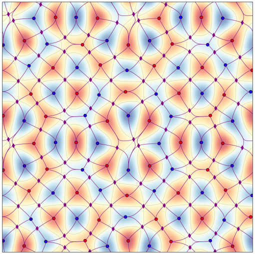
Neumann Domains and Spectral Position



Ram Band
Technion, Israel

based on joint works with

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David Fajman (Vienna)
Alexander Taylor (Bristol)

Analysis and Geometry on Graphs and Manifolds, Potsdam, August 2017

Outline

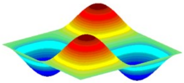
Nodal domains : Definitions

(M, g) 2d compact, connected Riemannian manifold (w\o boundary).

$-\Delta_g f_n = \lambda_n f_n$ with Dirichlet B.C. $f_n|_{\partial M} = 0$
or Neumann B.C. $\hat{n} \cdot \nabla f_n|_{\partial M} = 0$

Spectrum discrete and arranged non-decreasingly, $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \nearrow \infty$.
Orthonormal basis of eigenfunctions, $\{f_n\}_{n=1}^{\infty}$.

Example: M is flat torus with side length=2π



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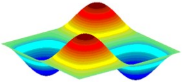

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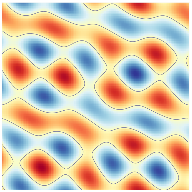

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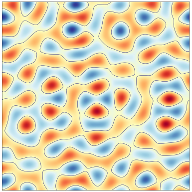

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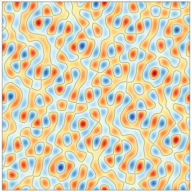

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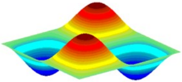

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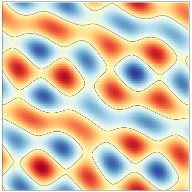

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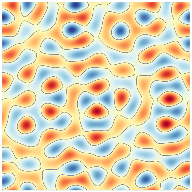

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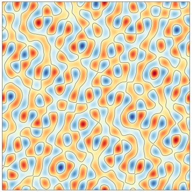

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Nodal set of f_n is $\mathcal{Z}(f_n) := \{x \in M \mid f_n(x) = 0\}$.

Nodal domains of f_n are the connected components of $M \backslash \mathcal{Z}(f_n)$.

$\nu_n := \#$ of nodal domains of f_n.

Courant's bound (1923)

Number of nodal domains of f_n is bounded by n, $\nu_n \le n$

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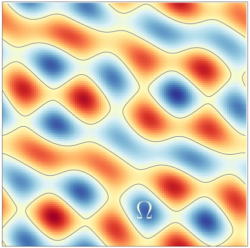

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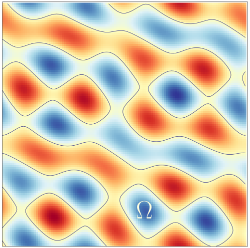

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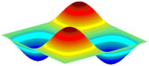
$f|_\Omega$ has a single nodal domain

$\Rightarrow$ $f|_\Omega$ is the 1^{st} eigenfunction of Ω (groundstate)

$\Rightarrow$ $\lambda_1(\Omega) = \lambda$.

"Topography" of the eigenfunction

Nodal domains - regions below\above sea level.



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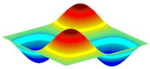
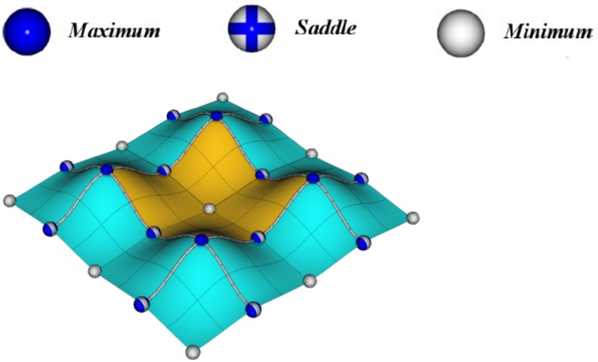


Neumann domains - where would a water droplet role to?



