Schrödinger operators over dynamical systems [1]

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[1]Es handelt sich nicht um ein Skriptum zur Vorlesung sondern nur um Notizen.

Contents

Organization

Chapter 2, 3 and 4 are based on a lecture “Aperiodische Ordnung” given by Daniel Lenz, a lecture “Ergodentheorie” given by myself and my PhD thesis. The content of Chapter 5 is advanced but standard and can be found in various books about spectral theory. In this notes I particularly used the lecture notes for “C*-algebras” given by Matthias Keller and the lecture notes for “Höhere Analysis” given by Daniel Lenz. Chapter 6 is again based on my PhD thesis and Chapter 7 is taken from the paper “Hölder continuity of the spectra for aperiodic Hamiltonians” by Beckus/Bellissard/Cornean.

There are still various persistent typos hidden in these notes (even from my eyes). Please leave a comment on moodle or by Email if you come across them. So, we may weed them out in a joint effort.

Literature

- Boto von Querenburg, *Mengentheoretische Topologie*, ISBN 978-3-642-56860-2, 2001 **(Topology)**
- M. Manetti, *Topology*, ISBN 978-3-319-16957-6, 2015 **(Topology)**
- J. Elstrodt, *Maß- und Integrationstheorie*, ISBN 978-3-642-17905-1, 2005 **(Measure Theory)**
- D. J. S. Robinson, *A course in the theory of groups*, ISBN 978-0-387-94461-6, 1995 **(Group theory)**
- M. Baake, U. Grimm, *Aperiodic Order*, Volume 1: A mathematical invitation, ISBN: 978-0-521-86991-1, 2013 **(Quasicrystals)**
- M. Baake, U. Grimm, *Aperiodic Order*, Volume 2: Crystallography and Almost Periodicity, ISBN: 978-1-1390-3386-2, 2017 **(Quasicrystals)**
- I. Satija, *Butterfly in the Quantum World*, The story of the most fascinating quantum fractal, ISBN: 978-1-6817-4117-8, 2016 **(Popular book)**
- D. Werner, *Funktionalanalysis*, ISBN 978-3-662-55406-7, 2000 **(Functional Analysis)**
- R. Carmona, J. Lacroix, *Spectral Theory of Random Schrödinger operators*, ISBN: 978-1-4612-8841-1 , 1990 **(Random Schrödinger Operators)**
- J. Weidmann, *Linear Operators in Hilbert spaces*, ISBN: 0-387-90427-1, 1980 **(Spectral theory)**
- M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, Vol. 1 & Vol. 2., 1980 **(Spectral theory)**
- P.D. Hislop, I.M. Sigal *Introduction to Spectral Theory: with applications to Schrödinger operators*, Applied Mathematical Science, Vol. 113 ISBN 0-387-94501-6, 1955 **(Spectral theory)**

Further interesting papers

Functional Analysis and the Lemma of Zorn

- Here you can find a work by Karagila discussing the Axiom of choice and its role in Functional Analysis. This document is designed for students.

Chabuty-Fell topology

- Fell - A Hausdorff Topology For The Closed Subsets Of A Locally Compact Non-Hausdorff space, 1962
- Fell - The dual spaces of C*-algebras, 1960
- Chabauty - Limite d'ensembles et géométrie des nombres, 1950
- Harpe - Spaces of closed subgroups of locally compact groups, 2008

Vietoris topology

- Vietoris - Bereiche zweiter Ordnung, 1922
- Michael - Topologies on spaces of subsets, 1951
- Beer - Topologies on closed and closed convex sets, 1983
- Kuratowski - Topology. Volumes I and II., 1966

Harper model - Almost-Mathieu operator

- Hofstadter - Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, 1976
- Bellissard - Lipshitz Continuity of Gap Boundaries for Hofstadter-like Spectra, 1993
- Avron, Simon - Stability of gaps for periodic potentials under variation of a magnetic field, 1985
- Rammal, Bellissard - Algebraic semi-classical approach to Bloch electrons in a magnetic field, 1990
- Helffer, Sjörstrand - Analyse semi-classique pour l'équation de Harper. II. Comportement semi-classique prés d'un rationnel, 1990

Convergence of the spectra and Delone dynamical systems

- Beckus/Bellissard - Continuity of the spectrum of a field of self-adjoint operators, 2016
- Beckus/Bellissard/De Nittis - Spectral continuity for aperiodic quantum systems I. General theory, 2018
- Beckus/Bellissard/Cornean - Hölder continuity of the spectra for aperiodic Hamiltonians, 2019
- Beckus/Pogorzelski - Delone dynamical systems and spectral convergence, 2020
- Beckus/Bellissard/De Nittis - Spectral continuity for aperiodic quantum systems: Applications of a folklore theorem, 2020
- Beckus - Spectral approximation of aperiodic Schrödinger operators

Here you can find an interesting movie about the Penrose tiling and Quasicrystals.

Required background

A solid background in the basic courses Analysis I-III and linear Algebra is required (in particular topology, measure theory, normed spaces (Banach spaces), Hilbert spaces (inner product).

What can you learn?

- Basic concepts in topological dynamical systems over discrete groups and the associated space of invariant probability measures
- Symbolic dynamical systems over a finite alphabet

- Basic knowledge in spectral theory (spectrum, resolvent as well as their basic properties)
- Approximation theory of self-adjoint bounded operators
- Introduction into the area of random Schrödinger operators respectively operator families over dynamical systems
- Interplay between dynamical and spectral properties

1. Motivation

The lecture presents the interplay of analysis, dynamics, probability, spectral theory and mathematical physics in the realm of solid state physics. The aim of the lecture is to introduce the interplay between spectral properties of operators and their underlying dynamics.

The first part of the lecture is devoted to topological dynamical systems and their associated invariant probability measures. More precisely, X is a compact metric space on which a group G (say e.g. $G = \mathbb{R}^d$ or $G = \mathbb{Z}^d$) acts, i.e. there is a continuous $\alpha : G \times X \to X$ satisfying

$$\alpha(e, x) = x \qquad \text{and} \qquad \alpha(g, \alpha(h, x)) = \alpha(gh, x),$$

where $e \in G$ is the neutral element of G. For instance, $e = 0$ and $gh := g + h$ if $G = \mathbb{R}^d$ or $G = \mathbb{Z}^d$. Then (X, G) is a dynamical system. We will study the space $\mathcal{J}$ of invariant closed subsets of X where $Y \subseteq X$ is invariant if $\alpha(g, y) \in Y$ for all $g \in G$ and $y \in Y$. Then (Y, G) is also a dynamical system. A particular focus is put on those dynamical subsystems of X that are minimal in terms of the partial order of inclusion on such dynamical subsystems.

We study probability measures μ on X that are invariant with respect to the G action, namely $\mu(gA) = \mu(A)$ for all measurable $A \subseteq X$. We will study the space of all invariant probability measures and we will prove that one always admits such measures if the group is amenable. Furthermore, we will analyze dynamical systems (X, G) that have exactly one invariant probability measures. Such systems are called uniquely ergodic. Meanwhile, we get a glimpse on the area of Ergodic theory.

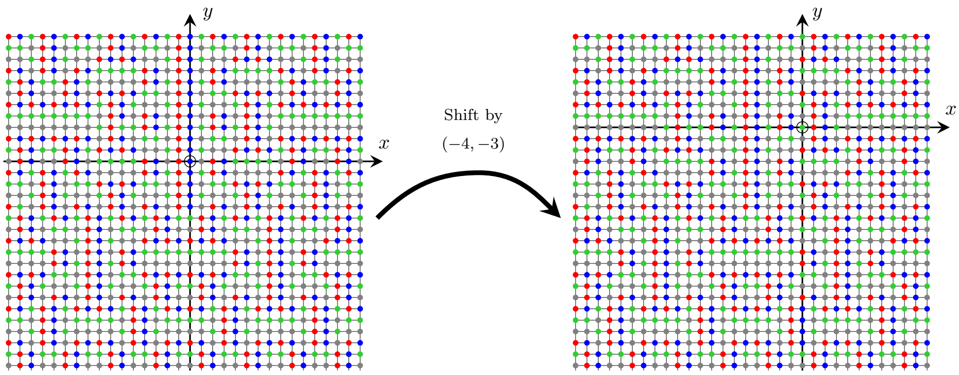

Figure 1. The set $\mathcal{A}$ consists of four elements (colors) and we consider an element in $\mathcal{A}^{\mathbb{Z}^2}$. The circled point indicates the origin $0 \in \mathbb{Z}^2$.

As one of the guiding examples, we will consider symbolic dynamical systems. Specifically, G is a discrete group, say $G = \mathbb{Z}^d$ and

$$X := \mathcal{A}^{\mathbb{Z}^d} = \left\{ w : \mathbb{Z}^d \to \mathcal{A} \right\}$$

equipped with product topology. Then $\mathbb{Z}^d$ acts on $\mathcal{A}^{\mathbb{Z}^d}$ by translation, namely

$$\alpha : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}, \quad \alpha(n, w)(m) := w(-n + m),$$

see e.g. Figure 1. In this setting, we provide a characterization of the topology of $\mathcal{J}$ and use this to find so called periodic approximations (see motivation below).

In the second part of the lecture, we will introduce basic concepts of spectral theory (with a view towards self-adjoint operators). Based on this, we will analyze operator families $A_X = (A_x)_{x\in X}$ over a dynamical system (X, G). Such operators are compatible with the dynamics (covariant) and they are strongly continuous in X. We will characterize concepts such as minimality by spectral properties of these operator families. Moreover, we construct approximations of the spectra of such operator families by appropriate approximations of the underlying dynamical systems.

1.1. Solid state physics and quasicrystals. This lecture seeks to provide you with some basic facts for random Schrödinger operators associated with dynamical systems. Such families of operators arise in solid state physics. In particular, they naturally pop up in the study of disordered media. Let us start to give some motivation based on solids.

A solid consists of atoms or molecules that are fixed in space. In a mathematical description such systems are assumed to be infinite in order to focus on properties of the solid and ignoring boundary effects. Depending on the species of the atoms and how they are distributed in space, the physical properties vary. A crystal is a solid material where these atoms are arranged in a highly ordered microscopic structure forming a lattice, see Figure 2. Such systems will be called periodic.

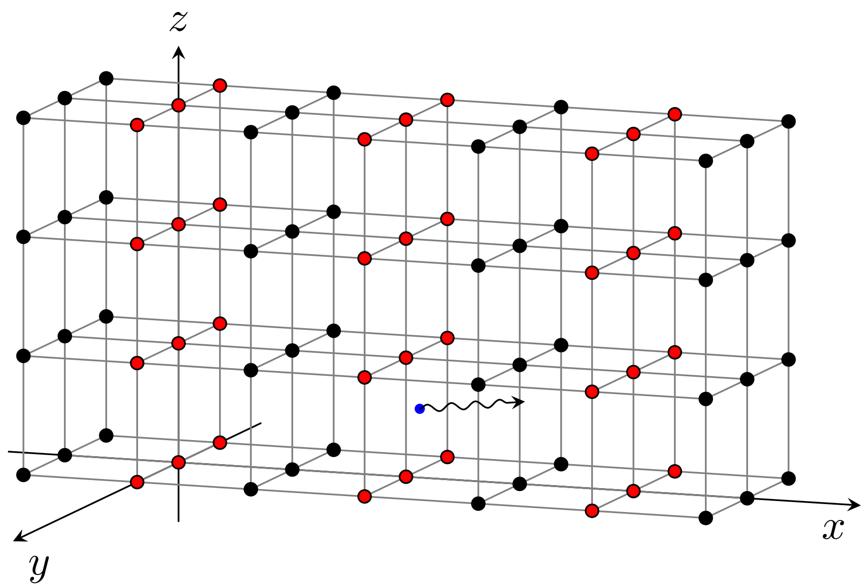

Figure 2. Two type of atoms (red and black) that are distributed on the lattice $\mathbb{Z}^3$ periodically with respect to the $\mathbb{Z}^3$-action.

One way of analyzing such solids is by X-ray crystallography where an X-ray beam is sent to the material from various directions, see Figure 3. These beams are diffracted into specific directions with certain intensities. These measurements are used to detect the localization of the atoms as well as their chemical bonds. In the case of a perfect crystal the measured peaks are sharp and admit only specific symmetries (see the crystallographic restriction theorem). Specifically, only n-fold symmetries for $n = 1, 2, 3, 4, 6$ are compatible with a periodic structure. In 1982 (published 1984), Dan Shechtmann (Technion, Haifa, Israel) discovered materials that had unexpected diffraction pattern with sharp peaks but ten-fold symmetry, which is incompatible with the lattice translation. In 2011, he was awarded with the Nobel prize for Chemistry for his discovery. Nowadays such materials are called quasicrystals. In the mathematical model, quasicrystals are modeled via point sets or tilings of the space. One of the most famous one is the Penrose tiling, see Figure 4. There are two models that are mainly considered to produce such tilings or point sets. One is the cut-and-project method and the second one are primitive substitutions. During the lecture, we will study substitutions in the one-dimensional situation.

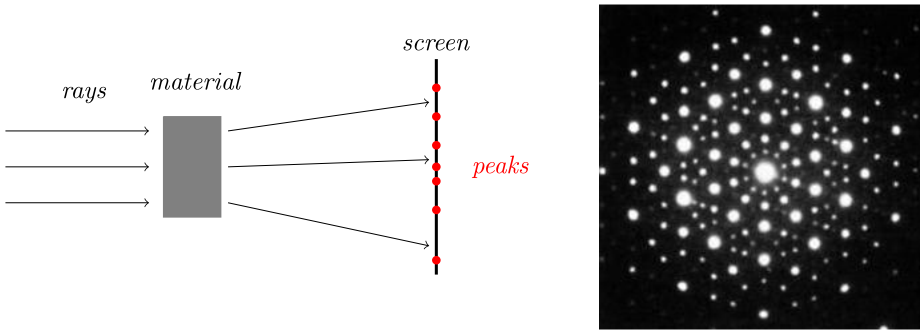

Figure 3. Left: Schematic idea for the diffraction experiment; Right: Diffraction pattern of a quasicrystal observed by D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, Metallic Phase with Long-Range Orientational, The American Physical Society, 1984.

In a mathematical description, there is a choice of an origin and we can move from one fixed origin to another by shifting it in our space (say $\mathbb{R}^d$). In the case of a periodic structure only a finite number of translations are relevant. On the other hand, the symmetry group of a quasicrystal or an amorphous solid can be very small. This leads us to study not a single one but rather the whole family of solids that is produced by shifting the origin even to infinity (here is a limit involved). This brings us naturally to the theory of topological dynamical systems that will be studied in the first part of this lecture. Then a dynamical system is a compact space X which represents the set of "all manifestations" of a fixed kind of order or disorder on a locally compact group G (e.g. $G = \mathbb{R}^d$).

In particularly, we will be interested in systems where the local atomic structure is everywhere the same, namely we don't admit impurities or defects in the material. Such dynamical systems are called minimal and we will learn a

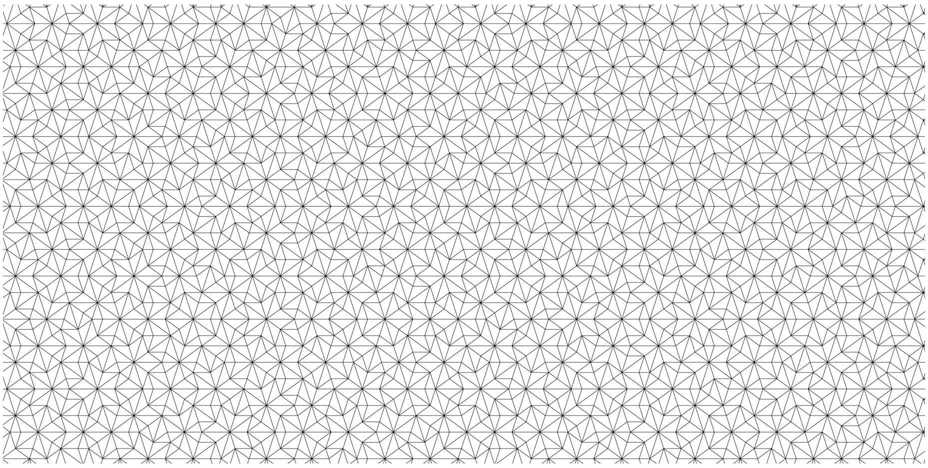

Figure 4. The Penrose tiling which is one of the toy models for quasicrystals. One can view the vertices as the position of atoms. In a discretization of the associated Hamiltonian, one is for instance interested in the spectral properties of a Schrödinger operator on the graph defined by this tiling. This picture was produced by Christian Scholz (Universität Potsdam).

equivalent formulation in the setting of symbolic dynamical systems reflecting this heuristically point of view. The terminology "minimal" comes from its definition that the topological dynamical system cannot be decomposed in a smaller one.

Besides that we will deal with probability measures on dynamical systems that are invariant with respect to the action of G. They allow us to study statistical properties of solids which is particularly interesting in the case of quasicristalls and amorphous solids. Here one is interested in properties that hold almost everywhere with respect to a given invariant probability measure. We will introduce the space of invariant measures and study basic properties of this space. In particular, we will prove a special case of the Banach-Alaoglu theorem. The measure theoretic counter part of minimality is that the measure is ergodic. The study of these objects is part of the lecture "Ergodic theory" and here we will only deal with the case where the dynamical properties forces that we have exactly one invariant probability measure (which is automatically ergodic). These systems are called uniquely ergodic.

The second part of the lecture focuses on operators $A_x, x \in X$, associated with dynamical systems (X, G). First we will provide an introduction to operators and their spectrum. Of particular interest are Schrödinger operators (or Hamiltonians) that describe the long time behavior of a particle within such a solid. The idea is that the geometry and dynamics of the underlying material will determine if a particle can freely move to infinity, stay in a bounded region or is doing something in between. Thus, one seeks to connect dynamical properties with spectral properties of such operators. As discussed before a single $x \in X$ might not admit a large symmetry group

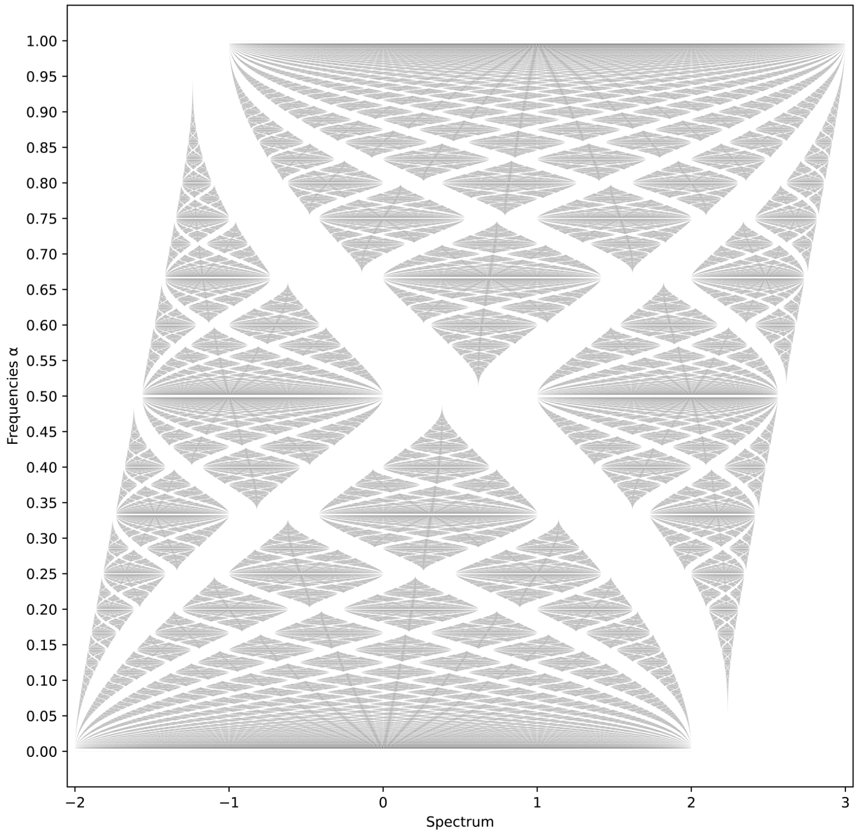

Figure 5. The Kohmoto butterfly plotted by Barak Biber (`https://github.com/DaAnIV/Quasiperiodic`, Technion, Israel). On each horizontal line the spectrum of certain one-dimensional periodic Schrödinger operators H_α is plotted for rational $\alpha \in [0,1]$. If these rational α's approach an irrational β, we get a Schrödinger operator for a certain quasicrystal. Then these rational approximations provide us some insight in the spectral nature of them.

while the whole dynamical system X is G-invariant. Similarly, the operator A_x may not have a large symmetry group and so one is interested to study the family of such operators simultaneously. Therefore, we make the assumptions on such operator families $A_X := (A_x)_{x\in X}$:

- A_X *is covariant or equivariant:* This reflects that shifting the origin from $x \in X$ to $gx \in X$ leads to a unitary transformation of the corresponding operators

$$A_x = U_g A_{gx} U_g^{-1}.$$

We will see that unitary transformations preserve spectral properties and so A_x and A_{gx} have the same spectral properties (we are independent of the choice of the origin).

- A_X *is strongly continuous:* This says that $A_x\psi$ should change continuously if $x \in X$ varies for any ψ (a small change in the configuration will only change slightly how our operator acts within a certain region).

We will see that (X, G) is a minimal dynamical system if and only if the spectrum of the associated operators are constant along $x \in X$.

The sharp peaks and the symmetry in the diffraction pattern that appear for crystals or quasicrystals reflects that the underlying solid is still ordered in a suitable sense. Due to this, the mathematical theory of quasicrystals is often referred to aperiodic order. In this sense quasicrystals are still very close to periodic structures. We will formally introduce a way to compare the distance between various dynamical systems. This in particular, enables us to talk about periodic approximations and so we will discuss the following questions:

- When does a dynamical system admit periodic approximations?
- Which dynamical and measure theoretic object are preserved by such an approximation process?
- Which spectral properties of an associated operator family are preserved if the underlying dynamical systems are approximated. In particular, we will focus on the spectrum as a set.

If time admits, we will also study some spectral properties of operators associated with crystals so periodic structures using that we have a large symmetry group. Having a suitable approximation theory at hand, this can be used to study spectral properties of other systems like quasicrystals. Their spectral properties can be very difficult and fascinating. See for instance Figure 5, where the spectrum of the operators $H_\alpha : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(H_\alpha\psi)(n) := \psi(n-1) + \psi(n+1) + \chi_{[1-\alpha,1)}(n\alpha \mod 1)\psi(n), \qquad n \in \mathbb{Z},$$

is plotted.

We note that the study of Hamiltonians associated with solids is best performed in the context of C^*-algebras. For instance any dynamical system (X, G) admits a C^*-algebra and their elements define operator families over (X, G). As it is, one is even led to a more general algebraic structure, viz C*-algebras of groupoids when studying certain quasicrystals modelled by tilings. But this is not considerd here.

2. Topological dynamical systems

A set G equipped with a composition $. : G \times G \to G$ and inversion $^{-1} : G \to G$ is called a *group* if

- the composition is associative, namely $(g_1.g_2).g_3 = g_1.(g_2.g_3)$ for all $g_1, g_2, g_3 \in G$,
- there exists a *unit* $e \in G$ satisfying $e.g = g.e = g$ for all $g \in G$,
- for all $g \in G$, we have $g.g^{-1} = g^{-1}.g = e$.

Furthermore, G is called a *topological group* if G is equipped with a topology such that the composition and inversion are continuous. For a subset $H \subseteq G$, we define

$$g.H := \{g.h \,|\, h \in H\}.$$

For the sake of simplicity, we will write $gh := g.h$ and $gH := g.H$ for $g, h \in G$ and $H \subseteq G$. A group G is called *abelian* if $gh = hg$ for all $g, h \in G$.

Example. *Let $G := \mathbb{R}^d$ with $x.y := x + y$ and $x^{-1} := -x$. Then $\mathbb{R}^d$ is an abelian topological group if equipped with the Euclidean topology.*

Example. *Let $G := \mathbb{Z}^d$ with $n.m := n + m$ and $n^{-1} := -n$. Then $\mathbb{Z}^d$ is a countable abelian group equipped with the discrete topology. If M is an invertible $d \times d$-matrix with real coefficients, then $M\mathbb{Z}^d := \{Mn \,|\, n \in \mathbb{Z}^d\}$ defines a discrete countable subgroup of $\mathbb{R}^d$.*

Example. *Consider the set of invertible matrices*

$$M_d(\mathbb{R}) := \left\{A := (a_{ij})_{i,j}\ d \times d - \textit{matrix with real coefficients and } \det(A) \neq 0\right\}$$

equipped with the usual matrix multiplication, inverse of a matrix and the Euclidean topology viewing elements of $M_d(\mathbb{R})$ as elements of $\mathbb{R}^{d^2}$. Then $M_d(\mathbb{R})$ defines a non-abelian topological group.

Example. *Consider the set of upper triangle matrices*

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

equipped with the usual matrix multiplication. If $g, h \in \mathbb{H}_3(\mathbb{R})$ then, $gh \in \mathbb{H}_3(\mathbb{R})$. Furthermore, each element of $\mathbb{H}_3(\mathbb{R})$ is invertible (determinant equal to one). Thus, $\mathbb{H}_3(\mathbb{R})$ is a topological subgroup of $M_d(\mathbb{R})$, which is called the (continuous) Heisenberg group*. Since*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix},$$

it is easy to see that $\mathbb{H}_3(\mathbb{Z})$ (which is the set of matrices in $\mathbb{H}_3(\mathbb{R})$ where $a, b, c \in \mathbb{Z}$) is a discrete subgroup of $\mathbb{H}_3(\mathbb{R})$. Then $\mathbb{H}_3(\mathbb{Z})$ is called (discrete) Heisenberg group*. Both groups are not abelian but amenable (see definition later). We will discuss in the Exercise session a different representation of the Heisenberg group and we will prove there that $\mathbb{H}_3(\mathbb{Z})$ is amenable.*

Example. *Let $S := \{a, b, a^{-1}, b^{-1}\}$. Consider all finite words made out of the symbols of S. A word $w := s_1 s_2 \dots s_n$ is called reduced if $s_i \neq s_{i+1}^{-1}$ for all i (with the convention that $(s_{i+1}^{-1})^{-1} = s_{i+1}$). Two words are identified if there* *corresponding reduced words coincide, so e.g.* $abaa^{-1}b^{-1} = a$. *The empty word is denoted by* $\epsilon = aa^{-1} = bb^{-1}$. *Let* $\mathbb{F}_2 := \mathbb{F}_S$ *be the set of all reduced words over* S. *A group action on* $\mathbb{F}_2$ *is defined by*

$$(s_1 s_2 \dots s_n)(t_1 t_2 \dots t_m) := s_1 s_2 \dots s_n t_1 t_2 \dots t_m,$$
$$(s_1 s_2 \dots s_n)^{-1} := s_n^{-1} s_{n-1}^{-1} \dots s_1^{-1}.$$

Then $\mathbb{F}_2$ *is a countable topological group (discrete topology) that is not abelian nor amenable. This group is called the* free group *with two generators* a *and* b.

Definition. *A* topological dynamical system (X, G) *is a (topological) group* G, *a compact metric space* (X, d) *and a continuous* action α *of* G *on* X, *namely a continuous map*

$$\alpha : G \times X \to X$$

satisfying

- $\alpha(e, x) = x$ *for all* $x \in X$,
- $\alpha(gh, x) = \alpha\big(g, (\alpha(h, x)\big)$ *for all* $g, h \in G$ *and* $x \in X$.

In this case we say that the group G *acts* ($G \curvearrowright X$) *on the space* X.

Remark. *For each* $g \in G$, $\alpha(g, \cdot) : X \to X$ *is a homeomorphism (bijective and bicontinuous) with inverse* $\alpha(g^{-1}, \cdot) : X \to X$. *The case* $G = \mathbb{Z}^d$ *will play a crucial role for us.*

For the sake of simplification, we will always use the following short notation

$$gx := \alpha(g, x).$$

Example. *Let* $X := \mathbb{T} := \{x \in \mathbb{C} \,|\, |x| = 1\}$ *be equipped with the Euclidean metric* d *and* $z \in \mathbb{T}$. *Then* $(\mathbb{T}, \mathbb{Z})$ *defines a dynamical system via* $\alpha(n, x) := z^n x$.

Example. *Let* $S := \{a, b, a^{-1}, b^{-1}\}$ *be equipped with the discrete topology. Recall that* $u \in \mathbb{F}_2$ *is always a reduced word. Let* $\omega : \mathbb{N} \to S \in S^{\mathbb{N}}$ *be a one-sided infinite word*

$$\omega(1)\omega(2)\omega(3)\omega(4)\dots.$$

Then ω *is called* reduced *if* $\omega(n) \neq \omega(n+1)^{-1}$ *for all* $n \in \mathbb{N}$. *Every element in* $\omega \in S^{\mathbb{N}}$ *has a unique reduced representation. Define*

$$X := \big\{\omega : \mathbb{N} \to S \text{ reduced}\big\} \subseteq S^{\mathbb{N}} := \prod_{n \in \mathbb{N}} S$$

equipped with the product topology (i.e. the coarsest topology (topology with the fewest open sets) on $S^{\mathbb{N}}$ *such that the maps* $\pi_n : X \to S, \pi_n(\omega) := \omega(n)$, $n \in \mathbb{N}$ *are continuous). Since* S *is finite (compact),* $S^{\mathbb{N}}$ *is a compact metrizable space where a metric is given by*

$$d(\omega, \rho) := \min\left\{1, \inf\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N},\, \omega|_{\{1,\dots,n\}} = \rho|_{\{1,\dots,n\}}\right\}\right\}.$$

It is straight-forward to check that $X \subseteq S^{\mathbb{N}}$ *is a closed subset and so* X *is a compact metric space with the induced metric* d. *With this at hand,* $\mathbb{F}_2$ *acts on* X *by*

$$\alpha : \mathbb{F}_2 \times X \to X, \quad \alpha(u, \omega) := u\, \omega(1)\omega(2)\omega(3)\dots,$$

*where $\alpha(u,\omega)$ is the unique reduced word. Then $(X,\mathbb{F}_2)$ is a dynamical system (**Exercise**). For instance, if $u := abba^{-1}$ and $\omega = ab^{-1}aab\ldots$ then*

$$\alpha(u,\omega) = \big(abba^{-1}\big)\big(ab^{-1}aab\ldots\big) = ab\underbrace{ba^{-1}ab^{-1}}_{=\epsilon}aab\ldots = ab\ aab\ldots.$$

We remark that elements in X are nothing but than infinite strings on the Cayley graph of the free group, which are also called geodesics. Then the set of all geodesics (so X) is called the boundary of this Cayley graph. Thus, one says that the free group acts on its boundary. The reader is referred to the book A course in the theory of groups *by D. J. S. Robinson in Chapter 2.*

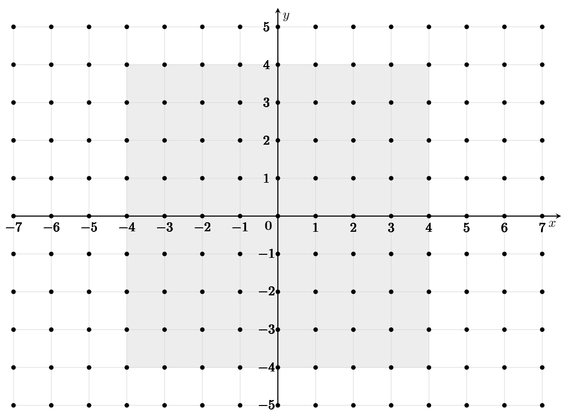

Figure 6. The cube Q_4 in gray in $\mathbb{Z}^2$

Example. *Let $\mathcal{A}$ be finite and G be a countable group (equipped with the discrete topology). Consider a sequence $K_n \subseteq G$, $n \in \mathbb{N}$, of finite sets satisfying*

- *$K_n \subsetneq K_{n+1}$ for all $n \in \mathbb{N}$ and*
- *$\bigcup_{n\in\mathbb{N}} K_n = G$.*

For instance, if $G = \mathbb{Z}^d$, we can set

$$K_n := Q_n := \left\{ g \in \mathbb{Z}^d \;\middle|\; \|g\|_\infty := \max_{1\le j\le d} |g_j| \le n \right\} = (-n-1, n+1)^d \cap \mathbb{Z}^d$$

the cubes of side length $2n+1$ around the origin, see Figure 6. Note that such a sequence (K_n) always exists as G is countable (why?). Consider the product space

$$\mathcal{A}^G := \prod_{g\in G} \mathcal{A} = \{w : G \to \mathcal{A}\}$$

equipped with the metric

$$d(\omega,\rho) := \min\left\{1, \inf\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \text{ be such that } \omega|_{K_n} = \rho|_{K_n} \right\}\right\}.$$

*Then $(\mathcal{A}^G, d)$ is a complete metric space (i.e. every Cauchy sequence admits a limits point in $\mathcal{A}^G$) and $\mathcal{A}^G$ is compact (**Exercise**). In particular, a sequence (ω_n) converges to ω if ω_n and ω coincide eventually on a large set* *around the origin (namely on a set K_n for n large enough), see e.g. Figure 7. The topology induced by d is nothing but than the product topology.*

Then

$$\alpha : G \times \mathcal{A}^G \to \mathcal{A}^G, \alpha(g,\omega)(h) := \omega(g^{-1}h),$$

*defines an action of G on $\mathcal{A}^G$ (**Exercise**), confer Picture 1. Such dynamical systems are called symbolic dynamical systems that we will study later separately in Chapter 3.*

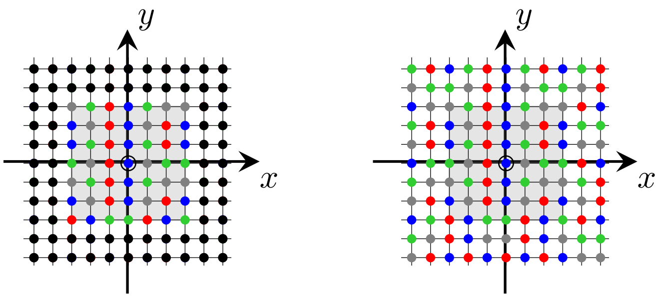

Figure 7. Two configurations in $\mathcal{A}^{\mathbb{Z}^2}$ where the black circle indicates the origin. They agree on the gray shaded box Q_3, namely $d(\omega,\rho) = \frac{1}{3}$ if $K_n := Q_n$. The larger this box the closer these elements are in the metric defined in the example.

Let us finish the section with a short discussion on so-called Haar measures.

Let X be a topological space. A measure μ on the Borel-σ-algebra of X is called *finite* if $\mu(X) < \infty$. Furthermore, μ is called *regular* if μ is *outer regular*, i.e.,

$$\mu(E) = \inf\{\mu(U) \mid U \text{ open with } E \subseteq U\}$$

and *inner regular*

$$\mu(E) = \sup\{\mu(K) \mid K \text{ compact with } K \subseteq E\}.$$

for all measurable $E \subseteq X$. If a measurable set $E \subseteq X$ satisfies the previous two identities, we say that E is *regular w.r.t.* μ. Furthermore a measure is called *locally finite* if for every $x \in X$, there is a neighborhood U of x such that $\mu(U) < \infty$. Then a measure μ on X is called a *Radon measure* if it is locally finite and regular.

For a topological group G, a measure μ on the Borel σ-algebra of G is called *(left) invariant (translation invariant)* if

$$\mu(gS) = \mu(S)$$

for all $g \in G$ and measurable $S \subseteq G$. Then Haar's theorem asserts that for every locally compact Hausdorff group there is, up to a positive multiplicative constant, a unique (nontrivial) Radon measure that is left-invariant and finite on compact subsets of G. This is the topic of a lecture in harmonic analysis.

You already know such a statement. In particular for $G = \mathbb{R}^d$ there is (up to a multiplicative constant) only one translation invariant measure:

the Lebesgue measure, see Analysis III. Also for $\mathbb{Z}^d$, the Haar measure is given by the counting measure $\lambda = \sum_{n\in\mathbb{Z}^d} \delta_n$. One actually can show that for all countable groups G the counting measure of G is a left-invariant Radon measure (**Exercise**). Based on this, we will restrict our further considerations on countable groups (equipped with the discrete topology). However, most of our considerations can be extended similarly to general locally compact Hausdorff groups using its Haar measure.

Week 1

2.1. The space of dynamical subsystems. Let (X, G) be a dynamical system, $g \in G$ and $Y \subseteq X$. We defined

$$gY := \{gy \,|\, y \in Y\}.$$

A subset $Y \subseteq X$ is called *invariant* if $gY \subseteq Y$ for each $g \in G$. It is immediate to see that an invariant satisfies $gY = Y$ for all $g \in G$. Examples of invariant subsets are *orbits*

$$Orb(x) := \{gx \,|\, g \in G\}$$

of an element $x \in X$.

If $Y \subseteq X$ is invariant, closed and non-empty, we can restrict the action of G onto Y. Specifically, (Y, G) itself defines a dynamical system. In light of this, define the set of all *dynamical subsystems* by

$$\mathcal{J} := \mathcal{J}(X, G) := \{Y \subseteq X \,|\, Y \text{ is closed, non-empty, invariant}\}.$$

Typical examples of elements of $\mathcal{J}$ are given by the orbit closures $\overline{Orb(x)}$. That $\overline{Orb(x)}$ is invariant follows by the continuity of the group action (**Exercise**). We call a dynamical system *topological transitive* if there exists an $x \in X$ such that $X = \overline{Orb(x)}$.

We will now show that this space can be equipped with a natural topology. Throughout this lecture we will analyze topological properties of this space $\mathcal{J}$ and show which spectral consequences they have for associated Schrödinger operators.

Since (X, d) is a compact metric space and each element on $\mathcal{J}$ is itself compact (as a closed subset of X), we have

$$\mathcal{J} \subseteq \mathcal{K}(X) := \{K \subseteq X \text{ compact}\}.$$

On $\mathcal{K}(X)$ we naturally consider the so-called Hausdorff metric (and its induced topology) induced by the metric d on X. In this way, $\mathcal{J}$ is naturally equipped with the subspace topology of $\mathcal{K}(X)$. For this purpose recall that the Hausdorff metric on $\mathcal{K}(X)$ is defined by

$$\delta_H(F, K) := \max\Big\{ \sup_{x\in F} \inf_{y\in K} d(x, y), \sup_{y\in K} \inf_{x\in F} d(x, y) \Big\},$$

and $(\mathcal{K}(X), \delta_H)$ is a compact metric space (**Exercise**).

Proposition 2.1 (Hausdorff topology). *Let (Z, d) be a locally compact metric space. The family of sets*

$$\mathcal{U}(F, \mathcal{O}) := \big\{K \in \mathcal{K}(Z) \,\big|\, K \cap F = \varnothing, K \cap O \neq \varnothing \text{ for all } O \in \mathcal{O}\big\},$$

where $F \subseteq Z$ is closed and $\mathcal{O}$ is a finite family of open subsets of Z, defines a base for the topology induced on $\mathcal{K}(Z)$ by the Hausdorff metric δ_H.

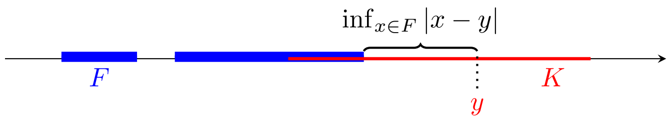

Figure 8. A sketch of the Hausdorff distance for $X = \mathbb{R}$ with $d(x,y) := |x-y|$.

Proof. $\mathcal{U}(F,\{Z\})$ is open: Let $F \subseteq Z$ be closed and $K_0 \in \mathcal{U}(F,\{Z\})$, namely $F \cap K_0 = \emptyset$. By compactness of K_0,

$$\varepsilon := \inf\{d(x,y) \,|\, x \in K_0, y \in F\} > 0.$$

Thus, if $K \in \mathcal{K}(Z)$ satisfies $\delta_H(K,K_0) < \varepsilon$ then $K \cap F = \emptyset$ follows and so $K \in \mathcal{U}(F,\{Z\})$.

$\mathcal{U}(\emptyset,\{O\})$ is open: Let $O \subseteq Z$ be open and $K_0 \in \mathcal{U}(\emptyset,\{O\})$. Then there exists an $x \in K_0 \cap O$. Since O is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ (open ball with radius ε around x) is contained in O. For every $K \in \mathcal{K}(Z)$ with $\delta_H(K,K_0) < \varepsilon$, we conclude that there is an $y \in K$ such that $d(x,y) < \varepsilon$ as $x \in K_0$. Thus, $K \cap O \neq \emptyset$ implying $K \in \mathcal{U}(\emptyset,\{O\})$.

Since each set $\mathcal{U}(F,\mathcal{O})$ is a finite intersection of sets of the form $\mathcal{U}(F,\{Z\})$ and $\mathcal{U}(\emptyset,\{O\})$, we deduce that $\mathcal{U}(F,\mathcal{O}) \subseteq \mathcal{K}(Z)$ is an open set in the topology induced by δ_H. It is left to show that every ball of radius ε around $K_0 \in \mathcal{K}(Z)$ in the Hausdorff metric contains such a set $\mathcal{U}(F,\mathcal{O})$ with $K_0 \in \mathcal{U}(F,\mathcal{O})$.

