The discrete Laplacian of a 2-simplicial complex

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Basic Concepts

Notion of Graphs

- Let $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ be a graph where :
  - $\mathcal{V}$ is the countable set of vertices.
  - $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges.
- $\mathcal{E}$ is without loops and symmetric :

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- The weights :
  - $c : \mathcal{V} \to (0, \infty)$.
  - $r : \mathcal{E} \to (0, \infty)$ such that $\forall e \in \mathcal{E},\ r(-e) = r(e)$.
- The degree of $x \in \mathcal{V}$ is :

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$$x_0 = x,\ x_n = y \text{ and, if } n \geq 1,\ \forall j,\ 0 \leq j \leq n-1 \Longrightarrow x_j \sim x_{j+1}.$$

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Triangulation

- A triangle is closed path of length 3. Let $\mathrm{Tr} \subseteq \mathcal{V}^3$ the set of all triangles of $\mathcal{K}$.
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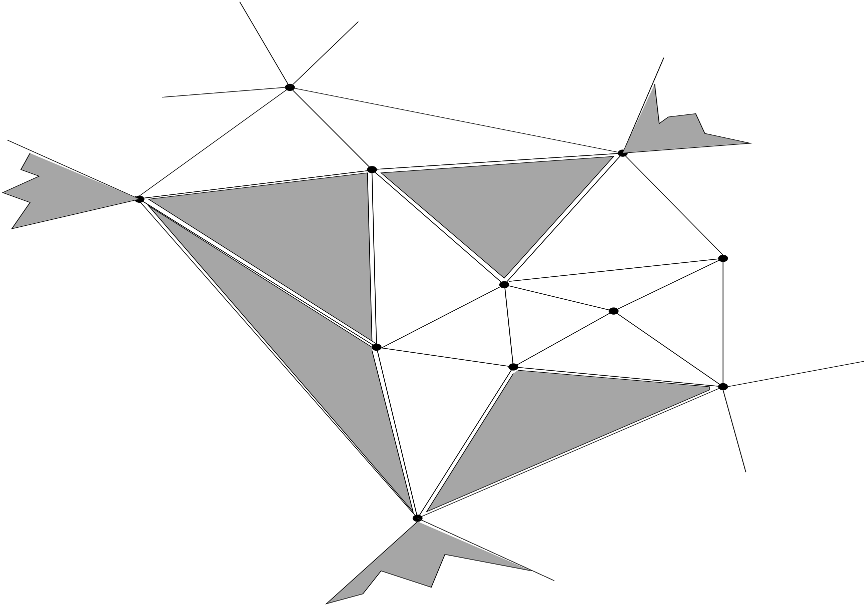

Figure : Triangulation

Functions spaces

- Let us give
  - $\mathcal{C}(\mathcal{V}) = \mathbb{C}^{\mathcal{V}}$.
  - $\mathcal{C}(\mathcal{E}) = \{\varphi : \mathcal{E} \to \mathbb{C},\ \varphi(-e) = -\varphi(e)\}$.
  - $\mathcal{C}(\mathcal{F}) = \{\phi : \mathcal{F} \to \mathbb{C},\ \phi(-\sigma) = -\phi(\sigma)\}$.

  The subsets of complex finite support functions are denoted respectively $\mathcal{C}_c(\mathcal{V})$, $\mathcal{C}_c(\mathcal{E})$ and $\mathcal{C}_c(\mathcal{F})$.

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  The subsets of complex finite support functions are denoted respectively $\mathcal{C}_c(\mathcal{V})$, $\mathcal{C}_c(\mathcal{E})$ and $\mathcal{C}_c(\mathcal{F})$.

- The Hilbert spaces are :
  - $$l^2(\mathcal{V}) := \{f \in \mathcal{C}(\mathcal{V}),\ \sum_{x\in\mathcal{V}} c(x)|f(x)|^2 < \infty\}$$

    with the inner product

    $$\langle f, g\rangle_{l^2(\mathcal{V})} := \sum_{x\in\mathcal{V}} c(x)f(x)\overline{g}(x).$$
  - $$l^2(\mathcal{E}) := \{\varphi \in \mathcal{C}(\mathcal{E}),\ \sum_{e\in\mathcal{E}} r(e)|\varphi(e)|^2 < \infty\}$$

    with the inner product

    $$\langle \varphi, \psi\rangle_{l^2(\mathcal{E})} := \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\varphi(e)\overline{\psi}(e).$$

$$l^2(\mathcal{F}) := \{\phi \in \mathcal{C}(\mathcal{F}), \sum_{\varpi \in \mathcal{F}} s(\varpi)|\phi(\varpi)|^2 < \infty\}$$

with the inner product

$$\langle \phi_1, \phi_2 \rangle_{l^2(\mathcal{F})} = \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}} s(x,y,z)\phi_1(x,y,z)\overline{\phi_2}(x,y,z).$$

