A Dynamical System on Metric Graphs and Hybrid Spaces and Number-Theoretic Problems

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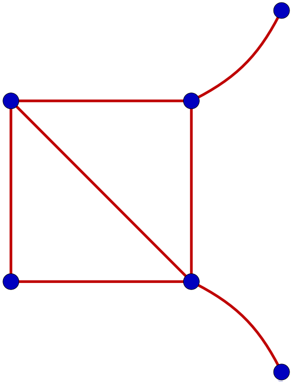
Outline

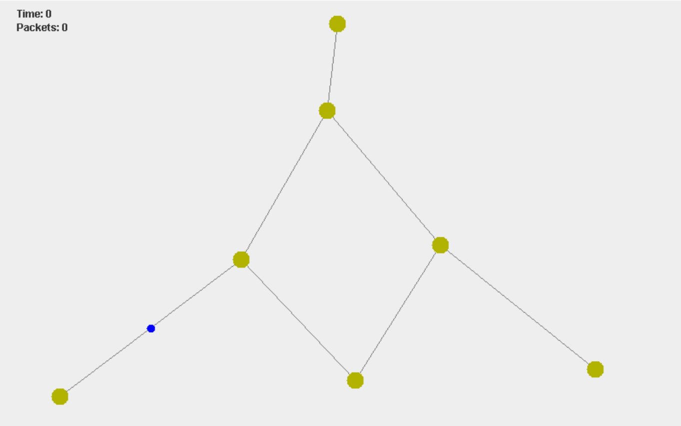
Outline

Metric graph

A metric graph here is a 1-dimensional CW complex with metrics on edges i.e. edges of this graph are regular smooth curves with finite length.



Dynamical system



Dynamical system

Let one point move along the graph at the initial time.

When k points come to the interior vertex of valence n at the same time, then n points are released, i.e. one point will correspond to one edge.

Reflection occurs in vertices of valence one.

Time for passing each individual edge (i.e. propagation time t_j) is fixed. It is assumed that there are no turning points on the edges.

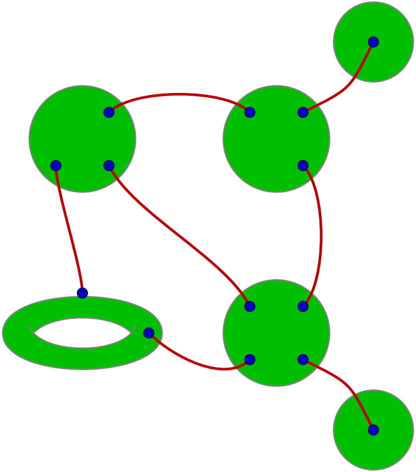
The problem is to analyze the asymptotic behavior of the number of such points on the graph as time increases.

Decorated graph

A decorated graph Γ_d is a singular space, obtained from a graph via replacing vertices with smooth manifolds M_k, $\dim M_k \leq 3$.

Edges γ_j of the graph are smooth parametrized curves, M_k are complete Riemannian manifolds.

Decorated graph



Dynamical system

Let a point move along a 1-dimensional edge of the decorated graph at the initial time. When the point comes to the end of the segment, the packet produces an expanding wavefront on the surface and a reflected point moving backwards on the edge.

The wavefront will travel in the manifold with unit speed along all geodesics emanating from the point of gluing i.e. it will form a circular wave front. When the front reaches another point of merging a new point starts to move along the corresponding edge, and produces a new wave front on the manifold (that emanates from the current point) and so on.

The problem is to analyze the asymptotic behavior of the number of such points on one-dimensional edges of the decorated graph as time increases. We will denote this number by $N(T)$.

Outline

The motivation

We study a time-dependent Schrödinger equation, in which the spatial variable varies within a decorated graph Γ_d.

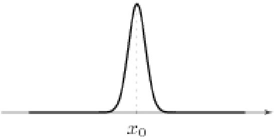
$$ih\frac{\partial\psi(x,t)}{\partial t} = \widehat{H}\psi(x,t) \tag{1}$$

We are interested in properties of asymptotic, as $h \to 0$ (semiclassical), solutions.

The main effect of the related "branching"of the space is in the multiple reflection at the vertices of the graph, which leads to the occurrence of nontrivial statistical phenomena.

Gaussian packet

These properties become especially clear when describing Gaussian packets, which were originally localized near a single point.



Differential operators on graphs

Each part of a decorated graph is Riemannian, so a Laplace–Beltrami operator on it can be defined, and a theory of selfadjoint extensions which classify how waves can be scattered or transmitted at the points where intervals are glued to the manifolds can be developed. This has been done by Brüning and Geyler.

Brüning J. and Geyler V. A. "Scattering on compact manifolds with infinitely thin horns." J. Math. Phys. 44, 2003, 371.

