Spectral gaps and discrete magnetic Laplacians

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1. Motivation and main result
2. Methods: Discrete spectral bracketing

Motivation and main result

- Periodic graphs: $\Gamma = \mathbb{Z}^r$ acts on infinite graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ such that $G = \widetilde{G}/\Gamma$ is finite
- (Combinatorial) Laplacian: $\Delta^{\widetilde{G}}\varphi(v) = \sum_{w \sim v}(\varphi(v) - \varphi(w))$
  $\rightsquigarrow \quad \sigma(\Delta^{\widetilde{G}}) \subset [0, 2d_\infty], \quad d_\infty = \sup_v \deg v \quad (< \infty)$ here)
- (Combinatorial) spectral gap: $S^{\widetilde{G}} = [0, 2d_\infty] \setminus \sigma(\Delta^{\widetilde{G}})$
  $\widetilde{G}$ has full (comb.) spectrum iff $S^{\widetilde{G}} = \emptyset$

Theorem (Fabila-Carrasco, Lledó, P (2017))

Assume $\widetilde{G}$ is a $\mathbb{Z}$-periodic tree. Then the following are equivalent:

(i) *$\widetilde{G}$ has full (comb.) spectrum ($S^{\widetilde{G}} = \emptyset$)*

(ii) *$\widetilde{G}$ is the $\mathbb{Z}$-lattice*

(iii) *$\widetilde{G}$ has no vertex of degree 1*

- Clear: (ii)$\Rightarrow$(i) (calculate); (ii)$\Rightarrow$(iii) (obvious); (iii)$\Rightarrow$(ii) (graph th.)
  We will show $\neg$(iii)$\Rightarrow\neg$(i)

Remarks on main result

- A related result holds for the standard Laplacian given by $(\Delta^{\widetilde{G},\mathrm{std}}\varphi)(v) = \frac{1}{\deg v}\sum_{w\sim v}(\varphi(v)-\varphi(w))$
  $\rightsquigarrow \quad \sigma(\Delta^{\widetilde{G},\mathrm{std}}) \subset [0,2], \qquad S^{\widetilde{G},\mathrm{std}} := [0,2]\setminus \sigma(\Delta^{\widetilde{G},\mathrm{std}})$
- Relation with full spectrum conjecture [HS04] for maximal abelian covering: ("all loops in G are unfolded") if $\widetilde{G}$ (or G) has no vertices of degree 1 then combinatorial or standard Laplacian has full spectrum (proven if all degrees are even — Euler path) or $\widetilde{G}$ is $(2k+1)$-regular with some additional property)
  $\rightsquigarrow$ we have shown the full spectrum conjecture for trees
- results and estimates on lengths of bands for periodic discrete gaps (see e.g. [KS14, KS15, KS17])

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Methods: Spectral ordering

- Let $b > 0$, $S^\pm$ self-adjoint in Hilbert space $\mathscr{H}^\pm$, $\dim \mathscr{H}^\pm = n^\pm < \infty$, $\sigma(S^\pm) \subset [0, b]$
- **Definition:** **(spectral ordering)** $S^- \preccurlyeq S^+$ iff $\lambda_k(S^-) \le \lambda_k(S^+)$ for all k (k-th eigenvalue) where $\lambda_k(S^\pm) = b$ if $k > n^\pm$ (maximal possible value)
- **Magnetic potential:** $\alpha \colon E \to \mathbb{R}$ with $\alpha(w, v) = -\alpha(v, w)$ ($E \subset V \times V$ such that $(v, w) \in E$ iff $(w, v) \in E$)
- (Combinatorial) magnetic Laplacian: $\Delta^G_\alpha \varphi(v) = \sum_{w \sim v} (\varphi(v) - \alpha(v, w)\varphi(w))$
- If G is a tree then $\Delta^G_\alpha \cong \Delta^G$
- Floquet theory: Let $\widetilde{G}$ be $\mathbb{Z}$-periodic tree then $\sigma(\Delta^{\widetilde{G}}) = \bigcup_\alpha \sigma(\Delta^G_\alpha)$ ($G = \widetilde{G}/\mathbb{Z}$) (and α can be supported on one edge only)

Methods: Discrete spectral bracketing

A discrete spectral bracketing result:

- Delete edges: $E_0 \subset E \quad \rightsquigarrow \quad G^- := G - E_0 := (V, E^-)$ with $E^- := E \setminus E_0$
- "Virtualise" vertices: $V_0 \subset V$, $G^+ := G - V_0 := (V^+, E)$, $V^+ := V \setminus V_0$ (some edges have now vertices not in G^+ anymore, virtual vertices, G^+ is a partial subgraph in G)

Theorem (Fabila-Carrasco, Lled'o, P (2017))

Choose $E_0 \subset E$, $V_0 \subset V$ and magnetic potential $\alpha \colon E \to \mathbb{R}$ such that

- *$G^- = G - E_0$ is a tree;* $\operatorname{supp} \alpha \subset E_0$*;*
- *$E_0 \subset \bigcup_{v \in V_0} E_v$ (edges in E_0 have at least one end in V_0)*

then $\Delta^{G^-} \preccurlyeq \Delta^G_\alpha \preccurlyeq \Delta^{G^+}$

Corollary

$J_k := [\lambda_k(\Delta^{G^-}), \lambda_k(\Delta^{G^+}]$, $J := \bigcup_k J_k$, *then* $\bigcup_\alpha \sigma(\Delta^G_\alpha) \subset J$ $(*)$.

Thank you for your attention!

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