Limit shape universality in cellular automata models on the Sierpinski gasket

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arXiv:1702.04017	C.–Huss–Sava-Huss–Teplyaev
arXiv:1702.08370	Huss–Sava-Huss
preprint soon	C.–Kudler-Flam

Cellular automata models & the limit shape universality conjecture

In this talk I will describe the following models: [recent results on *SG*]

1. Internal diffusion-limited aggregation (IDLA): [C.–Huss–Sava-Huss–Teplyaev '17]
2. Divisible sandpiles: [Huss–Sava-Huss '17]
3. Rotor-router aggregation: [C.–Kudler-Flam '17+]
4. Abelian sandpiles: [C.–Kudler-Flam '17+]

They all belong to the class of *abelian networks* introduced by Bond–Levine '13$\sim$'14.

The IDLA is a random growth model; all others are deterministic growth models.

Conjecture ("Folklore*" limit shape universality)

Given any fixed state space, *the limit shapes of clusters formed under the 4 models coincide.*

***Sources:** 2017 Bulletin AMS survey paper by Lionel Levine and Yuval Peres; 4 of Wolfgang Woess' PhD students (Huss, Sava-Huss, Bertacchi, Zucca); Antal Járai (*cf.* his sandpile lecture notes from the 2013 Cornell Probability Summer School.)

Status of its resolution. Covfefe??????????????

① **Internal** DLA [Meakin–Deutch '86, Diaconis–Fulton '88]

Set the first particle at the origin.
Each successive particle performs i.i.d. random walk started from the origin, and stops upon the first exit from the aggregate formed by the previous particles.

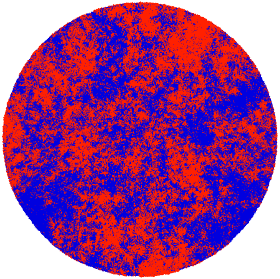


Figure by Lionel Levine

Limit shape theorem (Lawler–Bramson–Griffeath '92) IDLA clusters on $\mathbb{Z}^d$ fill Euclidean balls.

$$\forall\epsilon > 0 : \quad B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon)) \quad \text{for all sufficiently large } n\text{, w.p.1}$$

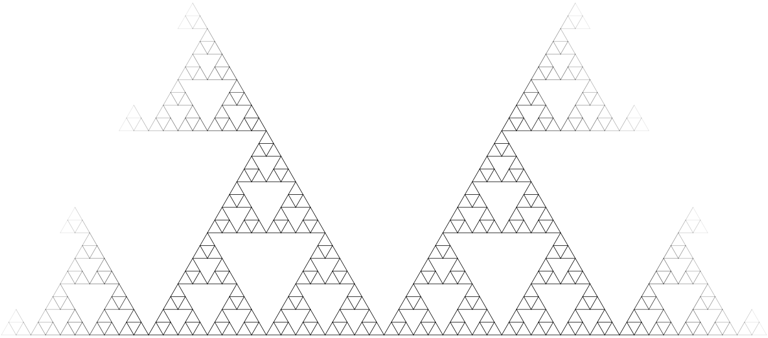
Corresponding limit shape theorems on many other state spaces: percolation clusters, (non)amenable groups, comb lattices, etc.

IDLA: known limit shape results

- Lawler–Bramson–Griffeath '92: On $\mathbb{Z}^d$, IDLA fills Euclidean balls w.p. 1.
- Lucas '12: Replace simple RWs by drifted RWs on $\mathbb{Z}^d$, IDLA cluster converges to a true heat ball in $\mathbb{R}^d$.
- Blachère–Brofferio '07: On discrete groups with exponential growth.
- Huss '09: On nonamenable graphs.
- Shellef '10: Inner bound for IDLA on supercrticial percolation cluster on $\mathbb{Z}^d$ is a ball.
- Duminil-Copin–Lucas–Yadin–Yehudayoff '13: Outer bound for IDLA on supercritical percolation cluster on $\mathbb{Z}^d$ is a ball.
- Huss–Sava '11: On comb lattices, IDLA fills "diamonds."