See also Kuchment P., Berkolaiko G. "Introduction to Quantum Graphs", Mathematical Surveys and Monographs, V. 186 AMS, 2014 and references therein.

Propagation of Gaussian packets

Proposition

Let Γ be a star graph, with the single vertex a of valence n. Let the initial data have the Gaussian form, where the point x_0 is on one of the egdes. Then the solution of

$$\widehat{H}\psi(x,t) = ih\frac{\partial\psi(x,t)}{\partial t} + O(h^{3/2})$$

is, in any finite time, a finite sum of Gaussian packets, i.e. functions of the form $\exp\left(\frac{iS^j(x,t)}{h}\right)\varphi^j(t)$, где

$S^j = S^j_0(t) + (x - x_j(t))S^j_1 + S^j_2(x - x_j(t))^2, \quad Im\, S^j_2 > 0.$

The packet that came to the vertex a is divided into n packets traveling on the edges. At each of the edges the solution is determined by two Hamiltonian systems.

The initial values for the amplitudes are determined by the explicit formulas for the "transmitted" and "reflected" Gaussian packets.

Outline

Statistics of the propagation of Gaussian packets

$$\Psi(x,t) = \sum_{j=1}^{N(t)} \exp\left(\frac{iS^j(x,t)}{h}\right)\varphi^j(t)$$

is a semiclassical solution of the Cauchy problem for the non-stationary Schrödinger equation with initial conditions of a special form.
We consider the asymptotical behavior of function $\Psi(x, t)$ as $t \to \infty$. Namely, we will see how the number of Gaussian packets $N(t)$ changes in time.

Remark

Let us consider the quantity t_j equal to the time, in which the Gaussian packet passes the j-th edge. The time of this propagation is determined by the solutions of the Hamiltonian system (with given initial data). So it is fixed if the initial conditions are fixed.

Asymptotics for $N(t)$

The number of paths on a graph is growing exponentially with time.

Question is: what will happen to the number of packets?

Asymptotics for $N(t)$

Theorem

Let graph Γ be compact. Suppose, moreover, that the numbers $t_1, \dots t_E$ are linearly independent over the $\mathbb{Q}$. Then the function $N(t)$ with increasing t can be represented as

$$N(t) = Ct^{E-1}(1 + o(1)), \tag{2}$$

where C is a positive constant.

Outline

The number of moving points and the number of lattice points

Correspondence has the following form: $\Gamma \to P \to N(t)$, where P is an area on a certain polyhedron for which we should find the number of lattice points. Here $N(t)$ stands for the number of moving points.

Lattice points and polynomials

Let $w_1, \ldots, w_k$ be fixed positive real numbers. We denote by $N_k(\lambda \,|\, w_1, \ldots, w_k)$ the number of natural solutions $(n_1, \ldots, n_k)$ of the inequality $\sum_{i=1}^{k} n_i w_i \leq \lambda$. The number of solutions of the equation $\sum_{i=1}^{k} n_i w_i = \lambda$ is taken with the weight $\frac{1}{2}$, that is, in points of discontinuity, the function $N_k(\lambda)$ is equal to half the sum of the limits on the left and on the right.

Lattice points and polynomials

As D. C. Spencer proved in his Cambridge thesis that for almost all $w_1, \ldots, w_k$ the function $N_k(\lambda|\, w_1, \ldots, w_k)$ is approximated by a Barnes' multiple Bernoulli polynomial $R_k(\lambda|\, w_1, \ldots, w_k)$, namely:

$$N_k(\lambda|\, w_1, \ldots, w_k) - R_k(\lambda|\, w_1, \ldots, w_k) = O((\log \lambda)^{k+\varepsilon}), \qquad (3)$$

as $\lambda \to \infty, \forall \varepsilon > 0$.

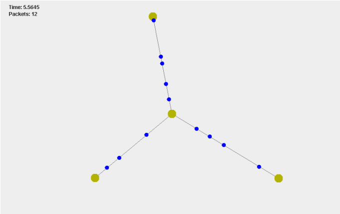
Lattice points and polynomials

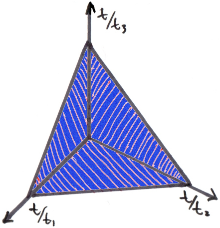
The role of these polynomials in the approximation of the number of integer points, up to some shifts of the argument, was independently discovered for the case of a polyhedron with rational vertices (see Chapter 14 of the book "Integer points in polyhedra", 2008 by A. Barvinok and article "The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes" by Pukhlikov A. V., Khovanskii A. G., 1992).
The polynomials are named "Todd polynomials" in these works, since these polynomials were introduced to describe a special case of characteristic classes.

