Quantum State Transfer on Graphs (with a potential)

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Analysis and Geometry on Graphs and Manifolds
Potsdam

August 4, 2017

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Joint work with Mark Kempton and S-T Yau (Harvard).

1. Perfect state transfer on graphs with a potential, Quant. Inf. Comput., **17** (2017) no. 3&4, 0303–0327
2. Pretty good quantum state transfer in symmetric spin networks via magnetic field, Quant. Inf. Proc. **16** (2017) no. 9, 210

Overview

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1. Introduction
2. State transfer with potential
3. Pretty Good State Transfer

Discrete Schrödinger equation

- $G(V, E)$ finite graph
- Δ is the (combinatorial) Laplace operator
- Study solutions $\varphi : V \to \mathbb{C}$ of

$$\frac{d}{dt}\varphi = i\Delta\varphi$$

- As usual, we can write the solution as

$$\varphi_t = e^{it\Delta}\varphi_0$$

- Since $e^{it\Delta}$ is unitary, $\|\varphi_t\|_2$ is preserved. If we normalize $\|\varphi_0\|_2 = 1$, then $|\varphi_t(v)|^2 : v \in V$ is a probability distribution on the vertices.

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Perfect State Transfer

- Quantum tunneling: the phenomenon that a quantum particle can cross a “potential barrier” even if it doesn’t have enough energy to do so in the classical sense.
- Discrete variant: fix two nodes $u, v \in V$. Let $\varphi_0 = \delta_u$, and measure $|\varphi_t(v)|^2$, the tunneling probability.
- We say that there is Perfect State Transfer (PST) from u to v in G if, for some value of t the tunneling probability is 1.

Question

Which graphs G and which pairs of nodes $u, v \in V(G)$ admit PST from u to v?

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Known examples and obstacles

- There is PST between the endpoints of a path of size 2 or 3, but not for any size $n \geq 4$.
- Certain Abelian Cayley graphs have PST.
- Product/join constructions.
- If a graph has PST between u and v then it can't have an automorphism taking $u \to v$ and another taking $u \to w$.
- For any d there are only finitely many graphs of maximum degree d that have PST.

Motivation from Quantum Communication

- Why is it called PST?
- Think of G as a network of spin-1/2 quantum particles where neighbors interact.
- I control node u, you control node v.
- I initialize the network by setting the particle at u to some specific quantum state. If there is PST then you will be able to read off the same exact state your node after a given time.
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Spectral characterization I

- Denote the eigenvalues of Δ by $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors by $\psi_1, \ldots, \psi_n$.
- PST at time t is clearly equivalent to

$$1 = \left| \left(e^{it\Delta} \right)_{uv} \right| = \left| \sum_{j=1}^{n} e^{it\lambda_j} \psi_j(u) \psi_j(v) \right|$$

- This can be estimated in two steps, using Caucy-Schwarz:

$$1 \overset{(1)}{\leq} \left| \sum_{j=1}^{n} |\psi_j(u)||\psi_j(v)| \right| \overset{(2)}{\leq} \sqrt{\sum_j \psi_j(u)^2 \sum_j \psi_j(v)^2} = 1.$$

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Spectral characterization II

- The second inequality is sharp if and only if $|\psi_j(u)| = |\psi_j(v)|$ for all j.
- In this case u and v are called strongly cospectral. This is a strengthening of the notion that u and v are cospectral, meaning that $G \setminus u$ and $G \setminus v$ have identical spectrum.
- Three types of eigenvalues: $L_+ \cup L_- \cup L_0 = \sigma(\Delta)$

$$L_0 = \{\lambda_j : \psi_j(u) = \psi_j(v) = 0\}$$
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Spectral characterization III

- For the triangle inequality to hold with equality, the requirement is that $e^{it\lambda_j}\psi_j(u)\psi_j(v)$ all have the same argument.
- This is equivalent to the existence of a $t \in \mathbb{R}$ such that

$$\forall j \in L_+ \forall k \in L_- : e^{it\lambda_j} = -e^{it\lambda_k}$$

- Yet another way to state this is what is called the rationality condition:

$$\frac{\lambda_a - \lambda_b}{\lambda_c - \lambda_d} = \frac{\text{odd}}{\text{even}}$$

whenever $a, c, d \in L_+$ and $b \in L_-$.

