Combinatorial curvature and isoperimetric constants on infinite planar graphs

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Analysis and Geometry on Graphs and Manifolds

July 03, 2017

G: a graph with the vertex set V and the edge set E such that

- G is undirected, connected, and simple;
- G is embedded into a 2-mainfold Ω locally finitely;
- every face of G (a component of $\Omega \setminus G$) is homeomorphic to the unit disk and the boundary of each face is homeomorphic to a circle or a straight line; and
- $3 \leq \deg v < \infty$ and $3 \leq \deg f \leq \infty$ for every $v \in V$ and $f \in F$, where F is set of faces of G.

In most cases we also assume that

- G is infinite and $\Omega = \mathbb{R}^2$; and
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Under these assumptions, one can check that G is 2-connected; that is, G minus a vertex still remains connected.

Basic Setting

- Note that in our setting two different faces may share more than one vertex without sharing an edge.
- But if we further assume that if the intersection of two faces is empty, or a vertex, or an edge, then G becomes a 3-connected graph; i.e., G minus any two vertices is connected.
- Such G is a tessellation graph or an edge graph of the plain tiling.

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Subgraph

A graph $S = (V(S), E(S))$ is called a *subgraph* of G if $V(S) \subset V$ and $E(G) \subset E$.

- A subgraph $S \subset G$ is called *induced* if for every $v, w \in V(S)$, $vw \in E$ implies $vw \in E(S)$.
- We define the *face set* $F(S)$ of $S \subset G$ as the *subset* of F such that $f \in F(S)$ if and only if $f \in F$ and f is a component of $\Omega \setminus S$.
- For the boundaries of S, we define

∂S: the set of edges connecting S to $G \setminus S$

$\operatorname{bd} S$: the set of edges in $E(S)$ that is incident to a face in $F \setminus F(S)$

$d_0 S$: the set of vertices in $V(S)$ which has a neighbor in $V \setminus V(S)$

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Isoperimetric Constants

Isoperimetric constants of G are defined by

$$\imath_p(G) := \inf_S \frac{|\partial S|}{\mathrm{Vol}\,(S)}, \qquad \imath(G) = \inf_S \frac{|\partial S|}{|V(S)|},$$

$$\jmath_0(G) = \inf_S \frac{|d_0 S|}{|V(S)|}, \qquad \jmath_1(G) = \inf_S \frac{|d_1 S|}{|V(S)|},$$

$$\imath^*(G) = \inf_S \frac{|\mathrm{bd}\, S|}{|F(S)|},$$

where $|\cdot|$ denotes the cardinality, $\mathrm{Vol}\,(S) = \sum_{v \in V(S)} \deg v$, and the infimums are taken over all finite subgraphs $S \subset G$.

Isoperimetric Constants

- Isoperimetric constants are discrete analogues of Cheeger's constant in Differential Geometry.
- We say that G satisfies a strong isoperimetric inequality if an isoperimetric constant is positive.
- One can check that
  - $\imath(G^*) = \imath^*(G)$, where G^* is the dual graph of G
  - $\jmath_0(G) = \jmath_1(G)/(1 + \jmath_1(G))$
  - $\imath_p(G) > 0 \Longleftrightarrow \imath(G) > 0$
  - $\jmath_0(G) > 0 \Longrightarrow \imath(G) > 0, \imath_p(G) > 0, \imath^*(G) > 0.$

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- For each $v \in V$, we define the *vertex curvature* at v by

$$\phi(v) := 1 - \frac{\deg v}{2} + \sum_{f : v \in V(f)} \frac{1}{\deg f}.$$

- For finite subgraph $S \subset G$, we define

$$\phi(S) = \sum_{v \in V(S)} \phi(v).$$

- Other combinatorial curvatures: face curvature, edge curvature, corner curvature, etc.

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Meaning of Combinatorial Curvature

- For each $f \in F$, we associate a Euclidean regular $(\deg f)$-gon with unit edge length, and paste them along edges exactly the way that the faces of G are pasted.
- Then the resulting surface Γ is a surface of *polyhedral* metric, which is a kind of Aleksandrov surfaces. Note that G is naturally embedded into Γ.
- Γ is locally Euclidean except at the points corresponding to the vertices of G, and the *total angle* at $v \in V(G) \subset \Gamma$ is

$$T(v) = \sum_{f : v \in V(f)} \frac{\pi(\deg f - 2)}{\deg f} = \pi \cdot \deg v - 2\pi \sum_{f : v \in V(f)} \frac{1}{\deg f},$$

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Combinatorial Gauss-Bonnet Theorem-Basic Form

Theorem (Combinatorial Gauss-Bonnet Theorem-Basic Form)

Suppose G is a finite connected simple graph embedded into a compact 2-manifold Ω. Then we have

$$\phi(G) = \sum_{v \in V(G)} \phi(v) = \chi(\Omega).$$

- There are other forms of Combinatorial Gauss-Bonnet Theorem in literature.
- Using the Combinatorial Gauss-Bonnet Theorem one can deduce many useful theorems. For example, using the basic form above, one can prove the famous theorem that every finite planar graph has a vertex of degree at most 5.