Lattice points and polynomials

In a recent paper "Lattice points in algebraic cross-polytopes and simplices" by Bence Borda (arXiv:1608.02417 [math.NT], 2016), the polynomials were rediscovered, but in this case, estimates of the number of integer points in the expanding simplex were carried out not for almost all real w_i, but for all algebraic w_i using the results of W. M. Schmidt.

An example of a polynomial corresponding to a metric tree





Statement

Consider a star graph with three edges e_1, e_2, e_3 with travel times t_1, t_2, t_3 such that rank $\{t_1, t_2, t_3\}$ over $\mathbb{Q}$ equals 3. Then

$$N(t) = \frac{1}{8}\frac{t_1+t_2+t_3}{t_1 t_2 t_3} t^2 + \frac{1}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right) t + O(\log^{2+\varepsilon} t).$$

A polynomial corresponding to a metric tree

Theorem

Let $\Gamma = (V, E)$ be a metric tree with edge lengths $t_1, \ldots, t_{|E|}$. Then for almost all $t_1, \ldots, t_{|E|}$: $N(T) = R(T) + O((\log T)^{|E|-1+\varepsilon})$, where $R(T) = R'(T) + R''(T)$ and polynomials R', R'' are defined as follows:

$$R'(T) = \sum_{E'} \sum_{I} \Big(\rho(E, \mathrm{end}(I)) - \rho(E', \mathrm{end}(I))\Big) R_{|E'|}\left(T + \sum_{e_i \in I} t_i \;\Big|\; \{2t_i\}_{e_i \in E'}\right),$$

where the first summation is taken over all subsets of edges $E' \subset E$ that form a subtree Γ' in Γ (with the same root as Γ), and the second summation is taken over all subsets of edges $I \subset E'$ that form a way (possibly an empty way) in the subtree Γ'.

$$R''(T) = \sum_{E'} \sum_{I} R_{|E'|-1}\left(T + \sum_{e_i \in I \setminus \mathrm{last}(I)} t_i - t_{\mathrm{last}(I)} \;\Big|\; \{2t_i\}_{e_i \in E' \setminus \mathrm{last}(I)}\right), \qquad (4)$$

where the first summation is taken over all subsets of edges $E' \subset E$ that form a subtree Γ' in Γ (with the same root as Γ), and the second summation is taken over all subsets of edges $I \subset E'$ that form a nonempty way in the subtree Γ' such that $\rho(E', \mathrm{end}(I)) > 1$.

A polynomial corresponding to a metric tree

Here H stands for the set of hanging edges of Γ, $up(e)$ is the set of edges that forms a way from the root to the end of an edge e (nearest to the root) ($e \notin up(e)$).

General formula for the leading coefficient

Theorem

Let Γ be a compact finite graph and consist of one connected component. Then for all linearly independent over $\mathbb{Q}$ numbers $t_1, \ldots t_E$, the leading coefficient, with increasing t, for the number of Gaussian packets can be determined by:

$$C = \frac{1}{2^{V-2}(E-1)!} \frac{\sum_{j=1}^{E} t_j}{\prod_{j=1}^{E} t_j}. \tag{5}$$

Here V is the number of vertices, and E is the number of edges.

Distribution of the packets

Let δ be a segment on an arbitrary edge of Γ.
N_δ is the number of packets, located on δ.

Theorem

For almost all incommensurable t_j

$$\lim_{t\to\infty} \frac{N_\delta(t)}{N(t)} = \frac{t_\delta}{\sum_j t_j}.$$

Second term

Theorem

Let $\Gamma = (V, E)$ be a tree with linearly independent over $\mathbb{Q}$ edge lengths $t_1, \ldots, t_{|E|}$. Then

$$N_2 = \frac{1}{(|E|-2)!\, 2^{|E|-2} \prod\limits_{e_i \in E} t_i} P_2(t_1, \ldots, t_{|E|}),$$

where

$$P_2(t_1, \ldots, t_{|E|}) = -\frac{1}{2} \sum_{e \in E} \sum_{e_i \in E \setminus (up(e) \cup e)} t_e t_i - \frac{1}{2} \sum_{e \in E \setminus H} t_e^2 +$$

$$+ \sum_{e \in H} \sum_{f \in E \setminus e} t_e t_f.$$

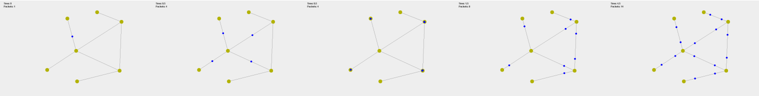
Second term

Corollary

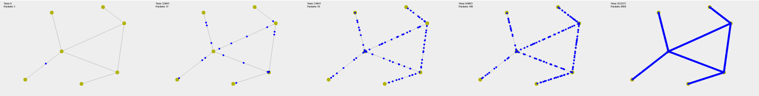
The tree Γ and its root (starting point) are uniquely determined by P_2 i.e. a quadratic form of edge lengths.