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Schrödinger equation with a potential

- Can one improve the situation by perturbing the system?
- Applying magnetic fields to the spin particles corresponds to perturbing the diagonal of Δ. So let us denote the new Hamiltonian by $H = \Delta + W$ where W is a diagonal matrix.

Question

Given a graph $G(V, E)$ and nodes $u, v \in V$, is there a potential W that induces PST between u and v?

- Casaccino, Lloyd, Mancini, Severini, *Int. J. Quant. Inf.* (2009) conjectured that the answer is yes.
- They proved it for G a complete graph, or a complete graph without the uv edge.
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Theorem (Kempton-L-Yau)

There is no potential W that induces PST between the endpoints of a path of size $n \geq 4$.

Outline:

1. Potential has to be symmetric.
2. Rationality condition implies all eigenvalues must be rational.
3. Express spectral moments as polynomials in the potential to show inductively that the potential has to take rational values.
4. Delicate but elementary mod 2 number theory argument.

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Twin nodes

Theorem (Kempton-L-Yau)

Let $u, v \in G$ be "twin nodes", that is, they have the same neighbors. Then there is some potential W that induces PST between u and v.

Outline:

1. Cospectrality is automatic.
2. Exactly $n-2$ ratios to satisfy for the rationality condition.
3. Exactly $n-2$ parameters.
4. The congruence conditions are dense.
5. Strategy: use perturbation argument.
6. It's still quite difficult to prove that we have an open map from the potential to the ratios....

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Possible relaxations of PST

Define the tunneling probability as

$$tp_H(u,v) = \sup_{t\to\infty} \left| \left(e^{itH}\right)_{uv} \right|$$

Definition (Perfect asymptotic tunneling)

Let $H = H_c = \Delta + c \cdot W$. We say there is perfect asymptotic tunneling between u and v if

$$\lim_{c\to\infty} tp_{H_c}(u,v) = 1$$

This has been completely characterized using perturbation theory in our paper with Yong Lin and S-T Yau: Quantum tunneling on graphs, *Comm. Math. Phys.* **311** (2012) no. 1. 113–132

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Pretty good state transfer

Definition (Pretty good state transfer)

There is PGST from u to v if $tp_H(u, v) = 1$

- Strong cospectrality is still necessary.
- Rationality conditions are replaced by a much less strict Kronecker-type approximation condition involving integer linear combinations of the eigenvalues in L_+ and L_-.
- PGST is still pretty rare and hard to pin down. Between endpoints of a path it only happens if the number of nodes is 2^k (this is due to Banchi, Coutinho, Godsil, Severini).

Question

Can the potential help to create PGST?

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Graphs with an involution

- Difficulty: the potential can ruin cospectrality.
- Hence, we restrict our attention to graphs G with an involution $T : V \to V$ that also preserves the potential, and such that $u = Tv$. Then cospectrality is again automatic.
- Furthermore, there are two nice quotients H_+ and H_- (in the sense of Chris Joyner's talk) of H that yield the L_+ and L_- part of the spectrum.
- The eigenvalue condition is the following: there should not exist integers $l_1, \ldots, l_n$ such that a) $\sum_j l_j = 0$, b) $\sum_j l_j \lambda_j = 0$, and c) $\sum_{j \in L_+} l_j$ is odd at the same time.

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Transcendental potential

Theorem (Kempton-L-Yau)

Suppose T either has a fixed point or a fixed edge. Then there is a potential that induces PGST from u to v.
For a path graph it is possible to put potential only at the end-nodes.

- Second part explains the numerical observations of Casaccino-Lloyd-Mancini-Severini.
- Idea: choose values of the potential to be independent transcendental numbers.
- Then (light) Galois theory yields that $\sum l_j \lambda_j = 0$ and $\sum l_j = 0$ can only happen if $\sum_{j \in L_+} l_j = 0$.
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Can one find explicit potentials that induce PST?

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Is it possible to quantify the trade-off between the transfer time and the "cost" of the potential?

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The End

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