Z}^d$

$$\begin{aligned} \text{Hamiltonian} &= \text{(negative) Laplacian} + \text{multiplication operator} \\ &= \text{kinetic energy} + \text{potential energy (in Quantum Mechanics)} \\ (H\phi)(x) &= \sum_{y \text{ neighbour of } x} \big(\phi(x) - \phi(y)\big) + V(x)\phi(x) \end{aligned}$$

$\Rightarrow H$ selfadjoint and $\sigma(H) \subset \mathbb{R}$.

How to describe '*density of the spectrum*'?

Local procedure, using cubes $\Lambda = \mathbb{Z}^d \cap [-L, L]^d$ and $H_L = \mathbf{1}_\Lambda^* H \mathbf{1}_\Lambda$:

$$\mathbb{R} \ni E \to N_L(E) := \frac{\#\{\text{ eigenvalues of} H_L \leq E\}}{(2L+1)^d}$$

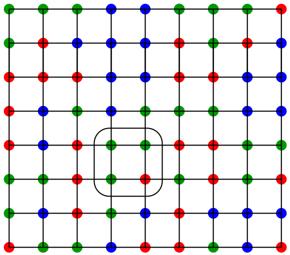
is a probability distribution function.

Well known:
Under appropriate ergodicity assumptions on potential or *coloring* $V : \mathbb{Z}^d \to \mathbb{R}$

$$\lim_{L\to\infty} N_L(E) =: N(E)$$

exists for almost all $E \in \mathbb{R}$ and is a distribution function.

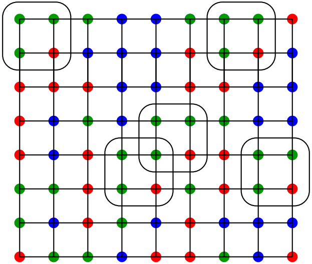
Assumptions on potential or *colouring* $V\colon \mathbb{Z}^d \to \mathcal{A}$



- finitely many values $|\mathcal{A}| < \infty$
- existence of pattern frequencies:

$P \in \mathcal{P}_r =$ all patterns of size $[-r, r]^d$

Assumptions on potential or *colouring* $V\colon \mathbb{Z}^d \to \mathcal{A}$

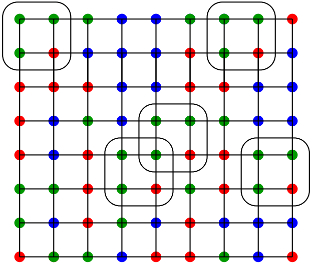


- finitely many values $|\mathcal{A}| < \infty$
- existence of pattern frequencies:

$P \in \mathcal{P}_r =$ all patterns of size $[-r, r]^d$

$$\nu_L(P) = \frac{\text{occurences of } P \text{ in } [-L, L]^d}{(2L+1)^d}$$

Assumptions on potential or *colouring* $V\colon \mathbb{Z}^d \to \mathcal{A}$



- finitely many values $|\mathcal{A}| < \infty$
- existence of pattern frequencies:

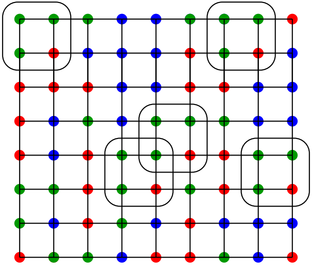
$P \in \mathcal{P}_r =$ all patterns of size $[-r, r]^d$

$$\nu_L(P) = \frac{\text{occurences of } P \text{ in } [-L, L]^d}{(2L+1)^d}$$

Assumption: $\forall\, r\, \forall\, P \in \mathcal{P}_r$ exists

$$\lim_{L\to\infty} \nu_L(P) =: \nu(P)$$

Assumptions on potential or *colouring* $V: \mathbb{Z}^d \to \mathcal{A}$



- finitely many values $|\mathcal{A}| < \infty$
- existence of pattern frequencies:

$P \in \mathcal{P}_r =$ all patterns of size $[-r, r]^d$

$$\nu_L(P) = \frac{\text{occurences of } P \text{ in } [-L, L]^d}{(2L+1)^d}$$

Assumption: $\forall\, r\, \forall\, P \in \mathcal{P}_r$ exists

$$\lim_{L\to\infty} \nu_L(P) =: \nu(P)$$

Theorem Lenz, Müller & Ves. 08

$$\lim_{n\to\infty} \|N_L - N\|_\infty = 0$$

In fact for any $r < L \in \mathbb{N}$

$$\sup_{E\in\mathbb{R}} \|N_L(E) - N(E)\|_\infty \leq const \left[\frac{1}{r} + \frac{r}{L} + \sum_{P\in\mathcal{P}_r} |\nu_L(P) - \nu(P)|\right]$$

Sum is finite since $\mathcal{A}$ finite.

Comment

Results does not require that the limiting distribution function is continuous.

Comment

Results does not require that the limiting distribution function is continuous.

Examples of colorings $V\colon \mathbb{Z}^d \to \mathcal{A}$ satisfying assumptions

- $V = \mathbf{1}_{\mathrm{VP}}$, where $\mathrm{VP} \subset \mathbb{Z}^d$ denotes the set of points visible form the origin
- $V(x)$, $x \in \mathbb{Z}^d$, i.i.d sequence of random variables $\Rightarrow$ almost surely satisfied

Comment

Results does not require that the limiting distribution function is continuous.

