On semi-linear elliptic inequalities on Riemannian manifolds

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Based on joint works with W.Hansen, I.Verbitsky, Y.Sun

Setup and problem statement

Let M be a connected Riemannian manifold, Δ – the Laplace-Beltrami operator on M.

Consider the equation

$$-\Delta u + \Phi(x)\, u^{\sigma} \underset{(\geq)}{=} f,$$

where $\Phi, f \in C(M)$, $f \geq 0$, $\sigma > 0$. Solution u should be *non-negative* and in $C^2(M)$.

Our goal: pointwise estimates of u.

We always assume that Δ has a positive Green function $G(x, y)$ on M, and use notation:

$$G\varphi(x) = \int_M G(x, y)\, \varphi(y)\, d\mu(y).$$

The estimates will be given in terms of the function

$$h = Gf.$$

Let $f \not\equiv 0$ so that $h > 0$. Assume in addition that $h < \infty$.

Linear case $\sigma = 1$

W. Hansen–Z.Ma 1990, AG–W.Hansen 2008: if $\Phi \geq 0$ and

$$-\Delta u + \Phi(x) u \geq f \quad \text{on } M,$$

then

$$u \geq h \exp\left(-\frac{1}{h} G(h\Phi)\right),$$

where $h = Gf$.

This implies the lower bound for the Green function G_Φ of $-\Delta + \Phi$:

$$G_\Phi(x,y) \geq G(x,y) \exp\left(-\frac{\int_M G(x,z) G(z,y) \Phi(z) d\mu(z)}{G(x,y)}\right).$$

In the case $\Phi \leq 0$ a similar estimate under additional assumptions was obtained by N.Kalton–I.Verbitsky 1999.

Main result

Theorem 1 (AG–I.Verbitsky, 2015) *Let $u \geq 0$ solve $-\Delta u + \Phi u^{\sigma} \geq f$ in M. Set $h = Gf$. Let $0 < h < \infty$ and let $G(h^{\sigma}\Phi)$ be well defined.*

(*i*) *If $\sigma = 1$ then*

$$u \geq h \exp\left(-\frac{1}{h} G(h\Phi)\right). \tag{1}$$

(*ii*) *If $\sigma > 1$ then*

$$u \geq \frac{h}{\left[1 + (\sigma - 1)\frac{1}{h} G(h^{\sigma}\Phi)\right]^{\frac{1}{\sigma-1}}}, \tag{2}$$

where the expression in square brackets is necessarily positive:

$$-(\sigma - 1)G(h^{\sigma}\Phi) < h. \tag{3}$$

(*iii*) *If $0 < \sigma < 1$ then*

$$u \geq h\left[1 - (1 - \sigma)\frac{1}{h} G\left(1_{\{u>0\}} h^{\sigma}\Phi\right)\right]_{+}^{\frac{1}{1-\sigma}}. \tag{4}$$

Estimates with boundary condition

Fix Ω – a relatively compact domain in M with smooth boundary. It suffices to prove (1)-(4) in Ω with G_Ω instead of G and with $h = G_\Omega f$.

New problem: let $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be positive and superharmonic in Ω. Set $f = -\Delta h$ and assume that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, $u \geq 0$, satisfies

$$-\Delta u + \Phi u^\sigma \geq f \quad \text{in } \Omega \quad \text{and} \quad u \geq h \text{ on } \partial\Omega. \tag{5}$$

Theorem 2 *Under the above condition, we have:*

(*i*) *If* $\sigma = 1$ *then* $u \geq h \exp\left(-\frac{1}{h} G_\Omega(h\Phi)\right)$.

(*ii*) *If* $\sigma > 1$ *then*

$$u \geq \frac{h}{\left[1 + (\sigma - 1)\frac{1}{h} G_\Omega(h^\sigma \Phi)\right]^{\frac{1}{\sigma-1}}},$$

where necessarily $-(\sigma - 1) G_\Omega(h^\sigma \Phi) < h$.