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Let G be a connected simple infinite graph embedded into $\mathbb{R}^2$. The following works were done independently.

Theorem (Żuk, 1997)

If $\phi(v) < 0$ for every $v \in V(G^*)$, then $\imath_p(G) > 0$.

Theorem (Woess, 1998)

If $\overline{\phi}(G) := \limsup_{|V(S)| \to \infty} \dfrac{\phi(S)}{|V(S)|} < 0$, *where the limit superior is taken over all connected and finite subgraphs S, then $\imath_p(G) > 0$.*

Theorem (Higuchi, 2001)

If $\phi(v) < 0$ for every $v \in V(G)$, then $\imath^*(G) > 0$.

Negatively Curved Graphs

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Negatively Curved Graphs

We also have

Theorem (O., 2014)

Suppose G and G^* are 2-connected simple infinite planar graphs. Then

1. $\jmath_0(G) > 0 \iff \jmath_0(G^*) > 0$
2. *If* $\overline{\phi}(G) := \limsup_{|V(S)| \to \infty} \frac{\phi(S)}{|V(S)|} < 0$ *and G is 3-connected, then* $\jmath_0(G) > 0$. *Therefore in this case we also have* $\imath(G) > 0$, $\imath^*(G) > 0$, $\jmath_0(G^*) > 0$, $\imath(G^*) > 0$, *and* $\imath^*(G^*) > 0$.
3. *If $\imath^*(G) > 0$ and the face degrees of G are bounded by above, then G is Gromov hyperbolic.*

Theorem (O., and Seo, 2016)

The above result can be extended to planar graphs with more than one end.

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Positively Curved Graphs

Theorem (DeVos and Mohar, 2007. Myer's Theorem for Tessellation)

Suppose G is a connected simple graph embedded into a 2-manifold without boundary. If $\phi(v) > 0$ *for every* $v \in V(G)$*, then G is finite. Moreover, if G is different from prisms or antiprisms, then* $|V(G)| \leq 3444$*.*

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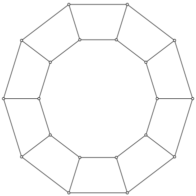
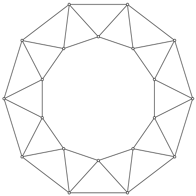


(c) Prism



(d) Antiprism

Positively Curved Graphs

Theorem (DeVos and Mohar, 2007. Myer's Theorem for Tessellation)

Suppose G is a connected simple graph embedded into a 2-manifold without boundary. If $\phi(v) > 0$ for every $v \in V(G)$, then G is finite. Moreover, if G is different from prisms or antiprisms, then $|V(G)| \leq 3444$.

Question(asked by DeVos and Mohar): what is the best constant C_0 that can take the place of 3444 in the previous statement?

The following are the known bounds for C_0.

$138 \leq C_0$ (Réti, Bitay, and Kosztolányi , 2007)

$C_0 < 580$ (Zhang, 2008)

$208 \leq C_0$ (Nicholson and Sneddon, 2008)

$C_0 \leq 380$ (O., 2017)

Thus we know that $208 \leq C_0 \leq 380$, but the exact value of C_0 is not known yet.

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Positively Curved Graphs

The following theorems are also worth to mention:

Theorem (Chen, 2009)

Suppose G is a connected simple infinite graph embedded into a 2-manifold without boundary. If $\phi(v) \geq 0$ for every $v \in V$, then $\phi(v) = 0$ for all $v \in V$ except finitely many.

Theorem (Hua and Su, preprint)

Under the same assumption as above, we have

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(p, q)-Regular Graphs

Let G be a simple graph embedded into $\mathbb{R}^2$ such that $\deg v = p$ and $\deg f = q$ for every $v \in V$ and $f \in F$, where p, q are natural numbers greater than or equal to 3. Such G will be called a (p, q)-*regular graph*.

CASE I: $\dfrac{1}{p} + \dfrac{1}{q} > \dfrac{1}{2}$

In this case G is finite and one of the platonic solids; i.e., G is one of the tetrahedron, the octahedron, the icosahedron, the cube, or the dodecahedron.