Examples of colorings $V\colon \mathbb{Z}^d \to \mathcal{A}$ satisfying assumptions

- $V = \mathbf{1}_{\mathrm{VP}}$, where $\mathrm{VP} \subset \mathbb{Z}^d$ denotes the set of points visible form the origin
- $V(x)$, $x \in \mathbb{Z}^d$, i.i.d sequence of random variables $\Rightarrow$ almost surely satisfied

Actually, Theorem applies to abstract fields

$$\mathcal{P}(\mathbb{Z}^d) \ni \Lambda \mapsto f_\Lambda \in \mathcal{B} \text{ Banach space}$$

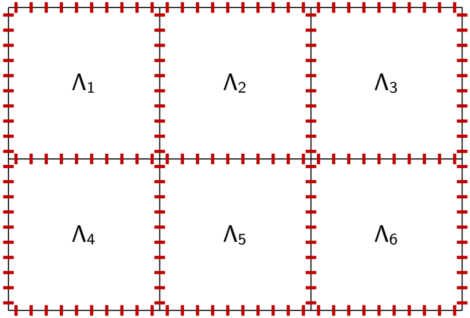
satisfying

- almost additivity with respect to finite Λ
- equi-variance with respect to colouring V, respectively patterns

Almost addittivity (semi locality, low complexity)

Let $\Lambda_1, \ldots, \Lambda_N \subset \mathbb{Z}^d$ finite, disjoint and $\Lambda := \Lambda_1 \cup \ldots \cup \Lambda_N$

$$\Rightarrow \qquad \Big\| f_\Lambda - \sum_{j=1}^{N} f_{\Lambda_j} \Big\|_{\mathcal{B}} \leq \text{const.} \sum_{j=1}^{N} \sharp \partial \boldsymbol{\Lambda}_{\mathbf{j}}$$



True for the eigenvalue countig functions due to interlacing theorem for eigenvalues.

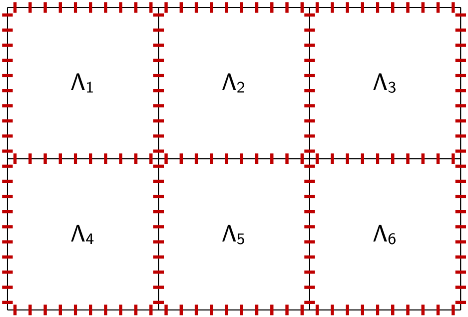
Each removed edge gives a rank one perturbation.

Almost addittivity (semi locality, low complexity)

Let $\Lambda_1, \ldots, \Lambda_N \subset \mathbb{Z}^d$ finite, disjoint and $\Lambda := \Lambda_1 \cup \ldots \cup \Lambda_N$

$$\Rightarrow \sup_{\omega \in \mathcal{A}^{\mathbb{Z}^d}} \left\| f_\Lambda(\omega) - \sum_{j=1}^N f_{\Lambda_j}(\omega) \right\|_{\mathcal{B}} \leq \text{const.} \sum_{j=1}^N \sharp \partial \boldsymbol{\Lambda_j}$$

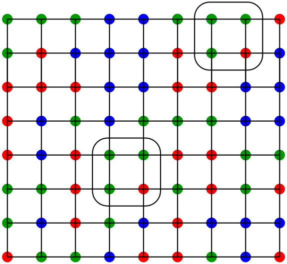
for all patterns/configurations $\omega \in \mathcal{A}^{\mathbb{Z}^d}$, if coloring generated by stochastic field



True for the eigenvalue countig functions due to interlacing theorem for eigenvalues.

Each removed edge gives a rank one perturbation.

Equi-variance with respect to colouring V

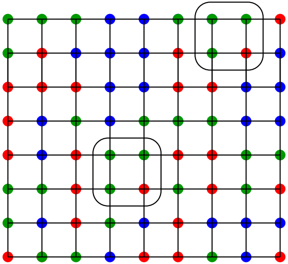


$$\text{same patterns} \quad \Rightarrow \quad \text{same values}$$

$$V|_{\Lambda} = V|_{\Lambda+k} \quad \Rightarrow \quad f_{\Lambda} = f_{\Lambda+k}$$

Obviously true for the eigenvalue countig functions.

Equi-variance with respect to colouring V

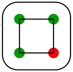
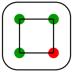


same patterns $\Rightarrow$ same values

$$V|_\Lambda = V|_{\Lambda+k} \quad \Rightarrow \quad f_\Lambda = f_{\Lambda+k}$$

Obviously true for the eigenvalue countig functions.

Consequence: f induces a function $\tilde{f}$: patterns $\rightarrow \mathcal{B}$

$\tilde{f}($  $)$ is well defied without knowing where  is located in $\mathbb{Z}^d$

Questions

Does uniform convergence theorem & error estimate hold

- for Laplacians of more general groups that $\mathbb{Z}^d$?
- for potential fields or colorings where the set of values $\mathcal{A}$ is infinite or even uncountable?

