Optimal Hardy-type inequality for elliptic operators: The continuum case

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Joint works with
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The setting

Consider a second-order linear elliptic operator P with real coefficients in divergence form

$$Pu := -\mathrm{div}\left[A(x)\nabla u + u\tilde{b}(x)\right] + b(x)\cdot\nabla u + c(x)u,$$

which is defined in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (or more generally, on a smooth noncompact weighted Riemannian manifold Ω of dimension n, where $\mathrm{d}\nu := m\,\mathrm{d}x$ is a given measure).

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P is symmetric if $\tilde{b} = b$. In this case, P is in fact a Schrödinger-type operator

$$Pu = -\mathrm{div}\big(A\nabla u\big) + \big(c - \mathrm{div} b\big)u = -\Delta_A + V(x).$$

Agmon's problem

Problem (Agmon (1982))

Given a symmetric elliptic operator P in $\mathbb{R}^n$, find a continuous, nonnegative function W which is 'as large as possible' such that for some neighborhood of infinity $\Omega_R = \{|x| > R\}$ the following inequality holds

$$\int_{\Omega_R} P\varphi\,\overline{\varphi}\,\mathrm{d}\nu \geq \int_{\Omega_R} W(x)|\varphi|^2\,\mathrm{d}\nu \qquad \forall\,\varphi \in C_0^\infty(\Omega_R).$$

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Agmon used such W to measure the decay of solutions of the equation $Pu = \lambda u$ in $\mathbb{R}^n$ via the celebrated Agmon's metric

$$\mathrm{d}s^2 := W(x)\sum_{i,j=1}^{n} a_{ij}(x)\,\mathrm{d}x_i\,\mathrm{d}x_j, \quad \text{where } \left[a_{ij}\right] := A^{-1}.$$

The decay is given in terms of W and a function h satisfying

$$|\nabla h(x)|_A^2 < W(x) \qquad \text{a.e. } \Omega.$$

Features of Hardy inequality $W(x) = \frac{C_H}{|x|^2}$

Let $\Omega^\star := \mathbb{R}^n \setminus \{0\}$. Consider the celebrated Hardy inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \,\mathrm{d}x \geq \lambda \int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star), \tag{1}$$

where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$.

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where $\lambda \leq 1$ and $C_H := \left(\frac{n-2}{2}\right)^2$. It has the following important features:

(a) $P = -\Delta - \frac{C_H}{|x|^2}$ is *critical* in $\Omega^\star$, i.e., for any $V(x) \gneqq \frac{C_H}{|x|^2}$ the inequality

$$\int_{\Omega^\star} |\nabla\varphi|^2 \, \mathrm{d}x \geq \int_{\Omega^\star} V(x) |\varphi(x)|^2 \, \mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\Omega^\star)$$

is not valid. In particular, $\lambda = 1$ is the best constant for (1).

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(c) The corresponding Rayleigh-Ritz variational problem

$$\inf_{\varphi \in \mathcal{D}^{1,2}(\Omega^\star)} \left\{ \frac{\int_{\Omega^\star} |\nabla \varphi|^2 \, \mathrm{d}x}{\int_{\Omega^\star} \frac{C_H}{|x|^2} |\varphi(x)|^2 \, \mathrm{d}x} \right\}$$

admits no minimizer.

Criticality theory

Definition

Let P be a general, second-order elliptic operator on a domain $\Omega \subset \mathbb{R}^n$ (or on a noncompact manifold Ω), $n \geq 2$.

- *P is nonnegative* ($P \geq 0$) *in* Ω if the equation $Pu = 0$ in Ω admits a global positive (super)solution.

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- If $P \not\geq 0$ in Ω, then P is *supercritical in* Ω.

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Remarks

1. In the symmetric case, $P \geq 0$ iff the quadratic form associated to P is nonnegative on $C_0^\infty(\Omega)$ (i.e. $\int_\Omega P\varphi\overline{\varphi}\,\mathrm{d}\nu \geq 0 \quad \forall\, \varphi \in C_0^\infty(\Omega)$) (the Allegretto-Piepenbrink theorem).

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4. If P is critical in Ω, then the equation $Pu = 0$ admits a unique positive (super)solution ψ in Ω, called the *(Agmon) ground state of P in Ω*.

Optimal Hardy-weight: Features (a)–(c)

We assume that $x_0 = 0 \in \Omega$, and denote $\Omega^* := \Omega \setminus \{0\}$.

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Let P be subcritical in Ω. We say that $W \geq 0$ is an *optimal Hardy-weight for P in Ω^** if $P - W$ has the following properties:

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(c) Denote the ground states of $P - W$ and $P^{\star} - W$ in Ω^* by ψ and $\psi^{\star}$. Then $\psi\psi^{\star}$ is not $W\mathrm{d}\nu$-integrable in any fixed neighborhood of either 0 or $\bar{\infty}$ (P is said to be *null-critical in $\Omega^{\star}$*).