In many examples, the IDLA cluster shapes coincide with level sets of the Green's function $x \mapsto G(o,x)$.
However it is NOT always true, *e.g.* on the comb lattice.

The (double-sided) graphical Sierpiński gasket

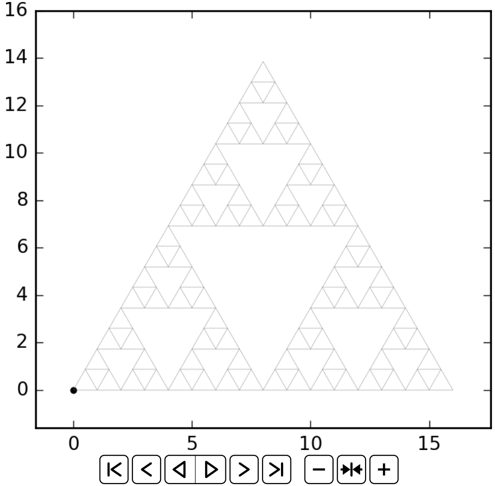


Volume: $|B(x,r)| \asymp r^{d_H}$, Expected exit time: $E_x[\tau_{B(x,r)}] \asymp r^{d_W}$.

Green's function: $G(x,y) \asymp d(x,y)^{d_W - d_H}$.

Hausdorff dim $d_H = \dfrac{\log 3}{\log 2}$, Walk dim $d_W = \dfrac{\log 5}{\log 2}$.

IDLA on the (one-sided) graphical *SG*



Simulations by Jonah Kudler-Flam

Limit shape of IDLA on SG

$\mathcal{I}(k)$: IDLA cluster consisting of k particles launched from the origin o.
$B_o(r)$: Closed ball of radius r in the *graph metric* centered at o.

Theorem (C.–Huss–Sava-Huss–Teplyaev. `arXiv:1702.04017`)

For every $\epsilon > 0$,

$$B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$$

holds for all sufficiently large n, with probability 1.

This theorem says that IDLA on SG fills balls in the graph metric, but does not provide the rate of convergence.
A more quantitative statement:

$$B_o(n-\phi_-(n)) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n+\phi_+(n)),$$

where $\phi_\pm(n)$ are $o(n)$ functions.

On $\mathbb{Z}^d$, the functions $\phi_\pm(n)$ were rigorously identified by Lawler '95, Asselah–Gaudillière '13 (2x), and Jerison–Levine–Sheffield '13, '14.

On SG this is an open question.

Proof ingredients [C.–Huss–Sava-Huss–Teplyaev '17]

Inner bound: $B_o(n(1-\epsilon)) \subset \mathcal{I}(|B_o(n)|)$.

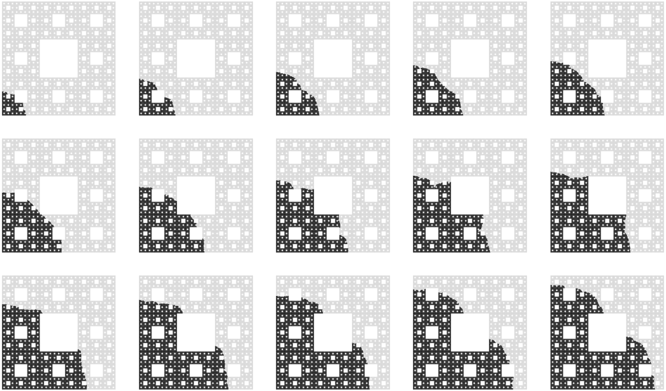
- Establish a discrete mean value inequality over balls $B_o(n)$.
  Given $\epsilon > 0$, for all suff. large n and all $z \in B_o(n(1-\epsilon))$:
  $$\frac{1}{|B_o(n)|} \sum_{y \in B_o(n)} G^n(y,z) \leq G^n(o,z).$$
  Use **divisible sandpiles**. [Done by Huss–Sava-Huss '17]
- Green's function $G(x,y)$ and exit time $\mathbb{E}_x[\tau_{B_x(r)}]$ estimates: well-known on SG.
- Implement the above into the machinery of Lawler–Bramson–Griffeath '92.