Laplace on discrete group, more precisely, on Cayley graph

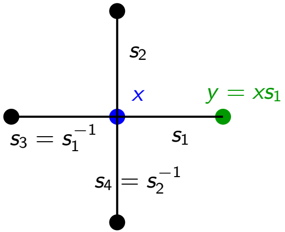
G discrete group, $S = S^{-1} \subset G \setminus \{\mathrm{id}\}$ finite, symmetric generating set

adjacency relation $x \sim y \Leftrightarrow x^{-1}y \in S$ defines Cayley graph $\mathrm{Cay}(G, S)$

Laplace on discrete group, more precisely, on Cayley graph

G discrete group, $S = S^{-1} \subset G \setminus \{\mathrm{id}\}$ finite, symmetric generating set

adjacency relation $x \sim y \Leftrightarrow x^{-1}y \in S$ defines Cayley graph $\mathrm{Cay}(G, S)$

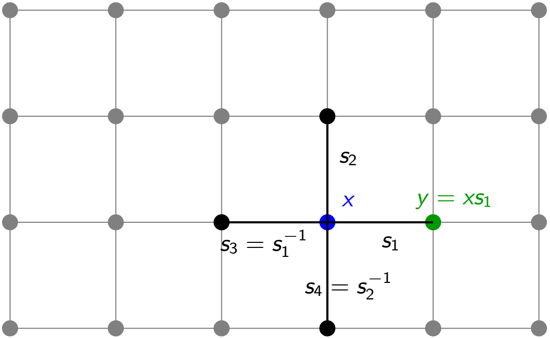


neighbours of x in $\mathbb{Z}^2$

Laplace on discrete group, more precisely, on Cayley graph

G discrete group, $S = S^{-1} \subset G \setminus \{\mathrm{id}\}$ finite, symmetric generating set

adjacency relation $x \sim y \Leftrightarrow x^{-1}y \in S$ defines Cayley graph $\mathrm{Cay}(G, S)$



neighbours of x in $\mathbb{Z}^2$

Cayley graph of $\mathbb{Z}^2$

$\mathrm{Cay}(\mathbb{Z}^2, \{s_1, s_2, s_3, s_4\})$

Hamilton operator on Cayley graph Cay(G, S)

$$(H\phi)(x) = \sum_{y \in G, y \sim x} \big(\phi(x) - \phi(y)\big) + V(x)\phi(x)$$

Hamilton operator on Cayley graph Cay(G, S)

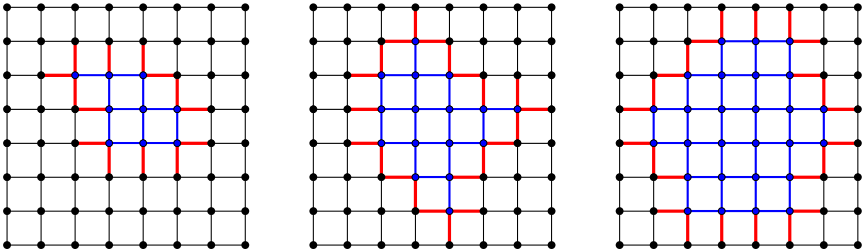
$$(H\phi)(x) = \sum_{y \in G, y \sim x} (\phi(x) - \phi(y)) + V(x)\phi(x)$$

Standing assumption: Amenability

Cay(G, S) amenable $\Leftrightarrow$ exists Følner sequence $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \ldots G$

$$\Leftrightarrow \Lambda_j \neq \emptyset \text{ finite and } \lim_{j\to\infty} \frac{|\partial\Lambda_j|}{|\Lambda_j|} = 0$$

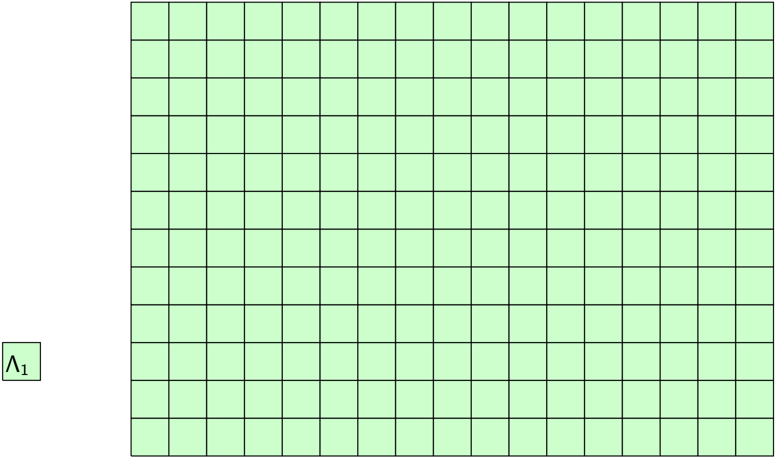
where $\partial\Lambda := \{(x, y) \mid \Lambda \ni x \sim y \notin \Lambda\}$