Let $K_0 \in \mathcal{K}(Z)$ and $\varepsilon > 0$. Let

$$F := \{x \in Z \,|\, \mathrm{dist}(x,K_0) \geq \varepsilon\}$$

where $\mathrm{dist}(x,K_0) := \inf_{y\in K_0} d(x,y)$. By definition $F \subseteq Z$ is closed and $F \cap K_0 = \emptyset$. By compactness of K_0 there are $x_1,\ldots,x_n \in K_0$ such that

$$K_0 \subseteq \bigcup_{j=1}^{n} O_j \qquad \text{where } O_j := B_{\frac{\varepsilon}{2}}(x_j).$$

It is immediate by construction that $K_0 \in \mathcal{U}(F,\{O_1,\ldots,O_n\})$ and if $K \in \mathcal{U}(F,\{O_1,\ldots,O_n\})$, then $\delta_H(K,K_0) < \varepsilon$. □

Remark. *(a) We point out that the topology defined on $\mathcal{K}(Z)$ does not need that Z is a metric space. The topology induced by the open set $\mathcal{U}(F,\mathcal{O})$, where $F \subseteq Z$ is closed and $\mathcal{O}$ is a finite family of open subsets of Z, is called* Vietoris topology.

(b) There is another topology that plays a crucial role in C^*-algebras, which is the so called Fell-topology *or* Chabauty-Fell topology *which is defined on closed (or compact) subsets of a topological space Z and a base for the Chabauty-Fell topology is given by $\mathcal{U}(K,\mathcal{O})$ where $K \subseteq Z$ is compact and $\mathcal{O}$ is a finite family of open subsets of Z. Clearly, the Chabauty-Fell topology and the Vietoris topology coincide if Z is compact but in general they are different (**Exercise**). Such topologies are also called hit-and-miss topologies.*

(c) If Z is compact, then $\mathcal{K}(Z)$ is compact in the Vietoris topology.

(d) The cases where $Z := \mathbb{R}$ or $Z := X$ (a compact space on which some group G acts) play a crucial role in this lecture. In order to distinguish the metric and avoiding to heavy notation we will use d_H for the Hausdorff metric on $\mathcal{K}(\mathbb{R})$ induced by the Euclidean metric and δ_H for the Hausdorff metric on $\mathcal{K}(X)$ induced by the metric d on X.

PROPOSITION 2.2. *The set $(\mathcal{J}, \delta_H)$ is a compact metric space.*

PROOF. Since X is compact, the space $\mathcal{K}(X)$ of all compact subsets of X is a compact metric space if equipped with the Vietoris topology respectively the Hausdorff metric δ_H (**Exercise**). Thus, it suffices to show that $\mathcal{J} \subseteq \mathcal{K}(X)$ is a closed subset.

Let $Y_n \in \mathcal{J}$ be convergent in the Hausdorff metric to a compact $Y \in \mathcal{K}(X)$. Clearly, the empty set is isolated in $\mathcal{K}(X)$ and so $Y \neq \emptyset$ follows as $Y_n \neq \emptyset$. It is left to show that Y is invariant.

Assume by contradiction, that Y is not invariant. Thus, there is a $y \in Y$ and $g \in G$ such that $gy \notin Y$. Since the group action is continuous and open, there exist open subsets U of X such that

$$y \in U, \quad gy \in gU \quad \text{and} \quad Y \cap \overline{gU} = \emptyset.$$

Consequently, $\mathcal{U}(\overline{gU}, \{U\})$ is a (Vietoris) open neighborhood of $Y \in \mathcal{K}(X)$ invoking the previous Proposition 2.1. Hence, there exists an $n \in \mathbb{N}$ such that $Y_n \in \mathcal{U}(\overline{gU}, \{U\})$, i.e. , $U \cap Y_n \neq \emptyset$ and $\overline{gU} \cap Y_n = \emptyset$. Thus, there is an $x \in U \cap Y_n$ implying $gx \in gU \cap Y_n$ as $Y_n \in \mathcal{J}$ is invariant, a contradiction. □

We continue studying dynamical properties of X. First we will consider the case when $\mathcal{J}$ is trivial, namely $\mathcal{J} := \{X\}$. The measure theoretic analog for dynamical systems are ergodic measures (see discussion below). Secondly, we will study invariant probability measures on X and the case when this space is trivial, namely that X admits only one invariant probability measure. Finally, we will compare these properties in this chapter.

2.2. Minimality. The concept of a minimal dynamical system is introduced.

DEFINITION. *A topological dynamical system (X, G) is called* minimal *if for each $x \in X$, the orbit $Orb(x) \subseteq X$ is dense in X (i.e. $\overline{Orb(x)} = X$ for all $x \in X$).*

In light of the previous definition $Y \in \mathcal{J}(X, G)$ is called *minimal*, if the corresponding dynamical system (Y, G) is minimal. Before showing the existence of minimal systems, we will provide a characterization.

PROPOSITION 2.3 (Characterization of Minimality). *Let (X, G) be a topological dynamical system. Then the following assertions are equivalent.*

(i) *(X, G) is minimal.*
(ii) *If $A \subseteq X$ is closed and invariant, then $A = X$ or $A = \emptyset$.*
(iii) *If $\emptyset \neq U \subseteq X$ is open, then $X = \bigcup_{g \in G} gU$.*

PROOF. (i) $\Longrightarrow$ (ii): Let $A \subseteq X$ be closed and invariant with $A \neq \emptyset$. We will show that $A = X$.

Since $A \neq \emptyset$, there is an $x \in A$. Then $Orb(x) \subseteq A$ since A is invariant. Thus,

$$X \overset{(i)}{=} \overline{Orb(x)} \subseteq \overline{A} = A \subseteq X$$

follows as A is closed. Hence, $X = A$ follows.

(ii) $\Longrightarrow$ (iii): Define

$$A := X \setminus \left(\bigcup_{g \in G} gU \right).$$

Then A is closed and invariant (as $\bigcup_{g \in G} gU$ is open and invariant). Thus, (ii) implies $A = \emptyset$ or $A = X$. Since $U \neq \emptyset$, the equality $A = X$ is absurd. Hence, $A = \emptyset$ follows proving (iii).

(iii) $\Longrightarrow$ (i): Assuming (iii), we have to show that the orbit for every $x \in X$ is dense. Therefore, let $U \subseteq X$ be open such that $U \neq \emptyset$. We have to show that there is an $h \in G$ such that $hx \in U$. Due to (iii), we have $x \in gU$ for some $g \in G$. Thus $hx \in U$ holds for $h = g^{-1}$. □

Remark. *(a) Terminology minimal: an order (see a reminder below) on the subsystems of (X, G) is given by inclusion. Clearly, a subsystem is minimal w.r.t. this order if and only if it is a minimal topological dynamical system.*

*(b) Given two minimal subsystems Y, Z of (X, G). Then either $Y \cap Z = \emptyset$ or $Y = Z$ holds **(Exercise)**.*

(c) We will see later another characterization of minimality in terms of the spectrum of associated Schrödinger operators.

Next we come to a structure result that every dynamical system has a minimal subsystems. In order to do so, we need Zorn's lemma. For the sake of completeness we recall it here.

Recall that a relation on a set A is a subset of $A \times A$. A relation $\leq$ on a set A satisfying

- for each $a \in A$, we have $a \leq a$, **(reflexiv)**
- if $a \leq b$ and $b \leq c$, then $a \leq c$, **(transitiv)**
- if $a \leq b$ and $b \leq a$, then $a = b$, **(antisymmetry)**

is called a *order*. Then the tuple $(A, \leq)$ is called an *ordered set*. Furthermore, $(A, \leq)$ is called *totally ordered* or *linearly ordered* if $a \leq b$ or $b \leq a$ holds for all pairs $a, b \in A$. Let $B \subseteq A$ and A be an ordered set. Then $a \in A$ is called *upper bound of B* if $b \leq a$ for all $b \in B$. Moreover, $b \in B$ is called *maximal element of B*, if $a \in B$ and $b \leq a$ imply $b = a$.

Lemma 2.4 (Zorn's lemma). *Consider an ordered set A such that each totally ordered subset $B \subseteq A$ has an upper bound in A, then A has at least one maximal element.*

Theorem 2.5 (Existence of Minimal Dynamical Systems). *Each topological dynamical system (X, G) has at least one minimal dynamical subsystem.*

Proof. Denote by $\mathcal{J}$ the set of all closed, non-empty, invariant subsets of X. The set $\mathcal{J}$ is not empty, since it contains X. We induce a partial ordering on $\mathcal{J}$ via $Y \geq Z :\Leftrightarrow Y \subseteq Z$. Let $(Y_i)_{i \in I}$ be a totally ordered subset in $\mathcal{J}$. We will show that $Y' := \bigcap_{i \in I} Y_i$ is non-empty, compact and invariant.

Every Y_i is closed. Hence, the intersection Y' is closed and, obviously, a subset of the compact space X. Thus, Y' is compact. It is left to show $Y' \neq \emptyset$. Assume by contradiction that $Y' = \emptyset$. Then, we have

$$X = X \setminus \bigcap_{i \in I} Y_i = \bigcup_{i \in I} X \setminus Y_i.$$

As every $X \setminus Y_i$ is open, they form an open cover of X. Thus, there is a finite subcover $X \setminus Y_{i_1}, \dots, X \setminus Y_{i_n}$. Without loss of generality we can assume $Y_{i_1} \subseteq \dots \subseteq Y_{i_n}$ (**Exercise**). Thus, we have $X = \bigcup_{k=1}^n X \setminus Y_{i_k}$ and, therefore,

$$\emptyset = X \setminus \left(\bigcup_{k=1}^n X \setminus Y_{i_k} \right) = \bigcap_{k=1}^n Y_{i_k}.$$

As the i_j's are ordered by size, and since (Y_i) is totally ordered, we have

$$\bigcap_{k=1}^n Y_{i_k} = Y_{i_1} \neq \emptyset.$$

This is a contradiction and so we have proven that Y' is compact and non-empty.

Moreover, we have

$$gY' = \bigcap_{i \in I} gY_i \subseteq \bigcap_{i \in I} Y_i$$

for all $g \in G$. Hence, the set Y' is invariant, and, thus an element of $\mathcal{J}$. Moreover, Y' is clearly an upper bound for $(Y_i)_{i \in I}$, namely $Y' \subseteq Y_i$ for all $i \in I$. Thus, we can apply Zorn's lemma and infer the existence of a maximal element Z in $\mathcal{J}$. We have to show that (Z, G) is minimal. Assume the opposite, i.e., there is a non-trivial, closed, invariant set $Y \subseteq Z$. Then, however, Y is an element of $\mathcal{J}$ with $Y \neq Z$ and $Y \geq Z$. This contradicts the maximality of Z. Hence, the theorem is proven. □

2.3. Linear bounded maps. In this section we will learn some basics in functional analysis. We refer the interested reader to the course Functional analysis that is currently given by Prof. M. Keller.

A *normed space* E is a vector space E over $\mathbb{C}$ equipped with a map $\|\cdot\| : E \to [0, \infty)$ satisfying

- $\|x\| = 0$ if and only if $x = 0$,
- $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in E$,
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

If E is a normed space then $d(x, y) := \|x - y\|$ defines a metric on E. A normed space is called a *Banach space* if (E, d) is complete as a metric space (every Cauchy sequence converges to an element in E). Note that by the conditions of a norm, the vector space actions are continuous (multiplication by constants, addition). If $(E, \|\cdot\|)$ is a normed space, then $\big| \|x\| - \|y\| \big| \leq \|x - y\|$ holds for all $x, y \in E$. This follows by the triangle inequality:

$$\|x\| \leq \|x - y\| + \|y\| \quad \text{and} \quad \|y\| \leq \|x - y\| + \|x\|.$$

Example. *Let X be a locally compact space and $C_0(X)$ be the set of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity. In particular, for* *each $\varepsilon > 0$, there is a compact $K \subseteq X$ such that $|f|_{X\setminus K}| \leq \varepsilon$. Then $C_0(X)$ gets a Banach space if equipped with the uniform norm $\|\cdot\|_\infty$ defined by*

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

Proposition 2.6. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{C}$ and $T : E \to F$ be linear (i.e. $T(\lambda x + y) = \lambda T(x) + T(y)$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$). Then the following assertions are equivalent.*

(i) *T is continuous,*
(ii) *T is continuous at 0,*
(iii) *there is an $C > 0$ such that $\|T(x)\| \leq C\|x\|$ holds for all $x \in E$,*
(iv) *T is uniformly continuous.*

Proof. This is an Exercise (Sheet 2). □

We define the set of all linear and bounded operators (maps) from a Banach space E to a Banach space F by

$$\mathcal{L}(E, F) := \left\{ T : E \to F \text{ linear, continuous } \right\}.$$

A vector space structure on $\mathcal{L}(E, F)$ over $\mathbb{C}$ is given by

$$(S + T)(x) := S(x) + T(x)\,, \qquad (\lambda T)(x) := \lambda T(x)$$

for $\lambda \in \mathbb{C}$, $x \in E$ and $S, T \in \mathcal{L}(E, F)$. If $F = E$, we use the notation $\mathcal{L}(E) := \mathcal{L}(E, E)$. Clearly, the identity operator $\mathrm{I}(x) := x$ is an element of $\mathcal{L}(E)$.

Proposition 2.7. *The map $\|\cdot\| : \mathcal{L}(E, F) \to [0, \infty)$ defined by*

$$\|T\| := \sup_{\|x\|_E \leq 1} \|Tx\|_F = \sup_{x\neq 0} \frac{\|Tx\|_F}{\|x\|_E}$$

defines a norm satisfying $\|Tx\|_F \leq \|T\|\|x\|_E$ for all $x \in E$. Furthermore, $(\mathcal{L}(X, Y), \|\cdot\|)$ defines a Banach space.

Proof. First note that the identity written in the definition of $\|T\|$ is obvious by the definition of a norm and since T is linear.

Let $\lambda \in \mathbb{C}$. Then

$$\|\lambda T\| = \sup_{\|x\|\leq 1} \|\lambda Tx\|_F = |\lambda| \sup_{\|x\|\leq 1} \|Tx\|_F = |\lambda|\ \|T\|$$

holds. In addition, $\|T\| = 0$ yields for all $x \in E \setminus \{0\}$ that $\frac{\|Tx\|_F}{\|x\|_E} = 0$ and so $\|Tx\|_F = 0$. Since $\|\cdot\|_F$ is a norm, we derive $Tx = 0$ and so $T = 0$ as $x \in E \setminus \{0\}$ was arbitrary. Furthermore, for $x \in E$ with $\|x\|_E \leq 1$, we get

$$\|(S + T)(x)\|_F = \|S(x) + T(x)\|_F \leq \|S(x)\|_F + \|T(x)\|_F \leq \|S\| + \|T\|.$$

Taking the supremum over all $x \in E$ with $\|x\|_E \leq 1$, we infer $\|S + T\| \leq \|S\| + \|T\|$.

Let $x \in E \setminus \{0\}$. Then

$$\|Tx\|_F = \frac{\|Tx\|_F}{\|x\|_E}\|x\|_E \leq \|T\|\|x\|_E$$

holds.

Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(E,F)$ be a Cauchy sequence in $\mathcal{L}(E,F)$, i.e. for all $\varepsilon > 0$, there is an $N_\varepsilon \in \mathbb{N}$ such that $\|T_n - T_m\| < \varepsilon$ if $n,m \geq N_\varepsilon$. For $x \in E$, $(T_n x)_{n\in\mathbb{N}} \subseteq F$ is also a Cauchy sequence in F as

$$\|T_n x - T_m x\|_F \leq \|T_n - T_m\| \|x\|_E.$$

Define $Tx := \lim_{n\to\infty} T_n x$ which converges as $(F, \|\cdot\|_F)$ is a Banach space. Clearly, the so defined map $T : E \to F$ is linear since

$$T(\lambda x + y) = \lim_{n\to\infty} T_n(\lambda x + y) = \lim_{n\to\infty} \lambda T_n(x) + T_n(y) = \lambda T(x) + T(y)$$

holds for all $x, y \in E$ and $\lambda \in \mathbb{C}$.

Next we show that T is continuous and $\|T_n - T\| \to 0$. Let $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ be given as above such that

$$\|T_n - T_m\| < \varepsilon, \qquad n, m \geq N_\varepsilon.$$

Let $x \in E$ with $\|x\|_E \leq 1$ and choose $m_0 = m_0(x,\varepsilon) \in \mathbb{N}$ such that $m_0 \geq N_\varepsilon$ and $\|T_m x - Tx\| \leq \varepsilon$ for $m \geq m_0$. Thus, for $n \geq N_\varepsilon$, we derive

$$\|T_n x - Tx\|_F \leq \|T_n x - T_{m_0} x\|_F + \|T_{m_0} x - Tx\|_F \leq \|T_n - T_{m_0}\| + \varepsilon \leq 2\varepsilon.$$

Hence, $\|T_n - T\| \leq 2\varepsilon$ is derived for all $n \geq N_\varepsilon$ by taking the supremum over $\|x\|_E \leq 1$. This shows $\|T_n - T\| \to 0$.

Fix $\varepsilon > 0$ and $n \geq N_\varepsilon$. Then

$$\|T\| \leq \|T - T_n\| + \|T_n\| \leq 2\varepsilon + \|T_n\| =: C(n,\varepsilon) < \infty$$

follows and so $\|Tx\|_F \leq C(n,\varepsilon)\|x\|_E$ holds. Thus, Proposition 2.6 yields that T is continuous, namely, $T \in \mathcal{L}(E,F)$. □

The norm on $\mathcal{L}(E,F)$ is called *operator norm*.

EXERCISE. *Let $T \in \mathcal{L}(E,F)$ be continuous, then*

$$\|T\| = \inf\left\{M > 0 \,\middle|\, \|Tx\|_F \leq M\|x\|_E \text{ holds for all } x \in E\right\}$$

holds. (simple exercise with supremum and infimum)

PROPOSITION 2.8. *Let E, F, G be Banach spaces over $\mathbb{C}$. Then the following statements hold.*

(a) *If $T \in \mathcal{L}(E,F)$ and $S \in \mathcal{L}(F,G)$, then $\|S \circ T\| \leq \|S\|\|T\|$ and $S \circ T \in \mathcal{L}(E,G)$.*
(b) *If $T \in \mathcal{L}(E,F)$, then*

$$\ker(T) := \{x \in E \,|\, T(x) = 0\}$$

is a closed subspace of E and

$$\operatorname{ran}(T) := \{T(x) \,|\, x \in E\} \subseteq F$$

is a subspace of F.

PROOF. (a) Clearly, $S \circ T \in \mathcal{L}(E,G)$ is a linear map. Since

$$\|S \circ T(x)\|_G = \|S(T(x))\|_G \leq \|S\|\|T(x)\|_F \leq \|S\|\|T\|\|x\|_E, \qquad x \in E,$$

holds by Proposition 2.7, Proposition 2.6 yields that $S \circ T$ is continuous. In addition, the latter estimate gives $\|S \circ T\| \leq \|S\|\|T\|$ by taking the supremum over all $x \in E$ with $\|x\|_E \leq 1$.

(b) Since (F, d_F) with $d_F(x,y) := \|x-y\|_F$ defines a metric space, we have $\{0\} \subset F$ is a closed subset (finite sets in a Hausdorff space are closed). Then $\ker(T) = T^{-1}(\{0\})$ (preimage) holds. Since T is continuous, the preimage of closed sets are closed. Hence, $\ker(T) \subseteq E$ is closed.

If $x, y \in \ker(T)$ and $\lambda \in \mathbb{C}$, then $T(\lambda x + y) = \lambda Tx + Ty = 0$ follows. Thus, $\lambda x + y \in \ker(T)$ if $x, y \in \ker(T)$ is derived implying that $\ker(T) \subseteq E$ is a closed subspace.

Let $x, y \in E$ and $\lambda \in \mathbb{C}$. Then

$$T(x + \lambda y) = T(x) + \lambda T(y)$$

implies that $\operatorname{ran}(T)$ is a subspace of F. □

Let $(E, \|\cdot\|)$ be a normed space that $E' := \mathcal{L}(E, \mathbb{C})$ equipped with the previous normed structure is called the *dual space of* E and elements of E' are called *(linear) functionals*.

2.4. The space of probability measures. In this section, the space of invariant probability measures over a dynamical system (X, G) is introduced. Moreover, we will show that the space of invariant probability measures is non-empty under suitable assumptions on the group

Remark. *current research questions: convergence of dynamical systems - convergence of associated measures*

For a topological space X, we denote by $\mathscr{B} := \mathscr{B}(X)$ the corresponding Borel σ-algebra. We define

$$\mathcal{M}(X) := \{\mu \text{ regular Borel measure}\}$$

the set of all regular Borel measures on X. We seek to topologize this set. In order to do so and for later purposes, Urysohn's lemma will be crucial, confer Lemma A.7. We will use the following notation. We write

$$K \prec f$$

if $K \subseteq X$ is compact, $f \in C_c(X)$ such that $0 \le f \le 1$ and $f(x) = 1$ for all $x \in K$. Furthermore, we write

$$f \prec V$$

if $V \subseteq X$ is open, $f \in C_c(X)$ be such that $0 \le f \le 1$ and *supp*$(f) \subseteq \overline{V}$ (i.e., $f(x) = 0$ if $x \notin V$).

Proposition 2.9 (Separation of measures by continuous functions). *Let X be a compact metric space and $\mu, \nu \in \mathcal{M}(X)$. If*

$$\int_X f \, d\mu = \int_X f \, d\nu$$

for all $f \in C(X)$, then $\mu = \nu$.

Proof. Since μ and ν are regular, it suffices to show $\mu(K) = \nu(K)$ for all compact $K \subseteq X$. Let $\varepsilon > 0$. Since ν is outer-regular, there exists an open set $U_\varepsilon \supseteq K$ such that $\nu(U_\varepsilon) \le \nu(K) + \varepsilon$. By Urysohn's lemma, there is an $K \prec f \prec U_\varepsilon$. Thus,

$$\mu(K) \overset{f(x) \ge 1_K}{\le} \int_X f \, d\mu = \int_X f \, d\nu \overset{f(x) \le 1_{U_\varepsilon}}{\le} \nu(U_\varepsilon) \le \nu(K) + \varepsilon.$$

Similarly, one shows $\nu(K) \leq \mu(K) + \varepsilon$ proving $\mu(K) = \nu(K)$ as $\varepsilon > 0$ was arbitrary. □

Week 2

With this at hand, we can define a topology on $\mathcal{M}(X)$. We will do so by saying when a sequence of measures converge. A sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}(X)$ converges to $\mu \in \mathcal{M}(X)$ if

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu$$

holds for all $f \in C_c(X)$. In this case, we write $\mu_n \rightharpoonup \mu$. Note that $C_c(X) = C(X)$ which will be used in the following.

A basis for this topology is given by

$$U\big(\{f_k\}_{k=1}^n, \{z_k\}_{k=1}^n, \varepsilon\big) := \left\{\nu \in \mathcal{M}(X) \,\middle|\, \left|\int_X f_k \, d\nu - z_k\right| < \varepsilon \text{ for all } 1 \leq k \leq n\right\}$$

where $f_1, \dots, f_n \in C(X)$, $z_1, \dots, z_n \in \mathbb{C}$ and $\varepsilon > 0$. Then a neighborhood basis of $\mu \in \mathcal{M}(X)$ is given by setting $z_k := \mu(f_k)$ for all $1 \leq k \leq n$. We leave it as an exercise to show that theses sets define a base for a topology on $\mathcal{M}(X)$.

This topology on $\mathcal{M}(X)$ is called the *vague topology* (which is the same as the weak-$*$ topology on $C(X)' := \mathcal{L}(C(X), \mathbb{C})$). The previous proposition asserts that the vague topology is Hausdorff. As we will see later, $C(X)$ admits a countable dense subset and so the vague topology is second countable.

We will be mostly interested in probability measures (or finite measures). A Borel measure is called *probability measure on* X if $\mu(X) = 1$. The set of all probability Borel measures is denote by

$$\mathcal{M}^1(X) := \big\{\mu \text{ probability (Borel) measure}\big\}.$$

We will show in the following that $\mathcal{M}^1(X) \subseteq \mathcal{M}(X)$. Thus, we can endow $\mathcal{M}^1(X)$ with the induced topology of $\mathcal{M}(X)$.

Proposition 2.10. *Let X be a compact metric space. If μ is a finite measure on the Borel-σ-algebra $\mathscr{B}$ of X, then μ is regular.*

Proof. Define

$$\mathscr{A} := \{A \in \mathscr{B} \text{ regular w.r.t. } \mu\}$$

We have to prove that $\mathscr{A} = \mathscr{B}$.

Step 1: ($\mathscr{A}$ is a σ-algebra) We have to prove that

- $X \in \mathscr{A}$;
- if $A \in \mathscr{A}$, then its complement is also in $\mathscr{A}$, namely $A^c \in \mathscr{A}$;
- if $A_i \in \mathscr{A}$ for $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} A_i \in \mathscr{A}$.

This is left as an exercise (Sheet 3).

Step 2: ($\mathscr{A} = \mathscr{B}$) Due to Step 1 and since $\mathscr{A} \subseteq \mathscr{B}$, it suffices to show that $\mathscr{A}$ contains all compact subsets of X as those generate the Borel σ-algebra. Let $K \subseteq X$ be compact. For indeed we have

$$\mu(K) = \sup\{\mu(F) \,|\, F \subseteq K, F \text{ compact}\}$$

by monotonicity of μ. For $n \in \mathbb{N}$, there is an $N_n \in \mathbb{N}$ and $x_1, \dots, x_{N_n} \in K$ such that

$$K \subseteq V_n := \bigcup_{j=1}^{N_n} B_{\frac{1}{n}}(x_j),$$

where $B_r(x)$ denotes the open ball. Set $U_n := \bigcap_{k=1}^{n} V_k$ which is open and $U_n \supseteq U_{n+1}$. Furthermore, it is straightforward to check that $K = \bigcap_{n\in\mathbb{N}} U_n$.

We know from standard measure theory arguments (see Appendix B) that if $A_j \in \mathcal{A}$ with $A_1 \supset A_2 \supset \dots$ and $\mu(A_1) < \infty$, then $\mu(\bigcap_{j\in\mathbb{N}} A_j) = \lim_{n\to\infty} \mu(A_n)$ holds. Thus,

$$\mu(K) = \mu\left(\bigcap_{n\in\mathbb{N}} U_n\right) = \lim_{n\to\infty} \mu(U_n)$$

as $\mu(U_1) \le \mu(X) < \infty$. Hence,

$$\mu(K) = \lim_{n\to\infty} \mu(U_n) \ge \inf\left\{\mu(U) \,\middle|\, K \subseteq U, U \text{ open}\right\} \ge \mu(K)$$

where the monotonoicity was used in the last step. Consequently, equality must hold in the last step proving that K is regular. ☐

Based on the Riesz-Markov representation theorem we can show that $\mathcal{M}^1(X)$ is compact. This is actually a consequence of a much more general statement, the Banach-Alaoglu theorem. The proof, which we will present here, transfers to the Banach-Alaoglu theorem for separable Banach spaces. In this case, we are not relying on Zorn's lemma and so not on the Axiom of choice. In order to do so, we first prove that $C(X)$ is separable if X is a compact metric space. Therefore, the Stone-Weierstrass theorem is used. This will also imply that $\mathcal{M}^1(X)$ is second countable.

A topological space Y is called *separable* if there is a countable dense subset in Y.

LEMMA 2.11. *Let X be a second countable space with countable basis $\{V_n\}_{n\in\mathbb{N}}$. Choose any $x_n \in V_n$ for each $n \in \mathbb{N}$. Then $Y := \{x_n \,|\, n \in \mathbb{N}\} \subseteq X$ is countable and dense.*

PROOF. Let $x \in X$. Then $\{V_n \,|\, x \in V_n\}$ defines a neighborhood basis of x. Thus, for each neighborhood U of x there is an $n_U \in \mathbb{N}$ such that $x \in V_{n_U} \subseteq U$. Since $x_{n_U} \in V_{n_U} \subseteq U$, we derive that in every neighborhood of x there is an element of Y. Thus, $x \in \overline{Y}$ implying $\overline{Y} = X$. ☐

With this at hand and the Stone-Weierstrass theorem we can prove the following.

THEOREM 2.12. *If (X, d) is a compact metric space then $C(X)$ is separable.*

PROOF. Since X is a compact metric space, it admits a countable base for the topology (balls with radius $\frac{1}{n}$). By the previous Lemma 2.11, there is a countable dense subset $Y \subseteq X$. Define

$$E := \left\{f \in C(X) \,\middle|\, f \text{ is a finite product of functions } d(\cdot, y)\,,\, y \in Y\right\} \cup \{1\}.$$

Denote by $alg(E)$ the algebra generated by E (i.e. all linear combinations of elements of E and products of such elements). We will first show that $alg(E)$ is a dense subalgebra of $C(X)$ by using the Stone-Weierstrass theorem. Then we will show that $alg(E)$ contains itself a countable dense subset $\tilde{E}$ implying that $C(X)$ is separable.

$alg(E) \subseteq C(X)$ dense: Clearly $alg(E)$ is a conjugation invariant subalgebra of $C(X)$ containing the constants. Furthermore, $alg(E)$ separates the points: For indeed, let $x, y \in X$ be different. Let $x' \in Y$ be such that $0 < d(x, x') < \frac{1}{3} d(x, y)$. Then $d(x', y) \geq \frac{2}{3} d(x, y)$ since otherwise

$$d(x, y) \leq d(x, x') + d(x', y) < \frac{1}{3} d(x, y) + \frac{2}{3} d(x, y) = d(x, y),$$

a contradiction. Let $f(z) := d(z, x') \in E$. Then $0 < f(x) < \frac{1}{3} d(x, y)$ while $f(y) > \frac{2}{3} d(x, y)$, namely $f(x) \neq f(y)$. Thus, the Stone-Weierstrass theorem implies that $alg(E) \subseteq C(X)$ is dense.

$alg(E)$ admits a countable dense subset: First note that E is countable as the countable union of countable sets. With this define the countable subset

$$\tilde{E} := \left\{ \sum_{k=1}^{N} (r_k + i s_k) g_k \,\middle|\, g_k \in E\,,\ r_k, s_k \in \mathbb{Q} \right\} \subseteq alg(E).$$

Note that if $f, g \in E$ then $f \cdot g \in E$ by definition. Recall that $alg(E)$ is nothing but the linear span of E. Hence, $\tilde{E}$ is dense in $alg(E)$ as

$$\left\{ r + is \,\middle|\, r, s \in \mathbb{Q} \right\}$$

is dense in $\mathbb{C}$.

Summing up, $alg(E)$ is dense in $C(X)$ and $\tilde{E}$ is countable and dense in $alg(E)$. Thus, $C(X)$ has a countable dense subset, namely it is separable. □

Corollary 2.13. *If X is a compact metric space then $\mathcal{M}^1(X)$ equipped with the vague topology is second countable.*

Proof. Let $\tilde{E} \subseteq C(X)$ be a dense and countable subset that exists by the previous Theorem 2.12. Recall that a basis on $\mathcal{M}^1(X)$ for the vague topology is given by

$$U\big(\{f_k\}, \{z_k\}, \varepsilon\big) := \left\{ \nu \in \mathcal{M}^1(X) \,\middle|\, \left| \int_X f_k \, d\nu - z_k \right| < \varepsilon \text{ for all } 1 \leq k \leq n \right\}$$

where $f_1, \ldots, f_n \in C(X)$, $\varepsilon > 0$ and $z_1, \ldots, z_n \in \mathbb{C}$. Since $\mu \in \mathcal{M}^1(X)$ satisfies $\mu(X) = 1$, we conclude

$$\left| \int_X f \, d\mu - \int_X g \, d\mu \right| \leq \| f - g \|_\infty .$$

Thus, the family

$$\left\{ U\big(\{f_k\}, \{q_k\}, 1/n\big) \,\middle|\, \begin{array}{c} n \in \mathbb{N},\ f_k \in \tilde{E} \text{ and } q_k = r_k + i s_k \text{ with} \\ r_k, s_k \in \mathbb{Q} \text{ for all } 1 \leq k \leq n \end{array} \right\}$$

defines a countable basis of the topology of $\mathcal{M}^1(X)$. □

Theorem 2.14 (Compactness of $\mathcal{M}^1(X)$). *Let X be a compact metric space. Then $\mathcal{M}^1(X)$ is sequentially compact in the vague topology and $\mathcal{M}^1(X) \subseteq \mathcal{M}(X)$ is a closed subset.*

Proof. Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}^1(X)$ be a sequence. Then

$$\varphi_n(f) := \int_X f\, d\mu_n$$

defines a sequence of linear functionals in $C(X)'$. We will first show that the functionals $(\varphi_n)_{n\in\mathbb{N}}$ admit a convergent subsequence and its limit point (of this subsequence) is denoted by φ. Then, we will use the Riesz-Markov representation theorem to deduce that the corresponding subsequence of $(\mu_n)_{n\in\mathbb{N}}$ converges to a measure $\mu \in \mathcal{M}^1(X)$ satisfying $\varphi(f) = \int_X f\, d\mu$.

Step 1: Due to the previous theorem, there are $(f_k)_{k\in\mathbb{N}} \subseteq C(X)$ which are dense in $C(X)$. We will make a Cantor diagonalization argument, which is only sketched here. The sequence $\left(\int_X f_1\, d\mu_n\right)_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers as the μ_n's are probability measures and $\|f_1\|_\infty < \infty$. Hence, there is a subsequence $(\mu_n^{(1)})_{n\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that $\left(\int_X f_1\, d\mu_n^{(1)}\right)_{n\in\mathbb{N}}$ converges.

Now the sequence $\left(\int_X f_2\, d\mu_n^{(1)}\right)_{n\in\mathbb{N}}$ is also a bounded sequence of complex measures and thus $(\mu_n^{(1)})_{n\in\mathbb{N}}$ admits a subsequence $(\mu_n^{(2)})_{n\in\mathbb{N}}$ such that $\left(\int_X f_2\, d\mu_n^{(2)}\right)_{n\in\mathbb{N}}$ converges. It is worth mentioning that $\left(\int_X f_1\, d\mu_n^{(2)}\right)_{n\in\mathbb{N}}$ still converge since $(\mu_n^{(2)})_{n\in\mathbb{N}}$ is a subsequence of $(\mu_n^{(1)})_{n\in\mathbb{N}}$.

We continue in this manner to get for each $i \in \mathbb{N}$ a subsequence $(\mu_n^{(i)})_{n\in\mathbb{N}}$ such that $\left(\int_X f_j\, d\mu_n^{(i)}\right)_{n\in\mathbb{N}}$ is convergent for all $1 \le j \le i$. Consider the diagonal sequence $(\mu_n^{(n)})_{n\in\mathbb{N}}$, then $\left(\int_X f_j\, d\mu_n^{(n)}\right)_{n\in\mathbb{N}}$ converges for each $j \in \mathbb{N}$.

Let $f \in C(X)$. Then there is a sequence $(f_{k_j})_{j\in\mathbb{N}}$ such that $\|f - f_{k_j}\|_\infty \to 0$. In particular $(f_{k_j})_{j\in\mathbb{N}}$ is a Cauchy sequence. Thus, for each $\varepsilon > 0$, there is $i_\varepsilon \in \mathbb{N}$ such that $\|f_{k_i} - f_{k_j}\|_\infty < \varepsilon$ whenever $i, j \ge i_\varepsilon$. Hence,

$$\left| \int_X f_{k_i}\, d\mu_n^{(n)} - \int_X f_{k_j}\, d\mu_n^{(n)} \right| \le \|f_{k_i} - f_{k_j}\|_\infty < \varepsilon\,, \qquad n \in \mathbb{N},\, i, j \ge i_\varepsilon,$$

follows, namely we have uniform convergence and so

$$\lim_{j\to\infty} \lim_{n\to\infty} \int_X f_{k_j}\, d\mu_n^{(n)} = \lim_{n\to\infty} \int_X \lim_{j\to\infty} f_{k_j}\, d\mu_n^{(n)} = \lim_{n\to\infty} \int_X f\, d\mu_n^{(n)}$$

exists. Define

$$\varphi : C(X) \to \mathbb{C}, \quad \varphi(f) := \lim_{n\to\infty} \int_X f\, d\mu_n^{(n)}\,.$$

Then φ is linear, bounded (as $|\varphi(f)| \le \|f\|_\infty$) and $\varphi(\mathbf{1}) = 1$ as $\mu_n^{(n)}$ is a probability measure.

Step 2: By the Riesz-Markov representation theorem (Theorem D.5), there exists a $\mu \in \mathcal{M}^1(X)$ (as $\varphi(\mathbf{1}) = 1$) such that $\varphi(f) = \int_X f\, d\mu$. By definition of φ, we have

$$\lim_{n\to\infty} \int_X f\, d\mu_n^{(n)} = \varphi(f) = \int_X f\, d\mu.$$

Thus, $(\mu_n^{(n)})_{n\in\mathbb{N}}$ converges in the vague topology to $\mu \in \mathcal{M}^1(X)$. Since $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}^1(X)$ was an arbitrary sequence, we have shown that $\mathcal{M}^1(X)$ is sequentially compact.

Step 3: Since $\mathcal{M}(X)$ is a Hausdorff space (Proposition 2.9 about the separation of measures by continuous functions), $\mathcal{M}^1(X) \subseteq \mathcal{M}(X)$ is a closed subset as it is sequentially compact by Step 1 and 2. □

REMARK. *For indeed the sequentially compactness would hold for every $\mathcal{M}^c(X)$ with $c > 0$.*

COROLLARY 2.15. *Let X be a compact metric space. Then $\mathcal{M}^1(X)$ is a compact, metrizable space.*

PROOF. It is a fundamental fact in topology that in a second countable Hausdorff space, the space is sequentially compact if and only if it is compact, confer Appendix A. Thus, $\mathcal{M}^1(X)$ is a compact, second countable Hausdorff space by the previous considerations. Such topological spaces are always metrizable (Urysohn's metrization theorem). □

2.5. Invariant probability measures. As we have seen in the last section the space of probability measures $\mathcal{M}^1(X)$ on a compact set X is a compact space. Furthermore, $\mathcal{M}^1(X)$ is nonempty as $\delta_x \in \mathcal{M}^1(X)$ for any $x \in X$. Clearly, δ_x is a finite Borel measure and so it is a regular Borel measure as discussed before in Proposition 2.10.

DEFINITION. *Let (X, G) be a topological dynamical system. A measure $\mu \in \mathcal{M}^1(X)$ is called* G-invariant *if for all measurable $A \subseteq X$ and $g \in G$, we have*

$$\mu(gA) = \mu(A).$$

The invariance of a measure can be characterized by the invariance of integrals over a suitable class of test functions. Therefore, the action of G onto X pushes forward to an action onto the set of measurable functions $f : X \to \mathbb{C}$. Specifically, for $g \in G$ and $f : X \to \mathbb{C}$ measurable, define

$$g.f := f \circ g^{-1} : X \to \mathbb{C}.$$

Then $g.f$ is measurable and if f is continuous $g.f$ is as well continuous (as a composition of measurable/continuous maps). In the following, we will use the notation

$$\mu(f) := \int_X f \, d\mu$$

for all continuous/integrable functions $f : X \to \mathbb{C}$.

PROPOSITION 2.16 (Invariance of the integral of bounded functions). *Let (X, G) be a dynamical system and $\mu \in \mathcal{M}^1(X)$. Then the following assertions are equivalent.*

(i) *μ is G-invariant.*

(ii) *For every $f \in C(X)$ and $g \in G$, we have*

$$\int_X f \, d\mu = \int_X g.f \, d\mu\,.$$

(iii) *For every $f \in L^\infty(X, \mu)$ and $g \in G$, we have*

$$\int_X f \, d\mu = \int_X g.f \, d\mu\,.$$

PROOF. (i)⇒(ii): Let $A \subseteq X$ be measurable and consider the characteristic function $1_A : X \to \{0,1\}$. Then

$$\int 1_A \, d\mu = \mu(A) = \mu(gA) = \int 1_{gA} \, d\mu = \int 1_A \circ g^{-1} \, d\mu = \int g.1_A \, d\mu.$$

Let $f \in C(X)$ be nonnegative. Then f is uniformly continuous as X is compact. Standard arguments give that there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions ($f : X \to \mathbb{C}$ is simple if $f = \sum_{i=1}^n c_i 1_{A_i}$ for $c_i \in \mathbb{C}$ and $A_i \subseteq X$ measurable) such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$ and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} \|g.f_n - g.f\|_\infty = 0 \quad \text{and} \quad f_n(g^{-1}x) \leq f_{n+1}(g^{-1}x) \quad \text{for each } x \in X, n \in \mathbb{N},$$

for all $g \in G$. Thus the monotone convergence theorem (Beppo Levi, Theorem B.5) leads to

$$\lim_{n\to\infty} \int_X g.f_n \, d\mu = \int_X g.f \, d\mu \quad \text{and} \quad \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Since f_n is a simple function, the previous considerations assert

$$\int_X f_n \, d\mu = \int_X g.f_n \, d\mu\,, \qquad n \in \mathbb{N}$$

and so

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \int_X g.f_n \, d\mu = \int_X g.f \, d\mu$$

follows.