Outer bound: $\mathcal{I}(|B_o(n)|) \subset B_o(n(1+\epsilon))$.

- Exploit the *abelian* property of the IDLA process.
- We adapt the algorithm of Duminil-Copin *et al.* '13, by freezing the IDLA process when *either* the particle attaches to the aggregate [STOP] *or* when it exits $B_o(n_j)$ [PAUSE]. (n_j is defined inductively.) Then relaunch the PAUSEd particles towards $B_o(n_{j+1})$. Repeat.
- With the following inputs, we can then implement the algorithm and use the IDLA inner bound to show there are no long outward tentacles, and hence gain control of the outer bound.
  - **Geometric** input: Volume growth of balls and of annuli in SG.
  - **Potential theoretic** input: Show that the Green's function $G^n(x,y)$ killed upon exiting $B_o(n)$ is $\geq C(\epsilon) > 0$ for all $x, y \in B_o((1-\epsilon)n)$. This follows from the **elliptic Harnack inequality** (proved by Kigami '01 on SG) and a chaining argument.

Interlude: IDLA on the graphical Sierpiński carpet (SC)

Limit shape status: Unclear.

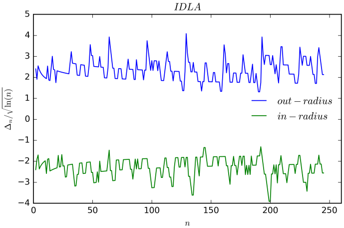


Simulations by Wilfried Huss (PhD thesis, TU Graz, 2010)

Huss: “There seems to be a family of limit shapes as opposed to one limit shape.”
Status of proof. All of our proofs on SG can be adapted to work on SC, except the harmonic measure (divisible sandpile) calculation. A delicate problem.
Known: hitting estimates of Brownian motions on square boundaries via Knight's and corner moves [Barlow–Bass '88∼'92].

Sublog fluctuations in the IDLA cluster on *SG*

In/out-radius (rescaled by $\sqrt{\log n}$) vs. expected radius in IDLA cluster.
($\geq$ 1000 runs for each value n of the expected radius)



Conjecture (C.–Kudler-Flam '17+)

$$\exists C > 0, \quad \forall n \in \mathbb{N}: \quad B_o(n - C\sqrt{\log n}) \subset \mathcal{I}(|B_o(n)|) \subset B_o(n + C\sqrt{\log n})$$

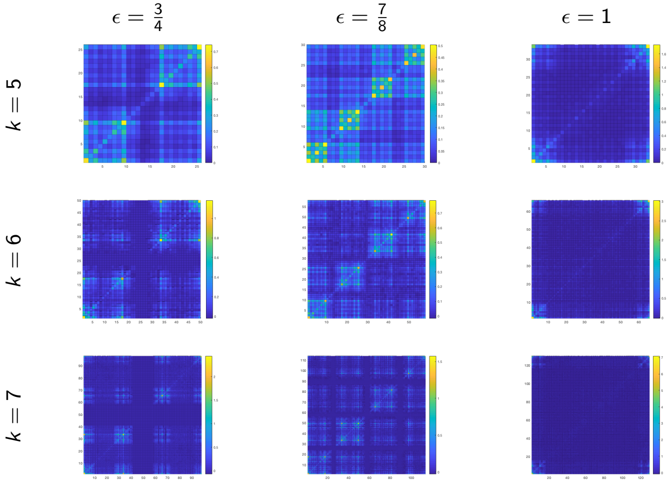
with probability 1.

Potential proof strategy is narrow and involves highly technical potential theoretic estimates, à la Asselah *et al.*

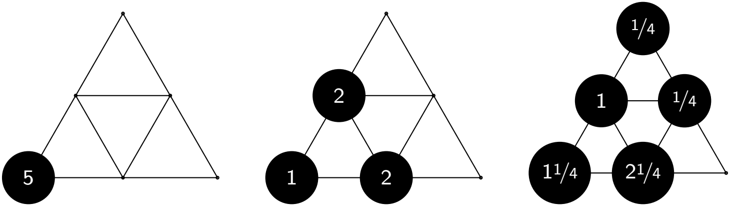
IDLA on SG: Conjectured form of a CLT [C.–Kudler-Flam '17+]

Run the IDLA with Poissonized time: $\mathcal{I}_{N(t)}$, $N(t)$ a rate-1 Poisson process, $t = |B_o(\epsilon 2^k)|$.
Pictured: Covariance of the "lateness function" (*cf.* Jerison–Levine–Sheffield '14).
(The covariance is nonnegative by the FKG inequality.)

