C*-Algebras

Winter semester 2016/17 **Prof. Dr. M. Keller**

Sheet 11

Let (b, c) be a graph over X and $\mathcal{L}$ the Laplacian

$$\mathcal{L}\varphi(x) = \sum_{x \in X} b(x, y)(\varphi(x) - \varphi(y)) + c(x)\varphi(x).$$

Furthermore, let

$$\deg : X \to [0, \infty), \quad x \mapsto \sum_{x \in X} b(x, y) + c(x).$$

and

$$\ell^2(X) = \sum_{x \in X} \{\varphi : X \to \mathbb{C} \mid \|\varphi\|_2 : (\sum_{x \in X} |\varphi(x)|^2 < \infty)^{1/2}\}$$

be the Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

(1) Show that the restriction L of $\mathcal{L}$ to $\ell^2(X)$ is a bounded selfadjoint operator if deg is a bounded function.
To this end show that

$$\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$$

and that the restriction of $\mathcal{L}$ to $C_c(X)$ is densely defined on $\ell^2(X)$, and bounded on $\ell^2(X)$ if and only if deg is a bounded function.
Hint: For the boundedness, show first that

$$\|\mathcal{L}\varphi\| = \sup_{\|g\|_2^2 = 1} \langle \mathcal{L}\varphi, g \rangle$$

From now on let $X = \mathbb{N}$ and $b(x, y) = 1$ if $|x - y| = 1$ and $b(x, y) = 0$ otherwise and let L be the restriction of $\mathcal{L}$ to $\ell^2(X)$.

(2) Show that 1_x is a cyclic vector for L in $\ell^2(X)$.

(3) Compute the spectrum of L.
Hint: Try to mimick the Fourier transform of $\mathbb{R}$.

(4) Compute the spectrum of $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by

$$S\varphi(n) = \varphi(n - 1), \qquad n \in \mathbb{Z}.$$