(*iii*) *If* $0 < \sigma < 1$ *then*

$$u \geq h\left[1 - (1 - \sigma)\frac{1}{h} G_\Omega\left(1_{\{u>0\}} h^\sigma \Phi\right)\right]_+^{\frac{1}{1-\sigma}}.$$

Approach to the proof of Theorem 2

Assume for simplicity that $u > 0$ and $h > 0$ in $\overline{\Omega}$. Assume first $h \equiv 1$. Then $f = -\Delta h = 0$ and

$$-\Delta u + \Phi u^{\sigma} \geq 0 \text{ in } \Omega, \quad u \geq 1 \text{ on } \partial\Omega.$$

Fix a C^2 function ϕ on (a interval of) $\mathbb{R}$ with $\phi' > 0$ and set

$$v = \phi^{-1}(u).$$

By the chain rule we have

$$\Delta u = \Delta\phi(v) = \phi'(v)\Delta v + \phi''(v)|\nabla v|^2,$$

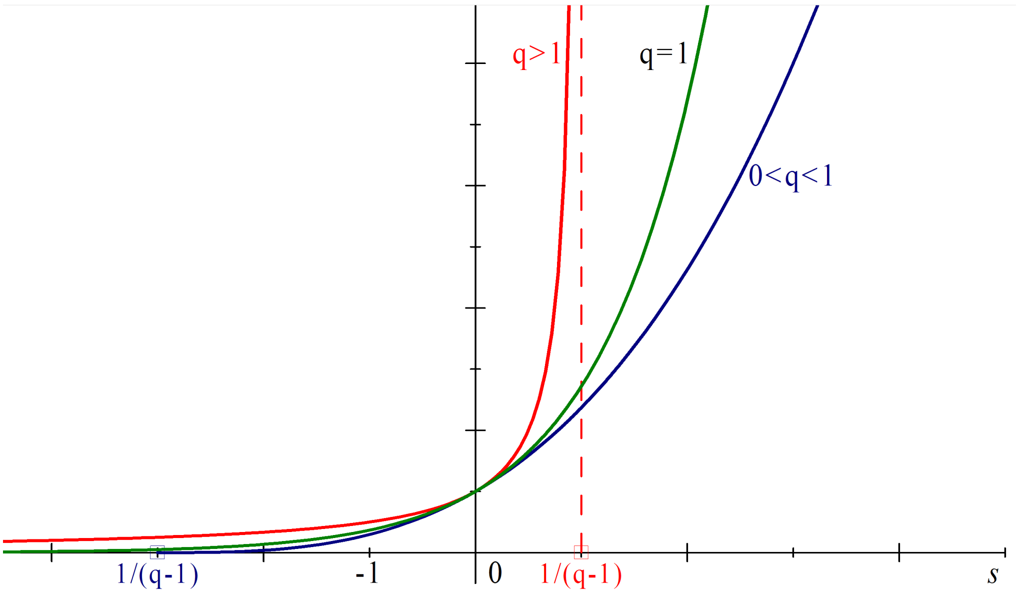
whence

$$-\Delta v = \frac{\phi'' |\nabla v|^2}{\phi'} - \frac{\Delta u}{\phi'} \geq \frac{\phi''}{\phi'} |\nabla v|^2 - \Phi \frac{\phi(v)^{\sigma}}{\phi'(v)}. \tag{6}$$

Choose ϕ to solve the initial value problem

$$\phi'(s) = \phi^{\sigma}(s), \quad \phi(0) = 1.$$

Hence, $\phi(s) = e^s$ if $\sigma = 1$, and $\phi(s) = [(1-\sigma)s + 1]^{\frac{1}{1-\sigma}}$ if $\sigma \neq 1$.



The inverse function ϕ^{-1} is always defined on $(0, +\infty)$.

The function ϕ is convex, and we obtain from (6)

$$-\Delta v \geq -\Phi \quad \text{in } \Omega. \tag{7}$$

Since on $\partial\Omega$ we have $v = \phi^{-1}(u) \geq \phi^{-1}(1) = 0$, it follows that

$$v \geq -G_\Omega \Phi \quad \text{in } \Omega$$

and, hence,

$$u \geq \phi\left(-G_\Omega \Phi\right) \quad \text{in } \Omega.$$

This yields the cases $(i) - (iii)$ of Theorem 2 in the case $h = 1$.

Indeed, in the case $\sigma = 1$ we have $\phi(s) = e^s$ and, hence,

$$u \geq \exp\left(-G_\Omega \Phi\right).$$

In the case $\sigma > 1$ we have $\phi(s) = [(1-\sigma)s + 1]^{-\frac{1}{\sigma-1}}$, which gives the estimate of (ii)

$$u \geq \frac{1}{[1 + (\sigma-1)G_\Omega \Phi]^{\frac{1}{\sigma-1}}}.$$

Similarly one treats the case $0 < \sigma < 1$.

For a general $h>0$, we use the *h-transform* of Δ in Ω:

$$\Delta^h := \frac{1}{h}\circ\Delta\circ h = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right) + \frac{\Delta h}{h} = L + \frac{\Delta h}{h},$$

where

$$L = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right)$$

is the *weighted* Laplacian associated with measure $d\tilde{\mu} = h^2 d\mu$.