Growth rate depends on metric properties

- Case of linearly dependent edges



- Case of linearly independent edges



Outline

Rank equals one

Statement

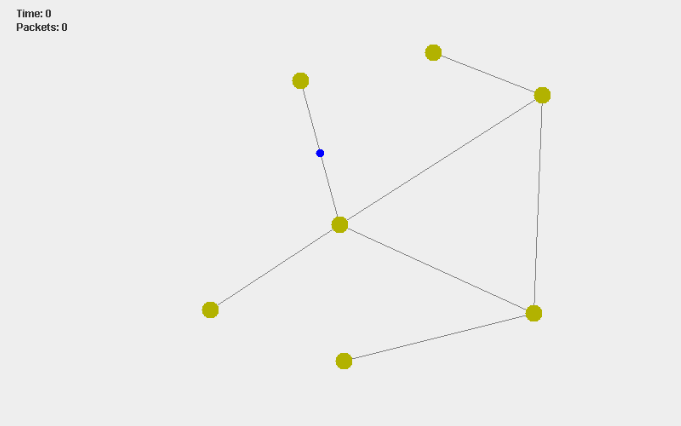
Suppose that $t_1, = n_1 t_0, \dots, t_E = n_E t_0$, where $n_i \in \mathbb{N}$ and $GCD(n_1, \dots, n_E) = 1$. Then, at some time $N(T)$ will cease to grow and will be equal to

$$N(T) = 2 \sum_{i=1}^{E} n_i,$$

if there is a cycle with the length, which is not divisible by $2t_0$, or

$$N(T) = \sum_{i=1}^{E} n_i.$$

Rank equals one



Stabilization time

The natural question is at what instant of time the number of moving points (quasi-particles) stops to grow?

Let us call the time after which the number of points ceases to grow the "stabilization time" and denote it by t_{st}.

Frobenius number

Let us recall that the Frobenius number $Fr(n_1, \ldots, n_k)$ for a given set of positive integers $n_1, \ldots, n_k$ is the largest natural number that cannot be represented as a linear combination of $n_1, \ldots, n_k$ with nonnegative integer coefficients.

Stabilization time for star graph

Statement

The stabilization time for a star graph with edges with relatively prime lengths can be calculated as follows:

$$t_{st} = 2(max(t_1, t_2, ..., t_n) + Fr(t_1, t_2, ..., t_n)).$$

Corollary:

$$t_{st} \leq 2(max(t_1, t_2, ..., t_n) + max^2(t_1, t_2, ..., t_n)).$$

Graph with lengths 1

Statement

If in a graph with edges of the same length there is no cycle of odd length, then the stabilization time is calculated as the maximum distance from a starting point to another vertex minus 1, that is

$$t_{st} = \epsilon(v) - 1,$$

Where $\epsilon(v)$ is the eccentricity of the initial vertex.

Graph with lengths 1

Statement

If in a graph with edges of the same length there is a cycle of odd length, then the stabilization time can be found like this:

$$t_{st} = max(s_{1c} + c_{max} + s_{ci}) - 1,$$

where s_{1c} is the minimum distance from the initial vertex to a cycle of odd length, C_{max} is the maximum length of a path along the cycle of odd length, s_{ci} is the minimum distance from an odd length cycle to the farthest vertex.

Matrix approach

- Dynamics of moving points on a metric graph with natural lengths could be reduced to their propagation on a discrete graph having more vertices and edges.
- Each edge is replaced by two oriented edges with the opposite direction and length equal to the length of the corresponding edge of the metric graph.
- Each of the oriented edges is divided by new vertices into segments of length 1 (there will be no reflection of impulses in new vertices).

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Matrix approach

- The current state of the dynamical system on the graph could be fixed using the state vector, in which $2t_i$ coordinates are responsible for the state of the system on a segment of length t_i.
- Writing down what happens to a point on a given edge after applying our operator, we get a square matrix A of dimension $2\sum_{i=1}^{E} t_i$.
-

$$A \cdot v_0 = v_1$$

$$A^{t_{st}} \cdot v_0 = v_{t_{st}}$$

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- $$A \cdot v_0 = v_1$$
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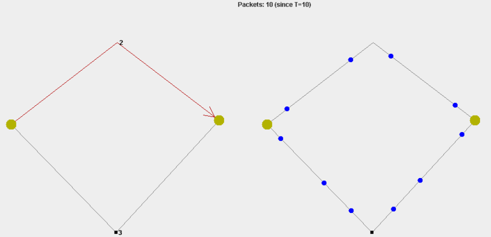
Matrix approach



Matrix approach

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Skolem–Mahler–Lech theorem

"The Skolem–Mahler–Lech theorem states that if a sequence of numbers is generated by a linear recurrence relation, then with finitely many exceptions the positions at which the sequence is zero form a regularly repeating pattern. Namely, this set of positions can be decomposed into the union of a finite set and finitely many full arithmetic progressions."