If $f \in C(X)$ is real-valued, then $f = f^+ - f^-$ where $f^+, f^- \in C(X)$ are nonnegative. Thus, $\mu(g.f) = \mu(f)$ follows by using linearity of the integral. Similarly, $\mu(g.f) = \mu(f)$ is derived if $f \in C(X)$ be complex-valued using $f = \Re(f) + i\Im(f)$.

(ii)⇒(iii): This is clear as we know from measure theory that $C(X)$ is dense in $L^\infty(X,\mu)$ if X is compact.

(iii)⇒(i): Let $A \subseteq X$ be measurable. Then $1_A \in L^\infty(X,\mu)$ holds. Thus, for every $g \in G$,

$$\mu(A) = \int_X 1_A \, d\mu = \int_X g.1_A \, d\mu = \int_X 1_{gA} \, d\mu = \mu(gA)$$

follows from (ii) proving (i). □

We define the set of all G-invariant measures by

$$\mathcal{M}^1(X,G) := \{\mu \in \mathcal{M}^1(X) \,|\, \mu \text{ is } G-\text{invariant}\} \subseteq \mathcal{M}^1(X).$$

We will first show that there always exists a G-invariant measure on (X,G). A sufficient condition to guarantee the existence of invariant measures is that the group G is amenable.

Define the symmetric difference of $A, B \subseteq G$ by $A\Delta B := A \setminus B \cup B \setminus A$. Furthermore, $\sharp K$ denotes the cardinality of a compact (and so finite) subset $K \subseteq G$ where G is a countable discrete group.

Definition (amenable, Følner sequence). *Let G be a countable group. A sequence $(F_n)_{n\in\mathbb{N}}$ of compact subsets of G is called* Følner sequence *if*

$$\lim_{n\to\infty} \frac{\sharp(F_n \Delta K F_n)}{\sharp F_n} = 0$$

holds for every $K \subseteq G$ compact. Then a group G is called amenable *if G admits a Følner sequence.*

Remark. *Note that*

$$\sharp(F_n \Delta K F_n) = \delta_G\big(F_n \Delta K F_n\big)$$

holds where δ_G is the counting measure. As we already have discussed, the counting measure is the Haar measure of a countable group. Thus, replacing the counting measure in general by the Haar measure will extend the notion to general locally compact groups.

Exercise. *Let G be a countable discrete group and $F_n \subseteq G$, $n \in \mathbb{N}$, be compact. Prove that $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence of the countable group G if and only if*

$$\lim_{n\to\infty} \frac{\sharp(F_n \cap K F_n)}{\sharp F_n} = 1$$

for all compact $K \subseteq G$.

Example. *If $G = \mathbb{Z}^d$ or $G = \mathbb{H}_3(\mathbb{Z})$ one can show that the sequence of closed balls $F_n := B_n(0)$ define a Følner sequence (one needs a left-invariant metric). In particular, groups of polynomial growth (the volume of balls grow polynomially) are amenable. We will see that $\mathbb{H}_3(\mathbb{Z})$ is amenable with a different Følner sequence (cubes) in the exercise session.*

Theorem 2.17 (Existence of invariant measures). *Let (X, G) be a topological dynamical system and G be a countable amenable group with Følner sequence $(F_n)_{n\in\mathbb{N}}$. Then there is at least one G-invariant probability measure on X. In particular, let $(x_n)_{n\in\mathbb{N}} \subseteq X$. Then any accumulation point of*

$$\mu_n := \frac{1}{\sharp F_n} \sum_{h\in F_n} \delta_{hx_n}$$

is an element of $\mathcal{M}^1(X, G)$.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$. Clearly, μ_n is a probability measure on X as δ_{x_n} is a probability measure for each $n \in \mathbb{N}$. As we have seen before, $\mathcal{M}^1(X)$ is sequentially compact. Thus, $(\mu_n)_{n\in\mathbb{N}}$ has at least one accumulation $\mu \in \mathcal{M}^1(X)$ (Theorem 2.14, Compactness of $\mathcal{M}^1(X)$). We will show that μ is a G-invariant measure.

Without loss of generality (by passing to a subsequence), we have $\mu_n \rightharpoonup \mu$. Let $g \in G$ and $f \in C(X)$. Then

$$\sum_{h\in F_n} g.f(hx_n) = \sum_{g^{-1}h\in g^{-1}F_n} f(g^{-1}hx_n) = \sum_{h\in g^{-1}F_n} f(hx_n)$$

holds. Thus,

$$\begin{aligned}|\mu(g.f)-\mu(f)| &= \lim_{n\to\infty}\frac{1}{\sharp F_n}\left|\sum_{h\in F_n} g.f(hx_n)-\sum_{h\in F_n} f(hx_n)\right| \\ &= \lim_{n\to\infty}\frac{1}{\sharp F_n}\left|\sum_{h\in g^{-1}F_n} f(hx)-\sum_{h\in F_n} f(hx)\right| \\ &\le \lim_{n\to\infty}\frac{1}{\sharp F_n}\sharp\big(F_n\Delta g^{-1}F_n\big)\|f\|_\infty\end{aligned}$$

Since $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence, we conclude

$$|\mu(g.f)-\mu(f)|\le \lim_{n\to\infty}\frac{\sharp\big(F_n\Delta g^{-1}F_n\big)}{\sharp F_n}\|f\|_\infty = 0.$$

Hence, $\mu(g.f)=\mu(f)$ holds for all $f\in C(X)$ and $g\in G$. Consequently, μ is G-invariant by Proposition 2.16 (Invariance of the integral of bounded functions). □

A subset $F\subseteq\mathcal{M}^1(X)$ is called convex if for every $0\le\lambda\le1$ and $\mu,\nu\in F$, we have $\lambda\mu+(1-\lambda)\nu\in F$.

Proposition 2.18. *Let (X,G) be a dynamical system. Then $\mathcal{M}^1(X,G)$ is a compact and convex set.*

Proof. Clearly, $\mathcal{M}^1(X)$ is convex as

$$\lambda\mu(X)+(1-\lambda)\nu(X)=\lambda+(1-\lambda)=1$$

for $\mu,\nu\in\mathcal{M}^1(X)$. In addition, $\mathcal{M}^1(X,G)$ is convex since

$$\begin{aligned}\big(\lambda\mu+(1-\lambda)\nu\big)(A) =&\lambda\mu(A)+(1-\lambda)\nu(A)\\ =&\lambda\mu(gA)+(1-\lambda)\nu(gA)\\ =&\big(\lambda\mu+(1-\lambda)\nu\big)(gA)\end{aligned}$$

if $\mu,\nu\in\mathcal{M}^1(X,G)$ and $A\subseteq X$ is measurable. Thus, we only need to show that $\mathcal{M}^1(X,G)$ is compact.

Let $(\mu_n)_{n\in\mathbb{N}}\subseteq\mathcal{M}^1(X,G)$ be a convergent sequence with limit point $\mu\in\mathcal{M}^1(X)$. Since μ_n is G-invariant, we have

$$\int_X f\,d\mu=\lim_{n\to\infty}\int_X f\,d\mu_n=\lim_{n\to\infty}\int_X g.f\,d\mu_n=\int_X g.f\,d\mu$$

for all $f\in C(X)$ and $g\in G$. Hence, μ is G-invariant by Proposition 2.16 (Invariance of the integral of bounded functions). Thus, $\mathcal{M}^1(X,G)$ is a closed subset of the compact space $\mathcal{M}^1(X)$. □

Remark. *One can actually show that the extreme points of $\mathcal{M}^1(X,G)$ are exactly the ergodic measures. Here an extreme point of $\mathcal{M}^1(X,G)$ is an element that cannot be written in a non-trivial way as a convex combination of other elements. Furthermore, an invariant measure $\mu\in\mathcal{M}^1(X,G)$ is called ergodic if every G-invariant subset $A\subseteq X$ satisfies*

$$\textit{either}\qquad \mu(A)=0\qquad \textit{or}\qquad \mu(X\setminus A)=0.$$

Specifically, every invariant set is either the whole space or the empty set (modulo measure zero sets). Thus, ergodicity of (X,G,μ) is the measure *theoretic analog of minimality of a dynamical system. We refer the reader to a course in Ergodic theory for further background.*

Week 3

2.6. Semi-continuity of the set of invariant measures. Let (X,G) be a dynamical system and $\mathcal{J}$ be the compact space of dynamical subsystems of (X,G). We will study continuity properties of the map

$$\mathcal{J} \to \mathcal{K}\big(\mathcal{M}^1(X,G)\big), \quad Y \mapsto \mathcal{M}^1(Y,G),$$

where $\mathcal{K}\big(\mathcal{M}^1(X,G)\big)$ is the space of compact subsets of $\mathcal{M}^1(X,G)$. Therefore, note first that $\mathcal{M}^1(Y,G)$ can be seen as a compact subset of $\mathcal{M}^1(X,G)$ by the following.

Exercise. *Let (X,G) be a dynamical system and $Y \in \mathcal{J}$. For $\mu \in \mathcal{M}^1(Y,G)$, define $\mu_X \in \mathcal{M}(X)$ by*

$$\mu_X(A) := \mu(Y \cap A)$$

for all measurable $A \subseteq X$. Prove the following assertions.

- *$\mu_X \in \mathcal{M}^1(X,G)$ is an invariant probability measure.*
- *The map $\iota : \mathcal{M}^1(Y,G) \to \mathcal{M}^1(X,G)$, $\mu \mapsto \mu_X$, is a continuous injective map.*
- *$\iota(\mathcal{M}^1(Y,G)) \subseteq \mathcal{M}^1(X,G)$ is a compact and convex subset.*

Remark. *Based on the latter exercise we will identify $\mu \in \mathcal{M}^1(Y,G)$ with $\iota(\mu) = \mu_X$ without mentioning ι explicitly.*

Furthermore, $\mathcal{M}^1(X,G)$ is a compact metrizable space by Corollary 2.15. Thus, we can equip the space $\mathcal{K}\big(\mathcal{M}^1(X,G)\big)$ with the induced Hausdorff metric (which is nothing but than the Chabauty-Fell topology), see Proposition 2.1.

In general,

$$\mathcal{J} \to \mathcal{K}\big(\mathcal{M}^1(X,G)\big), \quad Y \mapsto \mathcal{M}^1(Y,G),$$

is not continuous but we will see that this map is semi-continuous in a suitable sense. To this end, we need the concept of a support of a measure.

Definition. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Then the* support *of μ is defined by*

$$\operatorname{supp}(\mu) := \big\{x \in X \,\big|\, \mu(f) > 0 \text{ for all } f \in C(X) \text{ with } f \geq 0 \text{ and } f(x) > 0\big\}.$$

Exercise. *Let X be a compact metric space and $\mu \in \mathcal{M}(X)$. Prove the following assertions.*

(a) *The support* $\operatorname{supp}(\mu) \subseteq X$ *is closed.*
(b) *If (X,G) is a dynamical system and μ is G-invariant, then* $\operatorname{supp}(\mu)$ *is invariant.*

Recall that

$$\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$$

denotes the support of a function $f : X \to \mathbb{C}$.

THEOREM 2.19 (Semi-continuity of the invariant probability measures). *Let (X,G) be a dynamical system. Let $(Y_n) \subseteq \mathcal{J}$ be a convergent sequence of dynamical subsystems in $\mathcal{J}$ with limit point $Y \in \mathcal{J}$. If $\mu_n \in \mathcal{M}^1(Y_n,G)$ for all $n \in \mathbb{N}$, then every limit point μ of (μ_n) in $\mathcal{M}^1(X,G)$ is an element of $\mathcal{M}^1(Y,G)$. Specifically,*

$$\limsup_{n\to\infty} \mathcal{M}^1(Y_n,G) := \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{l\geq k} \mathcal{M}^1(Y_l,G)} \subseteq \mathcal{M}^1(Y,G)$$

and $\limsup_{n\to\infty} \mathcal{M}^1(Y_n,G)$ is a convex subset in $\mathcal{M}^1(Y,G)$.

REMARK. *We refer the reader to* [**BP20**] *for further background where this result was originally proven.*

PROOF. Without loss of generality, suppose that $\mu_n \rightharpoonup \mu \in \mathcal{M}^1(X,G)$ (otherwise pass to a subsequence using that $\mathcal{M}^1(X,G)$ is compact). Assume by contradiction that $\mu \notin \mathcal{M}^1(Y,G)$. It is elementary to check that if $\operatorname{supp}(\mu) \subseteq Y$, then $\mu \in \mathcal{M}^1(Y,G)$ must hold. Thus

$$Y \setminus \operatorname{supp}(\mu) \neq \emptyset$$

follows and so there is an $x \in \operatorname{supp}(\mu) \setminus Y$. Since X is a compact metric space, Urysohn's Lemma applies. Specifically, there exists an $f \in C(X)$ satisfying

$$0 \leq f \leq 1, \quad f(x) = 1 \quad \text{and} \quad \operatorname{supp}(f) \cap Y = \emptyset.$$

Thus, $x \in \operatorname{supp}(\mu)$ leads to $\mu(f) > 0$. By Proposition 2.1 the set

$$\mathcal{U}(\operatorname{supp}(f), \{X\}) := \{Z \in \mathcal{J} \,|\, Z \cap \operatorname{supp}(f) = \emptyset,\, Z \cap X \neq \emptyset\}$$

is a neighborhood of Y in $\mathcal{J}$. By the convergence of Y_n to Y, there is an $n_0 \in \mathbb{N}$ such that $Y_n \in \mathcal{U}(\operatorname{supp}(f), \{X\})$ for $n \geq n_0$. Since $\operatorname{supp}(\mu_n) \subseteq Y_n$, we derive

$$\operatorname{supp}(f) \cap \operatorname{supp}(\mu_n) = \emptyset, \qquad \text{for } n \geq n_0.$$

Consequently, $\mu_n(f) = 0$ follows for $n \geq n_0$.

On the other hand, the convergence of the measures $(\mu_n)_n$ to μ in the vague topology implies

$$0 = \lim_{n\to\infty} \mu_n(f) = \mu(f) > 0,$$

a contradiction. Thus, $\mu \in \mathcal{M}^1(Y,G)$ must hold if $Y_n \to Y$ in $\mathcal{J}$.

We have proven that every limit point μ of any sequence $\mu_n \in \mathcal{M}^1(Y_n,G)$ is automatically contained in $\mathcal{M}^1(Y,G)$. Since

$$\begin{aligned}\limsup_{n\to\infty} \mathcal{M}^1(Y_n,G) &:= \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{l\geq k} \mathcal{M}^1(Y_l,G)} \\ &= \{\mu \in \mathcal{M}^1(X,G) \,|\, \text{exists } \mu_{n_k} \in \mathcal{M}^1(Y_{n_k},G) \text{ with } \mu_{n_k} \rightharpoonup \mu\},\end{aligned}$$

the desired inclusion follows. □

REMARK. *(a) For this notion of limes superior (and similarly limes inferior), we forward the more interested reader to the terminology of Kuratowski convergence for compact subsets of a metric space.*
(b) If $\mathcal{M}^1(Y,G) = \{\mu\}$ holds in the previous theorem, then

$$\lim_{n\to\infty} \mathcal{M}^1(Y_n,G) = \limsup_{n\to\infty} \mathcal{M}^1(Y_n,G) = \mathcal{M}^1(Y,G)$$

*holds if $\mathcal{K}\big(\mathcal{M}^1(X,G)\big)$ is equipped with the induced Hausdorff topology (**Exercise**). Specifically, a base for a topology on $\mathcal{K}\big(\mathcal{M}^1(X,G)\big)$ is given by*

$$\mathcal{U}(F,\mathcal{O}) := \{M \in \mathcal{K}\big(\mathcal{M}^1(X,G)\big) \,\big|\, M \cap F = \emptyset,\ M \cap O \neq \emptyset \text{ for all } O \in \mathcal{O}\big\}$$

where $F \subseteq \mathcal{M}^1(X,G)$ is closed (=compact) and $\mathcal{O}$ is a finite family of open subsets of $\mathcal{M}^1(X,G)$. Dynamical systems (Y,G) satisfying $\mathcal{M}^1(Y,G) = \{\mu\}$ are called uniquely ergodic, which is studied in the following.

(c) We will later discuss also a semi-continuity of the spectra of operators, see Section 5.5.

(d) It is an open question which measures on $\mathcal{M}^1(Y,G)$ can be approximated in general, confer exercise session.

2.7. Unique Ergodicity. Since $\mathcal{M}^1(Y,G)$ is a convex set, it is an uncountable set in general (as soon as we have two different elements). In this section, we consider the case, where $\mathcal{M}^1(Y,G)$ is a singleton.

Definition. *A dynamical system (X,G) is called* uniquely ergodic *if it admits exactly one G-invariant measure, namely $\mathcal{M}^1(X,G) = \{\mu\}$.*

We already have seen in Sheet 3 that so-called periodic dynamical systems are uniquely ergodic. Therefore, recall the following definition.

Definition. *A dynamical system (Y,G) is called* periodic *if Y is minimal and finite.*

We point out that a minimal dynamical system is periodic if and only if $Y = Orb(x)$ is finite for some/any $x \in Y$.

Proposition 2.20. *Every periodic dynamical system (Y,G) is uniquely ergodic and minimal.*

Proof. This was proven in Sheet 3. □

We characterize unique ergodicity for amenable groups by the pointwise/uniform convergence of certain averages.

Theorem 2.21 (Characterization of unique ergodicity). *Let (X,G) be a dynamical system where G is an amenable group with Følner sequence $(F_n)_{n\in\mathbb{N}}$. The following are equivalent.*

(i) *For every $f \in C(X)$, the sequence of functions*

$$g_n : X \to \mathbb{C}, \quad x \mapsto \frac{1}{\sharp F_n} \sum_{h\in F_n} f(hx)$$

converges uniformly to a constant.

(ii) *For every $f \in C(X)$, the sequence*

$$g_n : X \to \mathbb{C}, \quad x \mapsto \frac{1}{\sharp F_n} \sum_{h\in F_n} f(hx)$$

converges pointwise to a constant independent of $x \in X$.

(iii) *There exists a $\mu \in \mathcal{M}^1(X,G)$ such that for every $f \in C(X)$ and all $x \in X$:*

$$\lim_{n\to\infty} \frac{1}{\sharp F_n} \sum_{h\in F_n} f(hx) = \int_X f\, d\mu.$$

(iv) (X, G) *is uniquely ergodic.*

Remark. *Let* $G = \mathbb{Z}$ *then* $F_n := \{1, \dots, n\}$ *defines a Følner sequence. The quantity*

$$\frac{1}{\sharp F_n} \sum_{h \in F_n} f(hx)$$

can be interpreted as a "time mean" of a certain "observable" f. *In light of this, (iii) asserts that the time mean converges to the space mean of the observable* f. *Such type of theorems are called Ergodic Theorems. In general the time mean does not converge for all* $x \in X$. *However, one can prove almost everywhere convergence (Birkhoff's Ergodic Theorem) or* L^2*-convergence (von Neumann's Mean Ergodic Theorem) w.r.t. an ergodic measure* μ. *This is subject of a lecture "Ergodic theory".*

Proof. (i)⇒(ii): This is obvious.

(ii)⇒(iii): Let $x \in X$ be fixed and set

$$\mu_n := \frac{1}{\sharp F_n} \sum_{h \in F_n} \delta_{hx}.$$

Define $\varphi : C(X) \to \mathbb{C}$ by

$$\varphi(f) := \lim_{n \to \infty} \mu_n(f) = \lim_{n \to \infty} \frac{1}{\sharp F_n} \sum_{h \in F_n} f(hx),$$

where the limit exists by (ii) and is independent of $x \in X$. Then φ is linear and bounded

$$|\varphi(f)| \le \|f\|_\infty.$$

Furthermore, φ is positive and $\varphi(\mathbf{1}) = 1$. Thus, Riesz-Markov representation theorem (Theorem D.5) implies that there exists a unique $\mu \in \mathcal{M}^1(X)$ such that

$$\varphi(f) = \int_X f \, d\mu$$

for all $f \in C(X)$. By construction, we have $\mu_n \rightharpoonup \mu$ and so $\mu \in \mathcal{M}^1(X, G)$ holds by Theorem 2.17 (Existence of invariant measures) since $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence. Thus, (iii) follows.

(iii)⇒(iv): Suppose $\mu \in \mathcal{M}^1(X, G)$ is such that

$$\lim_{n \to \infty} \frac{1}{\sharp F_n} \sum_{h \in F_n} f(hx) = \int_X f \, d\mu.$$

for all $x \in X$. Let $\nu \in \mathcal{M}^1(X, G)$. We will show that $\mu = \nu$. Since

$$g_n : X \to \mathbb{C}, \quad x \mapsto \frac{1}{\sharp F_n} \sum_{h \in F_n} f(hx)$$

converges pointwise to $\int_X f \, d\mu$ and since each term of the sequence is uniformly bounded by $\|f\|_\infty$, the Lebesgue's dominated convergence theorem yields

$$\begin{aligned}\int_X f(z)\,d\mu(z) &= \int_X \left(\int_X f(z)\,d\mu(z)\right) d\nu(y)\\ &= \int_X \lim_{n\to\infty} g_n(y)\,d\nu(y)\\ &= \lim_{n\to\infty} \int_X \frac{1}{\sharp F_n}\sum_{h\in F_n} f(hy)\,d\nu(y).\end{aligned}$$

Since ν is G-invariant, Proposition 2.16 leads to

$$\int_X \frac{1}{\sharp F_n}\sum_{h\in F_n} f(hy)\,d\nu = \frac{1}{\sharp F_n}\sum_{h\in F_n}\int_X h^{-1}.f(y)\,d\nu = \int_X f\,d\nu$$

for each $n\in\mathbb{N}$. Hence,

$$\int_X f\,d\mu = \int_X f\,d\nu$$

follows for all $f\in C(X)$. Then Proposition 2.9 (Separation of measures by continuous functions) implies $\mu=\nu$. As $\nu\in\mathcal{M}^1(X,G)$ was arbitrary, we conclude (iv).

(iv)⇒(i): This is left as an exercise (Sheet 4). □

2.8. Minimality and unique ergodicity. With this we can get to the structural connection between minimality and unique ergodicity.

THEOREM 2.22. *Let G be amenable and (X,G) be uniquely ergodic with G-invariant measure μ. Then (X,G) admits a unique closed minimal dynamical subsystem. This subset is given by the support of μ.*

PROOF. We first show that $\operatorname{supp}(\mu)$ is a minimal subset of X. Let $Y\subseteq\operatorname{supp}(\mu)$ be non-empty, invariant and closed. We have to show that $\operatorname{supp}(\mu)=Y$. Clearly (Y,G) is a dynamical system and since G is amenable, it admits an invariant measure $\nu\in\mathcal{M}^1(Y,G)$. Then ν can be seen as an invariant probability measure on X by setting $\nu_X(A):=\nu(A\cap Y)$, see Section 2.6. Since (X,G) is uniquely ergodic, we conclude $\nu_X=\mu$ and so

$$\operatorname{supp}(\mu)=\operatorname{supp}(\nu_X)\subseteq Y\subseteq\operatorname{supp}(\mu)$$

proving the claim.

Next, we show the uniqueness of the minimal set. Let $Y\subseteq X$ be minimal. Then there is (as before) a measure $\nu\in\mathcal{M}^1(Y,G)$ with extension $\nu_X\in\mathcal{M}^1(X,G)$ and

$$\operatorname{supp}(\mu)=\operatorname{supp}(\nu_X)\subseteq Y.$$

Since Y is minimal and $\operatorname{supp}(\mu)$ is a non-empty, closed, invariant subset of Y, we conclude $Y=\operatorname{supp}(\mu)$. □

COROLLARY 2.23. *Let (X,G) be a uniquely ergodic dynamical system where G is an amenable group. Then the following statements are equivalent.*

(i) *The dynamical system (X,G) is minimal.*
(ii) *We have $\mu(f)>0$ for all $f\in C(X)$ with $f\geq 0$ and $f\neq 0$.*

PROOF. By the previous theorem, (i) is equivalent to $X=\operatorname{supp}(\mu)$, which is clearly equivalent to (ii). □

Remark. *We note that dynamical systems that are minimal and uniquely ergodic are called* strictly ergodic *in the literature.*

3. Symbolic dynamical systems

We will focus in this chapter on a specific class of dynamical systems, so called symbolic dynamical systems. More precisely, we will focus on the product space $\mathcal{A}^G$ where $G := \mathbb{Z}^d$. Most of the considerations made in this chapter extend to the case where G is a countable group except the section on periodic approximations and substitutions where we consider symbolic dynamical systems defined by substitutions. This part is mainly developed for $G = \mathbb{Z}^d$ and it is a current subject of research to extend these notions to non-abelian groups such as the Heisenberg group. There is also a geometrical analog of substitutions if $G := \mathbb{R}^d$ that admit similar properties (Examples are the Penrose tiling or the Octogonal tiling) but this is not content of this lecture.

The aim of this chapter is to provide a different description of the topology on $\mathcal{J}$ associated with the dynamical system $(\mathcal{A}^G, G)$. In order to do so, the terminology of so-called dictionaries and the local pattern topology is discussed and we will see that this space is homeomorphic to $\mathcal{J}$, which provides us with an intuitive description of the topology on $\mathcal{J}$. Furthermore, we use this description to prove a sufficient condition on the existence of periodic approximations in the one-dimensional case $G = \mathbb{Z}$. In the case of substitution dynamical systems, these approximations can be defined recursively.

3.1. Dictionaries. Throughout this chapter, an *alphabet* $\mathcal{A}$ denotes a finite set equipped with the discrete topology. Elements of $\mathcal{A}$ are called *letters*. Furthermore, we only treat the case $G := \mathbb{Z}^d$ with inverse and composition given by

$$g^{-1} := -g \qquad \text{and} \qquad gh := g + h, \qquad g, h \in G.$$

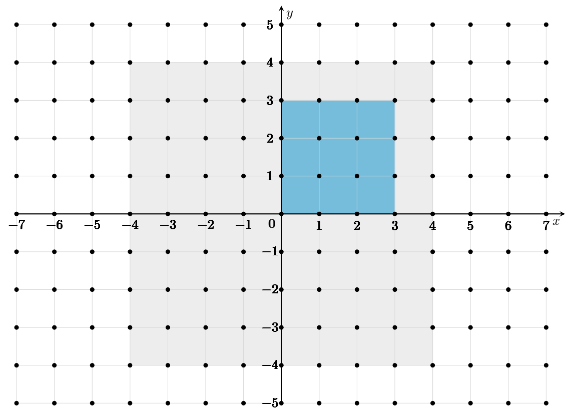

Figure 9. The cube Q_4 in gray and Q_3^+ in blue in $\mathbb{Z}^2$

Let

$$\mathcal{A}^G := \prod_{g \in G} \mathcal{A} = \{ w : G \to \mathcal{A} \}$$

be equipped with the product topology. In order to define a metric we define the closed cube

$$Q_r := \{ g \in G \mid \|g\|_\infty \le r \} \qquad \text{where } \|g\|_\infty := \max_{1 \le i \le d} |g_i|.$$

with side length $2r > 0$, see Figure 9.

LEMMA 3.1. *The product topology on $\mathcal{A}^G$ is induced by the (ultra) metric*

$$d(\omega,\rho) := \min\left\{1, \inf\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N} \text{ be such that } \omega|_{Q_n} = \rho|_{Q_n}\right\}\right\}.$$

Then $(\mathcal{A}^G, d)$ is a compact metric space and $(\mathcal{A}^G, G)$ is a dynamical system where the action is defined by

$$g\omega(h) := \omega(g^{-1}h), \qquad g, h \in G.$$

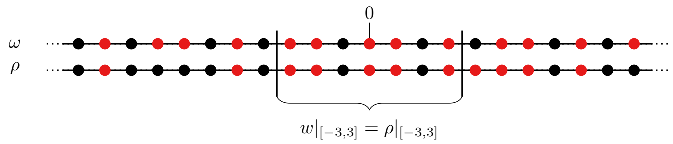

Figure 10. The distance of w and ρ equals $\frac{1}{3}$ for $G = \mathbb{Z}$.

PROOF. Since $\bigcup_{n\in\mathbb{N}} Q_n = \mathbb{Z}^d$ and $Q_n \subseteq Q_{n+1}$, this follows from a previous exercise (Sheet 1). ☐

REMARK. *Note that there are various choices for a metric on $\mathcal{A}^G$ that induces the product topology. However, this choice of the metric is particularly useful for the quantitative estimates between the spectra of Schödinger operators over such dynamical systems, see Chapter 8.*

Recall that $\Omega \subseteq \mathcal{A}^G$ is invariant if $g\Omega := \{g\omega \,|\, \omega \in \Omega\} \subseteq \Omega$ for all $g \in G$. Furthermore, the space of dynamical subsystems

$$\mathcal{J} := \mathcal{J}(\mathcal{A}^G, G) := \left\{\Omega \subseteq \mathcal{A}^G \,\middle|\, \Omega \text{ is closed, non-empty, invariant}\right\}$$

is equipped with the Hausdorff metric induced by the metric d on $\mathcal{A}^G$. In the literature, elements of $\mathcal{J}$ are often called *subshift* as G acts on $\mathcal{A}^G$ by shifting and elements of $\mathcal{J}$ are subsets of $\mathcal{A}^G$.

Define

$$\mathcal{A}^K := \{p : K \to \mathcal{A}\}$$

for any finite subset $K \subseteq G$. Clearly, $\mathcal{A}^K$ is finite as the alphabet is finite. Then an element in $\mathcal{A}^K$ is called a *pattern* with support K. If $F, K \subseteq G$ are finite we say, a *pattern $p \in \mathcal{A}^F$ occurs in a pattern $q \in \mathcal{A}^K$ (we write $p \prec q$)*, if there is an $h \in G$ such that

$$hF \subseteq K \qquad \text{and} \qquad p(g) = q(hg) \quad \text{for all } g \in F,$$

see e.g. Figure 11. Similarly, we say that a *pattern $p \in \mathcal{A}^K$ occurs in $\omega \in \mathcal{A}^G$* if there exists an $h \in G$ such that

$$(h\omega)|_K = p,$$

see e.g. Figure 11.

EXERCISE. *Prove that the relation $\prec$ is transitive, namely $p_1 \prec p_2$ and $p_2 \prec p_3$ imply $p_1 \prec p_3$.*

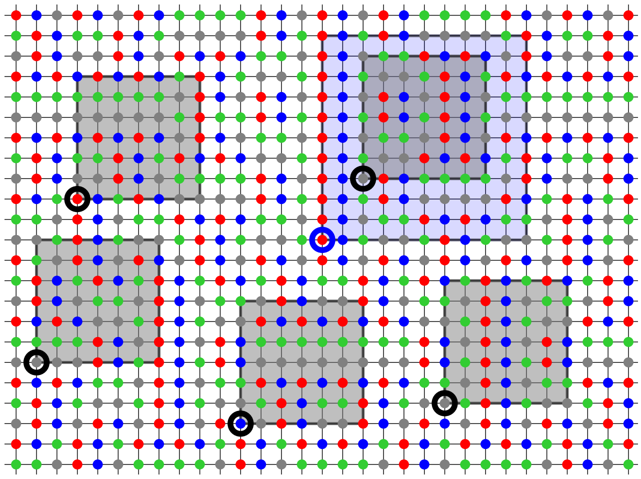

Figure 11. We see various patterns p (gray shaded areas) supported on Q_6^+ where the origin $0 \in Q_6^+$ is indicated by a black circle. The blue shaded patch is supported on Q_{10}^+ and one of the gray shaded patches occurs in the blue one.

A dictionary will be defined by a suitable collection of patterns. It is more convenient to consider only patterns defined on

$$Q_n^+ := \{g := (g_1, \ldots, g_d) \in G \,|\, g_i \geq 0 \text{ and } g_i \leq n\} = [0, n+1)^d \cap \mathbb{Z}^d$$

for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Note that Q_n^+ is nothing but Q_n intersected with the positive half quadrant in $\mathbb{R}^d$, see Figure 9. The set of all patterns with support in such boxes is denoted by

$$\mathcal{A}^* := \bigcup_{n \in \mathbb{N}_0} \mathcal{A}^{Q_n^+}.$$

Then a dictionary will be a suitable subset of $\mathcal{A}^*$. Before providing the formal definition, let us make a short excursion to the case $G = \mathbb{Z}$, in order to explain the key ideas.

A short excursion to the one-dimensional case $G = \mathbb{Z}$

Let $G = \mathbb{Z}$. Then $Q_n^+ = \{0, 1, 2, \ldots n\}$ holds and so $p \in \mathcal{A}^{Q_n^+}$ can be identified with a word $a_0 \ldots a_n$ with letters $a_i \in \mathcal{A}$ of length $n+1$. Due to this, we will introduce the short notation $\mathcal{A}^{n+1} := \mathcal{A}^{Q_n^+}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and call elements of $\mathcal{A}^n$ *patterns (or words) of lenth* $n \in \mathbb{N}$. For instance if $\mathcal{A} := \{a, b\}$, then

$$\begin{aligned}
\mathcal{A}^1 &= \{a, b\} = \mathcal{A}, \\
\mathcal{A}^2 &= \{aa, ab, ba, bb\}, \\
\mathcal{A}^3 &= \{aaa, aab, aba, baa, bba, bab, abb, bbb\}, \\
&\vdots
\end{aligned}$$

For the word $u := abaabaa$, the patterns/words $\{ab, ba, aa\}$ are all patterns of length 2 that occur in u but bb does not occur on u. In the literature one calls words that occurs in another word also *subwords*.

Two words $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$ can be concatenated

$$uv := u_1 \dots u_n v_1 \dots v_m \in \mathcal{A}^{n+m}.$$

With this at hand, we define $u^k := uu\dots u$ the kth time concatenation of the word u. For instance,

$$(abbaa)^3 = abbaa\ abbaa\ abbaa.$$

In dimension one, the so-called *empty word* ϵ is sometimes useful (recall example for the free group acting on its boundary in Chapter 2). In particular, with the previous notation at hand, we can say that $u \in \mathcal{A}^n$ occurs in $v \in \mathcal{A}^m$ if and only if there are $u_1, u_2 \in \mathcal{A}^* \cup \{\epsilon\}$ such that

$$v = u_1 u u_2.$$

Clearly, a necessary condition for this is that $n \le m$. For instance, consider $u := ab \in \mathcal{A}^2$ and $v := aabab \in \mathcal{A}^5$. Then u occurs in v since

$$v = a\ u\ ab \qquad \text{or} \qquad v = aab\ u\ \epsilon = aab\ u.$$

(the space between the words is only to indicate the possible splitting of v into u_1, u and u_2). In particular, we can choose $u_1 := a$ and $u_2 := ab$ or $u_1 := aab$ and $u_2 := \epsilon$. So we have seen that u actually appears twice in v at different positions.

Any element in $\omega \in \mathcal{A}^{\mathbb{Z}}$ can be represented by a two-sided infinite word

$$\omega := \dots\ \omega(-3)\ \omega(-2)\ \omega(-1)\ \big|\ \omega(0)\ \omega(1)\ \omega(2)\ \dots$$

where $\big|$ indicates the origin. Then the two-sided infinite concatenation of word $u, v \in \mathcal{A}^*$ is defined by

$$u^\infty | v^\infty := \dots u\ u\ u\ \big|\ v\ v\ v\ \dots$$

while $u^\infty := u^\infty | u^\infty$. The elements of the form u^∞ are particularly interesting as these are all periodic elements in $\mathcal{A}^{\mathbb{Z}}$, see later. We can also consider patterns that occur in an infinite $\omega \in \mathcal{A}^{\mathbb{Z}}$. For instance, ab, bb and ba occur in

$$\omega := (abb)^\infty = \dots abb\ abb\ \big|\ abb\ abb\ \dots \in \mathcal{A}^{\mathbb{Z}}$$

On the other hand, the word aa is not occurring in ω! In particular, all words of length 2 occurring in ω are given by $\{ab, bb, ba\}$ and similarly one can define the set of all subwords of a given length n. For $\omega := (abb)^\infty \in \mathcal{A}^{\mathbb{Z}}$, we get

$$\begin{aligned}
W(\omega) \cap \mathcal{A}^1 &= \{a, b\},\\
W(\omega) \cap \mathcal{A}^2 &= \{ab, bb, ba\},\\
W(\omega) \cap \mathcal{A}^{3n} &= \big\{(abb)^n, (bab)^n, (bba)^n\big\},\\
W(\Omega) \cap \mathcal{A}^{3n+1} &= \big\{(abb)^n a, (bab)^n b, (bba)^n b\big\},\\
W(\Omega) \cap \mathcal{A}^{3n+2} &= \big\{(abb)^n ab, (bab)^n ba, (bba)^n bb\big\},
\end{aligned}$$

for $n \in \mathbb{N}$.

In light of this, the set of all subwords of a given $\omega \in \mathcal{A}^{\mathbb{Z}}$ is denoted by $W(\omega)$. This set $W(\omega)$ is also called *dictionary*, *language* or *lexicon* (Wörterbuch in German) in the literature. They are very useful to characterize the topology on $\mathcal{J}$ for the dynamical systsem $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$. This is even the case for general countable groups G or point sets in a continuous group. However, we will only focus here to the case $G := \mathbb{Z}^d$ to explain the key ideas.

Week 4

Back to $G = \mathbb{Z}^d$

Let us define the notion of a dictionary.

DEFINITION. *A non-empty set $W \subseteq \mathcal{A}^*$ is called a* dictionary *if for every $n \in \mathbb{N}$ and $p \in W \cap \mathcal{A}^{Q_n^+}$, the following holds.*

(D1) *If $1 \leq k \leq n$ and $q \in \mathcal{A}^{Q_k^+}$ satisfies $q \prec p$, then $q \in W$.*

(D2) *There exists a $q \in W \cap \mathcal{A}^{Q_{n+2}^+}$ such that*

$$q(\mathbf{1}x) = p(x), \qquad \text{for all } x \in Q_n^+,$$

where $\mathbf{1} := (1, 1 \dots, 1) \in G$.

The set of all dictionaries is denoted by $Dic(\mathcal{A}^G)$.

REMARK. *Condition (D1) says that all patterns occurring in $p \in W$ are automatically elements of W. Then (D2) asserts that every pattern has an extension into any direction, compare Figure 12.*

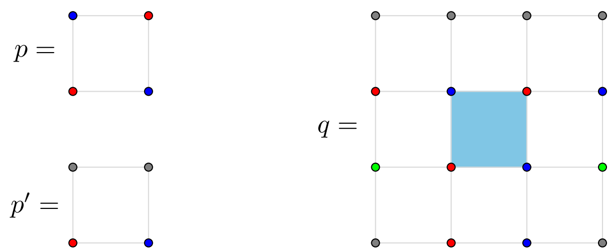

Figure 12. Both patterns p and p' occur in q. So if W is a dictionary with $q \in W$, then $p, p' \in W$ by (D1). The patterns p and q are given such that they satisfy (D2), namely $\mathbf{1}^{-1}q|_{Q_1^+} = p$. Specifically, p occurs in q at the blue shaded square as required in (D2).

PROPOSITION 3.2 (Dictionary of a subshift). *Let $\Omega \in \mathcal{J}$. Then the set*

$$W(\Omega) := \left\{ \omega|_{Q_n^+} \,\middle|\, n \in \mathbb{N},\, \omega \in \Omega \right\}$$

is a dictionary.

REMARK. *We denote by $W(\omega) := W\big(\overline{Orb(\omega)}\big)$ the dictionary associated with the orbit closure of $\omega \in \mathcal{A}^G$. Moreover, we have (**Exercise**)*

$$W(\omega) = \left\{ (g\omega)|_{Q_k^+} \,\middle|\, g \in G,\ k \in \mathbb{N}_0 \right\}.$$

PROOF. Clearly $W(\Omega)$ is non-empty. Let $p := \omega|_{Q_n^+} \in W(\Omega)$ for some $\omega \in \Omega$. Suppose $h \in G$ and $1 \leq k \leq n$ be such that $hQ_k^+ \subseteq Q_n^+$. Then

$$p(hx) = \omega(hx) = (h^{-1}\omega)(x), \qquad x \in Q_k^+$$

Since Ω is invariant, we conclude $h^{-1}\omega \in \Omega$ implying $h^{-1}\omega|_{Q_k^+} \in W(\Omega)$. Thus, $p|_{hQ_k^+} \in W(\Omega) \cap \mathcal{A}^{Q_k^+}$ follows, namely $W(\Omega)$ satisfies (D1).

Let $p := \omega|_{Q_n^+} \in W(\Omega)$ for some $\omega \in \Omega$. Then $q := (\mathbf{1}\omega)|_{Q_{n+2}^+} \in W(\Omega)$ satisfies

$$q(\mathbf{1}x) = (\mathbf{1}\omega)(\mathbf{1}x) = \omega(\mathbf{1}^{-1}\mathbf{1}x) = \omega(x) = p(x), \qquad x \in Q_n^+,$$

proving that $W(\Omega)$ fulfills (D2). □

Example. *Let $\mathcal{A} := \{a, b\}$ and $\omega \in \mathcal{A}^{\mathbb{Z}}$ be defined by*

$$\omega(n) := \begin{cases} a, & n \neq 0, \\ b, & n = 0. \end{cases}$$

The associated orbit closure $\overline{Orb(\omega)}$ is called one-defect. *Then*

$$W(\omega) \cap \mathcal{A}^{Q_n^+} = \left\{ba^n, aba^{n-1}, aaba^{n-2}, \ldots, a^n b, a^{n+1}\right\}$$

holds for each $n \in \mathbb{N}$.