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Aim: For general P and Ω find an optimal Hardy-weight W

The supersolution construction

Lemma (Supersolution construction)

Let v_j be two positive solutions (resp. supersolutions) of the equation $Pu = 0$, $j = 0, 1$, in a domain Ω. Then for any $0 \le \alpha \le 1$ the function

$$v_\alpha(x) := \big(v_1(x)\big)^\alpha \big(v_0(x)\big)^{1-\alpha}$$

is a positive solution (resp. supersolution) of the equation

$$\big[P - 4\alpha(1-\alpha)W(x)\big]u = 0 \qquad \text{in } \Omega,$$

Where

$$W(x) := \frac{Pv_{1/2}}{v_{1/2}} = \frac{|\nabla v|_A^2}{4v^2} \ge 0, \quad v := \frac{v_1}{v_0}, \quad |\xi|_A^2 := \xi \cdot A\xi.$$

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In particular, $P - W \ge 0$ in Ω.

Main result

Theorem

Let P be a subcritical operator in Ω, and let $G(x) := G_P^{\Omega}(x,0)$. Let u be a positive solution of the equation $Pu = 0$ in Ω satisfying

$$\lim_{x \to \bar{\infty}} v(x) = 0, \qquad \text{where } v(x) := \frac{G(x)}{u(x)}.$$

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Then $W := \frac{|\nabla v|_A^2}{4v^2}$ is an optimal Hardy-weight in $\Omega^\star$.

Furthermore, if P is symmetric and $W > 0$, then the spectrum and the essential spectrum of the operator $W^{-1}P$ on $L^2(\Omega^\star, W\mathrm{d}\nu)$ satisfy

$$\sigma(W^{-1}P) = \sigma_{\mathrm{ess}}(W^{-1}P) = [1, \infty).$$

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cf. Adimurthi-Sekar, Carron, Cowan, D'Ambrosio, Li-Wang, Cazacu-Zuazua, ...

On the condition $\lim_{x\to\bar{\infty}} \frac{G(x)}{u(x)} = 0$

Remark

By a result of A. Ancona (2002), if P is symmetric, or more generally if $G_P^\Omega(x,y) \asymp G_P^\Omega(y,x)$, then there exists $u > 0$ satisfying the equation $Pu = 0$ in Ω, and

$$\lim_{x\to\bar{\infty}} \frac{G(x)}{u(x)} = 0.$$

Proof that $\sigma(W^{-1}P) = \sigma_{\mathrm{ess}}(W^{-1}P) = [1, \infty)$

Using the supersolution construction

$$v_\alpha(x) = \big(G(x)\big)^\alpha \big(u(x)\big)^{1-\alpha}$$

with G and u not only for $0 \leq \alpha \leq 1$, but for $\alpha \in \mathbb{C}$ satisfying

$$4\alpha(1-\alpha) = \lambda, \qquad \text{where } \lambda \in \mathbb{R},$$

we get solutions of the equation $(W^{-1}P - \lambda)u = 0$ of the form

$$\varphi_\pm(\lambda, x) := (Gu)^{1/2} (G/u)^{\frac{\pm\sqrt{1-\lambda}}{2}}.$$

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Case 1: $\lambda \leq 1$.
Then $\varphi_\pm(\lambda, x) \geq 0$.

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Case 1: $\lambda \le 1$.
Then $\varphi_\pm(\lambda, x) \ge 0$.
Hence, the Allegretto-Piepenbrink theorem implies

$$\sigma(W^{-1}P) \subset [1, \infty).$$

$\sigma(W^{-1}P) = \sigma_{\mathrm{ess}}(W^{-1}P) = [1,\infty)$ Cont.

Recall:

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Case 2: $\lambda = 1$. Then there is only one such a solution, namely $\varphi_{+}(1, x) := (Gu)^{1/2}$.

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But there is another solution $\varphi_-(1, x) := (Gu)^{1/2} \log (G/u)$ which is not positive, but dominates $\varphi_+(1, x)$ near the ends of $\Omega^\star$.

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But there is another solution $\varphi_-(1, x) := (Gu)^{1/2} \log (G/u)$ which is not positive, but dominates $\varphi_+(1, x)$ near the ends of $\Omega^\star$.

Hence, by the (Khas'minskiĭ criterion for recurrency), $\varphi_+(1, x) := (Gu)^{1/2}$ is a ground state and $P - W$ is critical in $\Omega^\star$.

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Case 3: $\lambda \geq 1$.
In this case we have

$$|\varphi(\lambda, x)| \leq \varphi(1, x) = (Gu)^{1/2}.$$

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Hence, by a recent Shnol-type theorem of S. Beckus & Y.P., we have $\lambda \in \sigma(W^{-1}P)$.

Thus,

$$\sigma(W^{-1}P) = \sigma_{\rm ess}(W^{-1}P) = [1,\infty).$$

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6. Optimal Hardy inequalities for operators on graphs (will be presented tomorrow by Felix Pogorzelski's talk).

Thank you for your attention!