Conjecture about general asymptotics for $N(t)$

Let graph Γ be compact and finite. Then

$$N(t) = Ct^{rank_{\mathbb{Q}}\{t_j\}-1}(1 + o(1)), \tag{6}$$

where C is a positive constant.

Outline

Case of decorated graph: what is happening locally

Let Γ_d be a half-line, connected with the manifold M in a single point q.
An expanding wave front is formed when a packet reaches the end of the segment.

The packet entering the point of gluing produces on the surface the function which is localized near the circle formed by the endpoints of geodesics emitted from the point of gluing with any initial directions.

When this front reaches another point of gluing a new Gaussian packet emerges on the edge.

Details can be found in the article:

Chernyshev V. L., Shafarevich A. I. 2014 “Statistics of gaussian packets on metric and decorated graphs”, Philosophical transactions of the Royal Society A., Volume: 372, Issue: 2007, Article number: 20130145.

Outline

What is happening globally

- Equivalent graph is finite.
  The results valid for compact finite metric graphs are applicable.
- Equivalent graph is infinite.
  Subexponential or exponential growth for $N(t)$.

Outline

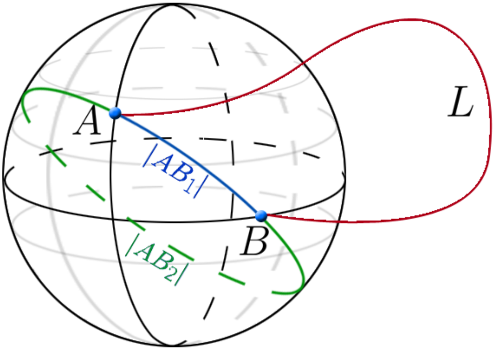
Let Γ_d be a compact decorated graph. For arbitrary finite t the solution of Cauchy problem has the form $\psi = \sum_j \psi_j + O(\sqrt{h})$, where ψ_j are the Gaussian packets.

Let $N(t)$ be the number of packets, localized on the edges of Γ_d (not on the manifolds).

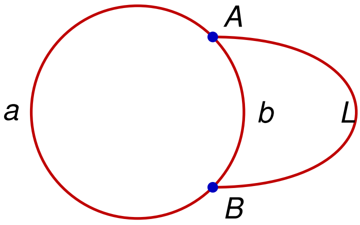
Let t_j be times of propagation of the trajectories along the edges of the graph and between gluing points on the manifolds.

Assumption: there is a finite number of times $t_1, \ldots, t_M$.

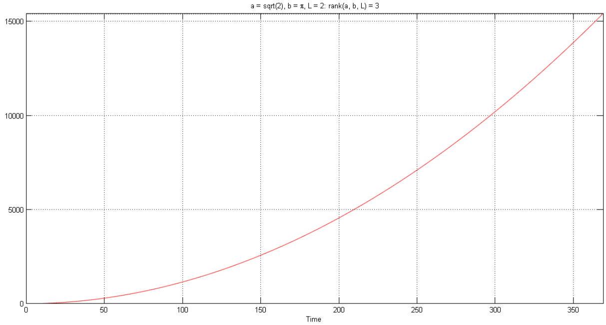
Decorated graph, obtained by gluing a segment to a sphere



Equivalent graph is finite



Rank equals three

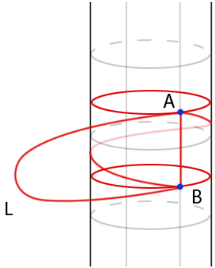


This result is in accordance with an asymptotic estimate:

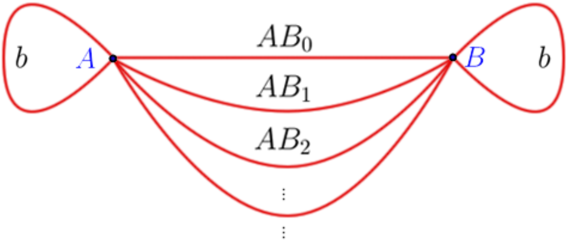
$$N(t) = Ct^{M-1} + o(t^{M-1}), M = 3.$$

Outline

Decorated graph, obtained by gluing a segment to a cylinder



Equivalent graph is infinite



Number of Gaussian packets on an edge

We are interested in the $N(T)$, i.e. the number of Gaussian packets on the edge at time T.