The set $Dic(\mathcal{A}^G)$ becomes a topological space if endowed with the *local pattern topology* defined by the following basis

$$\mathscr{V}(n, U) := \left\{W \in Dic(\mathcal{A}^G) \,\middle|\, W \cap \mathcal{A}^{Q_n^+} = U\right\}, \qquad n \in \mathbb{N}_0,\ U \subseteq \mathcal{A}^{Q_n^+}.$$

Exercise. *Let $U \subseteq \mathcal{A}^{Q_n^+}$ for $n \in \mathbb{N}$. Define*

$$U_k := \left\{q \in \mathcal{A}^{Q_k^+} \,\middle|\, \text{there is a } p \in U \text{ s.t. } q \text{ occurs in } p\right\}$$

for $0 \leq k \leq n$. Show that for all $W \in \mathscr{V}(n, U)$, we have $W \cap \mathcal{A}^{Q_k^+} = U_k$ for all $0 \leq k \leq n$. In particular, if $W_1 \cap \mathcal{A}^{Q_n^+} = W_2 \cap \mathcal{A}^{Q_n^+}$, then $W_1 \cap \mathcal{A}^{Q_k^+} = W_2 \cap \mathcal{A}^{Q_k^+}$ follows for all $0 \leq k \leq n$.

For the following recall, that a topological space Z is *disconnected* if it is the union of two disjoint nonempty subsets of Z. Otherwise Z is called connected. Furthermore, a subset $Y \subseteq Z$ is called *disconnected (resp. connected)*, if it is disconnected (resp. connected) as a topological space with the induced topology. The space Z is called *totally disconnected* if the only connected components of Z are singletons $\{x\} \subseteq Z$. If a topological space Z is Hausdorff and it admits a base $\mathscr{B}$ that consists only of clopen (i.e. open and closed) sets then Z is totally disconnected (**Exercise**). We already know an example of a totally disconnected space, namely $\mathcal{A}^G$.

Proposition 3.3. *The family*

$$\mathscr{B} := \left\{\mathscr{V}(n, U) \,\middle|\, n \in \mathbb{N}_0, U \subseteq \mathcal{A}^{Q_n^+}\right\}$$

of subsets of $Dic(\mathcal{A}^G)$ defines a base for a topology. The set $Dic(\mathcal{A}^G)$ equipped with this topology is second countable and Hausdorff. Furthermore, each $B \in \mathscr{B}$ is clopen and so $Dic(\mathcal{A}^G)$ is totally disconnected.

Proof. *$\mathscr{B}$ is a base:* The identity

$$Dic(\mathcal{A}^G) = \bigcup_{B \in \mathscr{B}} B$$

is trivial. We have to show that any non-trivial intersection of elements of $\mathscr{B}$ contains an element of $\mathscr{B}$. Therefore, let $U \subseteq \mathcal{A}^{Q_n^+}$ and $V \subseteq \mathcal{A}^{Q_m^+}$ for $n, m \in \mathbb{N}$. Without loss of generality assume $n \leq m$. If $\mathscr{V}(n, U) \cap \mathscr{V}(m, V) \neq \varnothing$, then there is an $W \in \mathscr{V}(n, U) \cap \mathscr{V}(m, V)$. Thus,

$$W \cap \mathcal{A}^{Q_n^+} = U \qquad \text{and} \qquad W \cap \mathcal{A}^{Q_m^+} = V$$

follows. Let $W' \in \mathscr{V}(m,V)$. Since $n \leq m$, the previous exercise yields

$$W' \cap \mathcal{A}^{Q_n^+} = W \cap \mathcal{A}^{Q_n^+} = U \qquad \text{since} \qquad W' \cap \mathcal{A}^{Q_m^+} = V = W \cap \mathcal{A}^{Q_m^+}.$$

Hence, $\mathscr{V}(m,V) \subseteq \mathscr{V}(n,U)$ follows implying $\mathscr{V}(n,U) \cap \mathscr{V}(m,V) = \mathscr{V}(m,V)$ if the intersection was non-empty and $n \leq m$. This proves that $\mathscr{B}$ is a base for a topology.

Second countable: Due to the finiteness of the alphabet $\mathcal{A}$, the set $\mathcal{A}^{Q_n^+}$ is finite for every $n \in \mathbb{N}$. Since the countable union of finite sets is countable $\mathscr{B}$ is countable. In particular, $Dic(\mathcal{A}^G)$ is second countable.

Hausdorff: Let W_1 and W_2 be two dictionaries. Then

$$\mathscr{V}(n, W_1 \cap \mathcal{A}^{Q_n^+}) \cap \mathscr{V}(n, W_2 \cap \mathcal{A}^{Q_n^+}) \neq \emptyset$$

holds if and only if $W_1 \cap \mathcal{A}^{Q_n^+} = W_2 \cap \mathcal{A}^{Q_n^+}$. This equality holds for all $n \in \mathbb{N}$ if and only if $W_1 = W_2$ implying the Hausdorff property.

Clopen: Consider some $n \in \mathbb{N}$ and $U \subseteq \mathcal{A}^{Q_n^+}$. Let W be in the closure $\overline{\mathscr{V}(n,U)}$, i.e. all neighborhoods of W intersect $\mathscr{V}(n,U)$. In particular, the set $\mathscr{V}(n, W \cap \mathcal{A}^{Q_n^+})$ is an open neighborhood of W. Thus, the intersection

$$\mathscr{V}(n, W \cap \mathcal{A}^{Q_n^+}) \cap \mathscr{V}(n,U)$$

is non-empty implying $U = W \cap \mathcal{A}^{Q_n^+}$. Hence, $W \in \mathscr{V}(n,U)$ follows and so $\mathscr{V}(n,U)$ is clopen. Specifically, the space $Dic(\mathcal{A}^G)$ is totally disconnected. □

Now we can prove one of the main theorems in this chapter where we characterize the topology on $\mathcal{J}$ by the local pattern topology.

THEOREM 3.4 (Local pattern topology). *The space $Dic(\mathcal{A}^G)$ and $\mathcal{J}$ are homeomorphic and the homeomorphism is given by*

$$\Phi : \mathcal{J} \to Dic(\mathcal{A}^G), \quad \Omega \mapsto W(\Omega).$$

REMARK. *We refer the reader to* [**Bec16, BBdN20**] *for further background where this result was originally proven.*

Let $\omega \in \mathcal{A}^G$ and $k \in \mathbb{N}$. Then any pattern $\omega|_{Q_k}$ can be identified with a pattern $p \in \mathcal{A}^{Q_{2k}^+}$ by setting

$$p(x) := \omega\big((k\mathbf{1})^{-1}x\big), \qquad x \in Q_{2k}^+,$$

where $k\mathbf{1} = (k,k,\dots,k) \in G = \mathbb{Z}^d$. We will make this identification without mentioning it explicitly, namely $\mathcal{A}^{Q_k} \cong \mathcal{A}^{Q_{2k}^+}$.

PROOF. Let $\Phi : \mathcal{J} \to Dic(\mathcal{A}^G)$ be the map defined by

$$\Phi(\Omega) := W(\Omega) := \bigcup_{\omega \in \Omega} W(\omega).$$

By definition, the empty set is not an element of $\mathcal{J}$ and so Φ is well-defined by Proposition 3.2. We know that $\mathcal{J}$ is compact and $Dic(\mathcal{A}^G)$ is Hausdorff. Thus, it suffices by standard topological arguments to prove that Φ is continuous and bijective, see e.g. the book *Mengentheoretische Topologie*, Satz 8.11, by B. von Querenburg.

Φ *injective:* Let $\Omega_1, \Omega_2 \in \mathcal{J}$ be distinct. Without loss of generality it can be assumed that $\Omega_1 \setminus \Omega_2 \neq \emptyset$. Let $\omega \in \Omega_1 \setminus \Omega_2$. We will show that there exists an $k \in \mathbb{N}$ such that

$$\omega|_{Q_k} \notin \Phi(\Omega_2) \qquad \text{while} \qquad \omega|_{Q_k} \in \Phi(\Omega_1).$$

Note that we use here the identification $\mathcal{A}^{Q_k} \cong \mathcal{A}^{Q_{2k}^+}$. Assume by contradiction that there is a sequence $\rho_n \in \Omega_2$, $n \in \mathbb{N}$, such that $\rho_n|_{Q_n} = \omega|_{Q_n}$. Hence, by definition of the product topology,

$$\omega = \lim_{n \to \infty} \rho_n \in \overline{\Omega_2} = \Omega_2,$$

because Ω_2 is closed, a contradiction.

Φ *surjective:* Let $W \in Dic(\mathcal{A}^G)$ be a dictionary and define

$$\Omega(W) := \left\{ \omega \in \mathcal{A}^G \,\middle|\, W(\omega) \subseteq W \right\}$$

We will show that $\Omega(W) \in \mathcal{J}$. If $\omega \in \Omega(W)$ and $g \in G$, then

$$W(\omega) = W\big(\overline{Orb(\omega)}\big) = W\big(\overline{Orb(g\omega)}\big) = W(g\omega)$$

follows proving that $g\omega \in \Omega(W)$. Thus, $\Omega(W)$ is invariant. If ω is an element of the closure of $\Omega(W)$, then for any pattern $\omega|_{hQ_m^+} \in W(\omega)$, there is an $n \in \mathbb{N}$ such that $hQ_m^+ \subseteq Q_n$. Since $\omega \in \overline{\Omega(W)}$, we conclude that there is an $\rho \in \Omega(W)$ with $d(\omega, \rho) < \frac{1}{n}$. Thus,

$$\omega|_{hQ_m^+} \prec \omega|_{Q_n} = \rho|_{Q_n} \in W$$

as $\rho \in \Omega(W)$ and using (D1) for W. Hence, $\omega \in \Omega(W)$ is concluded and so $\Omega(W)$ is invariant and closed.

In order to prove surjectivity of Φ, it is left to show that $\Omega(W)$ is non-empty and $\Phi(\Omega(W)) = W$.

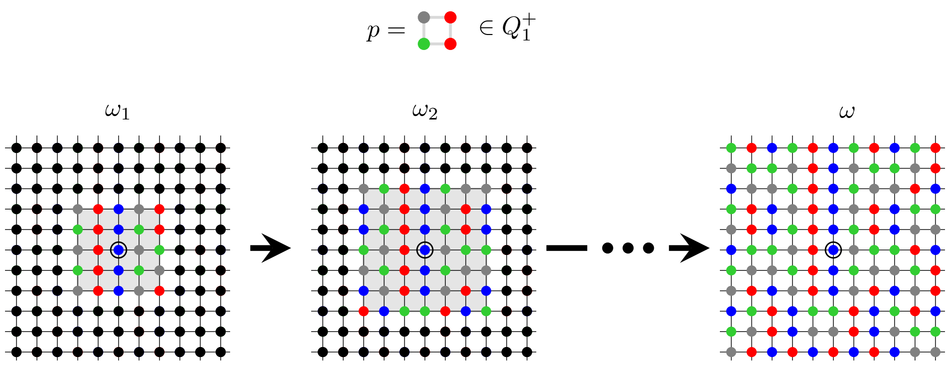

Figure 13. The gray shaded area denote the corresponding patterns p_i. Here the black points denote any fixed letter in $\mathcal{A}$.

Let $p \in W \cap \mathcal{A}^{Q_m^+}$ for some $m \in \mathbb{N}$. Due to (D2), there is a sequence $(p_n)_{n\in\mathbb{N}}$ of patterns such that (we use again the identification $\mathcal{A}^{Q_k} \cong \mathcal{A}^{Q_{2k}^+}$)

$$p \prec p_1, \qquad p_n \in W \cap \mathcal{A}^{Q_{m+n}} \qquad \text{and} \qquad p_{n+1}|_{Q_{m+n-1}} = p_n,$$

confer Figure 13. For each $n \in \mathbb{N}$, let $\omega_n \in \mathcal{A}^G$ be such that $\omega_n|_{Q_{m+n}} = p_n$, which trivially exists (why?). By construction, $(\omega_n)_{n\in\mathbb{N}}$ is convergent and denote its limit point by $\omega \in \mathcal{A}^G$. Since $\bigcup_{k\in\mathbb{N}} Q_k = G$, we conclude $W(\omega) \subseteq W$ by (D1), namely, $\omega \in \Omega(W)$.

Thus, $\Omega(W) \neq \varnothing$ follows. Moreover, $p \in W(\omega)$ holds by construction implying $W \subseteq \Phi(\Omega(W))$. The reverse inclusion $\Phi(\Omega(W)) \subseteq W$ holds by definition. Hence, $W = \Phi(\Omega(W))$.

Φ *continuous:* The set

$$O(p) := \{\omega \in \mathcal{A}^G \mid \omega|_{Q_n^+} = p\}$$

for $p \in \mathcal{A}^{Q_n^+}$ is clopen in the product topology of $\mathcal{A}^G$. Consider a non-empty open set $\mathscr{V}(n,U)$ in the local pattern topology where $n \in \mathbb{N}$ and $U := \{p_1, \dots, p_l\} \subseteq \mathscr{A}^{Q_n^+}$. In order to prove the continuity of Φ, the preimage $\Phi^{-1}(\mathscr{V}(n,U))$ needs to be open. Let

$$F := \bigcap_{j=1}^{l} \big(\mathcal{A}^G \setminus O(p_j)\big) = \big\{\omega \in \mathcal{A}^G \;\big|\, \omega|_{Q_n^+} \neq p_j \text{ for all } 1 \leq j \leq l\big\}$$

a closed subset of $\mathcal{A}^G$. Set $\mathcal{O} := \{O(p_1), \dots, O(p_l)\}$ a finite family of open subsets of $\mathcal{A}^G$. It suffices to show the equality

$$\begin{aligned} \mathcal{U}(F,\mathcal{O}) &:= \big\{\Omega \in \mathcal{J} \;\big|\; F \cap \Omega = \varnothing \text{ and } \Omega \cap O(p_j) \neq \varnothing \text{ for all } 1 \leq j \leq l\big\} \\ &= \Phi^{-1}(\mathscr{V}(n,U)). \end{aligned}$$

Let $\Omega \in \mathcal{U}(F,\mathcal{O})$. Since $\Omega \cap O(p_j) \neq \varnothing$ for all $1 \leq j \leq l$ it follows by definition of $W(\Omega)$ that

$$\{p_1, \dots, p_l\} \subseteq W(\Omega) \cap \mathscr{A}^{Q_n^+}.$$

Assume by contradiction that the latter inclusion is strict, then there is an $\omega \in \Omega$ with $\omega|_{Q_n^+} \neq p_j$ for all $1 \leq j \leq n$. Thus, $\omega \in F$ contradicting $F \cap \Omega = \varnothing$. Hence, $\Omega \in \Phi^{-1}(\mathscr{V}(n,U))$ follows implying $\mathcal{U}(F,\mathcal{O}) \subseteq \Phi^{-1}(\mathscr{V}(n,U))$.

It is left to show $\Phi^{-1}(\mathscr{V}(n,U)) \subseteq \mathcal{U}(F,\mathcal{O})$. Let $\Omega \in \Phi^{-1}(\mathscr{V}(n,U))$ meaning

$$W(\Omega) \cap \mathscr{A}^{Q_n^+} = \{p_1, \dots, p_l\}.$$

Since $p_j \in W(\Omega)$, we conclude $\Omega \cap O(p_j) \neq \varnothing$ by the definition of $W(\Omega)$. Now, assume by contradiction that $\Omega \cap F \neq \varnothing$. Then there is a $\omega \in \Omega$ such that $p := \omega|_{Q_n^+} \neq p_j$ for all $1 \leq j \leq l$. Thus,

$$p \in W(\Omega) \cap \mathscr{A}^{Q_n^+} \qquad \text{and} \qquad p \notin \{p_1, \dots, p_l\},$$

a contradiction with $\Omega \in \Phi^{-1}(\mathscr{V}(n,U))$. Consequently, we have proven that $\Omega \in \mathcal{U}(F,\mathcal{O})$. $\square$

Remark. *Since $\mathcal{J}$ is compact and metrizable, the latter theorem implies that $Dic(\mathcal{A}^G)$ is compact and metrizable. Indeed the compactness of $Dic(\mathcal{A}^G)$ can be checked directly by a Cantor-diagonalization argument.*

We highlight here that the dynamics is strongly involved in the topology on $\mathcal{J}$ (which is a bit hidden). For indeed the open sets $O(p_j)$ are only requiring that one observes the pattern p_j at a specific place. However, since Ω is invariant, we can always shift any pattern occurring in $\omega \in \Omega$ to the origin. This is crucial as we will see later when we deal with periodic approximations.

Corollary 3.5. *The space $\mathcal{J}$ is totally disconnected.*

Proof. This follows as $Dic(\mathcal{A}^G)$ and $\mathcal{J}$ are homeomorphic and $Dic(\mathcal{A}^G)$ is totally disconnected. □

Corollary 3.6. *A sequence $(\Omega_n)_n \subseteq \mathcal{J}$ converges to $\Omega \in \mathcal{J}$ if and only if, for every $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ such that*

$$W(\Omega) \cap \mathcal{A}^{Q_m^+} = W(\Omega_n) \cap \mathcal{A}^{Q_m^+}, \qquad n \geq n_m.$$

Proof. This follows immediately by the definition of the local pattern topology. □

Another consequence of the previous statement is the following characterization of minimality.

Theorem 3.7 (Minimality- bounded gaps). *Let $\Omega \in \mathcal{J}$ be a subshift of the dynamical system $(\mathcal{A}^G, G)$. Then the following are equivalent.*

(i) *The subshift (Ω, G) is minimal.*
(ii) *For every $p \in W(\Omega)$, there is an $m \in \mathbb{N}$ such that every $q \in W(\Omega) \cap \mathcal{A}^{Q_n^+}$ with $n \geq m$ satisfies that $p \prec q$.*
(iii) *We have $W(\Omega) = W(\overline{Orb(\omega)})$ for all $\omega \in \Omega$.*

Remark. *Let $\omega \in \Omega$ where Ω satisfies one of the previous conditions. Then $\Omega = \overline{Orb(\omega)}$ holds by minimality. The second condition (ii) says that patterns occur in bounded distance, namely for any pattern $p \in \mathcal{A}^{Q_k^+} \cap W(\Omega)$ there is an $m \in \mathbb{N}$ such that every pattern q with support Q_n^+ for $n \geq m$ contains a copy of the pattern p. Such ω are called* repetitive. *If m depends linearly on the size of p (namely $m = Ck$ for some $C > 0$), then ω is even called* linear repetitive. *Linear repetitive ω are interesting as their orbit closure $\overline{Orb(\omega)}$ turns out to be uniquely ergodic and minimal. We will later get to know the typical examples for linear repetitive configurations which are defined by substitutions. It is the content of current research studying these systems.*

Proof. (i)⇒(ii): Assume by contradiction that there exists a pattern $p \in W(\Omega) \cap \mathcal{A}^{Q_m^+}$ for some $m \in \mathbb{N}$ and a sequence of patterns

$$q_n := \omega_n|_{Q_{n+m}} \in W(\Omega) \cap \mathcal{A}^{Q_{n+m}} \qquad \text{with } \omega_n \in \Omega$$

such that p is not occurring in q_n, i.e. $p \nprec q_n$. Here we use again the identification $\mathcal{A}^{Q_k} \cong \mathcal{A}^{Q_{2k}^+}$.

Since Ω is compact, there is no loss of generality in assuming that $(\omega_n)_{n\in\mathbb{N}}$ converges to $\omega \in \Omega$ (otherwise pass to a subsequence). Then the convergence $\omega_n \to \omega$ means that for every $m \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that

$$\omega|_{Q_m} = \omega_n|_{Q_m}, \qquad n \geq n_0.$$

By construction, we have $p \notin W(\overline{Orb(\omega)})$ (otherwise it must appear in one pattern q_n).

Since $p \in W(\Omega)$, there is an $\rho \in \Omega$ such that $\rho|_{Q_m^+} = p$. Then $p \notin W(\overline{Orb(\omega)})$ leads to

$$\rho|_{Q_m^+} \neq \omega'|_{Q_m^+}$$

for all $\omega' \in \overline{Orb(\omega)}$, or equivalently

$$d(\rho, \omega') = \min\left\{1, \inf\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N} \text{ be such that } \omega'|_{Q_n} = \rho|_{Q_n}\right\}\right\} \geq \frac{1}{m}.$$

Thus, $\rho \notin \overline{Orb(\omega)}$ while $\overline{Orb(\omega)} = \Omega$ by minimality, a contradiction.

(ii)⇒(iii): This is clear as every $\omega \in \Omega$ has arbitrarily large patterns. More precisely, $W(\omega) \subseteq W(\Omega)$ holds as $\omega \in \Omega$.

For the reverse inclusion, let $p \in W(\Omega)$. By (ii), there is an $m \in \mathbb{N}$ such that all $q \in W(\Omega) \cap \mathcal{A}^{Q_n^+}$ with $n \geq m$ satisfy $p \prec q$. Thus, $q := \omega|_{Q_m^+} \in W(\Omega)$ holds since $\omega \in \Omega$. Consequently, $p \prec \omega|_{Q_m^+} \in W(\omega)$ follows by the previous considerations implying $W(\Omega) \subseteq W(\omega)$.

(iii)⇒(i): Let $\Phi : \mathcal{J} \to Dic(\mathcal{A}^G)$ be the homeomorphism defined in Theorem 3.4. If $W(\Omega) = W(\overline{Orb(\omega)})$ for all $\omega \in \Omega$, then

$$\Phi(\overline{Orb(\omega)}) = \Phi(\Omega)$$

holds for all $\omega \in \Omega$. Since Φ is injective, we conclude $\Omega = \overline{Orb(\omega)}$ for all $\omega \in \Omega$, namely Ω is minimal. □

3.2. Periodic elements. In this section we provide a characterization of periodic elements in $\mathcal{A}^G$. Therefore, recall that a periodic element in the one-dimensional case $G = \mathbb{Z}$ is given by v^∞ for some finite word v. We will provide the corresponding analog in the case $G = \mathbb{Z}^d$.

DEFINITION. *Let $(\mathcal{A}^G, G)$ be a dynamical system. An element $\omega \in \mathcal{A}^G$ is called*

- periodic *if the orbit $Orb(\omega)$ is finite.*
- aperiodic *if $g\omega = \omega$ for $g \in G$ imply that $g = e$.*

REMARK. *(a) Recall that a dynamical system (Y, G) is periodic if Y is minimal and finite. Then ω is periodic if and only if $\overline{Orb(\omega)}$ is periodic.*

*(b) We point out that if $\omega \in \mathcal{A}^G$ is not periodic, then it is not necessarily aperiodic. Specifically, there are $\omega \in \mathcal{A}^G$ such that ω is not periodic nor aperiodic if $G = \mathbb{Z}^d$ with $d \geq 2$ (**Exercise**). However, for $G = \mathbb{Z}$, every $\omega \in \mathcal{A}^{\mathbb{Z}}$ is either periodic or aperiodic (dichotomy) (**Exercise**).*

(c) Note that these notions are not unique in the literature. Specifically, elements that we call periodic are sometimes called strongly periodic and an aperiodic $\omega \in \mathcal{A}^G$ is also called non-periodic in the literature.

(d) The stabilizer $Stab(\omega)$ of $\omega \in \mathcal{A}^G$ is defined by $\{g \in G \,|\, g\omega = \omega\}$. It is straightforward to check that $Stab(\omega) \subseteq G$ is a subgroup (which in general is not normal). Furthermore $e \in Stab(\omega)$ holds always. Then ω is aperiodic if $Stab(\omega) = \{e\}$ and ω is periodic if the cardinality of the cosets $\{gStab(\omega) \,|\, g \in G\}$ is finite.

EXAMPLE. *For $\mathcal{A} := \{a, b\}$, consider the one-defect. Specifically, let $\omega \in \mathcal{A}^{\mathbb{Z}}$ defined by*

$$\omega(n) := \begin{cases} a, & n \neq 0, \\ b, & n = 0. \end{cases}$$

Then ω is aperiodic as

$$b = \omega(0) \neq \omega(0 + p) = a$$

for all $p \in \mathbb{Z} \setminus \{0\}$.

We will provide a characterization of periodic elements for $G = \mathbb{Z}^d$. Therefore, we need the following exercise.

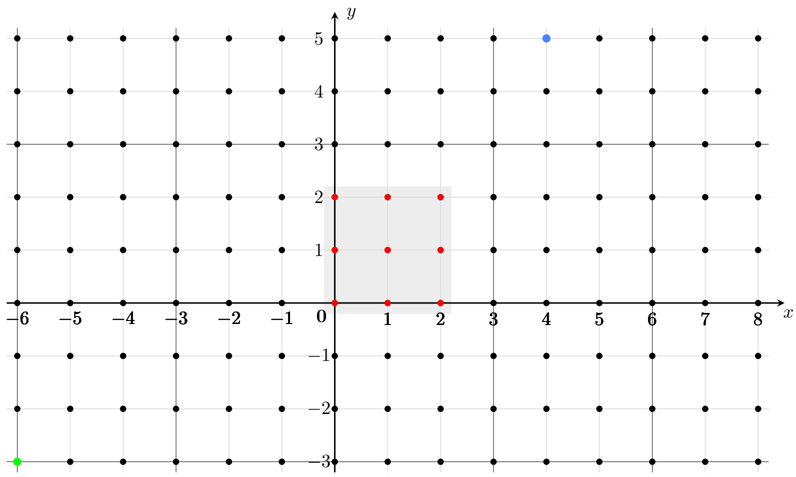

Figure 14. The gray shaded area (red points) denote the set Q_2. Then $g' = (3,3)$ and $g_h = (1,2)$ are the unique elements for the blue point at $(4,5)$. Similarly, $g' = (-6,-3)$ and $g_h = (0,0)$ are the unique elements for the green point at $(-6,-3)$.

Exercise. *For $m \in \mathbb{N}$, define*

$$m \cdot \mathbb{Z}^d := \left\{ (m \cdot g_1, \ldots, m \cdot g_d) \,\middle|\, (g_1, \ldots, g_d) \in \mathbb{Z}^d \right\}.$$

Prove that for all $n \in \mathbb{N}_0$, the equality

$$\mathbb{Z}^d = \bigsqcup_{g' \in (n+1) \cdot \mathbb{Z}^d} g' Q_n^+$$

holds where the union is a disjoint union.

Thus, for each $g \in G = \mathbb{Z}^d$, there is a unique $g' \in (n+1) \cdot \mathbb{Z}^d$ and a unique $g_n \in Q_n^+$ such that

$$g = g' g_n,$$

confer Figure 14 for an illustration.

For any $p \in \mathcal{A}^{Q_n^+}$, define $\omega_p := p^\infty \in \mathcal{A}^G$ by

$$\omega_p(g) := p(g_n), \qquad g \in G,$$

where $g_n \in Q_n^+$ is the unique element for $g \in G$ satisfying $g = g' g_n$ with $g' \in (n+1) \cdot \mathbb{Z}^d$. An example is given in Figure 15.

Proposition 3.8. *Let $\omega \in \mathcal{A}^G$ with $G = \mathbb{Z}^d$. Then the following is equivalent.*

(i) *ω is periodic.*

(ii) *There exists an $n \in \mathbb{N}_0$ and a $p \in \mathcal{A}^{Q_n^+}$ such that $\omega = \omega_p$.*

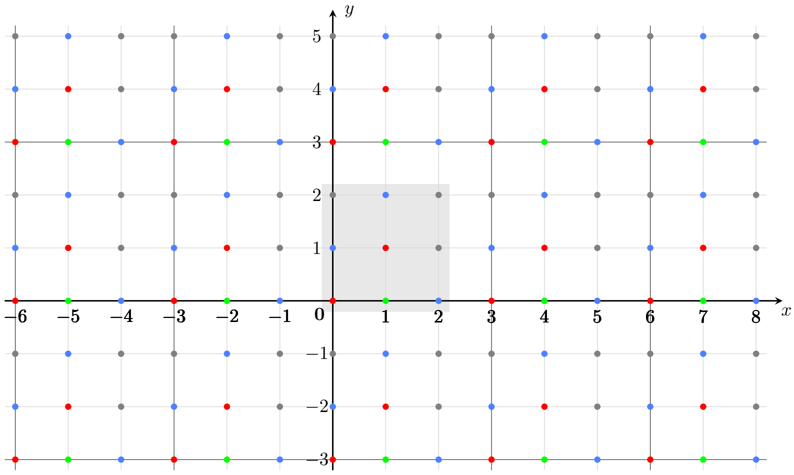

Figure 15. An example for an ω_p for Q_2^+. The pattern p is marked by the gray box.

Proof. This is left as an exercise. ☐

Week 5

3.3. Aperiodic subshifts. We provide here the notion of aperiodic subshifts.

Definition. *Consider the dynamical system $(\mathcal{A}^G, G)$. A subshift $\Omega \in \mathcal{J}$ is called*

- weakly aperiodic *if there exists an $\omega \in \Omega$ such that ω is aperiodic;*
- strongly aperiodic *if all elements of $\omega \in \Omega$ are aperiodic.*

An example of a weakly aperiodic but not strongly aperiodic subshift is discussed in the exercise session. Later we will provide also various examples for strongly aperiodic subshifts (substitutions for $G = \mathbb{Z}$).

Theorem 3.9. *Let $\Omega \in \mathcal{J}$ be a subshift over the alphabet $\mathcal{A}$. If Ω is weakly aperiodic and minimal, then Ω is strongly aperiodic.*

Proof. Since Ω is weakly aperiodic, there is an aperiodic $\omega \in \Omega$. Let $\rho \in \Omega$ and $h \in G$ be such that $h\rho = \rho$. We have to show that $h = e$. Minimality of Ω implies that there is a sequence $g_n \in \mathbb{N}$ such that $g_n\rho \to \omega$. Hence,

$$\omega = \lim_{n\to\infty} g_n\rho = \lim_{n\to\infty} g_n h\rho = \lim_{n\to\infty} hg_n\rho = h\omega$$

by using that $G = \mathbb{Z}^d$ is abelian and that the action of G on $\mathcal{A}^G$ is continuous. Since ω is aperiodic, we conclude that $h = e$ and so ρ is aperiodic. ☐

Remark. *Note this proof only works for abelian groups. In current research, it is studied when such properties also hold for general countable groups. For instance, the concept of substitutions (that we will see later) is extended to rational homogenuous groups including groups like the Heisenberg group. In this situation it becomes more involved to show that the corresponding subshift is strongly aperiodic if it is weakly aperiodic and minimal. It would be also interesting to find examples where the later statement fails.*

4. The one-dimensional case

In the following, we will restrict our considerations again to the one-dimensional case $G := \mathbb{Z}$. Recall that $Q_n^+ = \{0, 1, 2, \dots n\}$ holds and so $p \in \mathcal{A}^{Q_n^+}$ can be identified with a word $a_0 \dots a_n$ with letters $a_i \in \mathcal{A}$ of length $n+1$. In light of this, we introduced the notation $\mathcal{A}^{n+1} := \mathcal{A}^{Q_n^+}$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and we called elements of $\mathcal{A}^n$ *words (or patterns) of length* $n \in \mathbb{N}$.

Furthermore the concatenation of two words $u \in \mathcal{A}^n$ and $v \in \mathcal{A}^m$ was defined by

$$uv = u_1 \dots u_n v_1 \dots v_m \in \mathcal{A}^{n+m}.$$

Then

$$u^k := uu \dots u$$

is the kth time concatenation of the word u. In dimension one, the so-called *empty word* ϵ is the word of length zero. Then $u \in \mathcal{A}^n$ occurs in $v \in \mathcal{A}^m$ if and only if there are $u_1, u_2 \in \mathcal{A}^* \cup \{\epsilon\}$ such that

$$v = u_1 u u_2.$$

Clearly, a necessary condition for this is that $n \leq m$.

Any element in $\omega \in \mathcal{A}^{\mathbb{Z}}$ can be represented by a two-sided infinite word

$$\omega := \dots\ \omega(-3)\ \omega(-2)\ \omega(-1)\ \big|\ \omega(0)\ \omega(1)\ \omega(2)\ \dots$$

where $\big|$ indicates the origin. Then the two-sided infinite concatenation of word $u, v \in \mathcal{A}^*$ is defined by

$$u^\infty | v^\infty := \dots u\ u\ u\ \big|\ v\ v\ v\ \dots\ .$$

Furthermore, we set $u^\infty := u^\infty | u^\infty$.

4.1. Aperiodicity. Consider the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$. Recall that an $\omega \in \mathcal{A}^{\mathbb{Z}}$ is aperiodic if its stabilizer is trivial, i.e. $g\omega \neq \omega$ whenever $g \in \mathbb{Z} \setminus \{0\}$. On the other hand, $\omega \in \mathcal{A}^{\mathbb{Z}}$ is periodic, if its orbit $Orb(\omega)$ is finite. Due to Proposition 3.8, this is equivalent to the existence of an $v \in \mathcal{A}^L$ such that $\omega = v^\infty$. Clearly, $v = \omega(0) \dots \omega(L-1)$ must hold and

$$\omega(n) = \omega(n+L), \qquad n \in \mathbb{Z}.$$

Then L is called a *period* of ω (which is not unique!).

The next theorem provides a characterization of a weakly aperiodic subshift. In the case that Ω is minimal, this provides a characterization of strongly aperiodic subshifts. Before we need the following auxiliary statement.

LEMMA 4.1. *Let $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift. Suppose there are $L \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that*

$$\big(L^{-1}\omega\big)(m) = \omega(m) \qquad \textit{for all } m \leq k,\ \omega \in \Omega.$$

Then every element in Ω is periodic with period L.

PROOF. Assume by contradiction that there is an $\omega \in \Omega$ that is not periodic with period L. Hence, there is an $n \in \mathbb{Z}$ such that $\omega(n) \neq \omega(n+L)$. Clearly, $n > k$ must hold. Set $\rho := (k-n)\omega \in \Omega$. Then

$$\rho(k) = \omega(n) \neq \omega(n+L) = \rho(k+L) = (L^{-1}\rho)(k)$$

follows, a contradiction. □

Theorem 4.2 (Characterization of weakly aperiodicity). *Let $G = \mathbb{Z}$ and $\Omega \in \mathcal{J}$ be a subshift of $\mathcal{A}^{\mathbb{Z}}$. Then the following is equivalent.*

(i) *Ω is weakly aperiodic.*
(ii) *There are $\omega, \rho \in \Omega$ with $\omega \neq \rho$ and $\omega|_{\mathbb{N}_0} = \rho|_{\mathbb{N}_0}$.*
(iii) *There are $\omega, \rho \in \Omega$ with $\omega \neq \rho$ and $\omega|_{-\mathbb{N}} = \rho|_{-\mathbb{N}}$.*

Proof. We will only prove the equivalence between (i) and (ii), the equivalence between (i) and (iii) can be proven similarly. We remind the reader that if $\omega \in \mathcal{A}^{\mathbb{Z}}$ is not periodic then it is automatically aperiodic (which does not hold for general groups G), see the exercise session and discussion in Section 3.2.

(ii)⇒(i): We will show that ρ or ω cannot be both periodic. Assume by contradiction that ω and ρ are periodic with period $L_1 \in \mathbb{N}$ and $L_2 \in \mathbb{N}$. Then $L := L_1 L_2$ is a period of ρ and ω. Thus, $L\omega = \omega$ and $L\rho = \rho$ follows, see Figure 16. Thus, ω and ρ are uniquely determined by

$$\omega(0)\dots\omega(L-1) = \rho(0)\dots\rho(L-1).$$

Since $\omega|_{\mathbb{N}} = \rho|_{\mathbb{N}}$, we conclude that $\omega = \rho$, a contradiction.

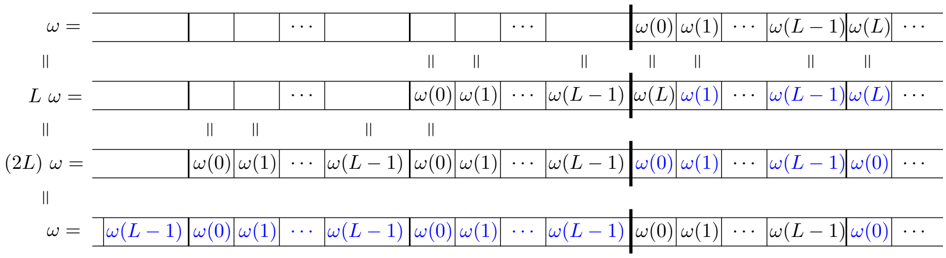

Figure 16. If $L\omega = \omega$ for some $L \in \mathbb{N}$, then ω is uniquely determined by $\omega(0)\dots\omega(L-1)$.

(i)⇒(ii): Consider the maps

$$P_+ : \Omega \to \mathcal{A}^{\mathbb{N}_0},\ \omega \mapsto \omega|_{\mathbb{N}_0}, \qquad \text{and} \qquad P_- : \Omega \to \mathcal{A}^{-\mathbb{N}},\ \omega \mapsto \omega|_{-\mathbb{N}}.$$

Then P_+ and P_- are continuous maps in the corresponding product topologies. Thus, their images

$$\Omega_+ := P_+(\Omega) \qquad \text{and} \qquad \Omega_- := P_-(\Omega)$$

are compact as continuous images of compact sets. Assume by contradiction that (ii) does not hold. Hence, P_+ is injective. Thus,

$$P_+ : \Omega \to P_+(\Omega)$$

is bijective and continuous. Since both spaces are compact and Hausdorff, we conclude that P_+ has a continuous inverse. Hence, the map

$$R : \Omega_+ \to \Omega_-, \quad \omega \mapsto P_-\big(P_+^{-1}(\omega)\big)$$

is continuous. Thus the values of $\omega \in \Omega$ on the right half-axis determine the values on the left hand side in a continuous way. To be more precise, consider the clopen set $U(a) := \{\omega \in \Omega_- \,|\, \omega(-1) = a\}$. Then the preimage $R^{-1}(U(a))$

is clopen as well (by continuity of R). Moreover, there are finitely many word $u_1, \dots, u_M \in \mathcal{A}^*$ such that

$$R^{-1}(U(a)) = \bigsqcup_{j=1}^{M} O(u_j) \qquad \text{(disjoint union)}$$

where $O(u) := \{\omega \in \Omega_+ \,|\, \omega|_{\{0,\dots,|u|-1\}} = u\}$ for $u \in \mathcal{A}^*$. Note that the u_j's can be chosen such that $|u_j| = |u_i|$ (Exercise). Since this works for each letter $a \in \mathcal{A}$ and since $\mathcal{A}$ is finite, we conclude that there is an $N \in \mathbb{N}$ (the maximum of the length of the u_j's) such that $\omega(0)\dots\omega(N)$ determines $\omega(-1)$ uniquely for all $\omega \in \Omega$. Thus, iteratively $v := \omega(0)\dots\omega(N)$ uniquely determines $\omega|_{-\mathbb{N}}$. Specifically, we get $v_1 := \omega(-1)\omega(0)\dots\omega(N-1)$ which is uniquely defined by v. Set $\omega_1 := 1^{-1}\omega$. Then $\omega_1(0)\dots\omega_1(N) = v_1$ which uniquely determines $\omega_1(-1) = \omega(-2)$. Iterating this argument proves that v uniquely defines $\omega|_{-\mathbb{N}}$.

Let $\omega \in \Omega$ and $v := \omega(0)\dots\omega(N)$. Set $k_1 := 1 + \sharp\mathcal{A}^{N+1}$. Then there is a $u \in \mathcal{A}^{N+1}$ that appears twice on $\omega|_{\{-k_1,\dots,N\}}$, see Figure 17. Specifically, the two copies of u start at i_1 and i_0 with $-k_1 \le i_1 < i_0 \le N$. Set $L_v := i_0 - i_1 \ge 1$.

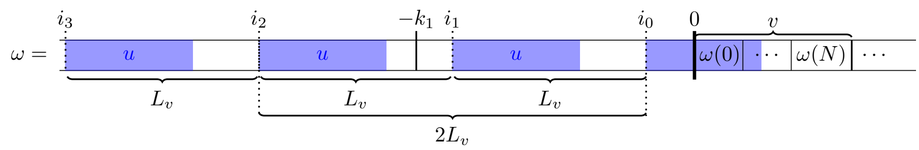

Figure 17. The blue box is the word u that repeats periodically to the left with period L_v.