Stable manifold
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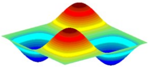

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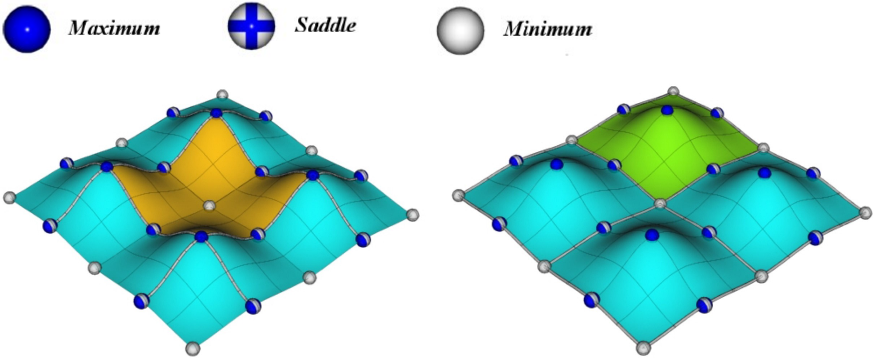

Stable manifold (of a minimum)

Unstable manifold (of a maximum)

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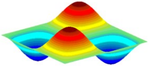

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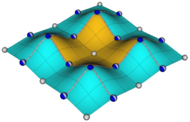

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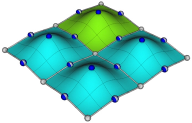

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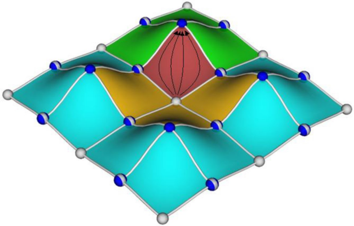

Their intersection
is a ***Neumann domain***

Figures by Attila Gabor Gyulassy.

Neumann domains : Definitions

(M, g) 2d compact, connected Riemannian manifold (w\o boundary).

$-\Delta_g f_n = \lambda_n f_n$ possibly with Dirichlet or Neumann boundary.

Define flow along integral curves of ∇f_n, $\varphi : \mathbb{R} \times M \to M$,

$$\partial_t \varphi(t, x) = -\nabla f_n\big|_{\varphi(t, x)} , \ \ \varphi(0, x) = x.$$

Assumption - f_n is a Morse function
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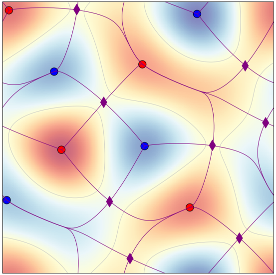

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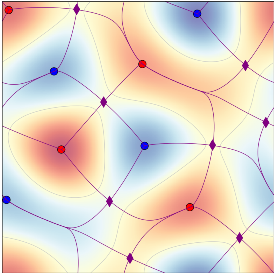

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Lemma (Basic Morse theory)

$\forall x \in M$ *both limits* $\lim_{t\to\pm\infty} \varphi(t; x)$ *exist and*

$$\lim_{t\to\pm\infty} \varphi(t; x) \in \mathcal{C}r(f)$$

$\mathcal{C}r(f) := \{x \in M \mid \nabla f|_x = 0\}$

$\mathcal{S}d(f) := \{r \in M \mid r \text{ is a saddle point of } f\}$

$\mathcal{M}in(f) := \{p \in M \mid p \text{ is a minimum of } f\}$

$\mathcal{M}ax(f) := \{q \in M \mid q \text{ is a maximum of } f\}$

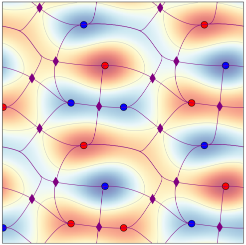
Neumann domains : Definitions

Definition (Stable and unstable manifolds)

For a critical point $x \in \mathcal{C}r\,(f)$

$$W^s(x) := \{y \in M \;\big|\; \lim_{t\to\infty} \varphi(t,\, y) = x\}$$

$$W^u(x) := \{y \in M \;\big|\; \lim_{t\to-\infty} \varphi(t,\, y) = x\}$$



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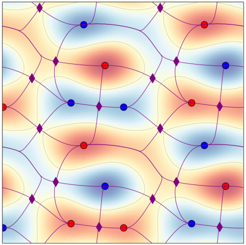


Definitions (Neumann domain and Neumann lines)

A ***Neumann domain*** is a connected component of $W^s\,(p) \cap W^u\,(q)$, where $p \in \mathcal{M}in\,(f)$, $q \in \mathcal{M}ax\,(f)$.