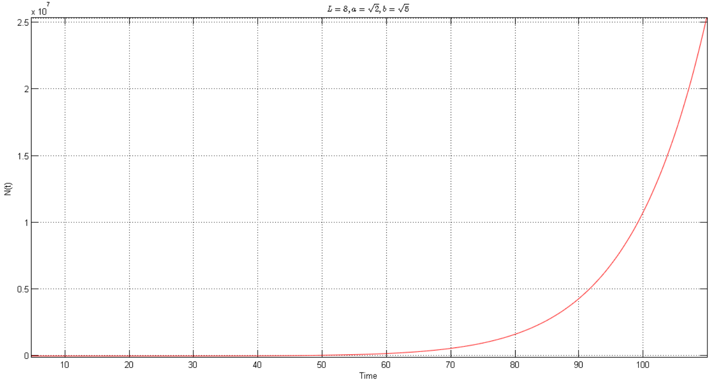
$N(T)$ equals the number of times that are less than T and have the form

$$t'n + \sum_{i=0}^{k} t_k n_k, n_k \geq 0, n \geq 0$$

where $t_k = \sqrt{(kb)^2 + a^2}$, $t' = b$, b is a distance between two points on the cylinder.

Here t_k are lengthes of geodesics connecting points A and B.

Subexponential growth of $N(t)$



Asymptotic for partition number

Let us denote by $p(n)$ the number of ways of writing n as a sum of positive integers.

S. Ramanujan, G.H. Hardy "Une formule asymptotique pour le nombre des partitions de n", Comptes Rendus, 164, 1917. pp. 35-38.

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), n \to \infty.$$

We will use results from the abstract multiplicative number theory in the style of J. Knopfmacher.

Knopfmacher, J. “Abstract Analytic Number Theory (2nd edition)”. New York, Dover Publishing. 1990.

An arithmetic semigroup G is a commutative monoid satisfying the following conditions: There exists a countable subset (finite or countably infinite) P of G, such that every element $a \neq 1$ in G has a unique factorisation of the form

$$x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where p_i are distinct elements of P, r may depend on x, and two factorisations are considered the same if they differ only by the order of the factors indicated. The elements of P are called the abstract primes of G. There exists a real-valued norm, i.e. mapping ∂ on G, such that

1. $\partial(1) = 1$,
2. $\partial(p) > 1$ for all $p \in P$,
3. $\partial(ab) = \partial(a)\partial(b)$ for all $a, b \in G$.

The total number $N_G(x)$ of elements $a \in G$ of norm $\partial(a) \leq x$ is finite, for each real $x > 0$.

Abstract prime number theorems

Considering asymptotic properties of arithmetical semigroup G one might begin with either $N_G(x)$ or $\pi_G(x)$ and investigate how assumptions about the asymptotic behaviour of one of these functions as $x \to \infty$ influences that of the other.

Any theorem of this kind can be called an “abstract prime number theorem”.

An example of the “abstract prime number theorem”

Statement

For any arithmetic semigroup G,
from $\pi_G(x) \sim \frac{ax^\delta}{\delta \ln x}$ as $x \to \infty$, where $\pi_G(x) =$ total number of elements $p \in P$ of norm $|p| \leq x$ follows that
there exist positive constants A, such that
$N_G(x) \sim Ax^\delta \ln^{a-1} x$ as $x \to \infty$,

Link with the Bose gas

Theorem (Nazaikinskii 2013)

Let $N(T)$ be the number of non-negative integer solutions of inequality $\sum_{i=1}^{\infty} \lambda_i N_i \le T$ and for sequence λ_j a counting function $\rho(\lambda) = \#\{j|\lambda_j \le \lambda\}$ has asymptotics

$$\rho(\lambda) = c_0 \lambda^{1+\gamma}(1 + O(\lambda^{-\varepsilon})), \varepsilon > 0. Then$$

$$\ln N(T) = (\gamma + 2) \left(\frac{c_0 \Gamma(\gamma + 2) \zeta(\gamma + 2)}{(\gamma + 1)^{\gamma+1}} \right)^{\frac{1}{\gamma+2}} T^{\frac{\gamma+1}{\gamma+2}} (1 + o(1))$$

as T goes to infinity.