What is the limit distribution?



② Divisible sandpiles [Levine–Peres '09]



Keep toppling vertices with > 1 sand and evenly distribute the excess to its neighbors, until every vertex has sand amount ≤ 1 ("stable").

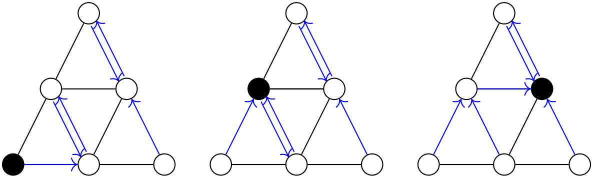
$\partial_I B_o(n) := \{x \in B_o(n) : \exists y \in (B_o(n))^c \text{ such that } x \sim y\}$. Inner boundary of $B_o(n)$
$b_n := |B_o(n)| - \frac{1}{2}|\partial_I B_o(n)|$.

Theorem (Huss–Sava-Huss. `arXiv:1702.08370`)

For any $m \geq 0$, let $n = \max\{k \geq 0 : b_k \leq m\}$. Then the sandpile cluster S_m ("firing set") on SG with initial mass m at o satisfies $B_o(n-1) \subset S_m \subset B_o(n)$.

The solution to the divisible sandpile problem yields effective estimate of the harmonic measure on spheres. Used as an input, in conjunction with arguments of Lawler–Bramson–Griffeath '92, to obtain the IDLA inner bound in [C.–Huss–Sava-Huss–Teplyaev '17].

Rotor(-router) walks



- Each vertex is equipped with an arrow ("rotor") pointing to one of the neighboring vertices.
- Assume the rotor mechanism is periodic and simple (e.g. *NWSENWSE* . . .).
- Rules of the walk: A walker starting at vertex x first rotates the rotor to the next orientation according to the fixed ordering, then moves to the neighboring vertex according to the new orientation. Continue.

Rotor-router aggregation = Aggregation formed by m rotor walkers (started at o).

(A more modest) Conjecture. The IDLA and rotor-router clusters have the same limit shape on *any* state space. Status of its resolution: covfefe.

Rotor-router aggregation on *SG*

$\mathcal{R}(m)$: cluster formed by launching m rotor walks from o.

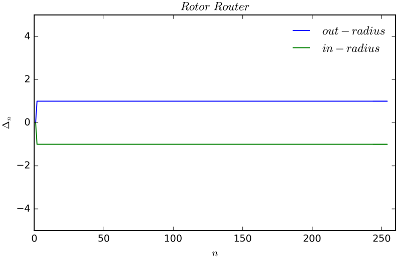
Theorem (C.–Kudler-Flam '17+)

For every $n \geq 1$,

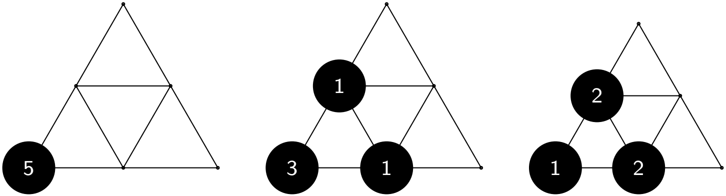
$$B_o(n-1) \subset \mathcal{R}(|B_o(n)|) \subset B_o(n+1)$$

regardless of the initial rotor configuration.