The set of ***Neumann lines*** is $\mathcal{N}\,(f) := \overline{\bigcup_{r\in\mathcal{S}d(f)} W^s(r) \cup W^u(r)}$.

Their union forms a cover of the whole manifold.

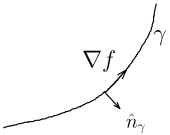
Rephrasing similar definitions from [Zelditch '13, McDonald, Fulling '14].

Neumann domains : Spectral position

Why ***Neumann*** domains?

Let $\gamma \subset \{\varphi(t;x)\}$ be part of a gradient flow line.

Then f's normal derivative at γ vanishes, $\hat{n}_\gamma \cdot \nabla f|_\gamma = 0$



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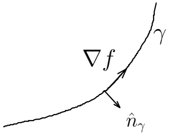
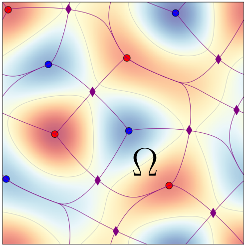




Let f be an eigenfunction of eigenvalue λ.

Let Ω be a single Neumann domain of f.

$\Rightarrow$ $f|_\Omega$ is an eigenfunction of Ω with Neumann boundary conditions.

What is the position of $f|_\Omega$ in the Neumann spectrum of Ω?

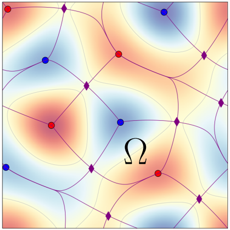
i.e., what is k in $\lambda_k(\Omega) = \lambda$?

Compare with similar question for a 'usual' (Dirichlet) nodal domain.

Neumann domains : basic properties

Theorem (B., Fajman '16)

1. $\overline{\Omega} \cap \mathcal{C}r\,(f) \subset \partial\Omega$
2. $\overline{\Omega} \cap (\mathcal{M}in\,(f) \cup \mathcal{M}ax\,(f)) = \{p, q\}$
3. Ω *is simply connected.*
4. $\overline{\Omega} \cap f^{-1}(0)$ *is a single non self-intersecting curve whose endpoints lie on* $\partial\Omega$.



The above shows that the "topography" of $f|_\Omega$ is relatively simple.

So...for a Neumann domain Ω
what is the position of $f|_\Omega$ in the spectrum of Ω?

Position of $f|_\Omega$ in Ω's Neumann spectrum

Let M be a 2d manifold and f an eigenfunction.

$$\mathrm{Pos}\left(f|_\Omega\right) := k \quad \Leftrightarrow \quad f|_\Omega \text{ is the } k^{\mathrm{th}} \text{ eigenfunction of } \Omega$$

Notation: The 0^{th} eigenfunction is the constant function.

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Fix M.

If $\forall f \ \ \forall \Omega$, $\mathrm{Pos}\left(f|_\Omega\right) = 1$ (so that $\lambda_1(\Omega) = \lambda$)
we can use the isoperimetric inequality $A(\Omega)\,\lambda_1(\Omega) \le \pi \lambda_1(\mathbb{D})$,
where $A(\Omega) :=$ Area of Ω and $\mathbb{D}$ is the unit disk [Szegö-Weinberger ('54-6)].

This gives estimates on the number of Neumann domains.

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Alternatively,
If $\text{Pos}\left(f|_\Omega\right) = 2$ then use $A(M)\,\lambda_2(M) \leq 2\pi\lambda_1(\mathbb{D})$
[Girouard, Nadirashvili, Polterovich '09]
or if , $\text{Pos}\left(f|_\Omega\right) = m$ then use $A(M)\,\lambda_m(M) \leq 8\pi m$ [Kröger '92].

All the above is applicable if $\exists m$ such that $\forall f \ \forall \Omega \quad \text{Pos}\left(f|_\Omega\right) \leq m$

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Proposition [B., Fajman '16]:
For the $2d$ flat torus, $\left\{\text{Pos}\left(f|_\Omega\right)\right\}_{f,\Omega}$ is not bounded.