By construction any word of length $N+1$ uniquely determines its letter to the left. Based on this, the word $\omega(i_1)\dots\omega(i_0) \in \mathcal{A}^{L_v}$ repeats periodically to the left as sketched in Figure 17. More precisely, we conclude

$$\big(L_v^{-1}\omega\big)(m) = \omega(m) \qquad \text{for all } m \le i_0.$$

Since there are only finitely many words $v \in \mathcal{A}^{N+1}$,

$$L := \prod_{v \in \mathcal{A}^{N+1}} L_v$$

is finite. Thus, any $\omega \in \Omega$ is L-periodic to the left, namely

$$\big(L^{-1}\omega\big)(m) = \omega(m) \qquad \text{for all } m \le -k_1,\ \omega \in \Omega.$$

Then the previous Lemma 4.1 implies that every $\omega \in \Omega$ is periodic with period L, contradicting (i). $\square$

4.2. Periodic approximations. With the characterization of the topology on $\mathcal{J}$ by the local pattern topology on dictionaries at hand, we can show that every minimal dynamical system of $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ admits periodic approximations in $\mathcal{J}$.

Definition. *A subshift $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ is called* periodically approximable, *if there are periodic $\Omega_n \in \mathcal{J}$ such that $\Omega_n \to \Omega$ in $\mathcal{J}$.*

Theorem 4.3 (Periodic approximations of minimal dynamical systems). *Every minimal subshift $\Omega \in \mathcal{J}$ of the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ is periodically approximable.*

Remark. *We refer the reader to* [**Bec16, BBdN20**] *for further background where this result was originally proven.*

We point out that this result holds only in dimension one, namely $G = \mathbb{Z}$. Specifically, for higher dimensions one cannot expect that minimality implies that the dynamical system admits periodic approximations. In higher dimensions (already for $\mathbb{Z}^2$) there are only few sufficient conditions on a subshift to be periodically approximable. Later we will learn about a sufficient condition for subshifts defined by substitution. This result extends to higher dimensions. For more general dynamical systems like Delone sets this question is also open.

Proof. Let $\Omega \in \mathcal{J}$ be minimal and $\omega \in \Omega$ be fixed. For $m \in \mathbb{N}$, we will define a periodic $\Omega_n \in \mathcal{J}$ such that

$$W(\Omega_n) \cap \mathcal{A}^m = W(\Omega) \cap \mathcal{A}^m, \qquad n \geq m.$$

This will imply that $\Omega_n \to \Omega$ in $\mathcal{J}$ by Corollary 3.6.

Let $n \in \mathbb{N}$. Since $W(\Omega) \cap \mathcal{A}^n$ is finite, the characterization (ii) of minimality (Theorem 3.7) implies that there is an $m_n \in \mathbb{N}$ such that every word $u \in W(\Omega) \cap \mathcal{A}^n$ occurs in $\omega|_{\{0,1,\dots,m_n\}}$. Note that $m_n \geq n$ must hold. Using again the characterization (ii) of minimality, we conclude that there is $k_n \in \mathbb{N}$ such that $k_n \geq m_n$ and

$$\omega(0)\omega(1)\dots\omega(n-1) = \omega(k_n)\omega(k_n+1)\dots\omega(k_n+n-1).$$

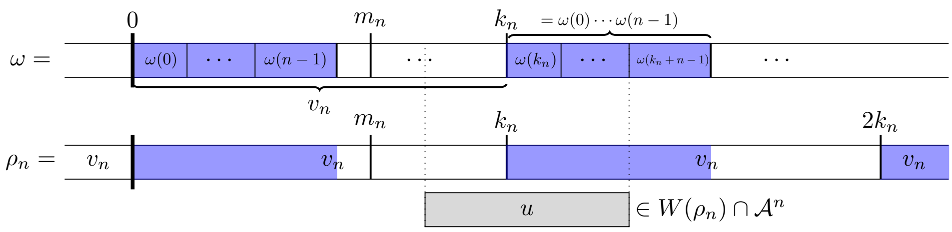

Figure 18. A sketch of the construction of the periodic approximations ρ_n. The word u is a generic element of $W(\rho_n) \cap \mathcal{A}^n$ that must occur in $v_n v_n$.

Define $v_n := \omega(0)\omega(1)\dots\omega(k_n - 1) \in \mathcal{A}^{k_n}$ and $\rho_n \in \mathcal{A}^{\mathbb{Z}}$ by

$$\rho_n := v_n^{\infty} = \dots\ v_n\ v_n\ v_n\ \big|\ v_n\ v_n\ v_n\ \dots\ .$$

Set $\Omega_n := Orb(\rho_n)$ which is periodic by definition. By construction, every $u \in W(\Omega) \cap \mathcal{A}^n$ occurs in $v_n \in W(\Omega_n)$ and so

$$W(\Omega_n) \cap \mathcal{A}^n \supseteq W(\Omega) \cap \mathcal{A}^n, \qquad n \in \mathbb{N},$$

follows. For the reverse inclusion, let $u \in W(\Omega_n) \cap \mathcal{A}^n$. By definition of ρ_n and since $|v_n| \geq n$, we conclude that u must occur in

$$v_n\ \big|\ v_n(0)v_n(1)\dots v_n(n-1) = \omega(0)\dots\omega(k_n-1)\ \big|\ \omega(k_n)\dots\omega(k_n+n-1).$$

Hence, $u \in W(\Omega)$ follows implying

$$W(\Omega_n) \cap \mathcal{A}^n = W(\Omega) \cap \mathcal{A}^n, \qquad n \in \mathbb{N}.$$

By Sheet 5, we conclude $W(\Omega_n) \cap \mathcal{A}^k = W(\Omega) \cap \mathcal{A}^k$ for all $1 \le k \le n$. Hence

$$W(\Omega_n) \cap \mathcal{A}^m = W(\Omega) \cap \mathcal{A}^m, \qquad n \ge m,$$

follows. Invoking Corollary 3.6, we conclude that $\Omega_n \to \Omega$ in $\mathcal{J}$ where each Ω_n is periodic. □

We will see in the exercise session that there are also non-minimal elements of $\mathcal{J}$ that admit periodic approximations (one-defect). However, not all subshifts are periodically approximable.

Example. *Let $\mathcal{A} := \{a, b\}$ and $w \in \mathcal{A}^{\mathbb{Z}}$ be defined by*

$$w(n) := \begin{cases} a, & n \le 0, \\ b, & n \ge 1, \end{cases} \qquad n \in \mathbb{Z}.$$

*Then $\Omega := \overline{Orb(w)}$ is an isolated point in $\mathcal{J}$ (**Exercise**). In particular, Ω does not admit periodic approximations.*

4.3. Substitutions. In this section, we will provide a specific way to define strongly aperiodic and minimal subshift by so called primitive substitutions. The main idea is that we define a map $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by substituting every letter by a word. This substitution is also denoted by S. Since the local patterns (words) are completely determined by the local rules of the substitution, we conclude the minimality (all patterns appear in a bounded distance). The aperiodicity is proven by using our previous result (Theorem 4.2). Furthermore, we construct recursively periodic approximations for such dynamical systems.

Definition. *A* substitution over the alphabet $\mathcal{A}$ *is a map*

$$S : \mathcal{A} \to \mathcal{A}^* := \bigcup_{n \in \mathbb{N}} \mathcal{A}^n.$$

Note that we require that S does not send any letter to the empty word. Let us first provide various examples.

Example (Fibonacci substitution). *The alphabet is given by $\mathcal{A} := \{a, b\}$ and S is defined by*

$$S(a) := ab, \qquad S(b) := a.$$

Example (Thue-Morse substitution). *The alphabet is given by $\mathcal{A} := \{a, b\}$ and S is defined by*

$$S(a) := ab, \qquad S(b) := ba.$$

Example (Period Doubling substitution). *The alphabet is given by $\mathcal{A} :=$ $\{a, b\}$ and S is defined by*

$$S(a) := ab, \qquad S(b) := aa.$$

Example (Golay-Rudin-Shapiro substitution). *The alphabet is given by $\mathcal{A} := \{a, b, c, d\}$ and S is defined by*

$$S(a) := ab, \qquad S(b) := ac, \qquad S(c) := db, \qquad S(d) = dc.$$

We can extend a substitution to a map $S : \mathcal{A}^* \to \mathcal{A}^*$ by applying it letterwise, namely

$$S : \mathcal{A}^* \to \mathcal{A}^*, \quad S(u_1 \dots u_n) = S(u_1)S(u_2)\dots S(u_n).$$

From these words and their subwords (which are called legal words), we can define a dictionary $W(S)$ and hence a subshift $\Omega(S)$.

DEFINITION. *Let S be a substitution over the alphabet $\mathcal{A}$. Then $v \in \mathcal{A}^*$ is called* legal (w.r.t. S) *if there is a letter $b \in \mathcal{A}$ and an $n \in \mathbb{N}$ such that $v \prec S^n(b)$. Furthermore, we define*

$$W(S) := \left\{ v \in \mathcal{A}^n \,\middle|\, n \in \mathbb{N},\ v \text{ is legal w.r.t. } S \right\}.$$

In general $W(S)$ is not a dictionary. For instance, the substitution $S(a) := a$ and $S(b) := b$ over the alphabet $\mathcal{A} := \{a, b\}$ satisfies $W(S) = \{a, b\}$. In particular, condition (D2) of a dictionary is not always satisfied. In light of this (and since we like to have minimal subshifts), we will focus on so-called primitive substitutions.

DEFINITION. *A substitution S over an alphabet $\mathcal{A}$ is called* primitive*, if there is an $L \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$, we have*

$$a \prec S^L(b).$$

In this case, we will also say that S is primitive with exponent L.

EXAMPLE (Fibonacci substitution). *For the Fibonacci substitution $S(a) := ab$, $S(b) := a$ over $\mathcal{A} := \{a, b\}$, we have that $b \nprec S(b)$. However,*

$$S^2(a) = S(ab) = S(a)S(b) = aba, \qquad S^2(b) = S(a) = ab$$

holds and so the Fibonacci substitution is primitive with exponent $L = 2$.

EXAMPLE (Thue-Morse substitution). *The Thue-Morse substitution $S(a) := ab$, $S(b) := ba$ over $\mathcal{A} := \{a, b\}$ is primitive with $L = 1$.*

EXAMPLE (Period Doubling substitution). *The Period doubling substitution $S(a) := ab$, $S(b) := aa$ over $\mathcal{A} := \{a, b\}$ is primitive with $L = 2$ since*

$$S^2(a) = S(ab) = ab\ aa \quad \text{and} \quad S^2(b) = S(aa) = ab\ ab.$$

EXAMPLE (Golay-Rudin-Shapiro substitution). *The Golay-Rudin-Shapiro substitution*

$$S(a) := ab, \qquad S(b) := ac, \qquad S(c) := db, \qquad S(d) = dc$$

*over $\mathcal{A} := \{a, b, c, d\}$ is primitive with $L = 3$. **(Exercise)***

PROPOSITION 4.4 (Primitive substitutions). *Let S be a primitive substitution over $\mathcal{A}$ with $\sharp\mathcal{A} \geq 2$. Then the following assertions holds.*

(a) $|S^n(a)| \to \infty$ *for all $a \in \mathcal{A}$.*
(b) *For all $u \in W(S)$, there is an $n_u \in \mathbb{N}$ such that $u \prec S^n(a)$ for all $n \geq n_u$ and each $a \in \mathcal{A}$.*
(c) *$W(S)$ is a dictionary.*

PROOF. Since S is primitive, there is an $L \in \mathbb{N}$ such that $a \prec S^L(b)$ for all $a, b \in \mathcal{A}$.

(a) This is left as an exercise (Sheet 6).

(b) The main idea of the proof is sketched in Figure 19. Let $u \in W(S)$. Then there is an $m \in \mathbb{N}$ and a letter $b \in \mathcal{A}$ such that $u \prec S^m(b)$. For $n_u := 3L + m$, we will show that for all $a \in \mathcal{A}$ and $n \geq n_u$, there are $v_1, v_2 \in \mathcal{A}^*$ non-empty words such that

$$v_1 u v_2 = S^n(a).$$

By (a), $|S^{2L}(a)| \geq 4$ holds for any letter $a \in \mathcal{A}$. Thus, we can choose a $d \in \mathcal{A}$ and two non-empty words $\tilde{v}$ and $\tilde{w}$ such that $S^{2L}(a) = \tilde{v} d \tilde{w}$ (namely $\tilde{v} \neq \epsilon$ and $\tilde{w} \neq \epsilon$). Hence,

$$S^{3L}(a) = S^L(\tilde{v})\, S^L(d)\, S^L(\tilde{w})$$

and $b \prec S^L(d)$ follow since S is primitive with exponent L. Let $n \geq 3L + m$ and $a \in \mathcal{A}$. Then the previous considerations imply that there are non-empty words $v := v(a,n)$ and $w := w(a,n)$ such that for

$$S^{n-m}(a) = v\, b\, w$$

as $n - m \geq 3L$. Consequently,

$$S^n(a) = S^m(v)\, S^m(b)\, S^m(w)$$

is derived. Since

$$\left|S^m(v)\right| \geq 1, \qquad \left|S^m(w)\right| \geq 1 \qquad \text{and} \qquad u \prec S^m(b),$$

we have proven the desired statement.

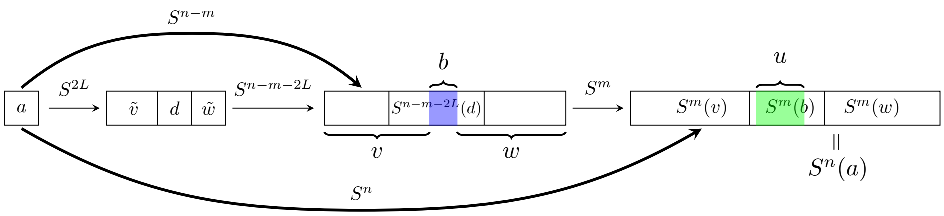

Figure 19. Sketch of the proof of (b) where $n \geq 3L + m$. Thus, $n - m - 2L \geq L$ holds and so S^{n-m-2L} must contain the letter b (marked in blue). The word u is marked with green.

(c) Since $S(a)$ is never the empty word, we conclude that $W(S)$ is non-empty (as it contains at least the letters occurring in $S(a)$ for any $a \in \mathcal{A}$). In addition, it is immediate from the definition that $W(S)$ satisfies (D1) as a pattern occurring in a legal pattern is automatically legal ($\prec$ is transitive).

Thus, it is left to show (D2). Let $u \in W(S) \cap \mathcal{A}^k$ for some $k \in \mathbb{N}$ and $a \in \mathcal{A}$. By (b), there is an $n_v \in \mathbb{N}$ and $v_1, v_2 \in \mathcal{A}^*$ (non-empty words) such that

$$v_1 u v_2 = S^{n_v}(a) \in W(S).$$

Thus, $W(S)$ satisfies (D2) as $v_1 u v_2$ is legal, $v_1 \neq \epsilon$ and $v_2 \neq \epsilon$.

□

Definition. *Let S be a primitive substitution, then $W(S)$ is the associated dictionary and*

$$\Omega(S) := \Phi^{-1}(W(S)) = \left\{\omega \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, W(\omega) \subseteq W(S)\right\}$$

is the associated subshift with the substitution S, where $\Phi : \mathcal{J} \to Dic(\mathcal{A}^G)$ was the homeomorphism defined in Theorem 3.4.

Remark. *We point out that primitivity of a substitution is not needed in order to guarantee that $W(S)$ is a dictionary and hence to define the associated subshift $W(S)$. More precisely, it suffices that $|S^n(a)| \to \infty$ for all $a \in \mathcal{A}$.*

Week 6

From the previous Proposition 4.4 (b), we conclude that $\Omega(S)$ is minimal if S is a primitive substitution.

Theorem 4.5 (Primitivity implies minimality). *Let S be a primitive substitution over the alphabet $\mathcal{A}$ with $\sharp\mathcal{A} \geq 2$. Then there is, for each $v \in W(S)$, an $l \in \mathbb{N}$ such that every $u \in W(S)$ with $|u| \geq l$ satisfies $v \prec u$. In particular, $\Omega(S)$ is minimal.*

Proof. The main idea is to choose the length of u so big that it must contain $S^{n_v}(a)$ of some letter $a \in \mathcal{A}$ (then it must contain v by definition of n_v). Since the alphabet $\mathcal{A}$ is finite, $|S^{n_v}(a)|$ is uniformly bounded. Here are the details:

Let $v \in W(S)$. By the previous Proposition 4.4 (b), there is an $n_v \in \mathbb{N}$ such that $v \prec S^n(a)$ for all $a \in \mathcal{A}$ and $n \geq n_v$. Define

$$N := \max\left\{n_v, \max_{a\in\mathcal{A}} \left|S^{n_v}(a)\right|\right\} \geq n_v.$$

Set $l := 3N$. Let $u \in W(S)$ be such that $|u| \geq l$. Then there is a $k \in \mathbb{N}$ with $k \geq n_v$ such that $u \prec S^k(b)$ for some $b \in \mathcal{A}$ applying the previous Proposition 4.4 (b) to u. Set $w := S^{k-n_v}(b) = w_1 \dots w_m$. Then

$$S^k(b) = S^{n_v}\big(S^{k-n_v}(b)\big) = S^{n_v}(w_1 \dots w_m) = S^{n_v}(w_1)S^{n_v}(w_2)\dots S^{n_v}(w_m).$$

Since $|S^{n_v}(w_j)| \leq N$ and $|u| \geq l = 3N$, there is a $1 \leq j \leq m$ such that $S^{n_v}(w_j) \prec u$. By definition of n_v, we have

$$v \prec S^{n_v}(w_j) \prec u$$

proving $v \prec u$.

The subshift $\Omega(S)$ is minimal by Theorem 3.7 (ii) and the previous considerations. □

Remark (Primitive substitutions define uniquely ergodic subshifts). *Recall that by Theorem 3.7, minimality of a subshift Ω is equivalent to the fact that every pattern (word) $v \in W(\Omega)$ appears with bounded distance. The subshifts defined by primitive substitutions satisfy that these bounded gaps can be bounded by the size of v. In particular, there is a constant $C > 0$ (only depending on S) such that every $v \in W(\Omega)$ occurs in any word $u \in W(\Omega)$ with*

$$|u| \geq C|v|.$$

Thus, we have a linear growth of these gaps when a word v appears again. Such subshifts are called linear repetitive. *It turns out that linear repetitive subshifts are even uniquely ergodic. In particular, every subshift $\Omega(S)$ for a primitive substitution S is uniquely ergodic. This is, however, not the content of this lecture.*

There is another way of describing the subshift $\Omega(S)$ of a primitive substitution by iterating the substitution on suitable elements on $\omega \in \mathcal{A}^{\mathbb{Z}}$. Recall that an element in $\omega \in \mathcal{A}^{\mathbb{Z}}$ can be represented as a two-sided infinite word

$$\omega = \quad \ldots \omega(-3)\,\omega(-2)\,\omega(-1)\,\big|\,\omega(0)\,\omega(1)\,\omega(2)\,\ldots$$

where the bar | indicates the origin. Then a substitution extends to a map $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ applied letterwise, namely

$$S(\omega) = \quad \ldots S\big(\omega(-3)\big)\,S\big(\omega(-2)\big)\,S\big(\omega(-1)\big)\,\big|\,S\big(\omega(0)\big)\,S\big(\omega(1)\big)\,S\big(\omega(2)\big)\,\ldots .$$

One could show that S is actually continuous (**Exercise**).

EXAMPLE. *Let $\mathcal{A} := \{a, b\}$ and S be the Fibonacci substitution. Let $\omega := a^{\infty} \in \mathcal{A}^{\mathbb{Z}}$. Then $\omega|_{\{k,k+1\}} = aa$ holds for all $k \in \mathbb{Z}$. Thus, $\omega|_{\{k,k+1\}}$ is legal for all $k \in \mathbb{N}$ as*

$$aa \prec S^3(a) = S^2(ab) = S(aba) = abaab.$$

Then we get

		period
$S^0(\omega) =$	$\ldots a\ a\ a\big\vert a\ a\ a \ldots$	***1***
$S^1(\omega) =$	$\ldots ab\ ab\ ab\big\vert ab\ ab\ ab \ldots$	***2***
$S^2(\omega) =$	$\ldots aba\ aba\ aba\big\vert aba\ aba\ aba \ldots$	***3***
$S^3(\omega) =$	$\ldots abaab\ abaab\ abaab\big\vert abaab\ abaab\ abaab \ldots$	***5***
$S^4(\omega) =$	$\ldots abaababa\ abaababa\ abaababa\big\vert abaababa\ abaababa\ abaababa \ldots$	***8***
$\vdots$	$\vdots$	$\vdots$

In particular, we notice that

$$\big(S^n(\omega)\big)(-1) = \begin{cases} a, & n \text{ even}, \\ b, & n \text{ odd}, \end{cases}$$

*and so $\big(S^n(\omega)\big)_{n\in\mathbb{N}}$ is not convergent in the product topology of $\mathcal{A}^{\mathbb{Z}}$. However, we can show that $S^{2n}(\omega)$ and $S^{2n+1}(\omega)$ are convergent in the product topology (**Exercise**). The key idea is that $|S^n(c)| \to \infty$ for $c \in \mathcal{A}$ and $S^2(a) = aba$ starts and ends with the letter a while $S^2(b) = ab$ ends with the letter b. In particular, we get two fixed points*

$$\rho_1 := \lim_{n\to\infty} S^{2n}(\omega) = \lim_{n\to\infty} \big(S^{2n}(a)\big)^{\infty}\big|\big(S^{2n}(a)\big)^{\infty}$$

$$\rho_2 := \lim_{n\to\infty} S^{2n+1}(\omega) = \lim_{n\to\infty} \big(S^{2n+1}(a)\big)^{\infty}\big|\big(S^{2n+1}(a)\big)^{\infty} = \lim_{n\to\infty} \big(S^{2n}(b)\big)^{\infty}\big|\big(S^{2n}(a)\big)^{\infty}$$

where

$$\rho_1(-1) = a, \qquad \rho_2(-1) = b \qquad \text{and} \qquad \rho_1|_{\mathbb{N}_0} = \rho_2|_{\mathbb{N}_0}.$$

Thus, Theorem 4.2 implies that $\Omega(S)$ is weakly aperiodic and so it is strongly aperiodic by Theorem 3.9 and Theorem 4.5 ($\Omega(S)$ is minimal since S is primitive. Furthermore, a subshift is strongly aperiodic if it is weakly aperiodic and minimal.)

Proposition 4.6. *Let S be a primitive substitution over the alphabet $\mathcal{A}$ with $\sharp\,\mathcal{A} \geq 2$. Let $\omega \in \mathcal{A}^{\mathbb{Z}}$ be such that $\omega|_{\{-1,0\}}$ is legal. Then any limit point $\rho \in \mathcal{A}^{\mathbb{Z}}$ of $\big(S^n(\omega)\big)_{n\in\mathbb{N}}$ is an element of $\Omega(S)$. In particular,*

$$W(\overline{Orb(\rho)}) = W(S) \qquad \text{and} \qquad \Omega(S) = \overline{Orb(\rho)}$$

holds.

Remark. *We remind the reader that $\mathcal{A}^{\mathbb{Z}}$ is compact and so $\big(S^n(\omega)\big)_{n\in\mathbb{N}}$ always admits a limit point.*

Proof. Let $n_k \in \mathbb{N}$ for $k \in \mathbb{N}$ be such that $\big(S^{n_k}(\omega)\big)_{k\in\mathbb{N}}$ converges to $\rho \in \mathcal{A}^{\mathbb{Z}}$. Thus, for each $m \in \mathbb{N}$, there is a $k_m \in \mathbb{N}$ such that

$$\rho|_{\{-m,\ldots,m\}} = \big(S^{n_k}(\omega)\big)|_{\{-m,\ldots,m\}}$$

for all $k \geq k_m$. Furthermore, we have

$$S^{n_k}(\omega)|_{\{-l_k,\ldots,r_k\}} = S^{n_k}\big(\omega|_{\{-1,0\}}\big)$$

where $l_k := |S^{n_k}(\omega(-1))|$ and $r_k := |S^{n_k}(\omega(0))| - 1$. Since $l_k \to \infty$ and $r_k \to \infty$ by the previous proposition, there is no loss in choosing k_m large enough such that

$$\rho|_{\{-m,\ldots,m\}} \prec S^{n_k}\big(\omega|_{\{-1,0\}}\big), \qquad k \geq k_m.$$

Recall that if $v \in W(\rho)$, then there is an $N \in \mathbb{Z}$ such that

$$\rho|_{\{N,N+1,\ldots,N+|v|-1\}} = v.$$

Thus, $v \prec S^{n_k}\big(\omega|_{\{-1,0\}}\big)$ follows for all k large enough by the previous considerations. Hence, v is legal since $\omega|_{\{-1,0\}}$ is legal. Thus, $W(\rho) \subseteq W(S)$ follows implying $\rho \in \Omega(S)$. Since $\Omega(S)$ is minimal by Theorem 4.5, this leads to $\Omega(S) = \overline{Orb(\rho)}$. Finally, $W(\overline{Orb(\rho)}) = W(S)$ follows as $\mathcal{J}$ and $Dic(\mathcal{A}^G)$ are homeomorphic by Theorem 3.4. $\square$

Exercise. *Let S be a primitive substitution over the finite alphabet $\mathcal{A}$ with $\sharp\,\mathcal{A} \geq 2$. Prove the following assertions.*

(a) *There is a $\rho \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{N}$ such that*

$$\rho|_{\{-1,0\}} = S^k(\rho)|_{\{-1,0\}}$$

and $\rho|_{\{-1,0\}}$ is legal.

(b) *If $\rho \in \mathcal{A}^{\mathbb{Z}}$ and $k \in \mathbb{N}$ satisfy $\rho|_{\{-1,0\}} = S^k(\rho)|_{\{-1,0\}} \in W(S)$, then the sequence $\big(S^{nk}(\rho)\big)_{n\in\mathbb{N}} \subseteq \mathcal{A}^{\mathbb{Z}}$ converges and its limit point*

$$\omega := \lim_{n\to\infty} S^{nk}(\rho) \in \Omega(S)$$

fulfills $S^k(\omega) = \omega$.

Elements $\omega \in \Omega(S)$ satisfying $S^k(\omega) = \omega$ for some $k \in \mathbb{N}$ are called *fixed points* (sometimes also k-periodic) of the substitution.

In general it is not clear that also the orbit closures $\overline{Orb(S^n(\omega))}$ converge to $\Omega(S)$ in $\mathcal{J}$. We will provide a sufficient condition next.

THEOREM 4.7 (Substitutions and approximations for the subshifts). *Let S be a primitive substitution over the alphabet $\mathcal{A}$ with $\sharp\mathcal{A} \geq 2$. Let $\omega \in \mathcal{A}^{\mathbb{Z}}$ be such that $\omega|_{\{k,k+1\}}$ is legal for all $k \in \mathbb{Z}$, namely $W(\omega) \cap \mathcal{A}^2 \subseteq W(S)$. Then*

$$\lim_{n\to\infty} \overline{Orb(S^n(\omega))} = \Omega(S).$$

REMARK. *We refer the reader to* [**Bec16, BBdN20**] *for further background where this result was originally proven.*

We have two statements, one that $\left(\overline{Orb(S^n(\omega))}\right)_{n\in\mathbb{N}}$ is convergent and secondly that it converges to $\Omega(S)$. Note however that this does not mean that $S^n(\omega)$ converges in the product topology on $\mathcal{A}^{\mathbb{Z}}$, confer the previous example about the Fibonacci substitution on page 63.

We point out that we know by Theorem 4.3 that every minimal subshift of $\mathcal{A}^{\mathbb{Z}}$ is periodically approximable. In particular, there is an periodic $\omega \in \mathcal{A}^{\mathbb{Z}}$ satisfying the conditions in the last theorem. Then the theorem tells us that we can define the periodic approximations recursively by applying the substitution. This is very important as recursive relations between periodic approximations are useful for numerics but also to analyze analytically the limit point. This is heavily used for the associated operators.

PROOF. Since $\mathcal{J}$ and $Dic(\mathcal{A}^G)$ are homeomorphic by Theorem 3.4, it suffices to show the convergence of the associated dictionaries. By the notion of the local pattern topology (see also Corollary 3.6) we have to show that for each $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ such that

$$W\big(S^n(\omega)\big) \cap \mathscr{A}^m = W(S) \cap \mathscr{A}^m, \qquad n \geq n_m.$$

Fix $m \in \mathbb{N}$. Since S is primitive, Proposition 4.4 (b) asserts that for each $v \in W(S) \cap \mathscr{A}^m$, there is an $n_v \in \mathbb{N}$ such that

$$v \prec S^n(a) \quad \text{for all} \quad a \in \mathcal{A},\ n \geq n_v.$$

Thus, $v \prec S^n(\omega(0))$ follows implying $v \in W\big(S^n(\omega)\big)$ if $n \geq n_v$. Set

$$n_m := \max\left\{ \max_{v\in W(S)\cap\mathscr{A}^m} n_v,\ \min\left\{ l \in \mathbb{N} \,\middle|\, |S^l(a)| \geq m \text{ for all } a \in \mathcal{A} \right\} \right\}.$$

Note that the maximum over v is finite as $\mathscr{A}^m$ is finite. Furthermore, the minimum exists and is finite since $|S^l(a)| \to \infty$ for each $a \in \mathcal{A}$ and $\mathcal{A}$ is finite.

Then the previous considerations lead to

$$W\big(S^n(\omega)\big) \cap \mathscr{A}^m \supseteq W(S) \cap \mathscr{A}^m, \qquad n \geq n_m,$$

since then $n \geq \max_{v\in W(S)\cap\mathscr{A}^m} n_v$.

For the reverse inclusion, let $v \in W\big(S^n(\omega)\big) \cap \mathscr{A}^m$ with $n \geq n_m$. This means that there is an $N \in \mathbb{Z}$ such that

$$\big(S^n(\omega)\big)|_{\{N,N+1,\dots,N+|v|-1\}} = v.$$

Then by the definition of $S^n : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, we have

$$S^n(\omega) = \quad \dots S^n\big(\omega(-2)\big)\, S^n\big(\omega(-1)\big) \,\big|\, S^n\big(\omega(0)\big)\, S^n\big(\omega(1)\big) \dots$$

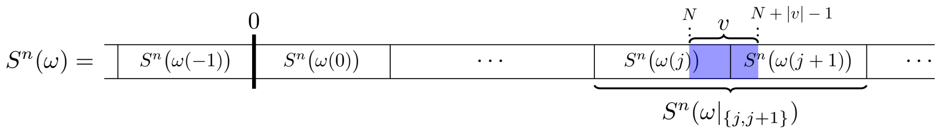

Figure 20. Sketch of the proof for the inclusion $W\big(S^n(\omega)\big) \cap \mathscr{A}^m \subseteq W(S) \cap \mathscr{A}^m$ for all $n \geq n_m$.

holds. For $n \geq n_m$, we conclude that $|S^n(\omega(j)| \geq m$ for all $j \in \mathbb{Z}$. Thus, $|v| = m$ yields that there is an $j \in \mathbb{Z}$ such that

$$v \prec S^n\big(\omega(j)\big)\ S^n\big(\omega(j+1)\big) = S^n\big(\omega|_{\{j,j+1\}}\big).$$

Since $\omega|_{\{j,j+1\}}$ is legal by assumption, we conclude that v is legal, namely $v \in W(S)$. This proves that $W\big(S^n(\omega)\big) \cap \mathscr{A}^m \subseteq W(S) \cap \mathscr{A}^m$ for all $n \geq n_m$. $\square$

We will finish this section by providing periodic approximations for various subshifts defined by primitive substitutions using the previous result. In particular, we will provide explicit periodic subshifts $\Omega_n = Orb(\omega_n)$ such that $\Omega_n \to \Omega(S)$.

Recall that periodic subshifts and subshifts $\Omega(S)$ defined by primitive substitutions are uniquely ergodic (Remark “Primitive substitutions define uniquely ergodic subshifts” after Theorem 4.5). In particular, we have

$$\mathcal{M}^1(\Omega_n, \mathbb{Z}) = \{\mu_n\},\ n \in \mathbb{N}, \qquad \text{and} \qquad \mathcal{M}^1(\Omega(S), \mathbb{Z}) = \{\mu\}.$$

Since $\Omega_n \to \Omega(S)$ in $\mathcal{J}$, Theorem 2.19 (see also Sheet 4) implies that $\mu_n \rightharpoonup \mu$ in the vague topology on $\mathcal{M}^1(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$.

We will see later that the convergence of subshifts also yields the convergence of spectra of associated operator families. This is the content of the next chapters.

Example (Periodic approximations for the Fibonacci substitution). *Let $\mathcal{A} := \{a, b\}$ and consider the Fibonacci substitution $S(a) := ab$, $S(b) := a$. Recall that $\Omega(S)$ is strongly aperiodic, see example on page 63. The periodic configuration $\omega := a^\infty \in \mathcal{A}^{\mathbb{Z}}$ satisfies*

$$W(\omega) \cap \mathcal{A}^2 = \{aa\} \subseteq W(S)$$

as $S^3(a) = S^2(ab) = S(aba) = abaab$. Hence, $W(\omega_n) \to W(S)$ holds by the previous Theorem 4.7 where

$$\omega_n := S^n(\omega) = \big(S^n(a)\big)^\infty, \qquad n \in \mathbb{N}.$$

Moreover ω_n is periodic with period bounded by $|S^n(a)|$ (which are the Fibonacci numbers). In particular, we have found explicit periodic approximations of the Fibonacci subshift $\Omega(S)$. It is worth pointing out that (ω_n) is not convergent in the product topology of $\mathcal{A}^{\mathbb{Z}}$ but it admits a convergent subsequence, see example on page 63.

Example (Periodic approximations for the Thue-Morse substitution). *Let $\mathcal{A} := \{a, b\}$ and consider the Thue-Morse substitution $S(a) := ab$, $S(b) := ba$. We have*

$$S^2(a) = S(ab) = abba \qquad \textit{and} \qquad S^2(b) = S(ba) = baab$$

and so $aa, ba \in W(S)$. Thus, the limits

$$\begin{aligned}\rho_1 &:= \lim_{n\to\infty} \big(S^{2n}(a)\big)^{\infty} \big| \big(S^{2n}(a)\big)^{\infty},\\ \rho_2 &:= \lim_{n\to\infty} \big(S^{2n}(b)\big)^{\infty} \big| \big(S^{2n}(a)\big)^{\infty}\end{aligned}$$

exist by the previous exercise. Furthermore, $\rho_1, \rho_2 \in \Omega(S)$, $\rho_1|_{\mathbb{N}_0} = \rho_2|_{\mathbb{N}_0}$ and

$$\rho_1(-1) = a \neq b = \rho_2(-1).$$

Hence, $\Omega(S)$ is weakly aperiodic by Theorem 4.2. Since $\Omega(S)$ is also minimal by Theorem 4.5, Theorem 3.9 implies that $\Omega(S)$ is strongly aperiodic.

The periodic configuration $\omega := b^{\infty} \in \mathcal{A}^{\mathbb{Z}}$ satisfies

$$W(\omega) \cap \mathcal{A}^2 = \{bb\} \subseteq W(S)$$

as $S^2(a) = S(ab) = abba$. Hence,

$$\omega_n := S^n(\omega) = \big(S^n(b)\big)^{\infty}, \qquad n \in \mathbb{N},$$

are periodic and $W(\omega_n) \to W(S)$ by the previous Theorem 4.7. In particular, we have found explicit periodic approximations of the Thue-Morse subshift $\Omega(S)$. Similarly one can show that $\omega'_n := \big(S^n(a)\big)^{\infty}$ satisfies

$$\lim_{n\to\infty} W(\omega'_n) = W(S).$$

Example (Periodic approximations for the Period Doubling substitution). *Let $\mathcal{A} := \{a, b\}$ and consider the Period Doubling substitution $S(a) := ab$, $S(b) := aa$. We have*

$$S^2(a) = S(ab) = abaa \qquad \text{and} \qquad S^2(b) = S(aa) = abab$$

and $aa, ba \in W(S)$. Thus, the limits

$$\begin{aligned}\rho_1 &:= \lim_{n\to\infty} \big(S^{2n}(a)\big)^{\infty} \big| \big(S^{2n}(a)\big)^{\infty},\\ \rho_2 &:= \lim_{n\to\infty} \big(S^{2n}(b)\big)^{\infty} \big| \big(S^{2n}(a)\big)^{\infty}\end{aligned}$$

exist by the previous exercise. Furthermore, $\rho_1, \rho_2 \in \Omega(S)$, $\rho_1|_{\mathbb{N}_0} = \rho_2|_{\mathbb{N}_0}$ and

$$\rho_1(-1) = a \neq b = \rho_2(-1).$$

Hence, $\Omega(S)$ is weakly aperiodic by Theorem 4.2. Since $\Omega(S)$ is also minimal by Theorem 4.5, Theorem 3.9 implies that $\Omega(S)$ is strongly aperiodic.

The periodic configuration $\omega := a^{\infty} \in \mathcal{A}^{\mathbb{Z}}$ satisfies

$$W(\omega) \cap \mathcal{A}^2 = \{aa\} \subseteq W(S)$$

as $S(b) = aa$. Hence,

$$\omega_n := S^n(\omega) = \big(S^n(a)\big)^{\infty}, \qquad n \in \mathbb{N},$$

are periodic and $W(\omega_n) \to W(S)$ by the previous Theorem 4.7. In particular, we have found explicit periodic approximations of the Period Doubling subshift $\Omega(S)$. Since $\big(S^{n+1}(b)\big)^{\infty} = \omega_n$, we conclude that

$$\lim_{n\to\infty} W\Big(\big(S^{n+1}(b)\big)^{\infty}\Big) = W(S).$$

The Golay-Rudin-Shapiro substitution is left as an exercise.

REMARK. *Not all substitutions lead to weakly/strongly aperiodic subshifts. Let* $\mathcal{A} := \{a, b\}$ *and define* $S : \mathcal{A} \to \mathcal{A}^*$ *by*

$$S(a) := aba \qquad \text{and} \qquad S(b) := bab.$$

Then S *is a primitive substitution but*

$$\Omega(S) := \{(ab)^\infty, (ba)^\infty\},$$

which is not aperiodic (but minimal and uniquely ergodic, ***Exercise****).*

5. Spectrum of bounded linear operators

We provide a short summary of basic notions and statements in spectral theory of self-adjoint operators. Therefore recall what we discussed in Section 2.3 "Linear bounded maps". You can find a more detailed discussion and all proofs in the Appendix E. With this at hand, we discuss consequences on the convergence of the spectra w.r.t. the Hausdorff metric d_H on $\mathbb{C}$ if $\mathcal{L}(H)$ is equipped with various topologies.

5.1. Spectrum and resolvent set. Let E be a Banach space. Then $I \in \mathcal{L}(E)$ denotes the identity operator on E, namely, $Ix := x$ for $x \in E$. In the following we will write $AB := A \circ B$ for $A, B \in \mathcal{L}(E)$. An operator $A \in \mathcal{L}(E)$ is called *invertible* if there is a $B \in \mathcal{L}(E)$ such that $AB = BA = I$. Then B is called the *inverse* of A that we denote by A^{-1}.

Definition. *Let E be a Banach space. For $A \in \mathcal{L}(E)$, define the* resolvent set *by*

$$\rho(A) := \{z \in \mathbb{C} \,|\, A - zI \text{ bijective with continuous inverse}\}$$

and its spectrum

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

The map

$$R_A : \rho(A) \to \mathcal{L}(H), \quad R_A(z) := (A - zI)^{-1},$$

is called resolvent.

Remark. *(a) In classical mechanics, the possible results of measurements are given precisely by the possible values of the observable functions. In quantum mechanics, the spectrum of an operator (i.e., the generalized eigenvalues) appears as the possible results of measurements of the associated observables. This corresponds to the transition from the spectrum in commutative algebras to the spectrum in non-commutative algebras.*

(b) About the term spectrum*: In optics, acoustics and harmonic analysis it is common practice to decompose objects in eigenfunctions (i.e., waves) to eigenvalues (i.e., frequencies). This is known as the spectral decomposition or as frequency analysis.*

(c) About the term resolvent*: For a bounded operator $A \in \mathcal{L}(E)$ on a Banach space, one has $\lambda \in \rho(A)$ if and only if the equation $(A - \lambda I)x = y$ has a unique solution for all y, i.e., the equation can be uniquely* resolved *with regard to x. In light of this we seek to figure out when we can solve such "generalized equation systems".*

(d) In the lecture "Functional Analysis" you will get to know the bounded inverse theorem (also called inverse mapping theorem). This states that every bijective $A \in \mathcal{L}(E, F)$ admits an inverse $A^{-1} \in \mathcal{L}(F, E)$ which is bounded where E and F are some Banach space. Specifically, $A : E \to F$ is a surjective continuous linear operator and so the open mapping theorem asserts that it maps open sets to open sets. Thus $(A^{-1})^{-1}(U)$ is open for $U \subseteq E$ open, namely A^{-1} is continuous. Note that this result is wrong if E is just a normed space.

(e) With a slight abuse of notation, we will use $\lambda := \lambda I$ for all $\lambda \in \mathbb{C}$.