In/out-radius (no rescaling!) vs. expected radius in rotor-router aggregation
($\geq$ 1000 realizations of the initial rotor configuration for each value n of the expected radius)



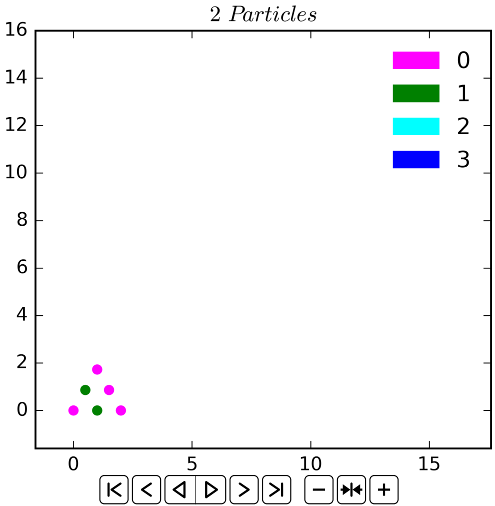
④ Abelian sandpiles [Bak–Tang–Wiesenfeld '88]



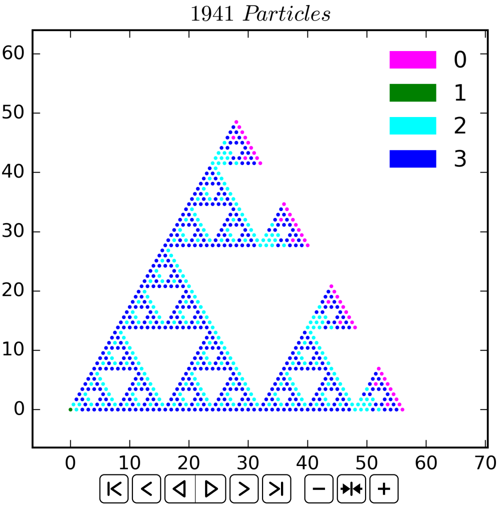
Whenever the # of sand grains at vertex $x \geq \deg(x)$, topple x by emitting one grain to each of the neighbors of x. Continue toppling until every vertex v has # of grains $< \deg(v)$ ("stable"). The order of topplings does not matter (abelian property).

Previous literature on *SG*:

- Physicists (late '90s): Daerden–Vanderzande, Daerden–Priezzhev–Vanderzande numerically studied avalanche statistics, found it fits a power law modulated by log-periodic oscillations.
- Fairchild–Haim–Setra–Strichartz–Westura (2013∼14 Cornell math REU): Established the power law of the diameter-to-mass scaling (Diameter $= \mathcal{O}(m^{1/d_H})$), identified the sandpile group.

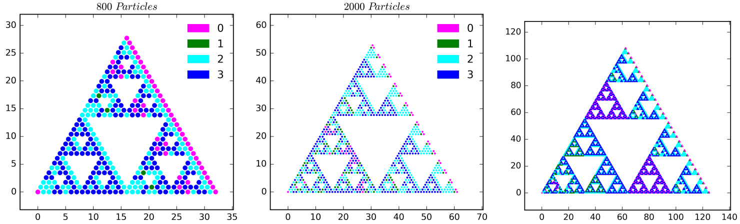


Simulations by Jonah Kudler-Flam



Simulations by Jonah Kudler-Flam

Cluster shapes for abelian sandpiles on SG: Exact ball



Starting from m sand grains at o, define:
The **receiving set** $S(m)$ = set of vertices which have received sands during the topplings.
The **firing set** $A(m)$ = set of vertices which have toppled. (Clearly $A(m) \subset S(m)$.)

Proposition (Receiving set is a perfect ball on SG)

For every $m \in \mathbb{N}$, there exists a unique $r_m \in \mathbb{N}$ such that $S(m) = B_o(r_m)$.

Proof. Induct on the sequence of configs of suitably paused abelian sandpiles in annuli $\{B_o(2^{n+1}) \setminus B_o(2^n)\}_{n=1}^{\infty}$. Relies strongly on the nested structure and self-similarity of SG.