Pos $(f|_\Omega)$ on the torus

Proposition (B., Fajman '16)

For the 2d flat torus, $\{Pos(f|_\Omega)\}_{f,\Omega}$ is not bounded.

Proof.

Let $\mathbb{T} = [0,1] \times [0,1]$ be the unit flat torus.
Assume by contradiction $\forall f \ \forall \Omega$ Pos $(f|_\Omega) \leq m$.
Fix some f with eigenvalue λ.

$$\forall \Omega \quad A(\Omega)\lambda_m(\Omega) \leq 8\pi m \quad \Rightarrow \quad A(\Omega)\lambda \leq 8\pi m$$

Summing over all Ω of f (denote by μ their number):

$$A(\mathbb{T})\lambda \leq 8\pi m \cdot \mu \quad \Rightarrow \quad \mu \geq \frac{1}{8\pi m}\lambda \tag{1}$$

$\mathrm{Pos}\left(f|_\Omega\right)$ on the torus

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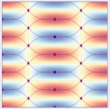
Set $\quad f(x,y) = \cos(2\pi n_x x)\cos(2\pi n_y y)$
$\lambda = 4\pi^2\left(n_x^2 + n_y^2\right)\ , \quad \mu = 8 n_x n_y.$
Choosing $n_x = 1$, $n_y \gg 1$ contradicts (1).



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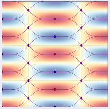
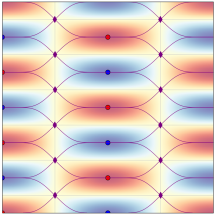


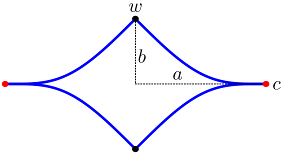
□

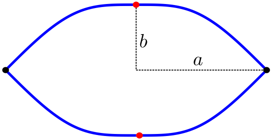
Remark: The above also shows that bounded $\mathrm{Pos}\left(f|_\Omega\right)$ implies $\mu_n \to \infty$.
This holds even if $\mathrm{Pos}\left(f|_\Omega\right)$ is bounded for a positive proportion of Ω's.

A tale of two Neumann domains

Back to the integrable function: $f(x,y) = \cos\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2b}y\right)$







We saw $\mathrm{Pos}\left(f|_\Omega\right) \underset{b\to 0}{\longrightarrow} \infty$.

For which of the domains above the position is unbounded?

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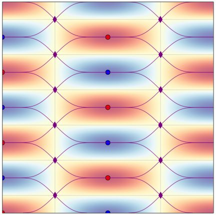

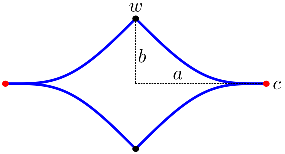

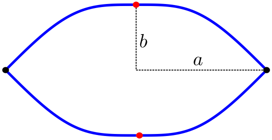

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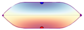
It is the lens-like domain $\Omega_{\text{lens}} =$ 

What about the star-like domain, Ω_{star}?

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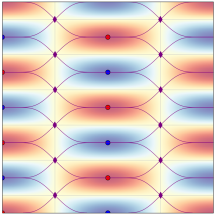

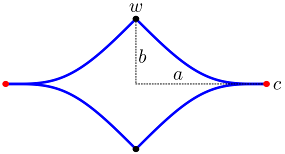

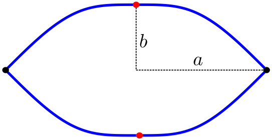

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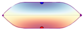
It is the lens-like domain $\Omega_{\text{lens}} =$ 

What about the star-like domain, Ω_{star}?

Theorem (B., Egger, Taylor)

$\forall a$, $\exists b_a$, such that $\forall b < b_a$,
the eigenfunction $f(x,y) = \cos\left(\frac{\pi}{2a}x\right)\cos\left(\frac{\pi}{2b}y\right)$ satisfies $Pos(f|_{\Omega_{star}}) = 1$.

area-to-perimeter ratio

Define the area-to-perimeter ratio of a Neumann domain Ω of eigenfunction f with eigenvalue λ

(following Elon, Gnutzmann, Joas, Smilansky '07)

$$\rho(\Omega) := \frac{A(\Omega)\sqrt{\lambda}}{L(\Omega)},$$

with $A(\Omega) :=$ Area and $L(\Omega) :=$ Perimeter length.