amenability of Cay(G, S) independent of choice of S

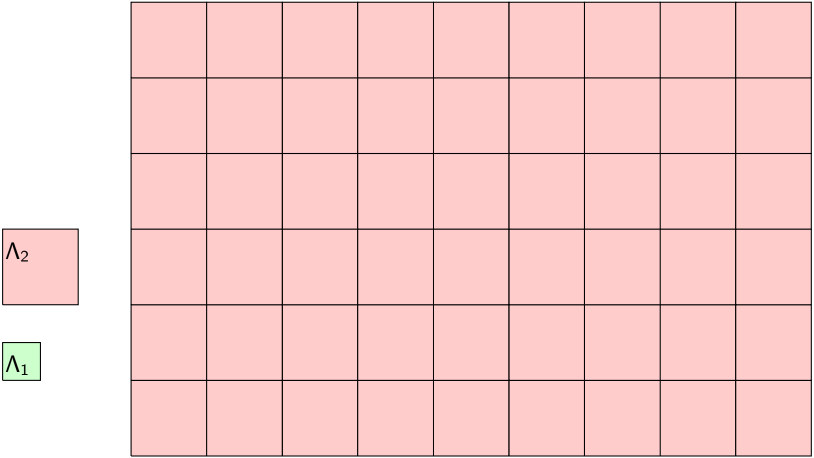
Temporary assumption: Symmetric tiling

finite $\Lambda \in \mathcal{P}(G)$ tiles G symmetricaly $\Leftrightarrow$ there is set $C = C^{-1} \subseteq G$ such that $G = \dot{\bigcup}_{g \in C} \Lambda g$.



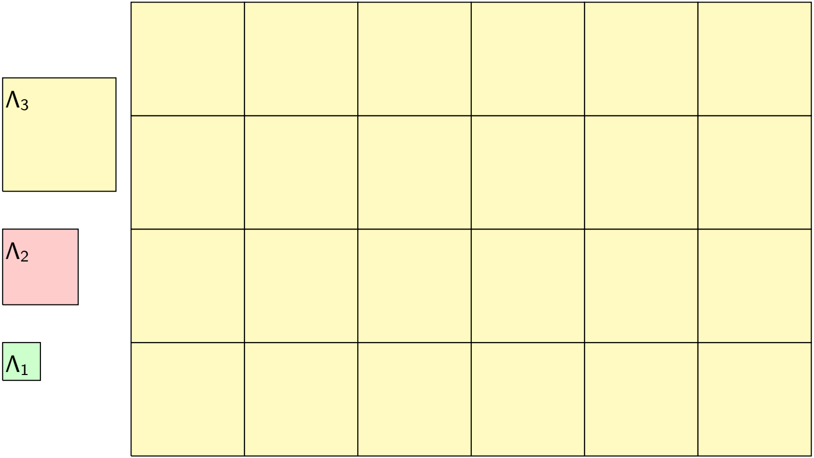
Temporary assumption: Symmetric tiling

finite $\Lambda \in \mathcal{P}(G)$ tiles G symmetricaly $\Leftrightarrow$ there is set $C = C^{-1} \subseteq G$ such that $G = \dot{\bigcup}_{g \in C} \Lambda g$.



Temporary assumption: Symmetric tiling

Assume existence of Foelner sequence of symmetric tiles.



Theorem Lenz, Schwarzenberger, Ves. 2011

Let $V\colon G \to \mathcal{A}$ have frequencies and $f : \mathcal{P}(G) \to \mathcal{B}$ be V equi-variant and almost-additive. Let (Λ_n) be a Følner sequence such that each Λ_n symmetrically tiles G. Let (U_j) be arbitrary Følner sequence.

Then the Banach space limit

$$\overline{f} := \lim_{j\to\infty} \frac{f_{U_j}}{|U_j|}$$

exists

Theorem Lenz, Schwarzenberger, Ves. 2011

Let $V\colon G \to \mathcal{A}$ have frequencies and $f : \mathcal{P}(G) \to \mathcal{B}$ be V equi-variant and almost-additive. Let (Λ_n) be a Følner sequence such that each Λ_n symmetrically tiles G. Let (U_j) be arbitrary Følner sequence.

Then the Banach space limits

$$\overline{f} := \lim_{j\to\infty} \frac{f_{U_j}}{|U_j|} = \lim_{n\to\infty} \sum_{P\in\mathcal{P}(\Lambda_n)} \nu(P) \frac{\tilde{f}(P)}{|\Lambda_n|}$$

exist and coincide.

For $j, n \in \mathbb{N}$ the difference

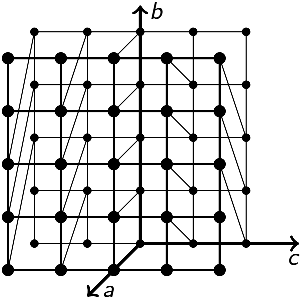
$$\left\| \frac{f_{U_j}}{|U_j|} - \sum_{P\in\mathcal{P}(\Lambda_n)} \nu(P) \frac{\tilde{f}(P)}{|\Lambda_n|} \right\|_{\mathcal{B}}$$

is bounded above by

$$const. \left[\frac{|\partial\Lambda_n|}{|\Lambda_n|} + \frac{|\partial^{diam(\Lambda_n)} U_j|}{|U_j|} + \sum_{P\in\mathcal{P}(\Lambda_n)} \left| \frac{\nu_{U_j}(P)}{|U_j|} - \nu(P) \right| \right]$$

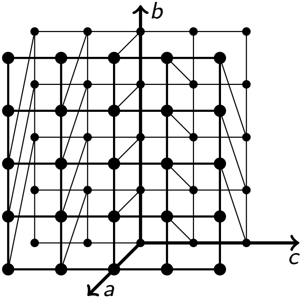
How restrictive is the assumption of having a symmetric tiling Følner sequence?