EXAMPLE (Operators on a finite-dimensional space). *For a matrix* $A \in \mathbb{C}^{n\times n}$, *we have*

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(A-\lambda) = 0\} = \{\textit{eigenvalues of } A\}.$$

EXERCISE. *Let* $E := \ell^2(\mathbb{N})$, $k \in \mathbb{N}$ *and consider the operator* $A_k \in \mathcal{L}(\ell^2(\mathbb{N}))$ *defined by* $A_k := \widehat{\chi_{[0,k]}}$, *namely the multiplication operator where* $\chi_{[0,k]} : \mathbb{N} \to \{0,1\}$ *is the characteristic function of the interval* $[0,k]$. *More precisely, we have*

$$(A_k\psi)(n) = \begin{cases} 0, & n > k, \\ \psi(n), & n \le k. \end{cases}$$

Prove that $\sigma(A_k) = \{0,1\}$ *for all* $k \in \mathbb{N}$. *Note that* A_k *is actually a projection, i.e.* $A_k^2 = A_kA_k = A_k$.

PROPOSITION 5.1 (Neumann series). *Let* E *be a Banach space and* $A \in \mathcal{L}(E)$ *with* $\limsup_{n\to\infty} \|A^n\|^{1/n} < 1$ *be given. Then* $(I-A)$ *is invertible and*

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

In particular, $(I-A)$ *is invertible if* $\|A\| < 1$.

REMARK. *(a) The formula should not come as a surprise. It is well known that the geometric series* $\sum_{n\ge1} q^n$ *equals* $\frac{1}{1-q}$ *for* $|q| < 1$. *As in analysis the geometric series, the Neumann series plays a crucial role in spectral theory.*

(b) If $\limsup_{n\to\infty} \|A^n\|^{1/n} < 1$, *then* $(I+A)$ *is also invertible with inverse*

$$(I+A)^{-1} = \sum_{k=0}^{\infty} (-A)^n.$$

(c) The identity operator $I \in \mathcal{L}(E)$ *is always invertible. In this sense the lemma is a stability result.*

THEOREM 5.2 (Basic properties of the spectrum). *Let* $A \in \mathcal{L}(H)$. *Then the spectrum* $\sigma(A)$ *is a compact and non-empty subset of* $\mathbb{C}$ *and the resolvent* R_A *is analytic (i.e. it can be locally developed into a norm-convergent power series). In particular,* R_A *is continuous and* $\sigma(A) \subseteq B_{\|A\|}(0)$ *holds.*

REMARK. *This is indeed a result "over* $\mathbb{C}$*" and which is not a coincidence: On* $\mathbb{R}$, *a corresponding result is already wrong for matrices (e.g. the rotation on* $\mathbb{R}^2$ *has no real eigenvalues). The proof makes extensive use of complex analysis in the form of Liousville's theorem. This also does not come as a surprise. Already for matrices one uses complex analysis, in terms of Liouville's theorem or Rousseau's theorem, to prove the fundamental theorem of algebra (every non-constant polynomial with complex coefficients has at least one complex root) which yields the existence of eigenvalues (in* $\mathbb{C}$*).*

PROPOSITION 5.3 (Transformation and the spectrum). *Let* E, F *be Banach spaces,* $A \in \mathcal{L}(E)$ *and* $U \in \mathcal{L}(E,F)$ *be such that* U *is invertible. Then*

$$\sigma(A) = \sigma(UAU^{-1}) \quad \textit{and} \quad \rho(A) = \rho(UAU^{-1}).$$

PROOF. Clearly $\sigma(A) = \sigma(UAU^{-1})$ is equivalent to $\rho(A) = \rho(UAU^{-1})$. Thus, we will only show $\rho(A) = \rho(UAU^{-1})$. First note that $\lambda \in \mathbb{C}$ is an element of $\rho(A)$ if and only if $A - \lambda$ is invertible in $\mathcal{L}(E, F)$. Since U is invertible, this is equivalent to $U(A-\lambda)U^{-1}$ is invertible. This can be shown similarly as in the proofs of Proposition E.1 (Basic properties of invertible elements). Then the identity

$$U(A - \lambda)U^{-1} = UAU^{-1} - \lambda UU^{-1} = UAU^{-1} - \lambda$$

finishes the proof. $\square$

EXERCISE. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Prove that if p is a polynomial with complex coefficients then*

$$\sigma(p(A)) = \{p(\lambda) \mid \lambda \in \sigma(A)\}.$$

5.2. Spectral radius. We can say more about the location of the spectrum in terms of the operator norm. Using Fekete's Lemma (Lemma E.7), the limit

$$\lim_{n\to\infty} \|A^n\|^{1/n}$$

exists for all $A \in \mathcal{L}(E)$ where E is a Banach space.

DEFINITION (Spectral radius). *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then*

$$r(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

is called the spectral radius *of A.*

The following theorem of Beurling is remarkable because it connects algebra and topology. Algebra enters via invertibility in form of the spectrum and topology enters in form of the spectral radius.

THEOREM 5.4 (Location of Spectrum – Beurling's Theorem). *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then,*

$$\max_{\lambda\in\sigma(A)} |\lambda| = r(A).$$

REMARK. *The spectral radius $r(A)$ is the smallest number r such that $\sigma(A)$ is contained in $B_r(0)$. Hence, the name spectral radius.*

Recall that $R_A(z) = (A - z)^{-1}$ denotes the resolvent of an operator $A \in \mathcal{L}(E)$ and $z \in \rho(A)$.

PROPOSITION 5.5. *Let E be a Banach space, $A \in \mathcal{L}(E)$ and $z \in \rho(A)$. Then*

$$\sigma\big(R_A(z)\big) = \big(\sigma(A) - z\big)^{-1} := \left\{ \frac{1}{\lambda - z} \,\middle|\, \lambda \in \sigma(A) \right\} \tag{a}$$

and

$$r\big(R_A(z)\big) = \frac{1}{\operatorname{dist}(z, \sigma(A))} \tag{b}$$

hold.

PROOF. We will first show that (a) implies (b): By Beurling's Theorem (Theorem 5.4), we have

$$r\big(R_A(z)\big) = \max_{\lambda\in\sigma(R_A(z))} |z|.$$

Furthermore, $\big(\sigma(A)-z\big)^{-1}$ is nonempty and compact as $\sigma(A)$ is nonempty and compact by Theorem 5.2 (Basic properties of the spectrum). Thus,

$$\frac{1}{\operatorname{dist}(z,\sigma(A))} = \frac{1}{\min_{\lambda\in\sigma(A)}|\lambda-z|} = \max_{\lambda\in\sigma(A)}\left|\frac{1}{\lambda-z}\right| = r\big(R_A(z)\big)$$

follows using (a).

In order to prove (a), let $\lambda\in\mathbb{C}\setminus\{0\}$. Then a short computation leads to

$$\begin{aligned} R_A(z)-\lambda = R_A(z)-\lambda(A-z)R_A(z) &= \big((1+\lambda z)I-\lambda A\big)R_A(z) \\ &= \left(\left(z+\frac{1}{\lambda}\right)I-A\right)\lambda R_A(z). \end{aligned}$$

By assumption $\lambda R_A(z)$ is invertible. Using Proposition E.1 (Basic properties of invertible elements) $R_A(z)-\lambda$ is invertible if and only if $\left(z+\frac{1}{\lambda}\right)I-A$ is invertible, if and only if $z_0 := z+\frac{1}{\lambda}\in\rho(A)$. This is equivalent to $\lambda=\frac{1}{z_0-z}$ for some $z_0\in\rho(A)$. Hence, $\lambda\in\rho(R_A(z))$ if and only if $\lambda=\frac{1}{z_0-z}$ for some $z_0\in\rho(A)$. We claim that from this, the desired identity follows. Here are the details:

$\underline{\sigma\big(R_A(z)\big)\supseteq\big(\sigma(A)-z\big)^{-1}}$: We will work with the complements of these sets. Let $\lambda\notin\sigma\big(R_A(z)\big)$, then $\lambda\in\rho\big(R_A(z)\big)$. Since $\lambda\neq 0$, we conclude that $\lambda=\frac{1}{z_0-z}$ for some $z_0\in\rho(A)$ by the previous considerations. Since $\rho(A)\cap\sigma(A)=\varnothing$, we deduce that $\lambda\notin\big(\sigma(A)-z\big)^{-1}$.

$\underline{\sigma\big(R_A(z)\big)\subseteq\big(\sigma(A)-z\big)^{-1}}$: We will work with the complements of these sets. Let $\lambda\notin\big(\sigma(A)-z\big)^{-1}$. If $\lambda=0$, then clearly $\lambda\notin\sigma\big(R_A(z)\big)$ since $A-z\in\mathcal{L}(E)$ is the inverse of $R_A(z)$. Thus suppose $\lambda\neq 0$. Then $\frac{1}{\lambda}\notin\sigma(A)-z$ follows or equivalently $z+\frac{1}{\lambda}\in\rho(A)$. Using again the previous considerations, we derive that $\lambda\in\rho(R_A(z))$, namely $\lambda\notin\sigma\big(R_A(z)\big)$. ☐

Week 7

5.3. Linear bounded operators on Hilbert spaces. We seek to study a specific class of operators that are called self-adjoint. In order to do so, a short reminder on Hilbert spaces and inner products is provided.

Let $\mathcal{H}$ be vector space over $\mathbb{C}$. A map $\langle\cdot,\cdot\rangle:\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ is an *inner product* if

- it is *complex linear (antilinear)* in the first and linear in the second component, namely

  $$\langle\lambda x+y,z\rangle=\overline{\lambda}\langle x,z\rangle+\langle y,z\rangle\,,\qquad \langle x,\lambda y+z\rangle=\lambda\langle x,y\rangle+\langle x,z\rangle\,,$$

  hold for all $x,y,z\in\mathcal{H}$ and $\lambda\in\mathbb{C}$,
- it is *symmetric*, i.e., $\langle x,y\rangle=\overline{\langle y,x\rangle}$ for all $x,y\in\mathcal{H}$,
- it is *positive-definite*, i.e. if $x\neq 0$ then $\langle x,x\rangle>0$.

Then $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is called *Hilbert space* if $(\mathcal{H},\|\cdot\|)$ is a Banach space where $\|x\|:=\sqrt{\langle x,x\rangle}$.

EXAMPLE. *For $d \in \mathbb{N}$, the set $H := \mathbb{C}^d$ equipped with inner product*

$$\langle x, y \rangle := \sum_{j=1}^{d} \overline{x_j} y_j$$

defines a (finite dimensional) Hilbert space.

EXAMPLE. *Let X be a countable discrete set. Then*

$$\ell^2(X) := \left\{ \psi : X \to \mathbb{C} \;\middle|\; \sum_{x \in X} |\psi(x)|^2 < \infty \right\}$$

is a Hilbert space with inner product

$$\langle \psi, \phi \rangle = \sum_{x \in X} \overline{\psi(x)} \phi(x)$$

and induced norm $\|\psi\|_2 = \sqrt{\sum_{x \in X} |\psi(x)|^2}$. We leave the details as an exercise.

Recall that elements of $H' := \mathcal{L}(H, \mathbb{C})$ are called linear bounded functionals, see Section 2.3 and Appendix D.

THEOREM 5.6 (Riesz-Fréchet representation theorem). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then for every $y \in H$, the map*

$$F_y : H \longrightarrow \mathbb{C}, \quad x \mapsto \langle y, x \rangle$$

defines a continuous linear functional on H with $\|F_y\| = \|y\|$. Furthermore, the map

$$H \longrightarrow H', y \mapsto F_y,$$

is complex linear (i.e. $F_{\lambda x + \mu z} = \overline{\lambda} F_x + \overline{\mu} F_z$) and bijectiv. In particular, for every linear bounded functional φ on H, there is a unique $y \in H$ such that $\varphi = F_y$.

REMARK. *The important statement is the surjectivity, namely that each linear bounded functional is represented by an F_y.*

Using the Riesz-Fréchet representation theorem (previous Theorem 5.6), we can define the concept of an adjoint operator.

PROPOSITION 5.7 (Adjoint operator). *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$. Then there is a unique $A^* : K \to H$ satisfying*

$$\langle y, Ax \rangle_K = \langle A^* y, x \rangle_H$$

for all $x \in H$ and $y \in K$. Furthermore, we have

$$\|A^*\| = \|A\|.$$

DEFINITION. *Let H, K be Hilbert spaces and $A \in \mathcal{L}(H, K)$. Then we call A^* (defined in the previous proposition) the* adjoint operator *of A.*

PROPOSITION 5.8 ($*$ is an involution). *Let H and K be Hilbert spaces, $\lambda \in \mathbb{C}$ and $A, B \in \mathcal{L}(H, K)$. Then the following statements hold.*

(a) $(A + \lambda B)^* = A^* + \overline{\lambda} B^*$.
(b) $(AB)^* = B^* A^*$.
(c) $(A^*)^* = A$.
(d) *If $I \in \mathcal{L}(H)$ is the identity, then $I^* = I$.*

(e) $(A^*)^{-1} = (A^{-1})^*$ *if A is invertible.*

Proposition 5.9 (C^*-property). *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H,K)$. Then*

$$\|A\|^2 = \|A^*A\| = \|AA^*\|.$$

Remark. *The previous identity has a great structural impact. If one considers the space $\mathcal{L}(H)$ of all continuous linear operators on a Hilbert space, then*

- *$\mathcal{L}(H)$ is an algebra with norm and an involution $*$.*
- *The norm $\|\cdot\|$ is submultiplicative and $\mathcal{L}(H)$ is complete.*
- *We have $\|A\|^2 = \|A^*A\| = \|AA^*\|$.*

A normed algebra with these properties is called a C^*-algebra and so $\mathcal{L}(H)$ is a C^*-algebra. Indeed one can show that every C^*-algebra is a subalgebra of $\mathcal{L}(H)$ (where H might be a huge Hilbert sapce) via the GNS-construction. C^*-algebras play a crucial role in physics.

With the notion of adjoint operator at hand, we can define the following.

Definition. *Let H be a Hilbert space and $A \in \mathcal{L}(H)$. Then*

- *A is called* self-adjoint *if $A^* = A$.*
- *A is called* normal *if $A^*A = AA^*$.*
- *A is called* unitary *if $A^* = A^{-1}$.*

Example. *Let $H := \ell^2(\mathbb{Z})$ be the Hilbert and $m \in \mathbb{Z}$. Define*

$$L_m : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (L_m\psi)(n) = \psi(-m+n).$$

Clearly, L_m is a linear operator such that

$$\|L_m\psi\|_2^2 = \sum_{n\in\mathbb{Z}} |L_m\psi(n)|^2 = \sum_{n\in\mathbb{Z}} |\psi(-m+n)|^2 = \sum_{k\in\mathbb{Z}} |\psi(k)|^2 = \|\psi\|_2^2.$$

Thus, $L_m \in \mathcal{L}\big(\ell^2(\mathbb{Z})\big)$ with $\|L_m\| = 1$. A short computation leads to

$$\begin{aligned} \langle \psi, L_m\phi\rangle &= \sum_{n\in\mathbb{Z}} \overline{\psi(m-m+n)}\phi(-m+n) \\ &= \sum_{k\in\mathbb{Z}} \overline{\psi(m+k)}\phi(k) \\ &= \sum_{k\in\mathbb{Z}} \overline{(L_{-m}\psi)(k)}\phi(k) \\ &= \langle L_{-m}\psi, \phi\rangle \end{aligned}$$

Thus, $L_m^*\psi = L_{-m}\psi$ follows as the adjoint operator is uniquely determined by this identity by Proposition 5.7. Furthermore

$$(L_mL_m^*\psi)(n) = (L_m^*\psi)(-m+n) = \psi(m-m+n) = \psi(n)$$

holds and similarly $L_m^*L_m\psi = \psi$. Thus, L_m is unitary and so it is also normal.

Example. *Let*

$$H := L^2(\mathbb{R}) = \big\{\psi : \mathbb{R} \to \mathbb{C} \,\big|\, \psi \text{ measurable}, \ \int_{\mathbb{R}} |\psi(x)|^2 \, dx < \infty\big\}$$

be the Hilbert space of square integrable functions with inner product

$$\langle \psi, \phi\rangle := \int_{\mathbb{R}} \overline{\psi(x)}\phi(x) \, dx.$$

Let $t : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function and define $\widehat{t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\widehat{t}\psi)(x) := t(x)\psi(x).$$

Then $\widehat{t}$ is a linear bounded operator satisfying

$$\|\widehat{t}\| \leq \|t\|_\infty := \sup_{x \in \mathbb{R}} |t(x)| < \infty.$$

Furthermore, a short computation leads to

$$\begin{aligned} \langle \psi, \widehat{t}\phi \rangle &= \int_{\mathbb{R}} \overline{\psi(x)} t(x) \phi(x) \; dx \\ &= \int_{\mathbb{R}} \overline{\overline{t(x)}\psi(x)} \phi(x) \; dx \\ &= \langle \widehat{\overline{t}}\psi, \phi \rangle. \end{aligned}$$

Hence, $\widehat{t}$ is self-adjoint if and only if $t(x) = \overline{t(x)}$ for (Lebesgue) almost-every $x \in \mathbb{R}$, or equivalently if t is real-valued for Lebesgue almost-every $x \in \mathbb{R}$. It is elementary to check that $\widehat{t}$ is normal.

Our main focus will lie on self-adjoint bounded operators. Their spectrum is always contained in the real line.

Proposition 5.10 (Spectrum of self-adjoint operators). *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then $A - zI$ for each $z \in \mathbb{C} \setminus \mathbb{R}$ is bijective and the inverse $(A - zI)^{-1} \in \mathcal{L}(H)$ is bounded by $\frac{1}{|\Im(z)|}$. In particular, $\sigma(A) \subseteq \mathbb{R}$.*

Remark. *The inclusion $\sigma(A) \subseteq \mathbb{R}$ allows us in quantum mechanics to interpret the spectrum of a self-adjoint operator as possible measurements.*

The reverse of the previous statement is not true in general as can be easily seen from the example of suitable matrices (Exercise).

Theorem 5.11 (Spectral radius and norm for normal elements). *Let $A \in \mathcal{L}(H)$ be normal. Then,*

$$\|A\| = r(A) = \max\left\{ |\lambda| \,\middle|\, \lambda \in \sigma(A) \right\}.$$

Corollary 5.12. *Let $A \in \mathcal{L}(H)$ be self-adjoint. If for some $\varepsilon > 0$, there is an $\psi \in H$ such that $\|(A - \lambda)\psi\| \leq \varepsilon \|\psi\|$, then*

$$\sigma(A) \cap [\lambda - \varepsilon, \lambda + \varepsilon] \neq \varnothing.$$

Proof. Assume by contradiction that $[\lambda - \varepsilon, \lambda + \varepsilon] \subseteq \rho(A)$. Then the previous Theorem 5.11 and Proposition 5.5 lead to

$$\|R_A(\lambda)\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} < \frac{1}{\varepsilon}$$

using that the spectrum is closed. Thus, for all $\varphi \in H$, we conclude

$$\|\varphi\| = \left\| R_A(\lambda)(A - \lambda)\varphi) \right\| \leq \|R_A(\lambda)\| \, \|(A - \lambda)\varphi\| < \frac{1}{\varepsilon} \|(A - \lambda)\varphi\|,$$

contradicting the assumption that $\|(A - \lambda)\psi\| \leq \varepsilon \|\psi\|$ for some $\psi \in H$. $\square$

5.4. Approximate eigenfunctions - Weyl's criterion. We provide here a characterization of the spectrum and the (essential) spectrum by approximate eigenfunctions.

Let $A \in \mathcal{L}(H)$ be linear bounded operator. An element $\lambda \in \sigma(A)$ is called an *isolated eigenvalue*, if there is a $\psi \in H$ such that $A\psi = \lambda\psi$ and there is an $\varepsilon > 0$ such that $B_\varepsilon(\lambda) \cap \sigma(A) = \{\lambda\}$. Furthermore, an isolated eigenvalue $\lambda \in \sigma(A)$ has *finite multiplicity* if

$$\{\psi \in H \,|\, A\psi = \lambda\psi\}$$

is finite dimensional. Then the *discrete spectrum* $\sigma_{disc}(A)$ of A is defined by

$$\sigma_{disc}(A) := \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda \text{ is an isolated eigenvalue with finite multiplicity}\right\}.$$

With this at hand, the *essential spectrum of A* is defined by

$$\sigma_{ess}(A) := \sigma(A) \setminus \sigma_{disc}(A).$$

Since the discrete spectrum consists only of isolated points in $\mathbb{C}$, the essential spectrum $\sigma_{ess}(A)$ is a compact set. We point out that the essential spectrum can still contain an isolated eigenvalue $\lambda \in \sigma(A)$ but with infinite multiplicity or a limit point of a sequence of eigenvalues with finite multiplicity. Unlike $\sigma(A)$, the essential spectrum is stable under compact pertubations. This theorem is called Weyl's theorem which is not content of this lecture. We refer the interested reader to Section 14.2 in the book *Introduction to Spectral Theory: with applications to Schrödinger operators* by Hislop and Sigal

Theorem 5.13 (Weyl's criterion - spectrum). *Let $A \in \mathcal{L}(H)$ be a self-adjoint operator on a Hilbert space H. Then the following are equivalent.*

(i) $\lambda \in \sigma(A)$.

(ii) *There exists a sequence $\psi_n \in H$, $n \in \mathbb{N}$, such that $\|\psi_n\| = 1$ and*

$$\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0.$$

Proof. (ii)⇒(i): We prove the statement by contraposition. Let $\lambda \in \rho(A)$, then there is a constant $C > 0$ such that

$$\|R_A(\lambda)\varphi\| \le C\|\varphi\|$$

for all $\varphi \in H$. For $\psi \in H$, set $\varphi := (A-\lambda)\psi$. Then the previous considerations lead to

$$\|\psi\| = \|R_A(\lambda)\varphi\| \le C\|(A-\lambda)\psi\|.$$

Hence, there cannot exist a sequence $(\psi_n)_n \subseteq H$ satisfying (ii).

(i)⇒(ii): Let $\lambda \in \sigma(A) \subseteq \mathbb{R}$. Thus $A - \lambda$ is not bijective (if $A - \lambda$ would be bijective, it must be automatically continuous).

If $A - \lambda$ is not injective, then $\ker(A-\lambda) \neq \{0\}$. Thus, there is an $\psi \in \ker(A-\lambda) \setminus \{0\}$ with $\|\psi\| = 1$ as $\ker(A-\lambda)$ is a subspace, see Proposition 2.8. Define $\psi_n := \psi$ for all $n \in \mathbb{N}$. Then $\|\psi_n\| = 1$ and $(A-\lambda)\psi_n = 0$ proving (ii).

Suppose $A - \lambda$ is not surjective, namely $\mathrm{ran}(A-\lambda) \neq H$. Thus, either $\overline{\mathrm{ran}(A-\lambda)} \neq H$ or $\overline{\mathrm{ran}(A-\lambda)} = H$ while $\mathrm{ran}(A-\lambda) \neq H$.

In the first case $\overline{\operatorname{ran}(A-\lambda)} \neq H$, the orthogonal complement $\operatorname{ran}(A-\lambda)^{\perp}$ is non trivial, see Appendix 3. Then Proposition E.19 implies

$$\{0\} \neq \operatorname{ran}(A-\lambda)^{\perp} = \ker(A-\lambda)^{*} = \ker(A-\lambda)$$

as A is self-adjoint and $\lambda \in \mathbb{R}$. Thus, we are in the case where $A-\lambda$ is not injective and so (ii) follows by the previous considerations.

It is left to treat the case where $\overline{\operatorname{ran}(A-\lambda)} = H$ and $\operatorname{ran}(A-\lambda) \neq H$. We will show that if (ii) does not hold then $\operatorname{ran}(A-\lambda)$ is closed and so

$$\operatorname{ran}(A-\lambda) = \overline{\operatorname{ran}(A-\lambda)} = H,$$

which would be a contradiction. So assume (ii) does not hold, then there is a constant $C > 0$ such that

$$\|(A-\lambda)\varphi\| \geq C\|\varphi\|, \qquad \varphi \in H \tag{$*$}$$

Let $\psi \in H$. Since $\overline{\operatorname{ran}(A-\lambda)} = H$, there is a sequence $\psi_n = (A-\lambda)\varphi_n$ converging to ψ in H. By $(*)$, we conclude

$$\|\varphi_n - \varphi_m\| \leq \frac{1}{C}\|(A-\lambda)(\varphi_n - \varphi_m)\| = \frac{1}{C}\|\psi_n - \psi_m\|.$$

Since $\psi_n \to \psi$, we deduce that $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and we denotes its limit point by $\varphi := \lim_{n\to\infty} \varphi_n$. Then

$$\psi = \lim_{n\to\infty} \psi_n = \lim_{n\to\infty} (A-\lambda)\varphi_n = (A-\lambda)\varphi$$

as $A-\lambda$ is a continuous map. Thus, $\psi \in \operatorname{ran}(A-\lambda)$ follows namely $\operatorname{ran}(A-\lambda) = \overline{\operatorname{ran}(A-\lambda)}$, a contradiction. □

One can even characterize the essential spectrum by approximate eigenfunction that weakly converge to zero. Therefore, recall that a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq H$ converges weakly to zero in a Hilbert space H if and only if for all $\varphi \in H$,

$$\lim_{n\to\infty} \langle \psi_n, \varphi \rangle = 0.$$

Theorem 5.14 (Weyl's criterion - essential spectrum). *Let $A \in \mathcal{L}(H)$ be a self-adjoint operator on a Hilbert space H. Then the following are equivalent.*

(i) $\lambda \in \sigma_{ess}(A)$.
(ii) *There exists a sequence $\psi_n \in H$, $n \in \mathbb{N}$, such that $\|\psi_n\| = 1$, $(\psi_n)_{n\in\mathbb{N}}$ converges weakly to zero and*

$$\lim_{n\to\infty} \|(A-\lambda)\psi_n\| = 0.$$

Remark. *We point out that in the previous two theorems the condition $\|\psi_n\| = 1$ can always replaced by $\lim_{n\to\infty} \|\psi_n\| = 1$.*

Since we are only using the implication (ii)⇒(i), we refer to a spectral theory course for proving (i)⇒(ii). The proof is also given in Section 7.2 in the book *Introduction to Spectral Theory: with applications to Schrödinger operators* by Hislop and Sigal.

PROOF. (ii)⇒(i): Let $\psi_n \in H$, $n \in \mathbb{N}$, be such that $\|\psi_n\| = 1$, $(\psi_n)_{n\in\mathbb{N}}$ converges weakly to zero and

$$\lim_{n\to\infty} \|(A - \lambda)\psi_n\| = 0.$$

By Weyl's criterion for the spectrum (Theorem 5.13), we conclude that $\lambda \in \sigma(A)$. Assume by contradiction that $\lambda \in \sigma_{disc}(A)$ and let $\varphi_1, \ldots, \varphi_m \in H$ be the corresponding normalized eigenfunctions that span the eigenspace which are pairwise orthogonal. Specifically, we have $\langle \varphi_i, \varphi_j \rangle = 0$ if $i \neq j$. Furthermore, there is a $C > 0$ such that $\sigma(A) \cap \overline{B_C(\lambda)} = \{\lambda\}$. Define the eigenprojection $P_\lambda \in \mathcal{L}(H)$ onto the eigenspace of λ by

$$P_\lambda \psi := \sum_{j=1}^{m} \langle \varphi_j, \psi \rangle \varphi_j.$$

Then $I - P_\lambda$ is also a projection. Let $B := A - \lambda|_{\operatorname{ran}(I-P_\lambda)}$ be the restriction of $A - \lambda$ onto the range of $I - P_\lambda$. Then B is injective (otherwise we would have another eigenfunction for λ) and $\|B\psi\| \geq C\|\psi\|$ by Corollary 5.12. The second condition implies that the range of B is closed, confer proof of Theorem 5.13. Since B is also self-adjoint, Proposition E.19 leads to

$$\operatorname{ran}(B)^\perp = \ker(B^*) = \ker(B) = \{0\}.$$

Thus B is also surjective and so B is invertible.

Since (ψ_n) converges weakly to zero, we get $\lim_{n\to\infty} \|P_\lambda \psi_n\| = 0$ (as P_λ is a projection on a finite dimensional subspace).

Define $\phi_n := (I - P_\lambda)\psi_n$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|\phi_n\| = 1$ follows as $\|\psi_n\| = 1$ and $\|P_\lambda \psi_n\| \to 0$. Furthermore,

$$\|(A - \lambda)\phi_n\| = \|(I - P_\lambda)(A - \lambda)\psi_n\| \leq \|I - P_\lambda\| \|(A - \lambda)\psi_n\| \to 0,$$

where we used that A commutes with the eigenprojection P_λ. For indeed, we have for $\phi \in H$ that

$$\begin{aligned} AP_\lambda \phi = \sum_{j=1}^{m} \langle \varphi_j, \phi \rangle \lambda \varphi_j &\overset{\lambda\in\mathbb{R}}{=} \sum_{j=1}^{m} \langle \lambda\varphi_j, \phi \rangle \varphi_j = \sum_{j=1}^{m} \langle A\varphi_j, \phi \rangle \varphi_j \\ &\overset{A^*=A}{=} \sum_{j=1}^{m} \langle \varphi_j, A\phi \rangle \varphi_j = P_\lambda A \phi. \end{aligned}$$

On the other hand, $\phi_n \in \operatorname{ran}(I - P_\lambda)$ and $A - \lambda$ is invertible on $\operatorname{ran}(I - P_\lambda)$ implying $\|\phi_n\| \to 0$, a contradiction with $\|\phi_n\| \to 1$. □

In light of the previous theorem, a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq H$ is called a *Weyl sequence* or *singular sequence (w.r.t. λ)* if

- $\|\psi_n\| = 1$,
- $(\psi_n)_{n\in\mathbb{N}}$ converges weakly to zero,
- $\|(A - \lambda)\psi_n\|$ converges to zero if $n \to \infty$.

5.5. Convergence of the spectra. In this section, we study the spectral map

$$\Sigma : \mathcal{L}(H) \to \mathcal{K}(\mathbb{C}), \quad A \mapsto \sigma(A).$$

We address the question on which subsets of $\mathcal{L}(H)$, the spectral map is continuous. Here we will equip $\mathcal{L}(H)$ with various topologies. We have already seen in Corollary 5.12 that we can localize the spectrum of an $A \in \mathcal{L}(H)$ using the norm of the resolvent. Recall that the resolvent can be locally developed into a norm convergent power series $\sum_{n\in\mathbb{N}_0} a_n(A-z_0)^{-n-1}$, see Proposition E.3. Here we will prove that already the norms of certain polynomials $p(A)$ with degree 2 are sufficient to locate the spectrum if the operator A is self-adjoint.

As we have seen before the spectrum of an $A \in \mathcal{L}(H)$ is a compact non-empty subset of $\mathbb{C}$. The set $\mathcal{K}(\mathbb{C})$ of compact subsets of $\mathbb{C}$ is equipped Hausdorff metric

$$\delta_H(F,K) := \max\Big\{\sup_{x\in F} \operatorname{dist}(x,K), \sup_{y\in K} \operatorname{dist}(y,F)\Big\}$$

where $\operatorname{dist}(x,K) := \inf_{y\in K}|x-y|$. With this at hand, we can measure the distance of two spectra and we get:

Proposition 5.15 (Spectrum is norm continuous). *Let $A, B \in \mathcal{L}(H)$ be normal. Then*

$$d_H\big(\sigma(A),\sigma(B)\big) \;\le\; \|A-B\| \;\le\; 2\max\big\{\|A\|\,,\,\|B\|\big\}\,.$$

Proof. This is an exercise. □

Thus the spectral map is continuous if restricted to the normal elements of $\mathcal{L}(H)$ equipped with the operator norm. However, the operator norm preserves much more spectral properties and we address the question which is the coarsest topology on the self-adjoint elements of $\mathcal{L}(H)$ so that the spectral map Σ is continuous. To motivate this, let us consider the following example.

Example. *Consider the Hilbert space $H := \ell^2(\mathbb{Z})$ and $t : \mathbb{Z} \to \{0,1\}$. For $k \in \mathbb{Z}$, define $t_k : \mathbb{Z} \to \{0,1\}$ by*

$$t_k(n) := \begin{cases} t(n), & n \neq k, \\ 1, & n = k \text{ and } t(k) = 0, \\ 0, & n = k \text{ and } t(k) = 1. \end{cases}$$

Let $\delta_k \in \ell^2(\mathbb{Z})$ be the delta function (i.e. $\delta_k(n) = 0$ if $n \neq k$ and $\delta_k(k) = 1$). Then $\|\delta_k\| = 1$ and

$$\|\widehat{t} - \widehat{t}_k\| \ge \|(\widehat{t} - \widehat{t}_k)\delta_k\| = |t(k) - t_k(k)| = 1.$$

Thus, the norm distance between $\widehat{t}$ and $\widehat{t}_k$ is uniformly bounded from below. Indeed it follows from the previous considerations that any change of the function t (where only some values are flipped from 0 to 1 or vice versa are allowed) will be a large change in terms of the operator norm.

On the other hand, we will see later Chapter 8 that there are possible changes such that the spectra is varying continuously (and even Lipschitz continuously). In particular, for every $\varepsilon > 0$, there are $t : \mathbb{Z} \to \{0,1\}$ and $s : \mathbb{Z} \to$

$\{0,1\}$ *such that* $t \neq s$, $\|\widehat{t} - \widehat{s}\| \geq 1$ *but*

$$d_H\big(\sigma(\widehat{t}), \sigma(\widehat{s})\big) \leq \varepsilon.$$

Note that these changes will be in general not just a single flip. Indeed the Hausdorff topology on the invariant closed subsets of $\{0,1\}^{\mathbb{Z}}$ *becomes crucial.*

The following lemma is the key allowing us to treat self-adjoint elements. Therefore denote by $\overline{B}_r(x) = [x-r, x+r]$ the closed ball of radius $r > 0$ around $x \in \mathbb{R}$ in $\mathbb{R}$.

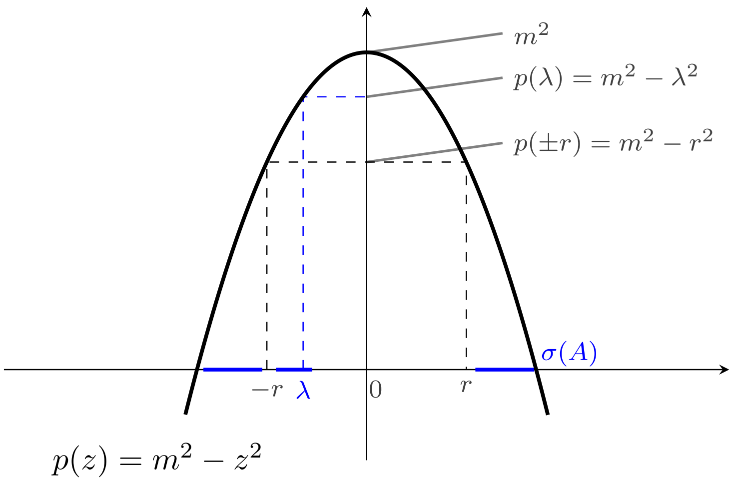

Figure 21. The idea of the proof of Lemma 5.16

Lemma 5.16. *Let* $A \in \mathcal{L}(H)$ *be self-adjoint,* $x \in \mathbb{R}$, $m > \|A\| + |x|$ *and* $p(z) = m^2 - z^2$. *Then the following holds.*

(a) *For* $r < m$, *we have*

$$\|p(A-x)\| < m^2 - r^2 \quad \Longleftrightarrow \quad \overline{B}_r(x) \cap \sigma(A) = \varnothing.$$

(b) *For* $r < m$, *we have*

$$\|p(A-x)\| \geq m^2 - r^2 \quad \Longleftrightarrow \quad \overline{B}_r(x) \cap \sigma(A) \neq \varnothing.$$

Proof. The two statements are equivalent and only (a) will be proven. Since A is self-adjoint, its spectrum is contained in the real line. Moreover, if q is any polynomial, then $\sigma(q(A)) = q(\sigma(A))$. Thus

$$p(\lambda - x) = m^2 - (\lambda - x)^2 \geq m^2 - \|A - x\|^2 \geq m^2 - (\|A\| + |x|)^2 > 0$$

follows for $\lambda \in \sigma(A)$, or, equivalently, $p(\mu) \geq m^2 - \|A-x\|^2 > 0$ for $\mu \in \sigma(A-x)$. Since A is self-adjoint and $m > \|A\| + |x|$, Theorem 5.11 (Spectral radius and norm for normal elements) leads to

$$\|p(A-x)\| = \sup_{\lambda \in \sigma(A-x)} \left|m^2 - \lambda^2\right| = \sup_{\lambda \in \sigma(A-x)} m^2 - \lambda^2.$$

We have

$$\begin{aligned}\overline{B}_r(x) \cap \sigma(A) = \varnothing \quad &\iff \quad [-r, r] \cap \sigma(A-x) = \varnothing \\ &\iff \quad |\lambda| > r \text{ for all } \lambda \in \sigma(A-x) \\ &\iff \quad m^2 - \lambda^2 < m^2 - r^2 \text{ for all } \lambda \in \sigma(A-x) \\ &\iff \quad \|p(A-x)\| = \max_{\lambda \in \sigma(A-x)} m^2 - \lambda^2 < m^2 - r^2,\end{aligned}$$

where the last equivalence holds since A is self-adjoint and its spectrum is compact (supremum is actually a maximum). □

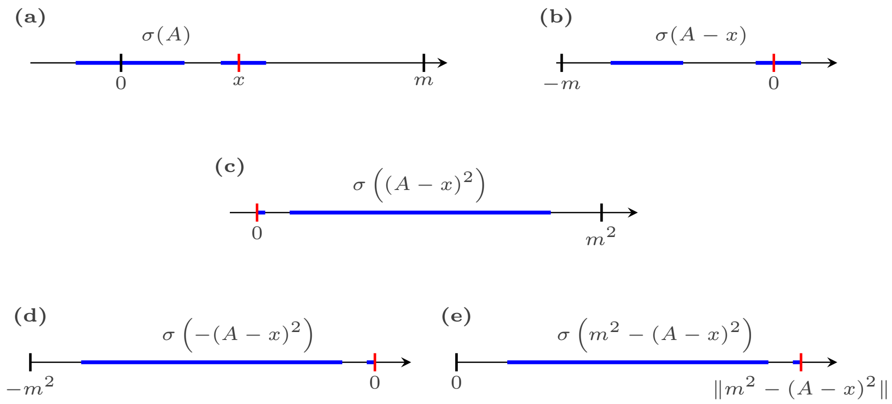

Figure 22. This figure sketches the idea as how these polynomials control the continuity of any spectral value. (a) The spectrum $\sigma(A)$ is contained in $[-m-|x|, m+|x|]$. (b) The spectrum is translated such that x gets the origin. (c) The spectrum is folded at the new origin by taking the square. This compresses and stretches the spectrum. (d) Multiplication by -1 reflects the spectrum to the negative part of the real line. (e) Translate the spectrum by m^2 such that it is contained in the positive part of the real line. Using the fact that $\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$ leads to the desired identity $p(x) = \|p(A)\|$ where $p(z) := m^2 - (z-x)^2$.

Definition ((p2)-continuous). *Let T be a topological space, $(H_t)_{t\in T}$ be a family of Hilbert spaces and $A_t \in \mathcal{L}(H_t)$, $t \in T$, be self-adjoint. Then $(A_t)_{t\in T}$ is called* (p2)-continuous *if*

$$N_p : T \to [0, \infty), \quad t \mapsto \|p(A_t)\|$$

is continuous for every polynomial $p(z) := p_2 z^2 + p_1 z + p_0$ with $p_2, p_1, p_0 \in \mathbb{R}$.

Theorem 5.17 (Convergence of the spectra for self-adjoint operators). *Let T be a topological space, $(H_t)_{t\in T}$ be a family of Hilbert spaces and $A_t \in \mathcal{L}(H_t)$, $t \in T$, be self-adjoint. Then the following statements are equivalent.*

(i) *The operator family $(A_t)_{t\in T}$ is (p2)-continuous.*

(ii) *The spectral map*

$$\Sigma : T \to \mathcal{K}(\mathbb{R}), \quad t \mapsto \sigma(A_t),$$

is continuous where the compact subsets of $\mathbb{R}$ are equipped with the Hausdorff metric.

REMARK. *We refer the reader to* [**BB16**] *for further background where this result was originally proven as well as* [**Bec16**] *for an extensive discussion.*

Week 8

PROOF. $(i) \Rightarrow (ii)$: Let $t_0 \in T$. Since $(A_t)_{t\in T}$ is (p2)-continuous, $t \mapsto \|A_t\|$ is continuous. Thus, there is a neighborhood U of t_0 such that $\sup_{t\in U} \|A_t\| \le \|A_{t_0}\| + 1$. Let $F \subseteq \mathbb{R}$ be closed and $\mathcal{O}$ be a finite family of open subsets of $\mathbb{R}$ such that

$$\sigma(A_{t_0}) \in \mathcal{U}(F, \mathcal{O}) := \{\sigma \in \mathcal{K}(\mathbb{R}) \,|\, \sigma \cap F = \varnothing, \sigma \cap O \neq \varnothing \text{ for all } O \in \mathcal{O}\}.$$

Set $m := 2\|A_{t_0}\| + 1$ and $K := F \cap [-\|A_{t_0}\| - 1, \|A_{t_0}\| + 1] \subseteq \mathbb{R}$, which is compact. Then for $t \in U$, we have $\sigma(A_t) \subseteq B_{\|A_{t_0}\|+1}(0)$ by Theorem 5.2. Thus, $F \cap \sigma(A_t) = \varnothing$ holds if and only if $K \cap \sigma(A_t) = \varnothing$.

