Power dissipation in fractal AC circuits

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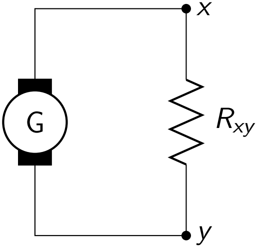
Passive linear networks. Resistors



Ohm's law

$$V_{xy} = I_{xy} R_{xy}.$$

Kirchoff's voltage law

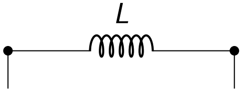
$$V_{xy} = v(x) - v(y),$$

$(v(x), v(y)) \in \mathbb{R}^2$ potential function.

Passive linear networks. Inductors and capacitors

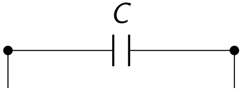
Time-dependent voltage $V(t)$ and current $I(t)$ functions.

Inductor



$$V(t) = L \frac{d}{dt} I(t).$$

Capacitor



$$I(t) = C \frac{d}{dt} V(t).$$

Frequency domain. Impedances

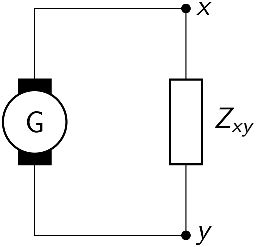
Fourier transform: $$\widehat{V}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v(t)e^{-i\omega t}dt.$$

Inductor: $$\widehat{V}(\omega) = i\omega L\,\widehat{I}(\omega) =: Z_L\,\widehat{I}(\omega),$$

Capacitor: $$\widehat{V}(\omega) = \frac{1}{i\omega C}\widehat{I}(\omega) =: Z_C\,\widehat{I}(\omega),$$

Resistor: $$\widehat{V}(\omega) = R\widehat{I}(\omega) =: Z_R\,\widehat{I}(\omega).$$

Ohm's law revisited



Ohm's law (complex-valued)

$$V_{xy}(\omega) = I_{xy}(\omega) Z_{xy}(\omega).$$

Kirchoff's voltage law

$$V_{xy}(\omega) = v(\omega, x) - v(\omega, y),$$

$(v(\omega, x), v(\omega, y)) \in \mathbb{C}^2$ potential function.

Power dissipation

From now on: frequency ω is fixed, φ phase shift.

$$V_{xy}(t) = |V_{xy}|e^{i\omega t}, \quad I_{xy}(t) = |I_{xy}|e^{i(\omega t - \varphi)}, \quad Z_{xy} = |Z_{xy}|e^{i\varphi}.$$

Average energy loss

$$\frac{1}{T}\int_0^T \Re(\mathrm{emf}_{xy}(t))\Re(I_{xy}(t))\,dt = \cdots = \frac{1}{2}|I_{xy}|^2\Re(Z_{xy}).$$

Power dissipation

Power dissipation of the potential $(v(x), v(y)) \in \mathbb{C}$

$$\mathcal{P}[v]_{Z_{xy}} = \frac{1}{2}\frac{\Re(Z_{xy})}{|Z_{xy}|^2}|v(x) - v(y)|^2.$$

Power dissipation in graphs

Let $\mathcal{G} = (V, E)$ be a finite graph, $\mathcal{Z} = \{Z_{xy},\ \{x, y\} \in E\}$ a network on $\mathcal{G}$ and $\ell(V) = \{v\colon V \to \mathbb{C}\}$. The quadratic form

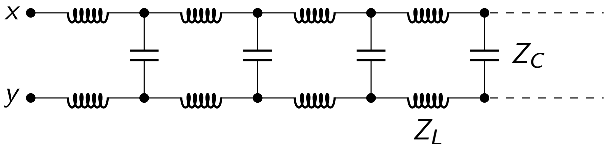
$$\mathcal{P}_{\mathcal{Z}}[v] = \frac{1}{2} \sum_{\{x,y\} \in E} \frac{\Re(Z_{xy})}{|Z_{xy}|^2} |v(x) - v(y)|^2$$

is the power dissipation in $\mathcal{G}$ associated with the network $\mathcal{Z}$.