- this is satisfied, if one finds a sequence of subgroups (G_n) and associated fundamental domains (F_n), which form a Følner sequence
- this is satisfied, if is residually finite, amenable group G (Krieger 2007 using Weiss 2001)
- example: Heisenberg group



How restrictive is the assumption of having a symmetric tiling Følner sequence?

- this is satisfied, if one finds a sequence of subgroups (G_n) and associated fundamental domains (F_n), which form a Følner sequence
- this is satisfied, if is residually finite, amenable group G (Krieger 2007 using Weiss 2001)
- example: Heisenberg group



- Not very restrictive.
- No discrete amenable groups known which violate condition.
- Still unsatisfactory condition.

Quasi-tilings

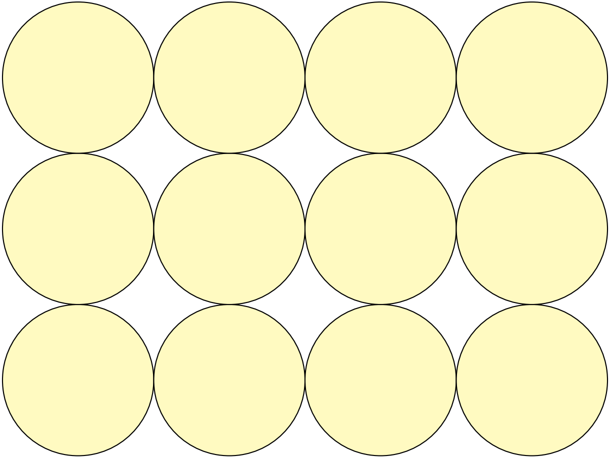


Figure: Ball quasi-tilings approximate $\mathbb{R}^d$ or $\mathbb{Z}^d$ very fast.

Quasi-tilings

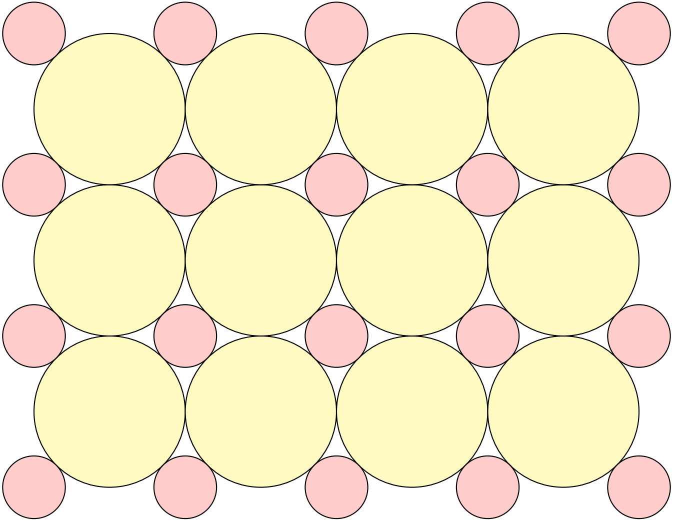


Figure: Ball quasi-tilings approximate $\mathbb{R}^d$ or $\mathbb{Z}^d$ very fast.

