# Quantum State Transfer on Graphs (with a potential)

#### Gabor Lippner

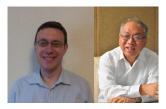
### Analysis and Geometry on Graphs and Manifolds Potsdam

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#### Joint work with Mark Kempton and S-T Yau (Harvard).

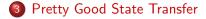


- 1. Perfect state transfer on graphs with a potential, Quant. Inf. Comput., 17 (2017) no. 3&4, 0303–0327
- Pretty good quantum state transfer in symmetric spin networks via magnetic field, Quant. Inf. Proc. 16 (2017) no. 9, 210













2 State transfer with potential



### Discrete Schrödinger equation

- G(V, E) finite graph
- $\Delta$  is the (combinatorial) Laplace operator
- Study solutions  $\varphi: V \to \mathbb{C}$  of

$$\frac{d}{dt}\varphi = i\Delta\varphi$$

As usual, we can write the solution as

$$\varphi_t = e^{it\Delta}\varphi_0$$

Since e<sup>it∆</sup> is unitary, ||φ<sub>t</sub>||<sub>2</sub> is preserved. If we normalize ||φ<sub>0</sub>||<sub>2</sub> = 1, then |φ<sub>t</sub>(v)|<sup>2</sup> : v ∈ V is a probability distribution on the vertices.

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### Perfect State Transfer

- Quantum tunneling: the phenomenon that a quantum particle can cross a "potential barrier" even if it doesn't have enough energy to do so in the classical sense.
- ► Discrete variant: fix two nodes u, v ∈ V. Let φ<sub>0</sub> = δ<sub>u</sub>, and measure |φ<sub>t</sub>(v)|<sup>2</sup>, the tunneling probability.
- We say that there is Perfect State Transfer (PST) from u to v in G if, for some value of t the tunneling probability is 1.

#### Question

Which graphs G and which pairs of nodes  $u, v \in V(G)$  admit PST from u to v?

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Which graphs G and which pairs of nodes  $u, v \in V(G)$  admit PST from u to v?

### Known examples and obstacles

- ► There is PST between the endpoints of a path of size 2 or 3, but not for any size n ≥ 4.
- Certain Abelian Cayley graphs have PST.
- Product/join constructions.
- If a graph has PST between u and v then it can't have an automorphism taking u → v and another taking u → w.
- ► For any *d* there are only finitely many graphs of maximum degree *d* that have PST.

### Motivation from Quantum Communication

- Why is it called PST?
- ► Think of G as a network of spin-1/2 quantum particles where neighbors interact.
- ▶ I control node *u*, you control node *v*.
- I initialize the network by setting the particle at u to some specific quantum state. If there is PST then you will be able to read off the same exact state your node after a given time.
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### Spectral characterization I

- Denote the eigenvalues of Δ by λ<sub>1</sub>,..., λ<sub>n</sub> and the corresponding eigenvectors by ψ<sub>1</sub>,..., ψ<sub>n</sub>.
- PST at time t is clearly equivalent to

$$1 = \left| \left( e^{it\Delta} \right)_{uv} \right| = \left| \sum_{j=1}^{n} e^{it\lambda_j} \psi_j(u) \psi_j(v) \right|$$

This can be estimated in two steps, using Caucy-Schwarz:

$$1 \stackrel{(1)}{\leq} \left| \sum_{j=1}^{n} |\psi_j(u)| |\psi_j(v)| \right| \stackrel{(2)}{\leq} \sqrt{\sum_j \psi_j(u)^2 \sum_j \psi_j(v)^2} = 1.$$

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### Spectral characterization II

- ► The second inequality is sharp if and only if |ψ<sub>j</sub>(u)| = |ψ<sub>j</sub>(v)| for all j.
- In this case u and v are called strongly cospectral. This is a strengthening of the notion that u and v are cospectral, meaning that G \ u and G \ v have identical spectrum.
- Three types of eigenvalues:  $L_+ \cup L_- \cup L_0 = \sigma(\Delta)$

$$L_{0} = \{\lambda_{j} : \psi_{j}(u) = \psi_{j}(v) = 0\}$$
  

$$L_{+} = \{\lambda_{j} : \psi_{j}(u) = \psi_{j}(v) \neq 0\}$$
  

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### Spectral characterization III

- For the triangle inequality to hold with equality, the requirement is that e<sup>itλ<sub>j</sub></sup>ψ<sub>j</sub>(u)ψ<sub>j</sub>(v) all have the same argument.
- This is equivalent to the existence of a  $t \in \mathbb{R}$  such that

$$\forall j \in L_+ \forall k \in L_- : e^{it\lambda_j} = -e^{it\lambda_k}$$

Yet another way to state this is what is called the rationality condition:

$$\frac{\lambda_a - \lambda_b}{\lambda_c - \lambda_d} = \frac{\mathsf{odd}}{\mathsf{even}}$$

whenever  $a, c, d \in L_+$  and  $b \in L_-$ .