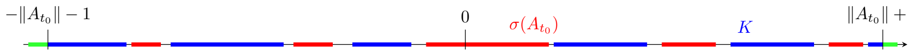

Figure 23. The blue lines denote the set K and the green lines together with the blue lines denotes the set F.

Since $K \cap \sigma(A_{t_0}) = \varnothing$ and both sets are compact (closed), there is an $0 < r(z) < \|A_{t_0}\|$ for each $z \in K$ such that $B_{r(z)}(z) \cap \sigma(A_{t_0}) = \varnothing$. Since $B_{r(z)}(z)$, $z \in K$, defines an open cover of K (which is compact), there are $z_1, \dots, z_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B_{r(z_i)}(z_i)$$

and

$$B_{r(z_i)}(z_i) \cap \sigma(A_{t_0}) = \varnothing \text{ for } i = 1, \dots, n.$$

Note that $\|A_{t_0} - z_i\| \le \|A_{t_0}\| + |z_i| \le m$. Thus, the previous Lemma 5.16 (a) yields

$$\|m^2 - (A_{t_0} - z_i)^2\| < m^2 - r(z_i)^2.$$

The (p2)-continuity implies that there is a neighborhood $U_i \subseteq U$ of t_0 such that

$$\|m^2 - (A_t - z_i)^2\| < m^2 - r(z_i)^2, \qquad t \in U_i.$$

Hence, the previous Lemma 5.16 (a) leads to $\sigma(A_t) \cap B_{r(z_i)}(z_i) = \varnothing$ for all $t \in U_i$. Set $U_F := \bigcap_{i=1}^n U_i$ which is an open neighborhood of t_0 (as a finite intersection of open sets). Then for $t \in U_F$, we get

$$\sigma(A_t) \cap K \subseteq \bigcup_{i=1}^{n} \big(B_{r(z_i)}(z_i) \cap \sigma(A_t)\big) = \varnothing.$$

Since $U_F \subseteq U$ and $\sup_{t\in U} \|A_t\| \le \|A_{t_0}\| + 1$, we derive $F \cap \sigma(A_t) = \varnothing$ for $t \in U_F$.

Let $O \in \mathcal{O}$. Then let $z \in \sigma(A_{t_0}) \cap O$ which exists as $\sigma(A_{t_0}) \in \mathcal{U}(F, \mathcal{O})$. Thus, there is an $r(z) > 0$ such that $B_{r(z)}(z) \subseteq O$ as O is open. Since $|z| \le \|A_{t_0}\|$, we conclude $\|A_{t_0} - z\| \le 2\|A_{t_0}\| \le m$. Hence, the previous Lemma 5.16 (b) implies

$$\|m^2 - (A_{t_0} - z)^2\| \ge m^2 - \frac{r(z)^2}{2} > m^2 - r(z)^2$$

as $z \in \sigma(A_{t_0})$. By the (p2)-continuity, there is an open neighborhood $U_O \subseteq U$ of t_0 such that

$$\|m^2 - (A_t - z)^2\| > m^2 - r(z)^2, \qquad t \in U_O.$$

Using the previous Lemma 5.16 (b), we conclude

$$\varnothing \neq B_{r(z)}(z) \cap \sigma(A_t) \subseteq O \cap \sigma(A_t)$$

for all $t \in U_O$.

Define $V := U_F \cap \bigcap_{O \in \mathcal{O}} U_O$ which is an open neighborhood of t_0 (as it is a finite intersection of such neighborhoods). If $t \in V$, then $\sigma(A_t) \cap F = \varnothing$ as $t \in U_F$ and for $O \in \mathcal{O}$, we have $\sigma(A_t) \cap O \neq \varnothing$ as $t \in U_O$. Thus, $\sigma(A_t) \in \mathcal{U}(F, \mathcal{O})$ for $t \in V$ proving the continuity of Σ.

$(ii) \Rightarrow (i)$: We have to show that $N_p : T \mapsto \mathbb{R}, N_p(t) := \|p(A_t)\|$, is continuous for each polynomial p. Using the exercise on page 71 and Theorem 5.11 (Spectral radius and norm of normal elements), we conclude

$$\begin{aligned} t \xmapsto{\Sigma} \sigma(A_t) \quad &\xmapsto{p} \quad p\big(\sigma(A_t)\big) = \sigma\big(p(A_t)\big) \\ &\xmapsto{|\cdot|} \quad \big\{|\lambda| \,\big|\, \lambda \in \sigma(p(A_t))\big\} \\ &\xmapsto{\sup} \quad \sup\big\{|\lambda| \,\big|\, \lambda \in \sigma(p(A_t))\big\} = \|p(A_t)\| \end{aligned}$$

where $p : \mathbb{C} \to \mathbb{C}$ and $|\cdot| : \mathbb{C} \to \mathbb{R}$ are continuous maps. Thus N_p is a composition of continuous maps invoking Sheet 2, and so it is continuous itself. □

Remark. *(a) Let $\|p\|_1 := |p_2| + |p_1| + |p_0|$ be the one norm of a polynomial $p(z) := p_2 z^2 + p_1 z + p_0$. Then a neighborhood base of $A_0 \in \mathcal{L}(H)$ for the* (p2)-topology *is given by*

$$\mathcal{U}_{\varepsilon,M}(A_0) := \big\{A \in \mathcal{L}(H) \,\big|\, \big|\|p(A_0)\| - \|p(A)\|\big| < \varepsilon \text{ for all } p \text{ with } \|p\|_1 \leq M\big\}$$

for $\varepsilon > 0$ and $M > 0$. The latter theorem says that Σ is continuous in the (p2)-topology. Clearly, the operator norm is finer than the (p2)-topology, while it has no relation with the strong operator topology accept in the finite dimensional case where the strong and operator norm topology coincide.

(b) It is remarkable that we can control the spectrum of operators living on different Hilbert spaces.

(c) The results extends to the setting of unitary operators and normal operators while different polynomials need to be considered. Specifically, if all $(A_t)_{t \in T}$ are unitary then N_p must be continuous for all

$$p(z) = 1 + Ez, \qquad E \in \mathbb{T} := \big\{y \in \mathbb{C} \,\big|\, |y| = 1\big\}.$$

For normal operators, N_p has to be continuous for all polynomials. This is related to the geometry of the spectra within $\mathbb{C}$. Specifically, unitary operators U have spectrum $\sigma(U) \subseteq \mathbb{T}$.

(d) If one can control the norms quantitatively then we can deduce even quantitative estimates on the Hausdorff distance of the spectra, see the following section.

One topology that is often considered on $\mathcal{L}(H)$ is the strong operator topology (note that this topology is not second countable in general).

Definition. *A sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(H)$* converges strongly *to $A \in \mathcal{L}(H)$ (we write $A_n \xrightarrow{str} A$) if*

$$\lim_{n\to\infty} A_n x = Ax \qquad \text{for all } x \in H.$$

Remark. *The strong convergence is a pointwise convergence while the operator norm convergence corresponds to a uniform convergence.*

Proposition 5.18. *Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(H)$ and $A \in \mathcal{L}(H)$ be such that $\|A_n - A\| \to 0$. Then $A_n \xrightarrow{str} A$ follows.*

Proof. Let $x \in H$. Then

$$\|A_n x - Ax\| = \|(A_n - A)x\| \le \|A_n - A\|\|x\|$$

follows leading to the desired result. □

In general the continuity of operators in the strong operator topology does not imply convergence of the spectrum as the following example shows

Example. *Let $H := \ell^2(\mathbb{N})$ and consider the operators $A_k \in \mathcal{L}(\ell^2(\mathbb{N}))$, $k \in \mathbb{N}$, defined by $A_k := \widehat{\chi_{[0,k]}}$, namely the multiplication operator where $\chi_{[0,k]} : \mathbb{N} \to \{0,1\}$ is the characteristic function of the interval $[0,k]$. Thus,*

$$(A_k\psi)(n) = \begin{cases} 0, & n > k, \\ \psi(n), & n \le k. \end{cases}$$

Then $\sigma(A_k) = \{0,1\}$, see Sheet 7. Let $I \in \mathcal{L}(\ell^2(\mathbb{N})$ be the identity operator. Then for every $\psi \in \ell^2(\mathbb{N})$, we conclude

$$\|(A_k - I)\psi\| = \sum_{n\ge k} |\psi(n)|^2 \overset{k\to\infty}{\longrightarrow} 0$$

as $\|\psi\| < \infty$ is square summable. Hence, A_k converges strongly to I. On the other hand, $\sigma(I) = \{1\}$ and so

$$\sigma(I) = \{1\} \subsetneq \{0,1\} = \lim_{k\to\infty} \sigma(A_k).$$

Note that A_k, $k \in \mathbb{N}$, and I are self-adjoint. However, we have seen that strong convergence does not imply the convergence of the spectrum in the Hausdorff metric d_H.

The last example has shown that we cannot expect continuity of the spectral map

$$\Sigma : \mathcal{L}(H) \to \mathcal{K}(\mathbb{C}),\ A \mapsto \sigma(A),$$

with respect to the strong operator topology. However, we observed an inclusion of the spectra ("semi-continuity") in the example. This is similar as the semi-continuity of the invariant probability measures

$$\limsup_{n\to\infty} \mathcal{M}^1(Y_n, G) \subseteq \mathcal{M}^1(Y, G) \qquad \text{if} \qquad Y_n \to Y \in \mathcal{J},$$

confer Theorem 2.19. In order to prove the semi-continuity of the spectra, we need the following.

Let T be a topological space. A function $f : T \to \mathbb{R}$ is called *lower semi-continuous* at $t_0 \in T$ if for every $r < f(t_0)$, there is a neighborhood U of t_0 such that $r < f(t)$ for all $t \in U$.

We denote by $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$, i.e. each $n \in \mathbb{N}$ is isolated and a neighborhood basis of the point $\infty \in \overline{\mathbb{N}}$ is given by

$$U_k := \left\{ t \in \overline{\mathbb{N}} \,\middle|\, t = \infty \text{ or } t \geq k \right\}, \qquad k \in \mathbb{N}.$$

LEMMA 5.19. *Let H be a Hilbert space and $A_t \in \mathcal{L}(H), t \in \overline{\mathbb{N}}$, be self-adjoint with $\sup_{n \in \mathbb{N}} \|A_n\| < \infty$ and $A_n \xrightarrow{str} A_\infty$. Then the map*

$$N_p : \overline{\mathbb{N}} \to [0, \infty), \quad N_p(t) := \|p(A_t)\|$$

is lower semi-continuous for all polynomials $p(z) := p_2 z^2 + p_1 z + p_0$ with $p_i \in \mathbb{C}$.

PROOF. This is an exercise. $\square$

PROPOSITION 5.20. *Let H be a Hilbert space and $A_t \in \mathcal{L}(H), t \in \overline{\mathbb{N}}$, be self-adjoint with $\sup_{n \in \mathbb{N}} \|A_n\| < \infty$. If $A_n \xrightarrow{str} A_\infty$, then*

$$\sigma(A_\infty) \subseteq \limsup_{n \to \infty} \sigma(A_n) := \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(A_m)}$$

REMARK. *We point out that* $\limsup$ *of these compact sets can be identified with*

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty} \sigma(A_m)} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \text{ exists } \lambda_{n_k} \in \sigma(A_{n_k}) \text{ with } \lambda_{n_k} \to \lambda \right\}.$$

*Moreover, if $\big(\sigma(A_n)\big)_{n \in \mathbb{N}}$ is convergent in the Hausdorff metric d_H, then it is straightforward to check that $\limsup_{n \to \infty} \sigma(A_n) = \lim_{n \to \infty} \sigma(A_n)$ (**Exercise**). This can be proven similarly as for G-invariant measures, confer Sheet 4.*

PROOF. All operators are self-adjoint and so $\sigma(A_n) \subseteq \mathbb{R}$. The set

$$\overline{\bigcup_{m=n}^{\infty} \sigma(A_m)} \subseteq \mathbb{R}$$

is closed and so $\limsup_{n \to \infty} \sigma(A_n)$ is a closed subset of $\mathbb{R}$ as the intersection of closed sets is closed. Thus, $\lambda \in \limsup_{n \to \infty} \sigma(A_n)$ if and only if for all $\varepsilon > 0$,

$$[\lambda - \varepsilon, \lambda + \varepsilon] \cap \limsup_{n \to \infty} \sigma(A_n) \neq \emptyset.$$

Let $\lambda \in \sigma(A_\infty)$. We will show that for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(A_n) \neq \emptyset$ for all $n \geq n_0$ proving the statement.

By assumption $m := 2 \sup_{t \in \overline{\mathbb{N}}} \|A_n\| < \infty$. Let $m > \varepsilon > 0$. Since $\lambda \in \sigma(A_\infty)$, we conclude

$$\left[\lambda - \frac{\varepsilon}{2}, \lambda + \frac{\varepsilon}{2}\right] \cap \sigma(A_\infty) \neq \emptyset.$$

Thus $\|A_t\| + |\lambda| \leq m$ holds for all $t \in T := \overline{\mathbb{N}}$. Define the polynomial $p(z) := m^2 - z^2$. Then

$$\|p(A_\infty - \lambda)\| \geq m^2 - \left(\frac{\varepsilon}{2}\right)^2 > m^2 - \varepsilon^2$$

follows by Lemma 5.16 (b). Since p is a polynomial up to degree 2 and $A_n \xrightarrow{\text{str}} A_\infty$, Lemma 5.19 implies that $t \mapsto \|p(A_t - \lambda)\|$ is lower semi-continuous. Hence, there is an $n_0 \in \mathbb{N}$ such that

$$\|p(A_n - \lambda)\| > m^2 - \varepsilon^2, \qquad n \geq n_0.$$

Applying again Lemma 5.16 (b), we conclude $\sigma(A_n) \cap [\lambda - \varepsilon, \lambda + \varepsilon] \neq \emptyset$ for all $n \geq n_0$. ☐

5.6. Hölder continuity of the spectra. We will show that quantitative estimates of the norms $t \mapsto \|p(A_t)\|$ of polynomials up to degree 2 lead to to quantitative estimates of the Hausdorff distance of the spectra and vice versa. It turns out that we loose some regularity in one direction.

Definition (Hölder-continuous). *Let (X, d_X) and (Y, d_Y) be two complete metric spaces. For $\alpha > 0$, a function $f : X \to Y$ is called* α-Hölder-continuous *if there is a constant $C_H(f) > 0$ such that*

$$d_Y(f(x), f(y)) \leq C_H(f) d_X(x,y)^\alpha, \qquad x, y \in X.$$

A 1-Hölder-continuous function f is also called Lipschitz continuous *with constant $C_L := C_L(f) := C_H(f) > 0$.*

Let $\mathcal{F}$ be a family of Hölder continuous functions $f : X \to Y$ with constant $C_H(f)$. This family $\mathcal{F}$ is called uniformly α-Hölder-continuous *if*

$$\sup_{f \in \mathcal{F}} C_H(f) < \infty.$$

Let (X, d_X) be a complete metric space. Recall that $\mathcal{K}(X) := \{K \subseteq X \text{ compact}\}$ is equipped with the Hausdorff metric d_H defined by

$$d_H(K,F) \;:=\; \max\left\{ \sup_{x \in K} \inf_{y \in F} d_X(x,y),\ \sup_{y \in F} \inf_{x \in K} d_X(x,y) \right\}, \quad K, F \in \mathcal{K}(X).$$

Lemma 5.21. *Let (X, d_X) and (Y, d_Y) be two complete metric spaces. Then the following assertions hold.*

(a) *If $f : X \to Y$ is an α-Hölder-continuous function with constant $C_H(f)$, then*

$$\widetilde{f} : \mathcal{K}(X) \to \mathcal{K}(Y),\ K \mapsto f(K),$$

is α-Hölder-continuous with the same constant $C_H(f)$ where $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are equipped with the corresponding Hausdorff metric.

(b) *The maximum* $\max : \mathcal{K}(\mathbb{R}) \to \mathbb{R}$ *and the minimum* $\min : \mathcal{K}(\mathbb{R}) \to \mathbb{R}$ *are Lipschitz continuous function with constant $C_L = 1$.*

Proof. (a) As f is continuous, it maps compact sets to compact sets. Thus, the map $\widetilde{f}$ is well-defined. Let $K, F \in \mathcal{K}(X)$. Then the estimate

$$\inf_{y \in F} d_Y(f(x), f(y)) \;\leq\; C_H(f) \inf_{y \in F} d_X(x,y)^\alpha$$

holds for each $x \in K$. By taking the supremum over $x \in K$ and interchanging the role of K and F the desired result follows.

(b) Let $F_1, F_2 \in \mathcal{K}(\mathbb{R})$. Then $\max(F_j)$ exists for $j = 1,2$ as F_j is bounded. Without loss of generality, assume that $\max(F_1) \le \max(F_2)$. Thus, the estimate

$$\left|\max(F_2) - \max(F_1)\right| = \max_{\mu\in F_2} \min_{\lambda\in F_1} |\mu - \lambda| \le d_H(F_1, F_2)$$

follows by using the definition of the Hausdorff metric. The case of the minimum is similarly treated. □

For $M > 0$, define

$$P_2(M) = \left\{p : \mathbb{C} \to \mathbb{C} \,\middle|\, p(z) = p_2 z^2 + p_1 z + p_0 \text{ with } p_0, p_1, p_2 \in \mathbb{R} \text{ and } \|p\|_1 \le M\right\}$$

where $\|p\|_1 := |p_0| + |p_1| + |p_2|$.

Definition ((p2)-α-Hölder-continuous). *Let (T, d) be a complete metric space, $(H_t)_{t\in T}$ be a family of Hilbert spaces and $A_t \in \mathcal{L}(H_t)$, $t \in T$, be self-adjoint such that $\sup_{t\in T} \|A_t\| < \infty$. Then $(A_t)_{t\in T}$ is called* (p2)-α-Hölder-continuous *if, for all $M > 0$, the family of maps*

$$N_p : T \to [0, \infty), \quad t \mapsto \|p(A_t)\|, \qquad p \in P_2(M),$$

is uniformly α-Hölder-continuous with constant $C_M := \sup_{p\in P_2(M)} C_H(N_p)$.

Theorem 5.22. *Let (T, d) be a complete metric space, $(H_t)_{t\in T}$ be a family of Hilbert spaces and $A_t \in \mathcal{L}(H_t)$, $t \in T$, be self-adjoint such that $m := \sup_{t\in T} \|A_t\| < \infty$.*

(a) *If $(A_t)_{t\in T}$ is a (p2)-α-Hölder-continuous field, then the map*

$$\Sigma : T \to \mathcal{K}(\mathbb{R}), \quad t \mapsto \sigma(A_t),$$

is $\alpha/2$-Hölder-continuous with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{R})$. In this case, the constant $C_H(\Sigma)$ of Σ is bounded by $\sqrt{C_{4m^2+2}}$.

(b) *If the map*

$$\Sigma : T \to \mathcal{K}(\mathbb{R}), \quad t \mapsto \sigma(A_t),$$

is α-Hölder-continuous with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{R})$ with Hölder constant $C_H(\Sigma)$, then $(A_t)_{t\in T}$ is a (p2)-α-Hölder-continuous family of self-adjoint operators. For $M > 0$, the Hölder constant C_M of the family of maps

$$N_p : T \to [0, \infty), \quad t \mapsto \|p(A_t)\|, \qquad p \in P_2(M),$$

is bounded by $(2m + 1)MC_H(\Sigma)$

Remark. *We refer the reader to* [**BB16**] *for further background where this result was originally proven as well as* [**Bec16**] *for an extensive discussion.*

Proof. (a): Let $s, t \in T$. According to the definition of the Hausdorff metric, it suffices to show

$$\inf_{\mu\in\sigma(A_t)} |\lambda - \mu| \le \sqrt{C_{4m^2+2}} \cdot d(s,t)^{\frac{\alpha}{2}}, \qquad \lambda \in \sigma(A_s),$$

since the case of interchanging s and t can be treated similarly. Since the left hand side is zero whenever $\lambda \in \sigma(A_t)$, there is no loss of generality in supposing that $\lambda \in \sigma(A_s) \setminus \sigma(A_t)$. Then the estimate

$$\|(A_t - \lambda)^2\| \overset{\text{self-adjoint}}{=} \|(A_t - \lambda)(A_t - \lambda)^*\| = \|A_t - \lambda\|^2 \le \left(\|A_t\| + |\lambda|\right)^2 \le 4m^2$$

follows using Proposition 5.9 (C^*-property).

For the polynomial $p(z) = 4m^2 - (z-\lambda)^2$, the estimate

$$\|p\|_1 \; = \; 1 + 2|\lambda| + 4m^2 - \lambda^2 \; = \; 4m^2 + 2 - (1-|\lambda|)^2 \; \le \; 4m^2 + 2$$

is derived implying $p \in P_2(4m^2+2)$. The fact that $\lambda \in \sigma(A_s)$ leads to

$$\|p(A_s)\| = \|4m^2 - (A_s - \lambda)^2\| = \sup_{\mu \in \sigma(A_s)} \left|4m^2 - (\mu - \lambda)^2\right| = 4m^2$$

using Theorem 5.11 (Spectral radius and norm of normal elements). Since $\lambda \notin \sigma(A_t)$, we conclude

$$\|p(A_t)\| \; = \; \sup_{\mu \in \sigma(A_t)} |p(\mu)| \; = \; 4m^2 - \left(\inf_{\mu \in \sigma(A_t)} |\mu - \lambda| \right)^2$$

using that $|\mu - \lambda|^2 \le 4m^2$ for all $\mu \in \sigma(A_t)$.

Consequently, the previous considerations lead to

$$\left(\inf_{\mu \in \sigma(A_t)} |\lambda - \mu| \right)^2 \; = \; \big| \|p(A_t)\| - \|p(A_s)\| \big| \; \le \; C_{4m^2+2} \cdot d(s,t)^\alpha$$

since $p \in P_2(4m^2+2)$. Hence, the map Σ is $\alpha/2$-Hölder-continuous with Hölder constant bounded by $\sqrt{C_{4m^2+2}}$.

(b): Since $m := \sup_{t \in T} \|A_t\|$ is finite, the spectrum $\sigma(A_t)$ is contained in the compact set $K := [-m, m] \subseteq \mathbb{R}$ for each $t \in T$.

Let $M > 0$ and consider a polynomial $p \in P_2(M)$, i.e., $p(z) = p_2 z^2 + p_1 z + p_0$ with $p_0, p_1, p_2 \in \mathbb{R}$ satisfying $\|p\|_1 := |p_0| + |p_1| + |p_2| \le M$. Then, for $x, y \in K$, the estimate

$$\big|p(x) - p(y)\big| \; \le \; \big(|p_1| + |p_2| \; |x+y|\big) \; |x-y| \; \le \; (2m+1)M \; |x-y|$$

holds and so p is Lipschitz continuous on $K = [-m, m] \subseteq \mathbb{R}$ with Lipschitz constant $(2m+1)M$. The absolute value $f : \mathbb{R} \to \mathbb{R}$, $z \mapsto |z|$, is a Lipschitz continuous function with Lipschitz constant 1.

Proposition 5.21 applies to $\widetilde{p}$ and $\widetilde{f}$ as p and f are Lipschitz-continuous maps. Thus, $\widetilde{p}$ and $\widetilde{f}$ are Lipschitz continuous as well as $\max : \mathcal{K}(\mathbb{R}) \to \mathbb{R}$. Recall that the norm map $T \ni t \mapsto \|p(A_t)\| \in [0, \infty)$ is represented by

$$T \ni t \xmapsto{\Sigma} \sigma(A_t) \xmapsto{\widetilde{p}} \sigma(p(A_t)) \xmapsto{\widetilde{f}} |\sigma(p(A_t))| \xmapsto{\max} \|p(A_t)\| \,.$$

Thus, N_p is a composition of Lipschitz continuous and a α-Hölder continuous function Σ implying the estimate

$$\big| \|p(A_t)\| - \|p(A_s)\| \big| \le (2m+1)M \cdot d_H(\sigma(A_s), \sigma(A_t)) \le (2m+1)M \cdot C \cdot d(s,t)^\alpha$$

for all $s, t \in T$. Then the supremum

$$C_M := \sup \big\{ C_H(N_p) \, \big| \, p \in P_2(M) \big\}$$

is bounded by $(2m+1)MC$. Consequently, the family of maps $T \ni t \mapsto \|p(A_t)\| \in [0, \infty)$, $p \in P_2(M)$, is uniformly α-Hölder-continuous, namely $(A_t)_{t \in T}$ is a (p2)-α-Hölder-continuous family of operators. □

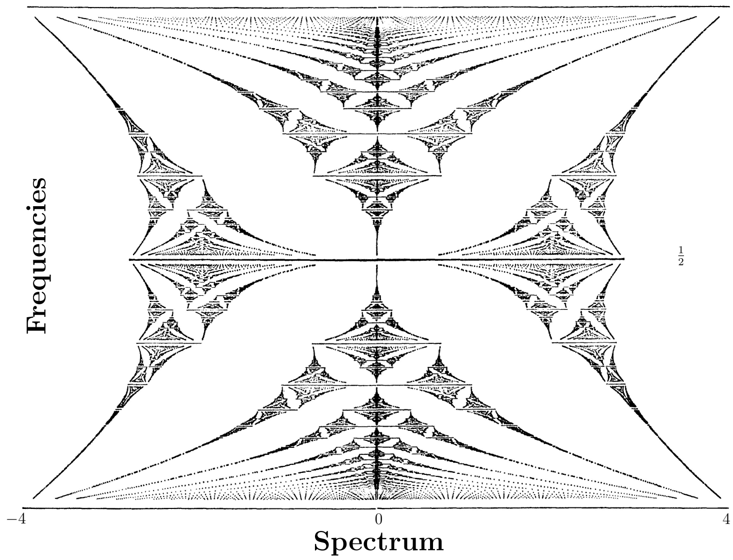

Figure 24. The spectrum of the Almost-Mathieu operators. This picture is taken from the work by [**Hof76**].

Remark. *In assertion (i), the family of spectra* $\sigma(A_t)$, $t \in T$, *only varies* $\alpha/2$*-Hölder-continuous and not* α*-Hölder-continuous. This estimate is optimal which can be seen along the following example. For* $\alpha \in [0,1]$*, let* $H_\alpha : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ *be defined by*

$$(H_\alpha\psi)(n) := \psi(n+1) + \psi(n-1) + 2\cos\big(2\pi n\alpha\big)\psi(n),$$

which is a self-adjoint operator with operator norm bounded by 4*. This family of operators is called Harper model or Almost-Mathieu operators (if the coupling constant is equal to one). It is straight forward to show that* $[0,1] \ni \alpha \mapsto H_\alpha$ *is strongly continuous but not norm continuous as* $\|H_\alpha - H_\beta\| = 2$*. However, it has been proven that*

$$[0,1] \ni \alpha \mapsto \sigma(H_\alpha)$$

is continuous and even $\frac{1}{2}$*-Hölder continuous* [**AS85**]*, namely*

$$d_H\big(\sigma(H_\alpha), \sigma(H_\beta)\big) \le C|\alpha - \beta|^{\frac{1}{2}}.$$

The numerical computations of the spectra for various rational angles is given in Figure 24 produced originally by D. R. Hofstadter. Based on his contribution, the picture is also called Hofstatder buttefly*. At the same time the norms are Lipschitz continuous* [**Bel94**]*. As it turns out the square root behavior of the spectra is optimal in this setting as it was observed that whenever the spectral gaps are closing then the square root behavior was observed* [**RB90**] *and* [**HS87**]*. In particular, the square root behavior is observed at places where the spectral gaps are closing. In view of that result Theorem 5.22 is optimal.*

Week 9

Let $A \in \mathcal{L}(H)$ be a self-adjoint bounded operator. Then every connected component (a,b) of $\mathbb{R} \setminus \sigma(A)$ is called a spectral gap. Thus, $a, b \in \sigma(A)$ with

$a < b$ but $(a,b) \cap \sigma(A) = \varnothing$. Note that there is only a countable number of gaps (why?).

Recall from the Definition of (p2)-α-Hölder continuity (see page 87) that $C_M := \sup_{p \in P_2(M)} C_H(N_p)$ where $C_H(N_p)$ is the Hölder constant of the map $N_p : T \to \mathbb{R}$, $N_p(t) := \|p(A_t)\|$, and $P_2(M)$ are all polynomials up to degree 2 with real coefficients such that $\|p\|_1 \leq M$.

PROPOSITION 5.23. *Let (T,d) be a complete metric space and $(A_t)_{t\in T}$ be a (p2)-α-Hölder-continuous family of bounded self-adjoint operators such that $m := \sup_{t \in T} \|A_t\| < \infty$.*

(a) *The supremum and the infimum*

$$T \ni t \mapsto \sup\big(\sigma(A_t)\big) \in \mathbb{R}, \qquad\qquad T \ni t \mapsto \inf\big(\sigma(A_t)\big) \in \mathbb{R}$$

are α-Hölder-continuous with constant C_{m+1}.

(b) *For $t_0 \in T$ and a gap (a_{t_0}, b_{t_0}) of $\sigma(A_{t_0})$, there exist an open neighborhood U_0 of t_0 and gaps (a_t, b_t) of $\sigma(A_t)$ for $t \in U_0$ such that the gaps $\big((a_t,b_t)\big)_{t\in U_0}$ are α-Hölder-continuous in the sense*

$$\max\big\{|a_s - a_t|, |b_s - b_t|\big\} \;\leq\; \frac{3C_{4m^2+2}}{|b_{t_0} - a_{t_0}|}\; d(s,t)^\alpha\,, \qquad s\,,\, t \in U_0\,.$$

PROOF. (a): For $t \in T$ and $\lambda \in \sigma(A_t)$, the inequality $m \pm \lambda \geq 0$ holds. Then the equations

$$\begin{aligned} \|m + A_t\| &= \sup_{\lambda \in \sigma(A_t)} |m + \lambda| &&= m + \sup_{\lambda\in\sigma(A_t)} \lambda\,, \\ \|m - A_t\| &= \sup_{\lambda \in \sigma(A_t)} |m - \lambda| &&= m - \inf_{\lambda\in\sigma(A_t)} \lambda\,, \end{aligned}$$

follows by Theorem 5.11 (Spectral radius and norm of normal elements). Thus,

$$\begin{aligned} \big|\sup\big(\sigma(A_t)\big) - \sup\big(\sigma(A_s)\big)\big| &= \big|\|m + A_t\| - \|m + A_s\|\big|, \\ \big|\inf\big(\sigma(A_t)\big) - \inf\big(\sigma(A_s)\big)\big| &= \big|\|m - A_t\| - \|m - A_s\|\big| \end{aligned}$$

follow. These polynomials $\mathbb{R} \ni z \mapsto m \pm z \in \mathbb{R}$ have 1-norm equal to $m+1$. Applying the (p2)-α-Hölder-continuity, the desired result is obtained.

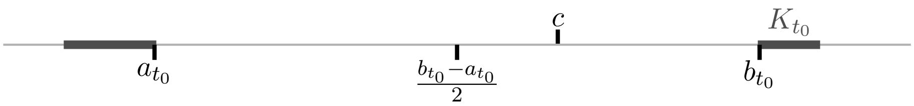

Figure 25. The choice of $c \in \mathbb{R}$ in Proposition 5.23 (b).

(b): Let $t_0 \in T$ and (a_{t_0}, b_{t_0}) be a gap of $\sigma(A_{t_0})$, namely

$$a_{t_0},\, b_{t_0} \in \sigma(A_{t_0}) \qquad \text{and} \qquad (a_{t_0}, b_{t_0}) \cap \sigma(A_{t_0}) = \varnothing.$$

Subdivide the interval (a_{t_0}, b_{t_0}) into six intervals of equal length

$$r := (b_{t_0} - a_{t_0})/6 \qquad \text{and set} \qquad c := a_{t_0} + 4r = b_{t_0} - 2r,$$

confer Figure 25. The field $(A_t)_{t\in T}$ is (p2)-continuous by assumption and so $\Sigma : T \to \mathcal{K}(\mathbb{R})$, $t \mapsto \sigma(A_t)$, is continuous, c.f. Theorem 5.17. Thus, there exists an open neighborhood U_0 of t_0 such that $d_H\big(\sigma(A_t), \sigma(A_{t_0})\big) < r$. By definition of the Hausdorff metric, we conclude for every $t \in U_0$, that there are $a_t, b_t \in \sigma(A_t)$ satisfying

$$a_t < b_t, \qquad |a_t - a_{t_0}| < r, \qquad |b_t - b_{t_0}| < r \qquad \text{and} \qquad (a_t, b_t) \cap \sigma(A_t) = \varnothing.$$

Consequently, for $t \in U_0$, the following inequalities hold:

$$\begin{array}{lllllll} (1) & b_t - c & = b_t - b_{t_0} + b_{t_0} - c & \Rightarrow\ r & < b_t - c & < 3r\,, \\ (2) & c - a_t & = c - a_{t_0} + a_{t_0} - a_t & \Rightarrow\ c - a_t & > 4r - r & = 3r\,. \end{array}$$

Thus, c is closer to b_t than to a_t. Hence, if $t \in U_0$, then

$$\inf_{\mu \in \sigma((A_t - c)^2)} \mu = \inf_{\lambda \in \sigma(A_t)} (\lambda - c)^2 = (b_t - c)^2$$

Since $c \in (a_{t_0}, b_{t_0})$, we conclude $|c| \le \|A_t\| \le m$ leading to

$$4m^2 - (b_t - c)^2 = \|4m^2 - (A_t - c)^2\|, \qquad t \in U_0.$$

This implies

$$\big|\|p(A_t)\| - \|p(A_s)\|\big| \ = \ \big|(b_t - c)^2 - (b_s - c)^2\big| \ = \ |b_t - b_s|\ |b_t + b_s - 2c|,$$

for all $s, t \in U_0$. Furthermore, the 1-norm of the polynomial $p(z) = 4m^2 - (z - c)^2$ is estimated by

$$\|p\|_1 \ = \ 1 + 2|c| + 4m^2 - c^2 \ = \ 4m^2 + 2 - (1 - |c|)^2 \ \le \ 4m^2 + 2\,.$$

In addition, (1) leads to

$$|b_t + b_s - 2c| > 2r = \frac{|b_{t_0} - a_{t_0}|}{3}$$

Combining the previous considerations, we conclude

$$|b_s - b_t| = \frac{\big|\|p(A_t)\| - \|p(A_s)\|\big|}{|b_t + b_s - 2c|} \le \frac{3C_{(4m^2+2)}}{|b_{t_0} - a_{t_0}|}\ d(s,t)^{\alpha}\,, \qquad s, t \in U_0\,.$$

Changing c into $a_{t_0} + 2r = b_{t_0} - 4r$ yields the same estimate of $|a_s - a_t|$ for $s, t \in U_0$. $\square$

6. Schrödinger operators over dynamical systems

Families of random operators arise in the study of disordered media. More precisely, one is given a (discrete) group G acting on a compact metric space X and a family of operators $(A_x)_{x\in X}$ on $\ell^2(G)$. Here, X represents the set of "all manifestations" of a fixed kind of disorder on the locally compact (countable) group G. One has two constraints on a family of operators $(A_x)_{x\in X}$

- translating the origin of coordinates is equivalent to shifting the atomic configuration in the background **covariance condition**
- continuity condition on the operator family, namely $x \mapsto A_x$ is strongly continuous, i.e. for all $\psi \in \ell^2(G)$, the norm $\|(A_x - A_y)\psi\|$ is small if x and y are close enough ("changing the configuration slightly doesn't change the energy of the function if A is a Hamiltonian")

We will make this more precise while focusing on the discrete model. More precisely, throughout this section we will only consider a countable discrete group G. Then

$$\ell^2(G) := \left\{ \psi : G \to \mathbb{C} \;\middle|\; \sum_{g\in G} |\psi(g)|^2 < \infty \right\}$$

is a Hilbert space with inner product defined by

$$\langle \psi, \phi \rangle := \sum_{g\in G} \overline{\psi(g)}\phi(g).$$

Then the induced norm by this inner product is given by

$$\|\psi\|_2 := \sqrt{\langle \psi, \psi \rangle} = \sqrt{\sum_{g\in G} |\psi(g)|^2}, \qquad \psi \in \ell^2(G).$$

The set of finitely supported functions

$$C_c(G) := \left\{ \psi : G \to \mathbb{C} \,\middle|\, \text{ there is a finite } K \subseteq G \text{ s.t. } \psi(g) = 0 \text{ for all } g \notin K \right\}$$

is dense in $\ell^2(G)$. The group G acts naturally on $\ell^2(G)$ by the *left-translation (shift)*

$$L_g : \ell^2(G) \to \ell^2(G), \quad (L_g\psi)(h) := \psi(g^{-1}h),$$

and by the *right-translation (shift)*

$$R_g : \ell^2(G) \to \ell^2(G), \quad (R_g\psi)(h) := \psi(hg),$$

LEMMA 6.1. *Let G be countable and $g, h \in G$. Then the L_g and R_g are unitary operators satisfying*

- $L_g^* = L_{g^{-1}}$ *and* $R_g^* = R_{g^{-1}}$,
- $\|L_g\| = 1 = \|R_g\|$,
- $L_g L_h = L_{gh}$ *and* $R_g R_h = R_{gh}$,
- $L_g R_h = R_h L_g$

PROOF. This is left as an Exercise. □

DEFINITION. *Let (X, G) be a dynamical system with G a discrete group. A family of operators $A_X := (A_x)_{x\in X}$ with $A_x \in \mathcal{L}(\ell^2(G))$ is called*

- covariant (or equivariant), *if*

$$A_{gx} = L_g A_x L_{g^{-1}}$$

for all $g \in G$ and $x \in X$.
- strongly continuous, *if for every $\psi \in \ell^2(G)$, the map*

$$X \to \ell^2(G), \quad x \mapsto A_x\psi$$

is continuous.
- bounded, *if $\|A\| := \|A_X\| := \sup_{x\in X} \|A_x\| < \infty$.*

If A_X is covariant, strongly continuous and bounded, then we call it an operator family over the dynamical system (X, G) *and define its spectrum by*

$$\sigma(A_X) := \overline{\bigcup_{x\in X} \sigma(A_x)}.$$

Furthermore, A_X is called self-adjoint *whenever A_x is self-adjoint for all $x \in X$.*

Remark. *(a) We defined the covariance condition by using the left-shift since we are working always with group actions from the left. If one would work with right action (right-invariant Haar measure etc.), then one needs to swap all left actions to right actions and vice versa.*

(b) The motivation of studying such families is that a single operator A_x does not admit enough symmetries to analyze it while the whole family is invariant by the action of G. Furthermore, the covariance encodes that we like to have a theory independent of the origin and unitary transformation preserve all spectral properties (see e.g. Proposition 5.3 for the spectrum as a set). In addition the continuity assumption encodes that the operators A_x continuously depend on the local structure of the configuration $x \in X$. Summing up, we require that the operator family is compatible with the G-action and the topology on X.

Exercise. *Let (X, G) be a topological transitive dynamical systems (namely there is an $x \in X$ with $X = \overline{Orb(x)}$). Prove that if a family of operators $A_X = (A_x)_{x\in X}$ is covariant, self-adjoint and strongly continuous, then A_X is also bounded.*

We already know an example of such a self-adjoint operator family over a dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$:

Example. *Let $G := \mathbb{Z}$, $\mathcal{A} \subseteq \mathbb{R}$ be a finite set and consider for $w \in \mathcal{A}^{\mathbb{Z}}$, the Schrödinger operator $H_w \in \mathcal{L}\big(\ell^2(\mathbb{Z})\big)$ defined by*

$$(H_\omega\psi)(n) := \psi(n-1) + \psi(n+1) + \omega(n)\psi(n).$$

Then $V : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}, V(\omega) := \omega(0)$ is a continuous map and $\omega(n) := V(n^{-1}\omega)$ and so $(H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}}}$ is a self-adjoint operator family over a dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ by Sheet 8. In particular, we can represent the operator H_ω by

$$H_w = R_1 + R_{-1} + \widehat{V}(w)$$

where $(\widehat{V}(w)\psi)(n) = V(n^{-1}w)\psi(n)$ is a multiplication operator.