Quasi-tilings

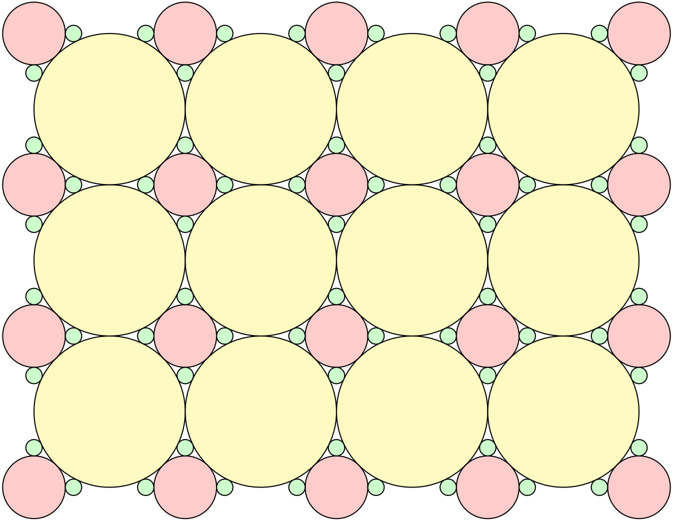
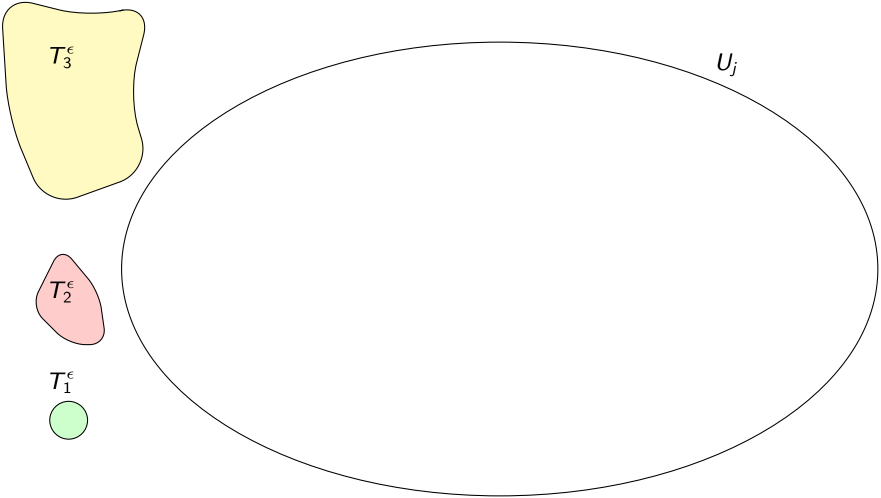
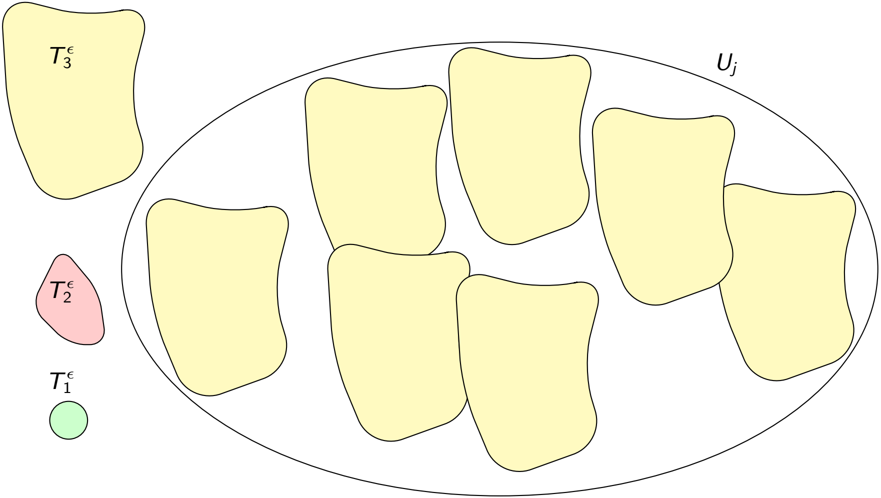


Figure: Ball quasi-tilings approximate $\mathbb{R}^d$ or $\mathbb{Z}^d$ very fast.

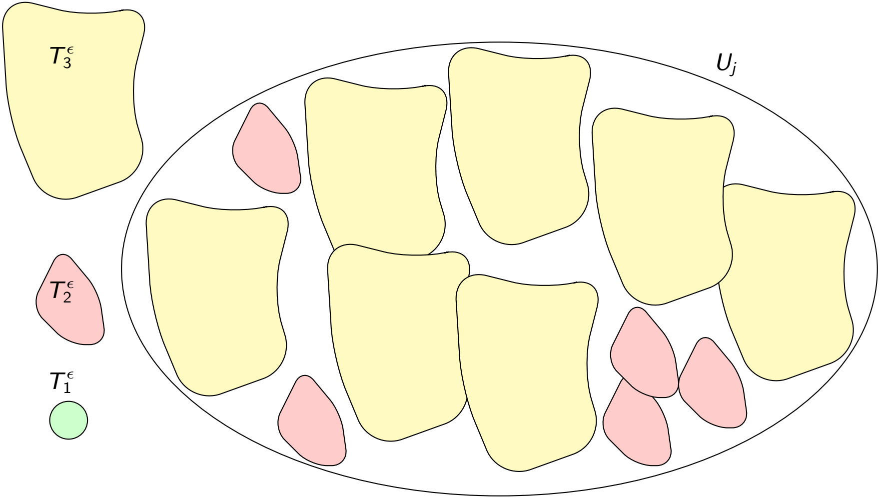
ϵ-Quasi tilings of Ornstein and Weiss



ϵ-Quasi tilings of Ornstein and Weiss

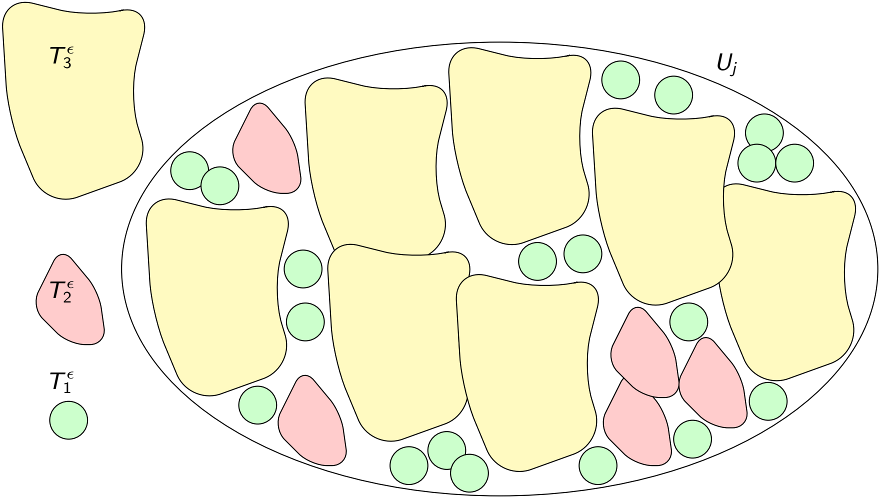


ϵ-Quasi tilings of Ornstein and Weiss



Leave at most ϵ-portion uncovered.
Have at most ϵ-portion overlap (within same generation).