Asymptotic estimate

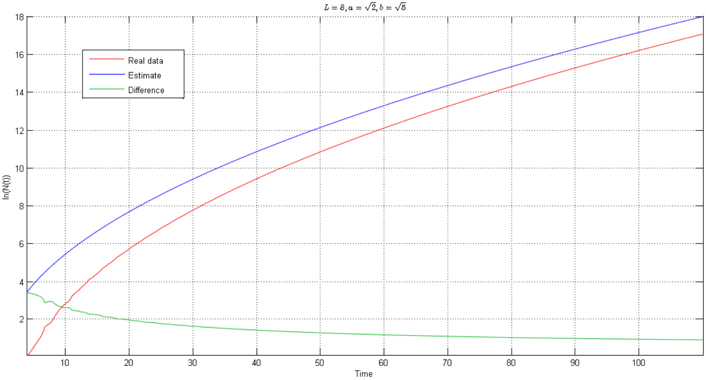
Statement

For a decorated graph obtained by attaching a segment to a flat cylinder, the following asymptotic estimate holds, as T goes to infinity:

$$\ln N(T) \leq \sqrt{\frac{2}{3b}}\,\pi\, T^{\frac{1}{2}}(1+o(1))$$

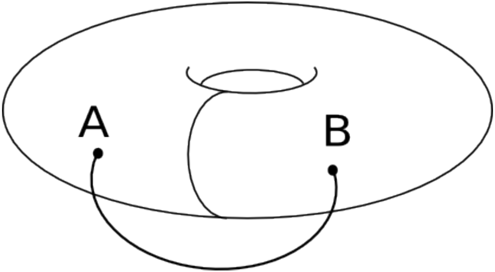
If times $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$ are linearly independent over $\mathbb{Q}$ then inequality turns into equality. For almost all real a and b times $\{t'\} \cup \{t_i\}_{i=0}^{\infty}$ are linearly independent over $\mathbb{Q}$.

Logarithm of $N(t)$ and its estimate

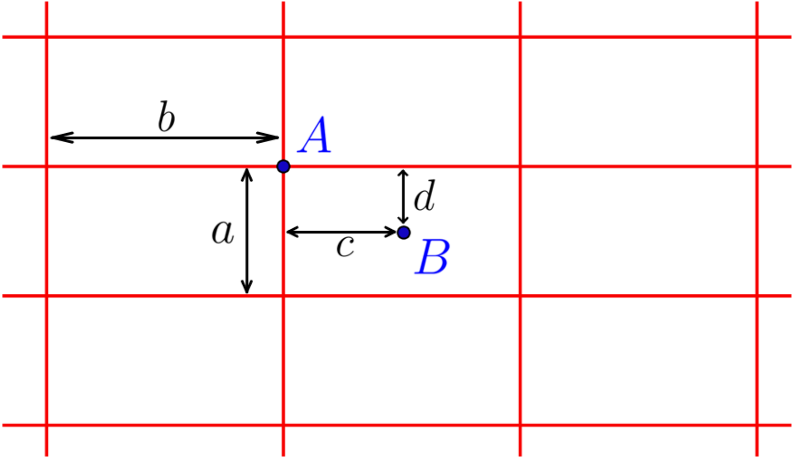


Decorated graph, obtained by gluing a segment to a torus

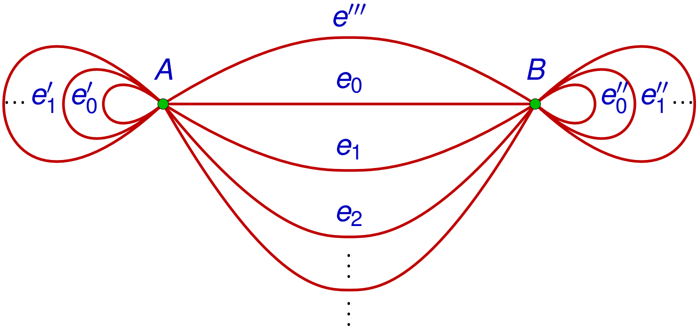
Let us take a flat torus with fundamental cycles of lengths a and b. Let us consider a fundamental rectangle with sides a, b and take points $A = (0, 0)$, $B = (c, d)$ on it. We glue a segment e''' with travel time t''' to points A, B.



Fundamental polygon of the torus



Equivalent graph is infinite



Asymptotic estimate

Statement

For a decorated graph obtained by attaching a segment to a flat 2-dimensional torus, the following asymptotic estimate holds, as T goes to infinity:

$$\ln N(T) \leq 3\left(\frac{5\pi}{8ab}\zeta(3)\right)^{\frac{1}{3}} T^{\frac{2}{3}}(1+o(1))$$

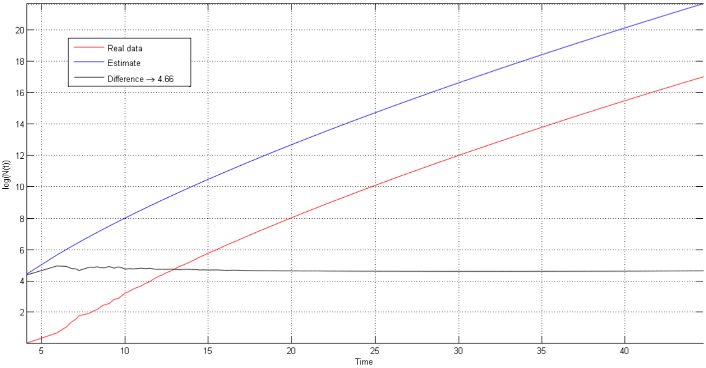
If $\{t_i'\}_{i=0}^{\infty} \cup \{t_i\}_{i=0}^{\infty}$ are linearly independent over $\mathbb{Q}$, then inequality turns into equality.