Part.2

The discrete Laplacian

- *The difference operator :* $d^0 : \mathcal{C}_c(\mathcal{V}) \longrightarrow \mathcal{C}_c(\mathcal{E})$ is given by

$$d^0(f)(e) = f(e^+) - f(e^-).$$

- *The co-boundary operator :* $\delta^0 : \mathcal{C}_c(\mathcal{E}) \longrightarrow \mathcal{C}_c(\mathcal{V})$ is given by

$$\delta^0(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e)\varphi(e).$$

- *The exterior derivative :* $d^1 : \mathcal{C}_c(\mathcal{E}) \longrightarrow \mathcal{C}_c(\mathcal{F})$ is given by

$$d^1(\psi)(x,y,z) = \psi(x,y) + \psi(y,z) + \psi(z,x).$$

- *The co-exterior derivative :* $\delta^1 : \mathcal{C}_c(\mathcal{F}) \longrightarrow \mathcal{C}_c(\mathcal{E})$ is given by

$$\delta^1(\phi)(e) = \frac{1}{r(e)} \sum_{x \in \mathcal{F}_e} s(e,x)\phi(e,x).$$

- *The Gauß-Bonnet operator on $\mathcal{T}$:* It is the symmetric operator of Dirac type $T : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$ given by

$$T(f, \varphi, \phi) = (\delta^0 \varphi, d^0 f + \delta^1 \phi, d^1 \varphi).$$

- *The discrete Laplacian on $\mathcal{T}$:*

$$\mathcal{L} := T^2 : \mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F}) \circlearrowleft$$

is given by

$$\mathcal{L}(f, \varphi, \phi) = (\delta^0 d^0 f, (d^0 \delta^0 + \delta^1 d^1)\varphi, d^1 \delta^1 \phi).$$

- We can write

$$\mathcal{L} := \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$$

where

  - $\mathcal{L}_0$ is the discrete Laplacian acting on $\mathcal{C}_c(\mathcal{V})$ given by

$$\mathcal{L}_0(f)(x) := \delta^0 d^0(f)(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e) d^0(f)(e).$$

  - $\mathcal{L}_1 = \mathcal{L}_1^- + \mathcal{L}_1^+ = d^0\delta^0 + \delta^1 d^1$ is the discrete Laplacian acting on $\mathcal{C}_c(\mathcal{E})$ given by

$$\mathcal{L}_1(\varphi)(x,y) = \underbrace{\frac{1}{c(y)} \sum_{e, e^+ = y} r(e)\varphi(e) - \frac{1}{c(x)} \sum_{e, e^+ = x} r(e)\varphi(e)}_{\mathcal{L}_1^-(\varphi)(x,y)}$$

$$+ \underbrace{\frac{1}{r(x,y)} \sum_{z \in \mathcal{F}_{(x,y)}} s(x,y,z) d^1(\varphi)(x,y,z)}_{\mathcal{L}_1^+(\varphi)(x,y)}.$$

- $\mathcal{L}_2$ is the discrete Laplacian acting on $\mathcal{C}_c(\mathcal{F})$ given by

$$\begin{aligned}\mathcal{L}_2(\phi)(x,y,z) &:= d^1\delta^1(\phi)(x,y,z)\\ &= \frac{1}{r(x,y)}\sum_{u\in\mathcal{F}_{(x,y)}} s(x,y,u)\phi(x,y,u)\\ &+ \frac{1}{r(y,z)}\sum_{u\in\mathcal{F}_{(y,z)}} s(y,z,u)\phi(y,z,u)\\ &+ \frac{1}{r(z,x)}\sum_{u\in\mathcal{F}_{(z,x)}} s(z,x,u)\phi(z,x,u),\end{aligned}$$

χ-completeness

- A graph $\mathcal{K} = (\mathcal{V}, \mathcal{E})$ is χ-complet if there exists an increasing sequence of finite sets $(\mathcal{B}_n)_{n\in\mathbb{N}}$ such that $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ and there exist related functions χ_n satisfying the following three conditions :
  - $\chi_n \in \mathcal{C}_c(\mathcal{V}),\ 0 \leq \chi_n \leq 1.$
  - $x \in \mathcal{B}_n \Rightarrow \chi_n(x) = 1.$
  - $\exists C > 0$ such that $\forall n \in \mathbb{N},\ x \in \mathcal{V}$

$$\frac{1}{c(x)} \sum_{e\in\mathcal{E}, e^{\pm}=x} r(e)|d^0\chi_n(e)|^2 \leq C.$$

C. Anné & N. Torki-Hamza
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Definition

A triangulation $\mathcal{T}$ is χ-complete, if

- $\mathcal{K}$ is χ-complet.
- $\exists M > 0, \ \forall n \in \mathbb{N}, \ e \in \mathcal{E}$, such that

$$\frac{1}{r(e)} \sum_{x \in \mathcal{F}_e} s(e,x) |d^0 \chi_n(e^-, x) + d^0 \chi_n(e^+, x)|^2 \leq M.$$

Proposition

Let $\mathcal{T}$ be a simple triangulation of bounded degree then $\mathcal{T}$ is a χ-complete triangulation.