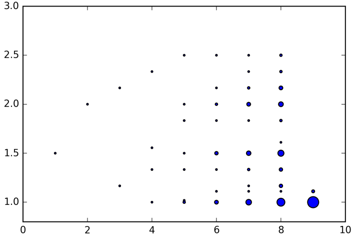
Radial jumps follow a log-periodic pattern

$m \mapsto r_m$ is a càdlàg jump function: Numerical findings by Kudler-Flam '17

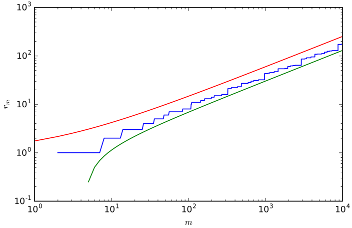
m	$r_m - r_{m-1}$
2	1
8	1
14	1
26	1
36	1
48	1
56	1
84	1
108	**2**
110	1
144	1
162	1
198	1
216	1
270	1
324	**5**

m	$r_m - r_{m-1}$
360	1
432	1
486	4
594	1
648	2
702	1
810	1
972	**11**
1080	1
1134	1
1296	2
1458	7
1782	1
1944	5
2106	2
2268	1

m	$r_m - r_{m-1}$
2430	1
2916	**22**
3240	1
3402	4
3888	4
4374	13
4698	1
5346	2
5832	8
6318	5
6804	2
7290	2
8748	**44**
9720	3



Strong shape theorem for abelian sandpiles on SG



Theorem (C.–Kudler-Flam '17+)

1. *For every $m \in \mathbb{N}$, $B_o(r_m - 1) \subset A(m)$ (firing set) $\subset B_o(r_m) = S(m)$ (receiving set) .*
2. *For every $\epsilon > 0$, $\frac{2}{9} - \epsilon \leq \frac{(r_m)^{d_H}}{m} \leq \left(\frac{3}{4}\right)^{d_H} + \epsilon$ holds for all sufficiently large m.*
3. *Furthermore let $r(x) = r_{\lfloor x \rfloor}$. Then*

$$r(x) = x^{1/d_H}[G(\log x) + o(1)] \quad \text{as } x \to \infty,$$

where G is a nonconstant, discontinuous, $(\log 3)$-periodic function.

Proof of ③. Uniform estimate of $r_{3m} - 2r_m$, combined with the renewal theorem.

Analytic connection among the cellular automata models

- Fix a connected graph and a vertex o. Initial configuration $m\delta_o$ ($m \in \mathbb{N}$).
- $u : V \to \mathbb{R}$ or $\mathbb{N}_0$ is the **odometer function**.

$$u(x) = \begin{cases} \#\text{ exits from } x \text{ in the aggregation process } (\mathbb{N}_0\text{-valued}) \\ \text{amount of mass emitted from } x \text{ in the divisible sandpile process } (\mathbb{R}\text{-valued}) \\ \#\text{ topples at } x \text{ in the abelian sandpile process } (\mathbb{N}_0\text{-valued}) \end{cases}$$

- $(\Delta u)(x) = \sum_{y \sim x} (u(y) - u(x))$ is the **graph Laplacian**.

Model	*Analytic problem*
IDLA	Proof may rely on solving the divisible sandpile problem
Rotor-router aggregation	"$\frac{1}{\deg}\Delta u = \mathbb{1}_{\mathcal{R}(m)} - m\delta_o$ on average" (Identity is exact if all rotors complete periodic rotations)
Divisible sandpiles	$u = \text{pt-wise inf}\{v : \; m\delta_o + \frac{1}{\deg}\Delta v \leq 1\}$
Abelian sandpiles	$u = \text{pt-wise inf}\{v : \; m\delta_o + \Delta v \leq (\deg - 1)\}$

- There is NO *a priori* boundary condition on the PDEs!
- *Least action principle* for the sandpile models: Harder to solve for the abelian sandpiles due to the integrality of the odometer function.

Cellular automata limit shapes: $\mathbb{Z}^d$, $d \geq 2$

For all models: Launch $|B_o(n)|$ walks at o.