Asymptotic estimate

The right side of the asymptotic estimate depends only on the parameters of the fundamental rectangle i.e. on a and b.

The parameters which determine the place of gluing, that is c and d, determine whether the asymptotic estimate is accurate or it is only an upper estimate.

Logarithm of $N(t)$ and its estimate



General result for manifolds with polynomial growth of number of geodesics.

Theorem

Let an interval with the travel time L be glued at two points A and B on the surface M. Suppose that for geodesics connecting A with A, B with B, A with B the following condition holds: the number $g(\lambda)$ of geodesics whose length is equal or less than λ equals

$$g(\lambda) = c_1 \lambda^{1+\gamma}(1 + O(\lambda^{-\varepsilon})), \varepsilon > 0.$$

Then

$$lnN(T) \leq (\gamma + 2) \left(\frac{3c_1 \Gamma(\gamma + 2) \zeta(\gamma + 2)}{(\gamma + 1)^{\gamma+1}} \right)^{\frac{1}{\gamma+2}} T^{\frac{\gamma+1}{\gamma+2}} (1 + o(1)).$$

Outline

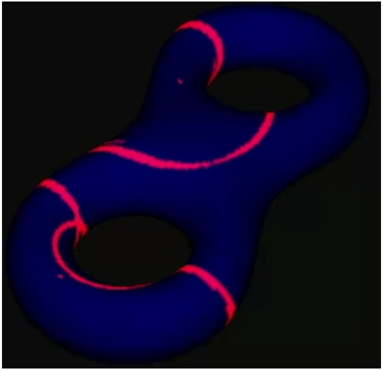
Positive entropy

For a compact surface of negative curvature the number $CF_T(x, y)$ of geodesics with length less than T from x to y satisfies

$$CF_T(x, y)/e^{hT} \to 1,$$

as $T \to \infty$.
A paper by R. Mañé on this subject in the Journal of Differential Geometry, Volume 45, Number 1 (1997), 74-93.

An example



Statement

Let Γ_d be a decorated graph, obtained by gluing a segment to a compact surface M at two points A and B. If M is a compact surface of negative curvature and has positive topological entropy h, then the following estimate for the number of Gaussian packets on the segment holds:

$$\ln N(T) \leq hT(1 + o(1)),$$

as T goes to infinity.

Theorem (Chernyshev, Minenkov, Nazaikinskii 2016)

For any arithmetic semigroup G,
from $\pi^{\#}_G(x) = b_0 x^\gamma e^x (1 + O(x^{-\delta})), b_0 > 0$, $\gamma > -1$ and $\delta \in (0, 1]$ follows that

$$\mathcal{N}^{\#}_G(x) = \left.\frac{e^{xs}\zeta_G(s)}{\sqrt{2\pi(\ln \zeta_G(s))''}}\right|_{s=\beta(x)} (1 + O(x^{-\kappa})), \qquad (7)$$

where $s = \beta(x) > 1$ is a unique solution of the equation

$$x + (\ln \zeta_G(s))' = 0, \qquad (8)$$

and $\kappa > 0$ ia a constant that satisfies

$$\kappa < \frac{\delta}{2 + \gamma}, \qquad \kappa \leq \frac{1 + \gamma}{2 + \gamma}. \qquad (9)$$

$$\zeta_G(s) = \sum_{a \in G} e^{-\partial(a)s}$$

Corollary

If all lengths are independent over $\mathbb{Q}$ except for a finite set and $\pi_G^{\#}(x) = e^x(b_0 + b_1 x^{-1} + O(x^{-2}))$, then

$$\ln \mathcal{N}_G^{\#}(x) = x + 2\sqrt{b_0 x} + \left(\frac{b_1}{2} - \frac{3}{4}\right)\ln x + O(1). \qquad (10)$$

In the opposite case, this equality turns into an upper bound.

Chernyshev V. L., Minenkov D. S., Nazaikinskii V. E. // Functional analysis and its applications. – 2016. Volume: 50 Issue: 4 Pages: 291-307.

Thank you for your attention!