- If $Z_{x,y}$, I_{xy}, v real, $\mathcal{P}_{\mathcal{Z}}(v) = \frac{1}{2} \sum_{\{x,y\} \in E} \frac{1}{Z_{xy}} (v(x) - v(y))^2$.

Power dissipation in an infinite network. The infinite ladder

Infinite ladder network [4]



If $\omega^2 LC < 4$, the characteristic impedance of the circuit satisfies

$$\Re(Z_{xy}^{\text{eff}}) > 0$$

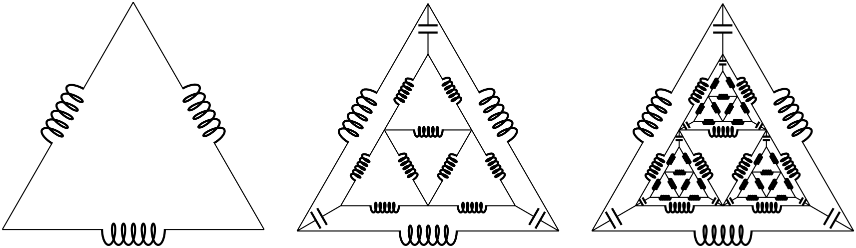
even though all elements in the circuit have purely imaginary impedances!

Questions

- Power dissipation on infinite (fractal-like) AC circuits?
- Definition? For which potentials?
- Harmonic potentials?
- Power dissipation measure?

The Feynman-Sierpinski ladder

Infinite network $\mathcal{Z}_{\mathsf{FS}} = \{Z_{xy},\ \{x, y\} \in E_\infty\}$.



Capacitors $Z_C = \frac{1}{i\omega C}$, inductors $Z_L = i\omega L$.

Theorem [3]: The effective impedance of the Feynman-Sierpinski ladder has positive real part whenever

$$9(4-\sqrt{15}) < 2\omega^2 LC < 9(4+\sqrt{15}) \qquad \text{(FC)}$$

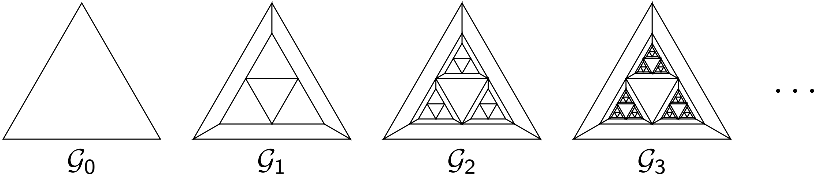
(filter condition).

In this case,

$$Z_{\mathsf{FS}}^{\mathsf{eff}} = \frac{1}{10\omega C}\left((9+2\omega^2 LC)i + \sqrt{144\omega^2 LC - 4(\omega^2 LC)^2 - 81}\right).$$

From infinite graphs to fractals

Underlying infinite graph structure $\mathcal{G}_\infty$ approximated by finite graphs $\mathcal{G}_n = (V_n, E_n)$, $n \geq 0$.



- $\pi : \mathcal{G}_\infty \to \mathbb{R}^2$
- $\pi(\mathcal{G}_0) \subseteq \pi(\mathcal{G}_1) \subseteq \ldots \subseteq \pi(\mathcal{G}_n) \subseteq \ldots$

The fractal Q_∞

The unique compact set $Q_\infty \subseteq \mathbb{R}^2$ such that

$$Q_\infty = \overline{\bigcup_{n \geq 0} \pi(\mathcal{G}_n)}^{\mathrm{Eucl}}$$

is a fractal quantum graph.