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Assume $\text{Pos}(f|_\Omega) = 1$ (so that $\lambda = \lambda_1(\Omega)$)

- Szegö-Weinberger $A(\Omega)\lambda_1(\Omega) \le \pi\lambda_1(\mathbb{D}) \Leftrightarrow \sqrt{A(\Omega)\lambda} \le \sqrt{\pi p^2}$, with $p \approx 1.8412$, (as the disk $\mathbb{D}$ as maximizer).

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Combining both gives

$$\rho(\Omega) = \frac{A(\Omega)\sqrt{\lambda}}{L(\Omega)} \leq \frac{p}{2} \approx 0.9206.$$

area-to-perimeter ratio

The area-to-perimeter ratio of a Neumann domain Ω

$$\rho(\Omega) := \quad \frac{A(\Omega)\sqrt{\lambda}}{L(\Omega)},$$

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The assumption $\mathrm{Pos}(f|_{\Omega}) = 1$ yields $\rho(\Omega) \leq \frac{p}{2} \approx 0.9206$.

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The area-to-perimeter ratio of a Neumann domain Ω

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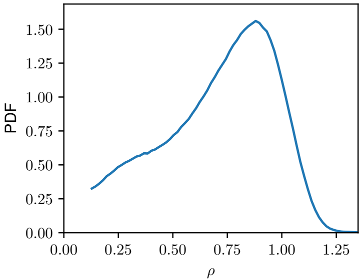
The assumption $\mathrm{Pos}(f|_\Omega) = 1$ yields $\rho\left(\Omega\right) \leq \frac{p}{2} \approx 0.9206$.

<u>*Numerical*</u> distribution of $\rho\left(\Omega\right)$ calculated for random eigenfunctions on the flat torus of eigenvalue=925 (mutiplicity=24)
(by Alexander Taylor)



Summary and Questions

Summary

An example with

1. Unobounded $\{\mathrm{Pos}\,(f|_\Omega)\}_{f,\Omega}$
2. But $\mathrm{Pos}\,(f|_\Omega) = 1$ for "half" of the Neumann domains.

Random eigenfunctions on the torus have $\mathrm{Pos}\,(f|_\Omega) > 1$ for a positive proportion of the Neumann domains.

Questions

1. Is $\{\mathrm{Pos}\,(f|_\Omega)\}$ generically bounded?
2. How is it distributed?
3. Conjecture: For all f, there is a positive proportion of Neumann domains with $\mathrm{Pos}(f|_\Omega) = 1$.
4. Using $\mathrm{Pos}\,(f|_\Omega)$ for estimates on the Neumann domain count.

After the coffee break...

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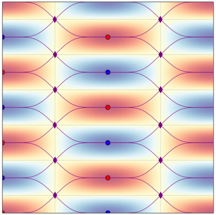
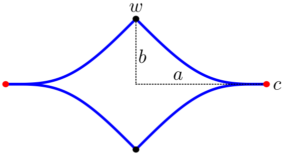

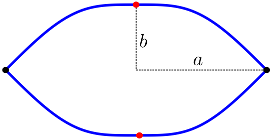

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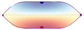
The position is unbounded for the lens-like domain $\Omega_{\text{lens}} =$ 

Whereas, for the star-like domain, Ω_{star} we have

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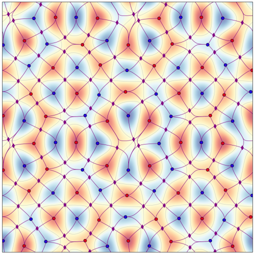
To be proven by Sebastian Egger after the coffee break.

Neumann Domains and Spectral Position



Ram Band
Technion, Israel

based on joint works with
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David Fajman (Vienna)
Alexander Taylor (Bristol)

Analysis and Geometry on Graphs and Manifolds, Potsdam, August 2017