ϵ-Quasi tilings of Ornstein and Weiss



Leave at most ϵ-portion uncovered.
Have at most ϵ-portion overlap (within same generation).

Quasi-Tiling

Theorem Pogorzelski, Schwarzenberger 11 based on Ornstein, Weiss 87

Let (U_j) and (S_n) be Følner sequences with $e \in S_n \subseteq S_{n+1}$, $(n \in \mathbb{N})$. For $0 < \epsilon \leq 1/10$ there are sets $T_i^\epsilon \in \{S_n \mid n \geq i\}$, $i = 1, \dots, N(\epsilon)$ which fulfil: for $j \geq j_0(\epsilon)$ we find center sets $(C_{i,j})_{i=1}^{N(\epsilon)}$ such that

- $T_i^\epsilon C_{i,j} \subseteq U_j \qquad (i \in \{1, \dots, N(\epsilon)\})$
- $\{T_i^\epsilon c\}_{c \in C_{i,j}}$ is ϵ-disjoint $\qquad (i \in \{1, \dots, N(\epsilon)\})$
- $\{T_i^\epsilon C_{i,j}\}_{i=1}^{N(\epsilon)}$ is a disjoint family
- $\left| \frac{|T_i^\epsilon C_{i,j}|}{|U_j|} - \eta_i(\epsilon) \right| \leq 2^{-N(\epsilon)-1}\epsilon \qquad (i \in \{1, \dots, N(\epsilon)\})$

where

- $N(\epsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil \qquad 0 < \epsilon \leq 1/10$
- $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\epsilon)-i} \qquad i = 1, \dots, N(\epsilon).$

Theorem Pogorzelski, Schwarzenberger 2011

For V with pattern frequencies $\nu(P)$, F almost additive & equi-variant, $(U_j) \subset G$ Følner sequence

$$\overline{f} = \lim_{j\to\infty} \frac{f_{U_j}}{|U_j|}$$

exists

Theorem Pogorzelski, Schwarzenberger 2011

For V with pattern frequencies $\nu(P)$, F almost additive & equi-variant, $(U_j) \subset G$ Følner sequence

$$\overline{f} = \lim_{j\to\infty} \frac{f_{U_j}}{|U_j|} = \lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\epsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(T_i^\epsilon)} \nu(P) \frac{\tilde{f}(P)}{|T_i^\epsilon|}$$

exists, where

- $N(\epsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon)\rceil \qquad 0 < \epsilon \leq 1/10$
- $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\epsilon)-i} \qquad i = 1, \ldots, N(\epsilon).$
- $(T_i^\epsilon)_{i=1}^{N(\epsilon)}$ is an ϵ-quasi-tiling (as before)
-

$$\left\| \overline{f} - \frac{f_{U_j}}{|U_j|} \right\| \leq const. \left[\epsilon + \sum_{i=1}^{N(\epsilon)} \eta_i(\varepsilon) \frac{b(T_i^\epsilon)}{|T_i^\epsilon|} + \sum_{i=1}^{N(\epsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(T_i^\epsilon)} \left| \frac{\nu_{U_j}(P)}{|U_j|} - \nu(P) \right| \right]$$

Potential or coloring V with uncountable many values in $\mathcal{A} \subset \mathbb{R}$

Recall error estimate of Lenz, Müller & Ves. 2008

$$\left\| \frac{1}{n^d} f_{\Lambda_n} - f \right\|_{\mathcal{B}} \leq const \left[\frac{1}{r} + \frac{r}{n} + \sum_{P \in \mathcal{P}_r} |\nu_n(P) - \nu(P)| \right] \qquad (*)$$

Sum does not make sense for uncountable $\mathcal{A}$, but we identify the total variation norm of the difference $\nu_n - \nu$.

$\Rightarrow$ convergence of pattern densities in total variation norm sufficient to extend the theorem.

Potential or coloring V with uncountable many values in $\mathcal{A} \subset \mathbb{R}$

Recall error estimate of Lenz, Müller & Ves. 2008

$$\left\| \frac{1}{n^d} f_{\Lambda_n} - f \right\|_{\mathcal{B}} \leq const \left[\frac{1}{r} + \frac{r}{n} + \sum_{P \in \mathcal{P}_r} |\nu_n(P) - \nu(P)| \right] \qquad (*)$$

Sum does not make sense for uncountable $\mathcal{A}$, but we identify the total variation norm of the difference $\nu_n - \nu$.

$\Rightarrow$ convergence of pattern densities in total variation norm sufficient to extend the theorem.

Unfortunately this, is not a natural assumption.

Glivenko–Cantelli Theory tells us

For an continuous probability measure ν on $\mathbb{R}$ the approximating empirical measures ν_n do not converge in total variation.
So, if $V(x)$ is an i.i.d. sequence of continuous random variables (*) does not converge to zero.

Monotonicity allows integration by parts

Assume that $\mathcal{B} = L^\infty(\mathbb{R})$ and $\omega_x \mapsto f_\Lambda(\omega)$ is monotone for each index $x \in \mathbb{Z}^d$.