The fractal K_∞

The set

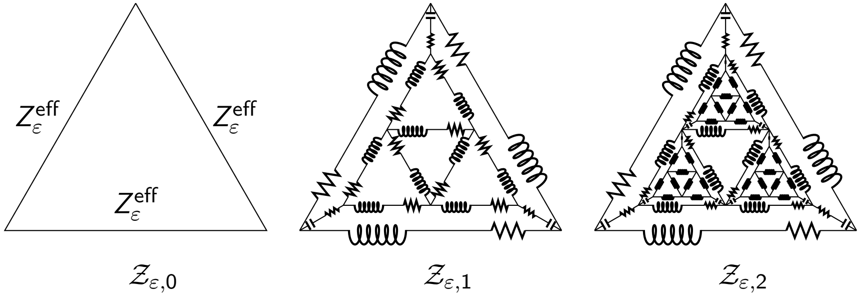
$$K_\infty = Q_\infty \setminus \bigcup_{n\geq 0} \pi(\mathring{E}_n)$$

is the union of countable many isolated points (nodes in $V_* := \bigcup_{n\geq 1} \pi(V_n)$) and a Cantor dust C_∞ (accumulation points).

- C_∞ can be identified with the Sierpinski gasket seen as the Martin boundary of a suitable Markov chain (Lau-Ngai [5], ...).

Networks on $\mathcal{G}_n$

$$\mathcal{Z}_{\varepsilon,n} = \{Z_{\varepsilon,xy} \mid \{x,y\} \in E_n\}, \qquad Z_{\varepsilon,xy} = Z_{xy} + \varepsilon.$$



(For completeness, $Z_\varepsilon^{\text{eff}} := \lim_{n\to\infty} Z_{\varepsilon,n}^{\text{eff}}$.)

Towards power dissipation in K_∞

The power dissipation in V_* associated with the Feynman-Sierpinski ladder is the quadratic form

$$\mathrm{P}_{\mathrm{FS}}[v] := \lim_{\varepsilon\to 0_+} \lim_{n\to\infty} \mathcal{P}_{\mathcal{Z}_{\varepsilon,n}}[v_{|V_n}],$$

where $\mathcal{P}_{\mathcal{Z}_{\varepsilon,n}} \colon \ell(V_n) \to \mathbb{R}$ is the power dissipation in $\mathcal{G}_n$ associated with $\mathcal{Z}_{\varepsilon,n}$.

- Theorem [3]: $\lim_{\varepsilon\to 0+} \lim_{n\to\infty} Z^{\mathrm{eff}}_{\varepsilon,n} = Z^{\mathrm{eff}}_{\mathrm{FS}}$.

$$\operatorname{dom} \mathsf{P}_{\mathsf{FS}} := \{ v \in \ell(V_*) \mid \mathsf{P}_{\mathsf{FS}}[v] < \infty \}$$

- meaningful functions in this set?
- extension of functions?

Harmonic functions

- A function $h \in \ell(V_*)$ is harmonic if for any $\varepsilon > 0$

$$\mathsf{P}_{\mathcal{Z}_{\varepsilon,0}}[h_{|_{V_0}}] = \mathsf{P}_{\mathcal{Z}_{\varepsilon,n}}[h_{|_{V_n}}] \qquad \text{for all } n \geq 0.$$

- Construction: harmonic extension rule [3].

Continuity of harmonic functions

Theorem (A.R.'17): Harmonic functions are continuous on V_*.

Corollary: Harmonic functions are well-defined on K_∞,

$$\mathcal{H}_{\mathsf{FS}}(K_\infty) = \{h\colon K_\infty \to \mathbb{C} \mid h_{|_{V_*}} \text{ harmonic on } V_*\}.$$

Power dissipation in the Feynman-Sierpinski ladder

The power dissipation in K_∞ associated with the Feynman-Sierpinski ladder is the quadratic form

$$\mathsf{P}_{\mathsf{FS}}[h] = \mathsf{P}_{\mathsf{FS}}[h_{|_{V_*}}], \qquad h \in \mathcal{H}_{\mathsf{FS}}(K_\infty).$$

Power dissipation measure

Theorem (A.R.'17): For each non-constant $h \in \mathcal{H}_{\mathsf{FS}}(K_\infty)$, power dissipation induces a continuous measure ν_h on K_∞ with $\operatorname{supp} \nu_h = C_\infty$.

Theorem (A.R.'17): The measure ν_h is singular with respect to the uniform self-similar measure on C_∞.