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Idea of proof

- Given $O \in \mathcal{V}$, let the ball $\mathcal{B}_n$ defined by :

$$\mathcal{B}_n = \{x \in \mathcal{V},\ d_{comb}(O, x) \leq n\}.$$

- We set the cut-off function $\chi_n \in \mathcal{C}_c(\mathcal{V})$ as follow :

$$\chi_n(x) := \left(\frac{2n - d_{comb}(O, x)}{n} \vee 0 \right) \wedge 1,\ \forall n \in \mathbb{N}^*.$$

i) If $x \in \mathcal{B}_n \Rightarrow \chi_n(x) = 1$ and $x \in \mathcal{B}_{2n}^c \Rightarrow \chi_n(x) = 0$.
ii) For $e \in \mathcal{E}$, we have that

$$|d^0\chi_n(e)| \leq \frac{1}{n} \left| d_{comb}(O, e^+) - d_{comb}(O, e^-) \right| = \frac{1}{n}.$$

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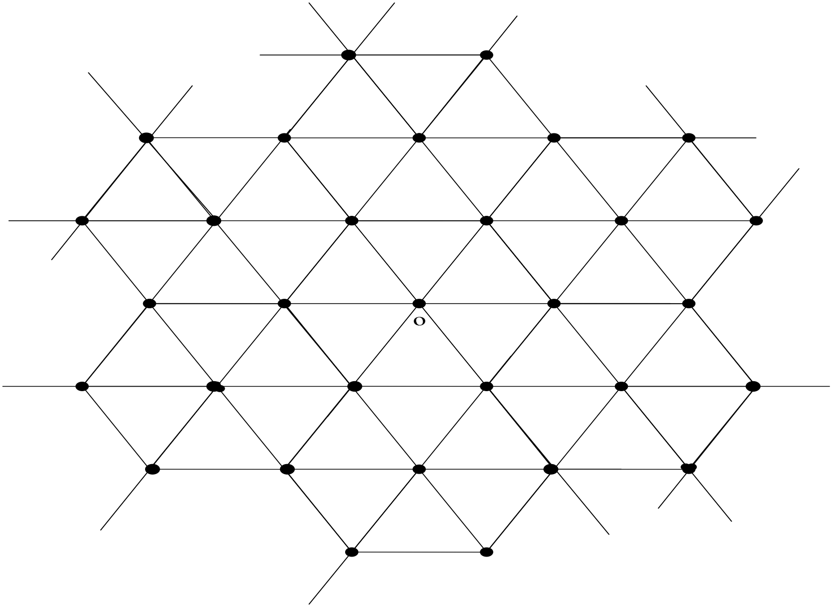

Figure : A 6-regular triangulation

Self-adjointness of the Laplacian

The idea of studying T rather than $\mathcal{L}$ comes back to :

- **P. R. Chernoff** in the case of complete manifolds

P.R. Chernoff
Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. 12, 401-414, 1973.

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Theorem

Let $\mathcal{T}$ be a χ-complete triangulation. Then

$$T \text{ is e.s.a} \Leftrightarrow \mathcal{L} \text{ is e.s.a}$$

This result is based on the

Theorem "Von Neumann"

Let T be a densely defined closed operator then TT^* and T^*T are self-adjoint.

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Let $\mathcal{T}$ be a χ-complete triangulation then the operator T is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.

Corollary

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Lemma

Let $\mathcal{T}$ be a χ-complete triangulation then

$$Dom(\overline{T}) = Dom(\overline{d^0}) \oplus \left(Dom(\overline{\delta^0}) \cap Dom(\overline{d^1})\right) \oplus Dom(\overline{\delta^1}).$$

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Let $\mathcal{K}$ be a χ-complet graph then $d^0 + \delta^0$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{V}) \oplus \mathcal{C}_c(\mathcal{E})$.

Proposition

Let $\mathcal{T}$ be a χ-complete triangulation then $d^1 + \delta^1$ is essentially self-adjoint on $\mathcal{C}_c(\mathcal{E}) \oplus \mathcal{C}_c(\mathcal{F})$.

Lemma

Let $\mathcal{T}$ be a χ-complete triangulation then

$$Dom(\overline{T}) = Dom(\overline{d^0}) \oplus \left(Dom(\overline{\delta^0}) \cap Dom(\overline{d^1})\right) \oplus Dom(\overline{\delta^1}).$$

Thanks