Model	*Shape theorem/conjecture*
IDLA	In/out-radius $\begin{cases} n \pm \mathcal{O}(\log n), & d=2 \\ n \pm \mathcal{O}(\sqrt{\log n}), & d \geq 3 \end{cases}$ [α,β,γ,δ]
Rotor-router aggregation	In-radius $n - c\log n$, out-radius $n + c' \log n$ [κ,ℓ] (c, c' indep of n)
Divisible sandpiles	In-radius $n - c$, out-radius $n + c'$ [κ] (c, c' indep of n)
Abelian sandpiles	Limit shape might not be an Euclidean ball? [κ] Rigorous upper/lower estimates available (with a gap) [κ,ι]

[α] Lawler–Bramson–Griffeath '92
[β] Lawler '95
[γ] Asselah–Gaudillière '13 (2x)
[δ] Jerison–Levine–Sheffield '13, '14
[κ] Levine–Peres '09
[ℓ] Levine–Peres '17
[ι] Fey–Levine–Peres '10

IDLA, RRA, and divisible sandpiles all fill Euclidean balls. Abelian sandpiles do not appear to fill Euclidean balls (due to the model's integrality) [See *e.g.* Levine-Peres '17 BAMS].

Abelian sandpile on $\mathbb{Z}^2$ [Levine–Peres '17 *BAMS*]

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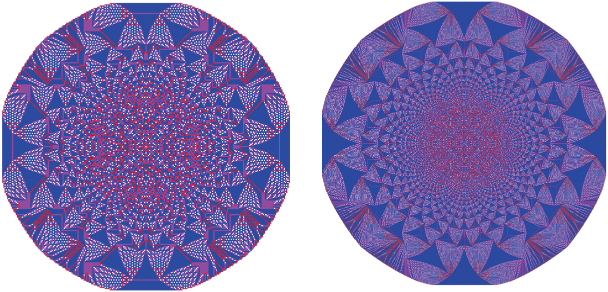

$n = 10^5$ $n = 10^6$

FIGURE 1. Sandpiles in $\mathbb{Z}^2$ formed by stabilizing 10^5 and 10^6 particles at the origin. Each pixel is colored according to the number of sand grains that stabilize there (white 0, red 1, purple 2, blue 3). The two images have been scaled to have the same diameter.

Cellular automata limit shapes: SG

For aggregation models: Launch $|B_o(n)|$ walks at o.
For sandpile models: Start with m chips at o.

Model	*Shape theorem/conjecture*
IDLA	In/out-radius $n \pm \mathcal{O}(\sqrt{\log n})$ [1,2]
Rotor-router aggregation	In/out-radius $n \pm 1$ [2]
Divisible sandpiles	In-radius $n_m - 1$, out-radius n_m [3] $(n_m = \max\{k \geq 0 : \|B_o(k)\| - \frac{1}{2}\|\partial_I B_o(k)\| \leq m\})$
Abelian sandpiles	Receiving set is an exact ball with radius $r_m = m^{1/d_H}(G(\log m) + o(1))$ [2] (G is nonconstant and $(\log 3)$-periodic)

[1] C.–Huss–Sava-Huss–Teplyaev '17
[2] C.–Kudler-Flam '17+
[3] Huss–Sava-Huss '17

Theorem (Limit shape universality on SG [C.–Huss–Kudler-Flam–Sava-Huss–Teplyaev '17+])

On SG, clusters in all 4 cellular automata models (single source at o) fill balls in the graph metric.

First (?) nontrivial state space (beyond $\mathbb{Z}$) **where the limit shape universality conjecture holds.**
Maybe SG is too special?! Finite ramification (many cut points), self-similarity, ...

Lots of open questions!

- How generic is this shape universality? Seems possible to extend to **nested fractals**.
  Example. Vicsek tree.
- Avalanche statistics and critical exponents in abelian sandpiles.
- Scaling limit of the abelian sandpile height functions
  On $\mathbb{Z}^d$ convergence in weak-* $L^\infty(\mathbb{R}^d)$ by Pegden–Smart '11. Apollonian structure proved by Levine–Pegden–Smart, *Ann. Math.* '17. I think convergence can be established on SG.
- Connections to other combinatorial & stat mech models: spanning trees/forests, complex-valued graph Laplacians
  Vector-bundle Laplacians $\leftrightarrow$ cycle-rooted spanning forests (Kenyon '11). Dhar's burning bijection (recurrent sandpile config $\leftrightarrow$ spanning trees). Also ties in with the geometry of AC circuits and regularized Laplacian determinants.

Thank you!