Monotonicity allows integration by parts

Assume that $\mathcal{B} = L^\infty(\mathbb{R})$ and $\omega_x \mapsto f_\Lambda(\omega)$ is monotone for each index $x \in \mathbb{Z}^d$.

Idea

monotonicity

$\rightsquigarrow$ weak differentiability

$\rightsquigarrow$ inegration by parts

$\rightsquigarrow$ replace TV norm by $\|.\|_\infty$ norm for distribution functions

Theorem Schumacher, Schwarzenberger & Ves. 17

$V_x : \Omega \to \mathbb{R}, x \in \mathbb{Z}^d$, i.i.d.random variables define random colouring,
$f : \mathcal{P}(\mathbb{Z}^d) \times \Omega \to \mathcal{B}$ equi-variant, almost additive, monotone coordinate-wise and

$$\|f_\Lambda\|_\infty \leq const\, |\Lambda|$$

$\Rightarrow$ exists isotone bounded function $f : \mathbb{R} \to [0, 1]$

$$\lim_{n\to\infty} \left\| \frac{1}{(2n+1)^d} f_{\Lambda_n} - f \right\|_\infty = 0 \qquad \text{for almost all } \omega$$

and explicit error estimates: geometric + probabilistic error.

Recall $\Lambda_n = \mathbb{Z}^d \cap [-n, n]^d$.

Theorem Schumacher, Schwarzenberger & Ves. 17

$V_x\colon \Omega \to \mathbb{R}, x \in \mathbb{Z}^d$, i.i.d.random variables define random colouring,
$f\colon \mathcal{P}(\mathbb{Z}^d) \times \Omega \to \mathcal{B}$ equi-variant, almost additive, monotone coordinate-wise and

$$\|f_\Lambda\|_\infty \leq const\, |\Lambda|$$

$\Rightarrow$ exists isotone bounded function $f\colon \mathbb{R} \to [0,1]$

$$\lim_{n\to\infty} \left\| \frac{1}{(2n+1)^d} f_{\Lambda_n} - f \right\|_\infty = 0 \qquad \text{for almost all } \omega$$

and explicit error estimates: geometric + probabilistic error.

Recall $\Lambda_n = \mathbb{Z}^d \cap [-n, n]^d$.

probabilistic error:
Large deviations type result $\rightsquigarrow$ fast stochastic convergense $\rightsquigarrow$ almost sure convergence

Extension to amenable groups G

Theorem Schumacher, Schwarzenberger & Ves. preprint 2017

$V_x \colon \Omega \to G, x \in \mathbb{Z}^d$, i.i.d.random variables define random colouring,
$f \colon \mathcal{P}(\mathbb{Z}^d) \times \Omega \to \mathcal{B}$ equi-variant, almost additive, monotone coordinate-wise and

$$\|f_\Lambda\|_\infty \leq const\, |\Lambda|$$

$\Rightarrow$ exists isotone bounded function $f \colon \mathbb{R} \to [0,1]$

$$\lim_{n\to\infty} \left\| \frac{1}{|\Lambda_n|} f_{\Lambda_n} - f \right\|_\infty = 0 \qquad \text{for almost all } \omega$$

for any Følner sequence (Λ_n) in G.

Extension to amenable groups G

Theorem Schumacher, Schwarzenberger & Ves. preprint 2017

$V_x \colon \Omega \to G, x \in \mathbb{Z}^d$, i.i.d.random variables define random colouring,
$f \colon \mathcal{P}(\mathbb{Z}^d) \times \Omega \to \mathcal{B}$ equi-variant, almost additive, monotone coordinate-wise and

$$\|f_\Lambda\|_\infty \leq const \, |\Lambda|$$

$\Rightarrow$ exists isotone bounded function $f \colon \mathbb{R} \to [0,1]$

$$\lim_{n\to\infty} \left\| \frac{1}{|\Lambda_n|} f_{\Lambda_n} - f \right\|_\infty = 0 \qquad \text{for almost all } \omega$$

for any Følner sequence (Λ_n) in G.

Error estimate involves not only (Λ_n) but also auxiliary ϵ-quasi tilings of Følner sets.

Extension to amenable groups G

Theorem Schumacher, Schwarzenberger & Ves. preprint 2017

$V_x \colon \Omega \to G, x \in \mathbb{Z}^d$, i.i.d.random variables define random colouring, $f \colon \mathcal{P}(\mathbb{Z}^d) \times \Omega \to \mathcal{B}$ equi-variant, almost additive, monotone coordinate-wise and

$$\|f_\Lambda\|_\infty \leq const\,|\Lambda|$$

$\Rightarrow$ exists isotone bounded function $f \colon \mathbb{R} \to [0,1]$

$$\lim_{n\to\infty} \left\| \frac{1}{|\Lambda_n|} f_{\Lambda_n} - f \right\|_\infty = 0 \qquad \text{for almost all } \omega$$

for any Følner sequence (Λ_n) in G.

Error estimate involves not only (Λ_n) but also auxiliary ϵ-quasi tilings of Følner sets.

New challenge here

Tiles in a quasi-tiling might overlap $\rightsquigarrow$ corresponding sample are not independent $\rightsquigarrow$ empirical measures for pattern frequencies are biased $\rightsquigarrow$ use resampling machinery to correct/estimate bias and apply Glivenko–Cantelli Theory.

Many thanks for your attention!