For function $\tilde{u} = \frac{u}{h}$ we have

$$-\Delta^h\tilde{u} = -\frac{1}{h}\Delta u \geq \frac{1}{h}\left(-\Phi u^\sigma + f\right) = -h^{\sigma-1}\Phi\tilde{u}^\sigma - \frac{\Delta h}{h}.$$

Setting $\tilde{\Phi} = h^{\sigma-1}\Phi$, we obtain that $\tilde{u}$ satisfies

$$-\Delta^h\tilde{u} + \tilde{\Phi}\tilde{u}^\sigma \geq -\frac{\Delta h}{h} \ \text{ in } \Omega, \qquad \tilde{u} \geq 1 \ \text{ on } \partial\Omega.$$

Now we use the same approach as in the case $h=1$, but for the operator Δ^h in place of Δ.

Set $v = \phi^{-1}(\tilde{u}) = \phi^{-1}(u/h)$ and compute $\Delta^h v$ as in (6). For the part $L = \frac{1}{h^2} \operatorname{div}(h^2 \nabla)$ of the operator Δ^h, computation is the same as for Δ. The part $\frac{\Delta h}{h}$ gives in the end an additional term so that instead of (7) we obtain

$$-\Delta^h v \geq -\tilde{\Phi} + \left(\frac{\phi(v) - 1}{\phi'(v)} - v \right) \frac{\Delta h}{h}.$$

Multiplying by h, we obtain

$$-\Delta(hv) \geq -h^\sigma \Phi + \left(\frac{\phi(v) - 1}{\phi'(v)} - v \right) \Delta h. \tag{8}$$

The convexity of ϕ implies

$$\frac{\phi(s) - 1}{\phi'(s)} - s \leq 0, \tag{9}$$

for any s in the domain of ϕ. Indeed, if $s > 0$ then $\exists \xi \in [0, s]$ such that

$$\frac{\phi(s) - 1}{s} = \frac{\phi(s) - \phi(0)}{s} = \phi'(\xi) \leq \phi'(s),$$

whence (9) follows. If $s<0$ then $\xi \in [s,0]$ such that

$$\frac{\phi(s)-1}{s}=\frac{\phi(s)-\phi(0)}{s}=\phi'(\xi)\geq\phi'(s),$$

which again implies (9) since $s<0$.

Since $\Delta h\leq 0$, we obtain

$$\left(\frac{\phi(v)-1}{\phi'(v)}-v\right)\Delta h\geq 0$$

and therefore by (8)

$$-\Delta(hv)\geq -h^{\sigma}\Phi \;\text{ in } \Omega.$$

On $\partial\Omega$ we have $v=\phi^{-1}(u/h)\geq\phi^{-1}(1)=0$, which implies

$$hv\geq -G_{\Omega}(h^{\sigma}\Phi) \;\text{ in } \Omega.$$

Dividing by h and applying ϕ, we obtain

$$u\geq\phi\left(-\frac{1}{h}G_{\Omega}(h^{\sigma}\Phi)\right) \;\text{ in } \Omega.$$

Existence of positive solutions

Let us ask for which values $\sigma > 1$ the inequality

$$\Delta u + u^{\sigma} \leq 0 \tag{10}$$

has a positive solution u on M (the case of $\Phi \equiv -1$). For example, in $\mathbb{R}^n$ with $n \leq 2$ any non-negative solution of (10) is 0 while for the case $n > 2$ (10) has a positive solution if and only if

$$\sigma > \frac{n}{n-2}$$

(Mitidieri and Pohozaev, 1998).

Let $d\,(x, y)$ be a distance function on M, not necessarily geodesic, but such that the metric balls

$$B(x,r) = \{y \in M, \quad d(x,y) < r\}\,.$$

are precompact open subsets of M. Set

$$V\,(x,r) = \mu\,(B\,(x,r))\,.$$

Theorem 3 (AG – Yuhua Sun, 2017) *Assume that, for some $x_0 \in M$,*

$$V(x_0, r) \simeq r^\alpha \quad \textit{for large } r \tag{V}$$

and

$$G(x, y) \simeq d(x, y)^{-\gamma} \quad \textit{for large } d(x, y), \tag{G}$$

where $\alpha > \gamma > 0$. Then, for any σ satisfying

$$1 < \sigma \leq \frac{\alpha}{\gamma},$$

the inequality

$$\Delta u + u^\sigma \leq 0 \tag{11}$$

has no positive solution in any exterior domain of M.