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### Schrödinger equation with a potential

- Can one improve the situation by perturbing the system?
- Applying magnetic fields to the spin particles corresponds to perturbing the diagonal of Δ. So let us denote the new Hamiltonian by H = Δ + W where W is a diagonal matrix.

#### Question

Given a graph G(V, E) and nodes  $u, v \in V$ , is there a potential W that induces PST between u and v?

- Casaccino, Lloyd, Mancini, Severini, Int. J. Quant. Inf. (2009) conjectured that the answer is yes.
- ► They proved it for *G* a complete graph, or a complete graph without the *uv* edge.
- They had numerical evidence for paths.

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### Paths

#### Theorem (Kempton-L-Yau)

There is no potential W that induces PST between the endpoints of a path of size  $n \ge 4$ .

- 1. Potential has to be symmetric.
- 2. Rationality condition implies all eigenvalues must be rational.
- 3. Express spectral moments as polynomials in the potential to show inductively that the potential has to take rational values.
- 4. Delicate but elementary mod 2 number theory argument.

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#### Twin nodes

#### Theorem (Kempton-L-Yau)

Let  $u, v \in G$  be "twin nodes", that is, they have the same neighbors. Then there is some potential W that induces PST between u and v.

- 1. Cospectrality is automatic.
- 2. Exactly n 2 ratios to satisfy for the rationality condition.
- 3. Exactly n 2 parameters.
- 4. The congruence conditions are dense.
- 5. Strategy: use perturbation argument.
- 6. It's still quite difficult to prove that we have an open map from the potential to the ratios....

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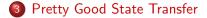
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2 State transfer with potential



Define the tunneling probability as

$$tp_H(u,v) = \sup_{t\to\infty} \left| \left( e^{itH} \right)_{uv} \right|$$

#### Definition (Perfect asymptotic tunneling)

Let  $H = H_c = \Delta + c \cdot W$ . We say there is perfect asymptotic tunneling between u and v if

$$\lim_{c\to\infty} tp_{H_c}(u,v)=1$$

This has been completely characterized using perturbation theory in our paper with Yong Lin and S-T Yau: Quantum tunneling on graphs, *Comm. Math. Phys.* **311** (2012) no. 1. 113–132

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### Pretty good state transfer

#### Definition (Pretty good state transfer)

There is PGST from u to v if  $tp_H(u, v) = 1$ 

- Strong cospectrality is still necessary.
- Rationality conditions are replaced by a much less strict Kronecker-type approximation condition involving integer linear combinations of the eigenvalues in L<sub>+</sub> and L<sub>-</sub>.
- PGST is still pretty rare and hard to pin down. Between endpoints of a path it only happens if the number of nodes is 2<sup>k</sup> (this is due to Banchi, Coutinho, Godsil, Severini).

#### Question

Can the potential help to create PGST?

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### Graphs with an involution

- Difficulty: the potential can ruin cospectrality.
- Hence, we restrict our attention to graphs G with an involution T : V → V that also preserves the potential, and such that u = Tv. Then cospectrality is again automatic.
- ► Furthermore, there are two nice quotients H<sub>+</sub> and H<sub>-</sub> (in the sense of Chris Joyner's talk) of H that yield the L<sub>+</sub> and L<sub>-</sub> part of the spectrum.
- The eigenvalue condition is the following: there should not exist integers l<sub>1</sub>,..., l<sub>n</sub> such that a) ∑<sub>j</sub> l<sub>j</sub> = 0, b) ∑<sub>j</sub> l<sub>j</sub>λ<sub>j</sub> = 0, and c) ∑<sub>j∈L+</sub> l<sub>j</sub> is odd at the same time.

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### Transcendental potential

#### Theorem (Kempton-L-Yau)

Suppose T either has a fixed point or a fixed edge. Then there is a potential that induces PGST from u to v. For a path graph it is possible to put potential only at the end-nodes.

- Second part explains the numerical observations of Casaccino-Lloyd-Mancini-Severini.
- Idea: choose values of the potential to be independent transcendental numbers.
- ► Then (light) Galois theory yields that ∑ l<sub>j</sub>λ<sub>j</sub> = 0 and ∑ l<sub>j</sub> = 0 can only happen if ∑<sub>j∈L+</sub> l<sub>j</sub> = 0.
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### Further directions

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Can one find explicit potentials that induce PST?

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*Is it possible to quantify the trade-off between the transfer time and the "cost" of the potential?* 

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#### The End

## Thank you!