Outlook

- Characterization of the domain.
- Generalization to more abstract spaces.
- Connection with Martin boundary.

References

- P. Alonso Ruiz, *Power dissipation in fractal Feynman-Sierpinski ac circuits*, Journal of Mathematical Physics **58** (2017), no. 7, 073503.
- O. Ben-Bassat, R. S. Strichartz, and A. Teplyaev, *What is not in the domain of the Laplacian on Sierpinski gasket type fractals*, J. Funct. Anal. **166** (1999), no. 2, 197–217.
- J.P. Chen, L. R. Rogers, L. Anderson, U. Andrews, A. Brzoska, A. Coffey, H. Davis, L. Fisher, M. Hansalik, S. Loew, and A. Teplyaev, *Power dissipation in fractal ac circuits*, Journal of Physics A: Mathematical and Theoretical **50** (2017), no. 32, 325205.
- R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics. Vol. 2: Mainly electromagnetism and matter*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964.
- K.-S. Lau and S.-M. Ngai, *Martin boundary and exit space on the Sierpinski gasket*, Science China Mathematics **55** (2012), no. 3, 475–494.

Thank you for your attention!

Self-similar measure on K_∞

Bernouilli measure μ on K_∞:

$$\mu(T_{w_1 \dots w_n}) = \mu_{w_1} \cdots \mu_{w_n}, \qquad \sum_{i=1}^{3} \mu_i = 1.$$

- $\operatorname{supp} \mu = C_\infty$,
- (C_∞, μ) is probability space,
- take $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$.

Singularity of power dissipation

Theorem (A.R.'17): Assume that for any non-constant $h \in \mathcal{H}_{\mathsf{FS}}(K_\infty)$ such that $h_{|v_0} = v_0$

$$x \mapsto \|D_{\mathsf{P}_0} M_n(x) \ldots M_1(x) v_0\|$$

is non-constant for some $n \geq 1$. Then, the measure ν_h is singular with respect to μ.

Random matrices

- Matrix representation of $\mathrm{P}_{\mathcal{Z}_0}$: $D^2_{\mathrm{P}_0} = \frac{\Re(Z^{\mathrm{eff}}_{\mathrm{FS}})}{2|Z^{\mathrm{eff}}_{\mathrm{FS}}|^2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$,
- matrices of the harmonic extension algorithm: A_1, A_2, A_3,
- for each $x \in C_\infty$ with $x = \bigcap_{n \geq 1} T_{w_1 \dots w_n}$: $M_n(x) := A_{w_n}$,
- $M_n(x)$ statistically independent w.r.t. μ.

Key lemma

Lemma: The measure ν_h is singular with respect to μ if for μ-a.e. $x \in C_\infty$

$$\lim_{n\to\infty} \frac{\nu_h(T_{w_1\dots w_n})}{\mu(T_{w_1\dots w_n})} = 0,$$

where $x = \bigcap_{n\geq 1} T_{w_1\dots w_n}$.

- Proof: generalized Lebesgue differentiation theorem ($(C_\infty, \mu, d_{\text{Euclidean}})$ is volume doubling.)

Sketch of proof

(Based on Bassat-Strichartz-Teplyaev '99 [2].)

- For each n-cell, $\nu_h(T_{w_1 \dots w_n}) = \|D_{\mathrm{P}_0} A_{w_n} \cdots A_{w_n} h_{|_{V_0}}\|^2$,
- since $M_n(x)$ i.i.d., Furtstenberg's Theorem yields

$$\limsup_{n \to \infty} \frac{1}{n} \log \|D_{\mathrm{P}_0} M_n(x) \dots M_1(x) h_{|_{V_0}}\| = \beta,$$

- prove $\beta < \frac{1}{2} \log 3$,
- $\frac{\nu_h(T_{w_1 \dots w_n})}{\mu(T_{w_1 \dots w_n})} = 3^n \|D_{P_0} M_{w_n}(x) \cdots M_{w_1}(x) h_{|_{V_0}}\|^2 \xrightarrow{n \to \infty} 0.$