If in addition d is the geodesic distance, M has bounded geometry, and (V) holds for all $x_0 \in M$, then, for any

$$\sigma > \frac{\alpha}{\gamma},$$

the inequality (11) has a positive solution on M.

Hence, the critical value of the exponent σ is equal to $\frac{\alpha}{\gamma}$.

Example 1

Let Γ be an infinite connected graph with a uniformly bounded degree. Let $d(x, y)$ be the graph distance on Γ and $V(x, r)$ – the volume function. Let the discrete Laplace operator on Γ have a positive Green function $G(x, y)$.

If Γ satisfies conditions (V) and (G) for some α and γ then we construct a manifold with the same properties by inflating Γ, that is, by replacing the edges of Γ by 2-dim cylinders. Since M has bounded geometry, the both parts of Theorem 3 apply in this case.

M.Barlow constructed in 2004 a fractal graph satisfying (V) and (G), for any pair (α, γ) such that

$$0 < \gamma \leq \alpha - 2.$$

Since γ can be arbitrarily small, the critical value $\frac{\alpha}{\gamma}$ can be arbitrarily large, unlike the Euclidean critical value $\frac{n}{n-2}$.

Example 2

Assume that $G(x,y)$ satisfies the following $3G$-inequality

$$\frac{1}{G(x,y)} \leq C\left(\frac{1}{G(x,z)} + \frac{1}{G(z,y)}\right)$$

for all $x,y,z \in M$ and some $C > 1$. Then the function $\rho(x,y) = \frac{1}{G(x,,y)}$ is a pseudo-distance on M. It follows that there exists a distance function $d(x,y)$ and $\gamma > 0$ such that

$$\rho(x,y) \simeq d(x,y)^{\gamma}.$$

Hence, we obtain

$$G(x,y) \simeq d(x,y)^{-\gamma},$$

that is, M satisfies (G).

Assume that (M,ρ) satisfies (V) that is

$$\mu\{y : \rho(x,y) < r\} \simeq r^{\alpha}. \tag{12}$$

Then, for metric balls $B(o,r)$ with respect to d, we obtain

$$\mu(B(x,r)) \simeq r^{\alpha\gamma}.$$

Hence, (M, d) satisfies (V) with $\tilde{\alpha} := \alpha\gamma$. Assuming in addition that all balls are precompact, we obtain by Theorem 3 that the critical value of σ is equal to $\frac{\tilde{\alpha}}{\gamma} = \alpha$.

Idea of the proof of Theorem 3

Assume that u is a positive solution in $M \setminus K$ of

$$\Delta u + u^\sigma \leq 0.$$

For any precompact open set $U \supset K$, we have

$$u \geq G_{\overline{U}^c}\left(u^\sigma\right) \quad \text{in } U^c. \tag{13}$$

If $u > 0$ then the superharmonicity of u implies the estimate

$$u \geq cG\left(\cdot, x_0\right) \quad \text{in } U^c, \tag{14}$$

for some $c > 0$. On the other hand, one can prove that, for any precompact open set $\Omega \subset M$,

$$\sup_\Omega (\Delta u + \lambda_1(\Omega) u) \geq 0,$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω. It follows that

$$\lambda_1(\Omega) \geq \inf_\Omega u^{\sigma-1}.$$

Combining this with (13) and (14), we obtain

$$\lambda_1(\Omega)^{\frac{1}{\sigma-1}} \geq c \inf_{x\in\Omega} \int_{U^c} G_{\overline{U}^c}(x,y)\, G^{\sigma}(y,x_0)\, d\mu(y)\,.$$

If $\sigma \leq \frac{\alpha}{\gamma}$ then we bring this inequality to contradiction by choosing Ω large enough and by applying the hypotheses (V), (G) to estimate all the quantities involved.

For the proof of the second part of Theorem 3, we construct a positive solution of the equation

$$\Delta u + u^{\sigma} + \lambda^{\sigma} f^{\sigma} = 0 \ \text{ in } M,$$

where f is a specifically chosen decreasing function and $\lambda > 0$ is small enough. This differential equation amounts to the integral equation

$$u(x) = \int_M G(x,y)\left(u^{\sigma}(y) + \lambda^{\sigma} f(y)^{\sigma}\right) d\mu(y),$$

and the latter is solved in a certain closed subset of $L^{\infty}(M)$ by observing that the operator in the right hand side is a contraction for small enough λ. Next, we improve the regularity properties of u in two steps: first show that u is Hölder and then that $u \in C^2$.