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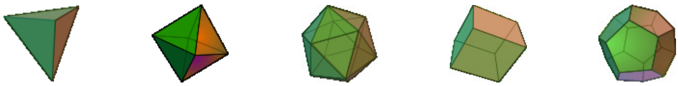

CASE II: $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{2}$

In this case G is infinite and one of the regular tilings of the plane; i.e., G is one of the regular triangulation of the plane, the square lattice, or the hexagonal honeycomb. Consequently, all isoperimetric constants of G are zero in this case.

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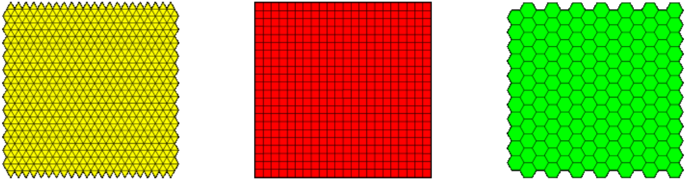

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Theorem (Häggström, Jonasson, and Lyons, (2002) & Higuchi and Shirai (2003))

Suppose G is a (p, q)-regular graph with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Then

$$\imath_p(G) = \frac{p-2}{p}\sqrt{1 - \frac{4}{(p-2)(q-2)}},$$

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Lower Bounds for Isoperimetric Constants

Let $\mathcal{G}(p, q)$ be the collection of (infinite) simple planar graphs such that $\deg v \geq p$ and $\deg f \geq q$. If $q = 3$, we will use the notation $\mathcal{G}(p, 3) = \mathcal{G}(p)$.

Theorem (Dodziuk, 1984)

If $G \in \mathcal{G}(7)$, then $\imath^*(G) \geq \frac{1}{26}$.

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If $G \in \mathcal{G}(p)$ for some $p \geq 7$, then $\imath^*(G) \geq \frac{p-6}{p-4}$.

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If $G \in \mathcal{G}(p)$ for some $p \geq 7$, then

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Conjecture (Lawrencenko, Plummer, and Zha, 2002)

If $G \in \mathcal{G}(p)$ for some $p \geq 7$, then $\imath^*(G) \geq \sqrt{\frac{p-6}{p-2}}$.

They also implicitly conjectured that if $G \in \mathcal{G}(p,q)$ for some p, q satisfying $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, then

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This is a very natural conjecture, since if $G \in \mathcal{G}(p,q)$, then it is negatively curved more than the (p,q)-regular graph, hence the isoperimetric constants of G would be greater than or equal to those of the (p,q)-regular graph.

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Sketch of Proof

The main lemma for the proof is the following.

Lemma (Main Lemma)

Suppose S is a finite subgraph of G and let S° be the induced graph with $V(S^\circ) = V(S) \setminus V(bd(S))$. Then

$$|V(bd(S))| \geq |V(bd(S^\circ))| + (pq - 2p - 2q)|V(S^\circ)| + 2q$$

if $V(S^\circ) \setminus V(bd(S^\circ)) \neq \emptyset$, and

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Using the main lemma repeatedly, one can associate S with a sequence a_n such that $a_0 \geq 1$, $a_k = |V(\mathrm{bd}\,(S))|$ for some $k \in \mathbb{N}$, and $a_n \geq a_{n-1} + (pq - 2p - 2q)(a_0 + a_1 + \cdots + a_{n-1}) + 2q$ for every $n \geq 2$, etc.

Then by studying the sequence a_n carefully, it is possible to obtain the sharp lower bounds for isoperimetric constants.

The main tools for the main lemma are some versions of Combinatorial Gauss-Bonnet Theorem.

Suppose S is a finite subgraph of G. Then one walks along $\mathrm{bd}\,(S)$ *from the outside* and get the total *left turns* (exterior curvatures), which we denote by $\tau_o(\mathrm{bd}\,(S))$. Note that this is the total left turns occurred on the boundary of the ϵ-neighborhood of $D(S) := \sum_{f \in F(S)} \overline{f}$. Similarly let $\tau_i(\mathrm{bd}\,(S))$ be the total left turns (interior curvatures) obtained when one walks along $\mathrm{bd}\,(S)$ *from the inside*.

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Theorem (Combinatorial Gauss-Bonnet Theorem involving turns on the boundary)

Suppose S is a finite subgraph of G. Then

(1) $\phi(S) = \chi(S) + \tau_o(bd(S))$

(2) $\phi(S^\circ) + \tau_i(bd(S)) = \chi(S) + C$.

By using (1) applied to S°, one can get a lower bound of the number of edges from S° to $\text{bd}(S)$, and then by using (2) one can obtain a lower bound of the number of vertices on $\text{bd}(S)$, which gives a proof for the main lemma.

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