The term $R_1 + R_{-1}$ is usually interpreted as a discrete version of a Laplacian. It is not difficult to see that for general $\omega \in \mathcal{A}^{\mathbb{Z}}$ (aperiodic), the operator

*R_m, $m \in \mathbb{Z}$, and $\widehat{V}(w)$ do not commute, namely $R_m\widehat{V}(w) \neq \widehat{V}(w)R_m$. On the other hand, the spectrum of $R_1 + R_{-1}$ can be easily computed by the discrete Fourier transform (you have seen this in the continuous case and we will discuss that in the upcoming chapter). Also we have $\sigma(\widehat{V}(w)) = \{V(\rho) \mid \rho \in \overline{Orb(\omega)}\}$ (**Exercise**). However, since both operators do not commute, the spectral properties of H_ω can be very wild. Compare for instance Figure 5 where $\mathcal{A} := \{0,1\}$ and $w_\alpha := \chi_{[1-\alpha,1)}(n\alpha \mod 1)$ or the Kohmoto butterfly in Figure 5.*

REMARK. *We refer to the first term usually to the "kinetic term" and the multiplication operator $\widehat{V}$ to the potential term. If one deals with higher orders of interaction (further sumands of the type $R_m + R_{-m}$), then these interaction terms can be also weighted (even depending on the underlying space), see below. It is even possible to pass to infinite interaction terms but then the interaction must decay so that we still have a bounded linear operator.*

If one considers such an operator family over a dynamical system, then it is natural to find relations between dynamical properties of (X,G) and spectral properties of $(A_x)_{x\in X}$. For instance one raises questions such as

- Is the spectrum $\sigma(A_x)$ independent of $x \in X$?
- If we don't have constancy of the spectrum, when can we have a.e. constancy w.r.t. to some $\mu \in \mathcal{M}^1(X,G)$?
- Are the spectral types (such as discrete, absolutely continuous and singular continuous part of the spectrum) of $\sigma(H_x)$ are constant (almost everywhere)? (The reader is referred to a course on Spectral theory for these terminologies. We will not discuss them here.)
- How does the spectrum changes if we change the underlying dynamics?

We will mainly focus on the first and last question.

PROPOSITION 6.2. *Let (X,G) be a dynamical system and $A_X = (A_x)_{x\in X}$ be a self-adjoint operator family over (X,G).*

(a) *The spectrum $\sigma(A_X)$ is a compact, non-empty subset of $\mathbb{R}$. In particular, $\sigma(A_X) \subseteq \overline{B}_{\|A\|}(0)$ holds where $\|A\| := \|A_X\|$.*

(b) *If $X = \overline{Orb(x)}$ for some $x \in X$ (i.e. X is topological transitive), then $\sigma(H_X) = \sigma(H_x)$.*

PROOF. (a) Let $x \in X$. By Theorem 5.2, we know that $\sigma(A_x)$ is a compact, non-empty subset of $\mathbb{R}$. Thus, $\sigma(A_X)$ is automatically non-empty and a closed subset of $\mathbb{R}$. Furthermore, Theorem 5.2 asserts that

$$\sigma(A_X) = \overline{\bigcup_{x\in X} \sigma(A_x)} \subseteq \overline{\bigcup_{x\in X} \overline{B}_{\|A_x\|}(0)} \subseteq \overline{B}_{\|A\|}(0).$$

Thus, $\sigma(A_X)$ is compact as a closed subset of the compact ball $\overline{B}_{\|A\|}(0)$.

(b) This is left as an exercise. □

6.1. Hamiltonians. We will introduce in this section a specific class of operator families over a dynamical system that we call Hamiltonians. Furthermore, we provide another characterization of when a dynamical system is minimal by a corresponding property of the spectra of operator families over this dynamical systems.

LEMMA 6.3. *Let (X,G) be a dynamical system and G be a discrete group. For a continuous $t : X \to \mathbb{C}$, the maps*

$$\widehat{t}(x) : \ell^2(G) \to \ell^2(G), \quad \big(\widehat{t}(x)\psi\big)(g) := t(g^{-1}x)\psi(g), \qquad x \in X,$$

are bounded linear operators on $\ell^2(G)$ with

$$\sup_{x\in X} \|\widehat{t}(x)\| \le \|t\|_\infty := \sup_{x\in X} |t(x)|.$$

Furthermore, the adjoint operator $\widehat{t}(x)^* \in \mathcal{L}(H)$ satisfies

$$\big(\widehat{t}(x)^*\psi\big)(g) = \overline{t(g^{-1}x)}\psi(g).$$

and the operator family $(\widehat{t}(x))_{x\in X}$ is strongly continuous, covariant and bounded.

REMARK. *For $t : X \to \mathbb{C}$, the map $f : G \to \mathbb{C}$ defined by $f(g) := t(g^{-1}x)$ is continuous and so $\widehat{t}(x)$ is a multiplication operator for each $x \in X$.*

PROOF. *$\widehat{t}$ is bounded and linear:* The linearity of $\widehat{t}(x)$ is trivial. For the boundedness, we observe

$$\|\widehat{t}(x)\psi\|_2^2 = \sum_{g\in G} |t(g^{-1}x)|^2 \; |\psi(g)|^2 \le \|t\|_\infty^2 \sum_{g\in G} |\psi(g)|^2 = \|t\|_\infty^2 \; \|\psi\|_2^2$$

proving $\sup_{x\in X} \|\widehat{t}(x)\| \le \|t\|_\infty$. This also proves that $\big(\widehat{t}(x)\big)_{x\in X}$ is a bounded family of operators as $\|t\|_\infty < \infty$ holds always (t is a continuous function on a compact space).

adjoint operator: For the adjoint operator, we get

$$\langle \psi, \widehat{t}(x)^*\phi\rangle = \langle \widehat{t}(x)\psi, \phi\rangle = \sum_{g\in G} \overline{t(g^{-1}x)\psi(g)}\phi(g) = \sum_{g\in G} \overline{\psi(g)} \, \Big(\overline{t(g^{-1}x)}\phi(g)\Big)$$

for all $\psi, \phi \in \ell^2(G)$ proving the identity for the adjoint operator.

strongly continuous: For the strong continuity, take first $\psi \in C_c(G)$. Then

$$\big\|\big(\widehat{t}(x) - \widehat{t}(y)\big)\psi\big\|_2 = \sqrt{\sum_{g\in G} |t(g^{-1}x) - t(g^{-1}y)|^2 \; |\psi(g)|^2}.$$

Since $t \circ g^{-1} : X \to \mathbb{C}$ is continuous (as composition of continuous functions) for each $g \in G$ and the sum is finite (as ψ has finite support), we conclude that

$$\lim_{y\to x} \|(\widehat{t}(x) - \widehat{t}(y))\psi\|_2 = 0$$

whenever $\psi \in C_c(G)$. Now it is left to show the latter identity extends to $\psi \in \ell^2(G)$.

We will show that for each $\psi \in \ell^2(G)$, $x \in X$ and all $\varepsilon > 0$, there is an open neighborhood U of x such that $\|(\widehat{t}(x) - \widehat{t}(y))\psi\|_2 < \varepsilon$ for all $y \in U$. Let $\psi \in \ell^2(G)$ and $\varepsilon > 0$. Then there is an $\phi \in C_c(G)$ such that

$$\|\psi - \phi\|_2 \le \frac{\varepsilon}{3\|t\|_\infty}$$

as $C_c(G) \subseteq \ell^2(G)$ is dense. By the previous considerations, there is a neighborhood U of x such that if $y \in U$, then $\|(\widehat{t}(x) - \widehat{t}(y))\phi\|_2 < \frac{\varepsilon}{3}$. Hence, the triangle inequality yields

$$\begin{aligned}
\left\|\big(\widehat{t}(x) - \widehat{t}(y)\big)\psi\right\|_2 &\leq \left\|\big(\widehat{t}(x) - \widehat{t}(y)\big)\phi\right\|_2 + \left\|\big(\widehat{t}(x) - \widehat{t}(y)\big)(\psi - \phi)\right\|_2 \\
&\leq \frac{\varepsilon}{3} + \big(\|\widehat{t}(x)\| + \|\widehat{t}(y)\|\big)\|\psi - \phi\|_2 \\
&\leq \frac{\varepsilon}{3} + 2\|t\|_\infty \ \|\psi - \phi\|_2 < \varepsilon
\end{aligned}$$

proving that $\big(\widehat{t}(x)\big)_{x\in X}$ is strongly continuous.

covariant: Let $g \in G$ and $x \in X$. Then

$$\begin{aligned}
(\widehat{t}(gx)\psi)(h) &= t\big((g^{-1}h)^{-1}x\big)\big(L_{g^{-1}}\psi\big)(g^{-1}h) \\
&= \big(\widehat{t}(x)L_{g^{-1}}\psi\big)(g^{-1}h) \\
&= \big(L_g\widehat{t}(x)L_{g^{-1}}\psi\big)(h)
\end{aligned}$$

holds for all $h \in G$ and $\psi \in \ell^2(G)$. This shows that the operator family $\big(\widehat{t}(x)\big)_{x\in X}$ is covariant. □

In the following, we denote by $d : G \times G \to [0,\infty)$ a left-invariant metric on our countable group that always exists. If, for instance, $G = \mathbb{Z}^d$, then $d(g,h) := \sqrt{\sum_{j=1}^d |g_j - h_j|^2}$ for $g,h \in \mathbb{Z}^d$.
The previous example (see also Sheet 8) about Schrödinger operators on $\ell^2(\mathbb{Z})$ leads to the following definitions.

Definition (Hamiltonian). *Let (X,G) be a dynamical system, $\mathcal{R} \subseteq G$ be finite and $t_h : X \to \mathbb{C}, h \in \mathcal{R}$, are continuous. Then the family of operators $H_X := (H_x)_{x\in X}$ defined by*

$$H_x := \sum_{h\in\mathcal{R}} \big(\widehat{t}_h(x)R_h + R_{h^{-1}}\widehat{t}_h(x)^*\big) \tag{H}$$

is called a Hamiltonian over (X,G) *with* finite range $\mathcal{R}$. *Furthermore,*

$$\|H_X\|_S := 2\sum_{h\in\mathcal{R}} \|t_h\|_\infty \sqrt{1 + \frac{d(e,h)^2}{2}}$$

is called the Schur norm *of H_X.*

Remark. *(a) Note that the 2 in the Schur norm comes from the fact that we have two summands $\widehat{t}_h(x)R_h + R_{h^{-1}}\widehat{t}_h(x)^*$. This norm measures the polynomial decay of the off-diagonal terms. Instead of choosing the term $\sqrt{1 + \frac{d(e,h)^2}{2}}$ one could also take $d(e,h)$ which is qualitatively behaving similarly at infinity. However, the first term is more convenient as it is greater or equal than one and so $\|H_X\| \leq \|H\|_S$ can be derived (see below). Furthermore, note that the Schur norm mainly plays a role when $\mathcal{R}$ is infinite, a case that we will discuss only shortly at the end.*

Proposition 6.4 (Basic properties of a Hamiltonian). *Let (X,G) be a dynamical system where G is a countable group. If H_X is a Hamiltonian over (X,G), then H_X is a self-adjoint operator family over (X,G) satisfying*

$$\|H_X\| \leq \|H_X\|_S < \infty.$$

Proof. Since $(AB)^* = B^*A^*$, it is elementary to see $H_x^* = H_x$ for all $x \in X$, namely H_X is self-adjoint. Since $\big(\widehat{t}_h(x)\big)_{x\in X}$ is strongly continuous, and the finite sum strongly continuous operator families is again strongly continuous (**Exercise**, use triangle inequality), we conclude that H_X is strongly continuous as $\mathcal{R}$ is finite.

Recall that $(\widehat{t}(x))_{x\in X}$ and $(\widehat{t}(x)^*)_{x\in X}$ are covariant by the previous Lemma 6.3. Furthermore, $L_h R_g = R_g L_h$ holds for all $g, h \in G$, see Lemma 6.1. Thus,

$$\begin{aligned} H_{gx} &= \sum_{h\in\mathcal{R}} \big(\widehat{t}_h(gx) R_h + R_{h^{-1}} \widehat{t}_h(gx)^*\big) \\ &= \sum_{h\in\mathcal{R}} \big(L_g \widehat{t}_h(x) L_{g^{-1}} R_h + R_{h^{-1}} L_g \widehat{t}_h(x)^* L_{g^{-1}}\big) \\ &= L_g \left(\sum_{h\in\mathcal{R}} \big(\widehat{t}_h(x) R_h + R_{h^{-1}} \widehat{t}_h(x)^*\big) \right) L_{g^{-1}} \\ &= L_g H_x L_{g^{-1}} \end{aligned}$$

follows proving that H_X is covariant. A short computation leads to

$$\begin{aligned} \|H_X\| &\leq \sum_{h\in\mathcal{R}} \big(\|\widehat{t}_h(x) R_h\| + \|R_{h^{-1}} \widehat{t}_h(gx)^*\| \big) \\ &\leq \sum_{h\in\mathcal{R}} \big(\|\widehat{t}_h(x)\| \ \|R_h\| + \|R_{h^{-1}}\| \|\widehat{t}_h(gx)^*\| \big) \\ &\leq 2 \sum_{h\in\mathcal{R}} \|t_h\|_\infty \end{aligned}$$

as $\|R_h\| = 1$. The latter term is estimated from above by the Schur norm $\|H_X\|_S$ since $\Big(1 + \frac{d(e,h)^2}{2}\Big)^{\frac{1}{2}} \geq 1$. Furthermore, $\|H_X\|_S$ is finite as a finite sum over finite summands using that $\|t_h\|_\infty < \infty$ for all $h \in \mathcal{R}$ (t_h is a continuous map on a compact space). □

We note that the Hamiltonians are a generalized version of a Schrödinger operator and this terminology is not standard in the literature. If $G = \mathbb{Z}$ and $\mathcal{R} = \{0, 1\}$, then such Hamiltonians are also called Jacobi operators. The terminology Hamiltonian comes from Quantum mechanics. Specifically, recall that the Hamilton function in classical mechanics represents the total energy of a system while its analog in Quantum mechanics is an operator representing the total energy of the quantum system.

Theorem 6.5 (Characterization of minimality). *Let (X, G) be a dynamical system. Then the following assertions are equivalent*

(i) *(X, G) is minimal.*

(ii) *For all continuous $V : X \to \mathbb{R}$, the self-adjoint operator family $\big(\widehat{V}(x)\big)_{x\in X}$ over the dynamical system (X, G) satisfies*

$$\sigma(\widehat{V}(x)) = \sigma(\widehat{V}(y)) \qquad \textit{for all } x, y \in X.$$

(iii) *For every self-adjoint operator family $A_X = (A_x)_{x\in X}$ over (X, G), we have*

$$\sigma(A_x) = \sigma(A_y)$$

for all $x, y \in X$.

PROOF. (i)⇒(iii): Let $x, y \in X$ and A_X be a self-adjoint operator family over (X, G). If (X, G) is minimal, then $\overline{Orb(x)} = X = \overline{Orb(y)}$ follows. Thus, the Proposition 6.2 leads to

$$\sigma(A_x) = \sigma(A_X) = \sigma(A_y).$$

(iii)⇒(ii): This is trivial as $\big(\widehat{V}(x)\big)_{x \in X}$ is an operator family over (X, G) by Lemma 6.3.

(ii)⇒(i): Assume by contradiction that (X, G) is not minimal. Thus there are $x, y \in X$ such that $y \notin \overline{Orb(x)}$. Thus, the Lemma of Urysohn (Lemma A.7) implies that there is a continuous $V : X \to [0, 1]$ such that $V(y) = 1$ and $V(z) = 0$ for all $z \in \overline{Orb(x)}$. Then

$$\big(\widehat{V}(x)\psi\big)(g) = V(g^{-1}x)\psi(g) = 0$$

for all $g \in G$, $\psi \in \ell^2(G)$. Hence, $\widehat{V}(x) = 0$ follows and so $\sigma\big(\widehat{V}(x)\big) = \{0\}$ (why?).

On the other hand,

$$\big(\widehat{V}(y)\delta_e\big)(e) = V(y)\delta_e(e) = 1$$

holds implying $\|\widehat{V}(y)\| \geq 1$. The spectral radius of $\widehat{V}(y)$ satisfies $r\big(\widehat{V}(y)\big) = \|\widehat{V}(y)\| \geq 1$ by Theorem 5.11 (Spectral radius and norm of normal operators) as $\widehat{V}(y)$ is self-adjoint. Thus, Beurling's Theorem 5.4 implies that there is an $\lambda \in \sigma\big(\widehat{V}(y)\big)$ such that

$$|\lambda| = r\big(\widehat{V}(y)\big) > 0.$$

Due to (ii), we have

$$\{0\} = \sigma\big(\widehat{V}(x)\big) = \sigma\big(\widehat{V}(y)\big) \ni \lambda \neq 0,$$

a contradiction. □

With this at hand, Weyl's criterion for the essential spectrum yields that for minimal dynamical systems, we have pure essential spectrum for any operator family over (X, G).

COROLLARY 6.6 (Minimal systems give rise to essential spectrum). *Let (X, G) be a minimal dynamical system and $A_X = (A_x)_{x \in X}$ be a self-adjoint operator family over (X, G). Then*

$$\sigma(A_X) = \sigma(A_x) = \sigma_{ess}(A_x)$$

for all $x \in X$.

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PROOF. Due to Theorem 6.5, we have $\sigma(A_X) = \sigma(A_x)$ for all $x \in X$ as (X, G) is minimal. Let $x \in X$ and $A_X = (A_x)_{x \in X}$ be a self-adjoint operator family over (X, G). By definition of the essential spectrum, we have $\sigma_{ess}(A_x) \subseteq \sigma(A_x)$. Thus, it suffices to show $\sigma(A_x) \subseteq \sigma_{ess}(A_x)$. Therefore, let $\lambda \in \sigma(A_x)$ be fixed.

We say a sequence $(g_n)_{n \in \mathbb{N}} \subseteq G$ *escapes to infinity* if for all compact $K \subseteq G$, there is an $n_K \in \mathbb{N}$ such that $g_n \notin K$ for all $n \geq n_K$. Since (X, G) is minimal, there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ escaping to infinity such that $\lim_{n\to\infty} g_n x = x$ (**Exercise**).

Since $\lambda \in \sigma(A_x)$, Weyl's criterion for the spectrum (Theorem 5.13) implies that there is a sequence $\psi_m \in \ell^2(G)$, $m \in \mathbb{N}$, such that $\|\psi_m\| = 1$ and $\lim_{m\to\infty} \|(A_x - \lambda)\psi_m\| = 0$. Without loss of generality, suppose

$$\|(A_x - \lambda)\psi_m\| \le \frac{1}{m}, \qquad m \in \mathbb{N}.$$

Using the covariance of A_X, we get

$$(A_{g_n x} - \lambda)\psi_m = (L_{g_n} A_x L_{g_n^{-1}} - \lambda L_{g_n} L_{g_n^{-1}})\psi_m = L_{g_n}\left((A_x - \lambda)L_{g_n^{-1}}\psi_m\right)$$

implying

$$\left\|(A_{g_n x} - \lambda)\psi_m\right\| = \left\|L_{g_n}\left((A_x - \lambda)L_{g_n^{-1}}\psi_m\right)\right\| = \left\|(A_x - \lambda)L_{g_n^{-1}}\psi_m\right)\|$$

where we used that L_{g_n} is unitary and so preserves the norm.

Since $(A_x)_{x\in X}$ is strongly continuous, we conclude that $A_{g_n x} \xrightarrow{\text{str}} A_x$. Thus, for each $m \in \mathbb{N}$, there is an $N_m \in \mathbb{N}$ such that

$$\|(A_{g_n x} - A_x)\psi_m\| \le \frac{1}{m}, \qquad n \ge N_m.$$

Let $K_m \subseteq G$ be compact for $m \in \mathbb{N}$ such that $K_m \subseteq K_{m+1}$ and $\bigcup_{m\in\mathbb{N}} K_m = G$, which always exists (why?). Since $\psi_m \in \ell^2(G)$, there is an $F_m \subseteq G$ compact such that $|\psi_m(g)| \le \frac{1}{m}$ for all $g \in G \setminus F_m$. Since additionally $(g_n)_{n\in\mathbb{N}}$ escapes to infinity, there is for each $m \in \mathbb{N}$ an $L_m \in \mathbb{N}$ such that

$$\left|\langle L_{g_n^{-1}}\psi_m, \delta_g\rangle\right| \le \frac{1}{m}, \qquad g \in K_m,\ n \ge L_m.$$

With this at hand, define $n(m) := \max\{L_m, N_m\}$ and $\varphi_m := L_{g_{n(m)}^{-1}}\psi_m$ for $m \in \mathbb{N}$. We claim that $(\varphi_m)_{m\in\mathbb{N}}$ is a Weyl sequence w.r.t. $\lambda \in \mathbb{R}$ of A_x. For indeed,

$$\begin{aligned}\|(A_x - \lambda)\varphi_m\| = \left\|(A_{g_{n(m)}x} - \lambda)\psi_m\right\| &\le \|(A_{g_{n(m)}x} - A_x)\psi_m\| + \|(A_x - \lambda)\psi_m\| \\ &\le \frac{2}{m} \xrightarrow{m\to\infty} 0\end{aligned}$$

follows as $n(m) \ge N_m$. Furthermore,

$$\|\varphi_m\| = \|L_{g_{n(m)}^{-1}}\psi_m\| = \|\psi_m\| = 1$$

holds as $L_{g_{n(m)}^{-1}}$ is unitary and so preserves the norm. Finally,

$$|\langle\varphi_m, \delta_g\rangle| = \left|\langle L_{g_{n(m)}^{-1}}\psi_m, \delta_g\rangle\right| \le \frac{1}{m}$$

holds for all $g \in K_m$. Since $K_m \subseteq K_{m+1}$ and $\bigcup_{m\in\mathbb{N}} K_m = G$, the latter yields that $(\varphi_m)_{m\in\mathbb{N}}$ converges weakly to zero (the functions δ_g, $g \in G$, are an orthonormal base in $\ell^2(G)$). Thus, $(\varphi_m)_{m\in\mathbb{N}}$ is a Weyl sequence w.r.t. $\lambda \in \mathbb{R}$ of A_x and so $\lambda \in \sigma_{ess}(A_x)$ by Weyl's criterion for the essential spectrum (Theorem 5.14). Hence, $\sigma(A_x) \subseteq \sigma_{ess}(A_x) \subseteq \sigma(A_x)$ is proven implying the desired result. $\square$

REMARK. *(a) As we discussed before, the dynamical system associated with a solid is minimal if the solid has no impurities/defects (each pattern appears with bounded gaps). So impurities are connected with isolated eigenvalues with finite multiplicity.*

(b) We have seen that the essential spectrum is "located at infinity".

(c) There are self-adjoint operator families over a dynamical system (X, G) *that have purely essential spectrum but* (X, G) *is not minimal. For instance, let* $(X, \mathbb{Z})$ *be a dynamical system. Then* $\Delta := R_1 + R_{-1}$ *is self-adjoint and its spectrum equals to* $[-2, 2]$*, see the subsequent chapter. Clearly there is no isolated element in the spectrum and so* $\sigma_{disc}(H_\omega) = \varnothing$*. On the other hand,* $(X, \mathbb{Z})$ *does not need to be minimal at all.*

7. Spectral theory of periodic Hamiltonians

In this chapter, we will get to know some basic tools to treat periodic Hamiltonians over $(X, \mathbb{Z}^d)$.

Consider the sphere (one-dimensional torus) $\mathbb{T} := [0, 2\pi)$ equipped with the normalized Lebesgue measure $\frac{dx}{2\pi}$ on the Borel-σ-algebra of $\mathbb{T}$. Then $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is the Hilbert space of square integrable measurable functions $f : \mathbb{T} \to \mathbb{C}$. We know that an orthonormal base $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is given by the functions f_n, $n \in \mathbb{Z}$ defined by

$$f_n(x) := e^{inx}, \qquad x \in \mathbb{T}.$$

Let us first consider the discrete Laplace operator on $\ell^2(\mathbb{Z})$. As it turns out, we can use the Fourier transform to compute its spectrum.

Proposition 7.1. *Let $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the discrete Laplacian defined by*

$$(\Delta\varphi)(n) := \varphi(n+1) + \varphi(n-1), \qquad \varphi \in \ell^2(\mathbb{Z}),\, n \in \mathbb{Z}.$$

Then $\sigma(\Delta) = [-2, 2]$ holds.

Proof. Let $\varphi \in \ell^2(\mathbb{Z})$ and define $\psi_N \in L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ by $\psi_N(x) := \sum_{n=-N}^{N} \varphi(n)e^{inx}$ for $n \in \mathbb{Z}$. For $M \geq N$, we have

$$\begin{aligned}
\|\psi_N - \psi_M\|_2^2 &= \int_0^{2\pi} \left(\overline{\sum_{N \leq |n| \leq M} \varphi(n)e^{inx}}\right)\left(\sum_{N \leq |k| \leq M} \varphi(k)e^{ikx}\right)\frac{dx}{2\pi} \\
&= \sum_{N \leq |n|, |k| \leq M} \overline{\varphi(n)}\varphi(k) \int_0^{2\pi} e^{i(k-n)x}\,\frac{dx}{2\pi} \\
&= \sum_{N \leq |n| \leq M} |\varphi(n)|^2
\end{aligned}$$

since

$$\int_0^{2\pi} e^{i(k-n)x}\,\frac{dx}{2\pi} = \begin{cases} 0, & k \neq n, \\ 1, & k = n. \end{cases}$$

holds and so $(\psi_N)_{N \in \mathbb{N}}$ is a Cauchy-sequence in $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$. With this at hand, define the discrete Fourier transformation $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ by

$$(\mathcal{F}\varphi)(x) := \lim_{N \to \infty} \sum_{n=-N}^{N} \varphi(n)e^{inx}$$

where the limit is taken in $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$. The limit exists by the previous considerations.

Let $\varphi, \psi \in C_c(\mathbb{Z})$, then

$$\begin{aligned}
\langle \mathcal{F}\varphi,\, \mathcal{F}\psi \rangle &= \int_0^{2\pi} \overline{\left(\sum_{n \in \mathbb{Z}} \varphi(n)e^{inx}\right)}\left(\sum_{k \in \mathbb{Z}} \psi(k)e^{ikx}\right)\frac{dx}{2\pi} \\
&= \sum_{n,k \in \mathbb{Z}} \overline{\varphi(n)}\psi(k) \int_0^{2\pi} e^{i(k-n)x}\,\frac{dx}{2\pi} \\
&= \sum_{n \in \mathbb{Z}} \overline{\varphi(n)}\psi(n) \\
&= \langle \varphi,\, \psi \rangle.
\end{aligned}$$

Since $C_c(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ is dense, we conclude $\langle \mathcal{F}\varphi, \mathcal{F}\psi\rangle = \langle \varphi, \psi\rangle$ for all $\varphi, \psi \in \ell^2(\mathbb{Z})$. Moreover, we claim that $\mathcal{F}$ has dense image in $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$, which can be proved as follows. Let $f \in L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ be such that $\langle f, \mathcal{F}\varphi\rangle = 0$ for all $\varphi \in C_c(\mathbb{Z})$. Hence,

$$0 = \langle f, \mathcal{F}\delta_m\rangle = \int_0^{2\pi} \overline{f(x)} \sum_{n\in\mathbb{Z}} \delta_m(n) e^{inx} \frac{dx}{2\pi} = \int_0^{2\pi} \overline{f(x)} e^{imx} \frac{dx}{2\pi} = \langle f, e^{im\cdot}\rangle$$

for all $m \in \mathbb{Z}$. Since $\{x \mapsto e^{imx} \,|\, m \in \mathbb{Z}\} \subseteq L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is an orthonormal basis, we conclude $f = 0$.

This proves that $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is unitary, namely $\mathcal{F}^* = \mathcal{F}^{-1}$. Note that $\mathcal{F}^{-1} : L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right) \to \ell^2(\mathbb{Z})$ is given by

$$(\mathcal{F}^{-1} f)(n) = \int_0^{2\pi} f(x) e^{-inx} \frac{dx}{2\pi}, \qquad f \in L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right), n \in \mathbb{Z}.$$

For $\varphi \in \ell^2(\mathbb{Z})$, we get

$$\begin{aligned}
(\mathcal{F}\Delta\varphi)(x) &= \sum_{n\in\mathbb{Z}} (\Delta\varphi)(n) e^{inx} \\
&= \sum_{n\in\mathbb{Z}} e^{inx} \big(\varphi(n+1) + \varphi(n-1)\big) \\
&= \sum_{m\in\mathbb{Z}} e^{i(m-1)x} \varphi(m) + \sum_{k\in\mathbb{Z}} e^{i(k+1)x} \varphi(k) \\
&= \sum_{n\in\mathbb{Z}} \varphi(n) \big(e^{i(n-1)x} + e^{i(n+1)x}\big)
\end{aligned}$$

by substituting $m = n + 1$ and $k = n - 1$. Since

$$e^{i(n-1)x} + e^{i(n+1)x} = 2e^{inx} \frac{e^{ix} + e^{-ix}}{2} = 2e^{inx} \cos(x),$$

we conclude

$$(\mathcal{F}\Delta\varphi)(x) = 2\cos(x) \sum_{n\in\mathbb{Z}} \varphi(n) e^{inx} = 2\cos(x)(\mathcal{F}\varphi)(x).$$

Hence, $\mathcal{F}\Delta\mathcal{F}^{-1} : L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right) \to L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is a multiplication operator

$$\big(\mathcal{F}\Delta\mathcal{F}^{-1} f\big) = (2\cos(x)) f(x).$$

We know that the spectrum of a multiplication operator on $L^2\left(\mathbb{T}, \frac{dx}{2\pi}\right)$ is given by the closure of the image of the function. In our case, we conclude

$$\sigma(\Delta) = \sigma\big(\mathcal{F}\Delta\mathcal{F}^{-1}\big) = \overline{\{2\cos(x) \,|\, x \in [0, 2\pi]\}} = [-2, 2]$$

invoking Proposition 5.3. □

We note that the property $L_g \Delta L_{-g} = \Delta$ holds for all $g \in \mathbb{Z}^d$ which is crucial here. The aim is to extend this result to operators that satisfy $L_g A L_{-g} = A$ for $g \in H$ with H being a subgroup of $\mathbb{Z}^d$ large enough. Specifically, we will show this for Hamiltonians over periodic dynamical systems as they admit such symmetries due to the periodicity of the underlying dynamical system.

Proposition 3.8 asserts that an $\omega \in \mathcal{A}^{\mathbb{Z}^d}$ is periodic if only if there is an $N \in \mathbb{N}_0$ and a $p \in \mathcal{A}^{Q_N^+}$ such that $\omega = \omega_p = p^\infty$ where

$$Q_N^+ := \left\{ g := (g_1, \dots, g_d) \in \mathbb{Z}^d \,\middle|\, g_i \geq 0 \text{ and } g_i \leq N \right\} = [0, N+1)^d \cap \mathbb{Z}^d.$$

Recall the notation

$$(N+1)\cdot\mathbb{Z}^d := \left\{\big((N+1)\cdot g_1,\dots,(N+1)\cdot g_d\big)\,\big|\,(g_1,\dots,g_d)\in\mathbb{Z}^d\right\},$$

which will play the role of the subgroup H, mentioned before. Then, for all $g\in(N+1)\cdot\mathbb{Z}^d$, we have

$$g\omega=\omega.$$

With this at hand, any corresponding Hamiltonian $H=(H_\rho)_{\rho\in\Omega}$ with $\Omega:=Orb(\omega)$ has a large symmetry group using the covariance. Specifically, we have

$$L_g H_\omega L_{-g} = H_{g\omega} = H_\omega, \qquad g\in N\cdot\mathbb{Z}^d.$$

As for the Laplace operator on $\ell^2(\mathbb{Z})$, this will help us to study the spectrum of H_ω.

Throughout this chapter, we focus on symbolic dynamical systems over a finite alphabet $\mathcal{A}$ and a periodic configuration $\omega\in\mathcal{A}^{\mathbb{Z}^d}$.

Remark. *As the previous discussion indicates, we will restrict our considerations only to the group $G=\mathbb{Z}^d$ (equipped with addition). In order to avoid confusions with the multiplication (like $(N+1)\cdot g$), we will always write the group multiplication by '+'. This is the case for the whole Chapter 7 as well as the following Chapter 8.*

7.1. Discrete Fourier transform. We like to use also the Fourier transform to compute the spectrum for Hamiltonians H_Ω associated with periodic dynamical systems $(\Omega,\mathbb{Z}^d)$.

Therefore, consider the d-dimensional torus $\mathbb{T}^d:=[0,2\pi)^d$ equipped with the normalized Lebesgue measure $\frac{dx}{(2\pi)^d}$ on the Borel-σ-algebra of $\mathbb{T}^d$. Then $L^2\left(\mathbb{T}^d,\frac{dx}{(2\pi)^d}\right)$ is the Hilbert space of square integrable functions $f:\mathbb{T}^d\to\mathbb{C}$. We know that an orthonormal base $L^2\left(\mathbb{T}^d,\frac{dx}{(2\pi)^d}\right)$ is given by the functions f_n, $n\in\mathbb{Z}^d$ defined by

$$f_n(x):=e^{in.x}, \qquad x\in\mathbb{T}^d,$$

where

$$x.y=\sum_{j=1}^d x_j y_j, \qquad x,y\in\mathbb{R}^d.$$

Specifically, we have

$$\langle f_n,f_m\rangle=\delta_{n,m}$$

and $\langle f,f_n\rangle=0$ for all $n\in\mathbb{Z}^d$ if and only if $f=0$.

In addition, we need the direct sum of Hilbert spaces. Therefore, let $\mathcal{H}_1,\dots,\mathcal{H}_M$ be Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_k$ and induced norm $\|\cdot\|_k$. Then

$$\bigoplus_{k=1}^M\mathcal{H}_k := \left\{ h\in\prod_{k=1}^M\mathcal{H}_k \,\middle|\, \sum_{k=1}^M\|h_k\|_k^2<\infty\right\}$$

gets a Hilbert space if equipped with the inner product

$$\langle h,f\rangle := \sum_{k=1}^M\langle h_k,f_k\rangle_k.$$

We will only consider the case where $M := \sharp Q_N^+$ and

$$\mathcal{H}_k := \mathcal{H} := \overline{\operatorname{Lin}\left\{f_n \,\middle|\, n \in (N+1)\cdot\mathbb{Z}^d\right\}} \subseteq L^2\left(\mathbb{T}^d, \frac{dx}{(2\pi)^d}\right)$$

for some fixed $N \in \mathbb{N}_0$, namely where the Hilbert space is independent of k. Then we will use the notation

$$\mathcal{H}(N) := \bigoplus_{k\in Q_N^+} \overline{\operatorname{Lin}\left\{f_n \,\middle|\, n \in (N+1)\cdot\mathbb{Z}^d\right\}}.$$

REMARK. *We note that the d-dimensional torus $\mathbb{T}^d$ can be identified with the quotient $\mathbb{R}^d/\mathbb{Z}^d$ and similarly, the quotient $\mathbb{R}^d/(N+1)\cdot\mathbb{Z}^d$ is identified with $\mathbb{T}_N^d := [0,(N+1)2\pi)^d$ for $N \in \mathbb{N}_0$. Then the Hilbert space*

$$\overline{\operatorname{Lin}\left\{f_n \,\middle|\, n \in (N+1)\cdot\mathbb{Z}^d\right\}} \subseteq L^2(\mathbb{T}^d, \frac{dx}{(2\pi)^d})$$

is isomorphic to the L^2-space $L^2(\mathbb{T}_N^d) := L^2\big(\mathbb{T}_N^d, \frac{dx}{((N+1)2\pi)^d}\big)$ by

$$U : \overline{\operatorname{Lin}\left\{f_n \,\middle|\, n \in (N+1)\cdot\mathbb{Z}^d\right\}} \to L^2\left(\mathbb{T}_N^d\right), (Uf_n)(x) := f_n(x).$$

It is straight forward to check that the Uf_n for $n \in (N+1)\cdot\mathbb{Z}^d$ are an orthonormal base for $L^2(\mathbb{T}_N^d)$ and that $\|Uf_n\| = \|f_n\| = 1$ (Exercise).

PROPOSITION 7.2. *Let $N \in \mathbb{N}_0$. Then the map $\mathcal{F}_N : C_c(\mathbb{Z}^d) \to \mathcal{H}(N)$ defined by*

$$(\mathcal{F}_N\varphi)_k(x) := \sum_{n\in(N+1)\cdot\mathbb{Z}^d} \varphi(k+n)e^{in.x}, \qquad k \in Q_N^+,$$

satisfies $\langle \mathcal{F}_N\varphi, \mathcal{F}_N\psi\rangle = \langle\varphi, \psi\rangle$. Furthermore, it extends uniquely to surjective isometry $\mathcal{F}_N : \ell^2(\mathbb{Z}^d) \to \mathcal{H}(N)$.

PROOF. In order to simplify the notation, set $\mathcal{F} := \mathcal{F}_N$. It is immediate that $\mathcal{F}$ is linear on $C_c(\mathbb{Z}^d)$. Let $\varphi, \psi \in C_c(\mathbb{Z}^d)$. Then a short computation leads to

$$\begin{aligned}\langle \mathcal{F}\varphi, \mathcal{F}\psi\rangle &= \sum_{k\in Q_N^+} \int_{\mathbb{T}^d} \overline{(\mathcal{F}\varphi)_k(x)}(\mathcal{F}\psi)_k(x)\frac{dx}{(2\pi)^d}\\ &= \sum_{k\in Q_N^+}\ \sum_{n,m\in(N+1)\cdot\mathbb{Z}^d} \overline{\varphi(k+n)}\psi(k+m)\int_{\mathbb{T}^d} \overline{e^{in.x}}e^{im.x}\frac{dx}{(2\pi)^d}.\end{aligned}$$

Since

$$\int_{\mathbb{T}^d} \overline{e^{in.x}}\, e^{im.x}\frac{dx}{(2\pi)^d} = \langle f_n, f_m\rangle = \delta_{n,m}$$

and

$$\mathbb{Z}^d = \bigsqcup_{n\in(N+1)\cdot\mathbb{Z}^d} n + Q_N^+,$$

we conclude

$$\langle \mathcal{F}\varphi, \mathcal{F}\psi\rangle = \sum_{k\in Q_N^+}\ \sum_{n\in(N+1)\cdot\mathbb{Z}^d} \overline{\varphi(k+n)}\psi(k+n) = \sum_{m\in\mathbb{Z}^d}\overline{\varphi(m)}\psi(m) = \langle\varphi, \psi\rangle.$$

This implies that $\mathcal{F}$ is an isometry as

$$\|\mathcal{F}\varphi\| = \sqrt{\langle\mathcal{F}\varphi, \mathcal{F}\varphi\rangle} = \sqrt{\langle\varphi,\varphi\rangle} = \|\varphi\|_2$$

holds for all $\varphi \in C_c(\mathbb{Z}^d)$.

Now we show that $\mathcal{F}$ is also surjective. Let $f \in \mathcal{H}(N)$ and $m \in \mathbb{Z}^d$. Then there is a unique $M_m \in (N+1)\cdot\mathbb{Z}^d$ and $k_m \in Q_N^+$ such that $m = M_m + k_m$ holds. A short computation leads to

$$\begin{aligned}0 = \langle f, \mathcal{F}\delta_m\rangle &= \sum_{k\in Q_N^+}\int_0^{2\pi}\overline{f_k(x)}\left(\sum_{n\in(N+1)\cdot\mathbb{Z}^d}\delta_{M_m+k_m}(k+n)e^{-in.x}\right)\frac{dx}{(2\pi)^d}\\ &= \int_0^{2\pi}\overline{f_{k_m}}(x)e^{iM_m.x}\frac{dx}{(2\pi)^d}\\ &= \langle f_{k_m}, f_{M_m}\rangle.\end{aligned}$$

Since this identity holds for all $m \in \mathbb{Z}^d$, we get $f_k = 0$ for all $k \in Q_N^+$ and so $f = 0$. Here we use that $\{f_n\}_{n\in(N+1)\cdot\mathbb{Z}^d}$ defines an orthonormal basis for the Hilbert space $\overline{\operatorname{Lin}\{f_n \mid n \in (N+1)\cdot\mathbb{Z}^d\}}$.

Thus, the image $\mathcal{F}(C_c(\mathbb{Z}^d)) \subseteq \mathcal{H}(N)$ is dense while $\|\mathcal{F}\varphi\|_2 = \|\varphi\|_2$. Thus, $\mathcal{F}$ can be uniquely extended to $\ell^2(\mathbb{Z}^d)$ (by denseness of $C_c(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$) such that $\mathcal{F}$ maps onto $\mathcal{H}(N)$. $\square$

Remark. *(a) We point out that $\mathcal{F}_N^{-1} : \mathcal{H}(N) \to \ell^2(\mathbb{Z}^d)$ is given by*

$$(\mathcal{F}_N^{-1}f)(m) = \int_{\mathbb{T}^d} f_{k_m}(x)e^{-iM_m\cdot x}\frac{dx}{(2\pi)^d}, \qquad f \in \mathcal{H}(N), m \in \mathbb{Z}^d.$$

where $k_m \in Q_N^+$ and $M_m \in (N+1)\cdot\mathbb{Z}^d$ are the unique elements satisfying $m = M_m + k_m$.

(b) Furthermore, we remark that the Hilbert space $\mathcal{H}(N)$ can be identified with the Hilbert space

$$\int_{\mathbb{T}_N^d}^{\oplus}\mathbb{C}^M\frac{dx}{((N+1)2\pi)^d}$$

with $M := \sharp Q_N^+$, which is called the direct integral. This notion can be seen as a generalization of a direct sum, which is a special case of a direct integral over a counting measure. We refer the reader to [**RS78**, Chapter XIII.16] *or* [**KR97**, Chapter 14] *for further background. This is in particular useful for the continuous model.*

(c) The (scaled) d-dimensional torus $[0,(N+1)2\pi)^d$ is often called the Brillouin zone *in solid state physics for periodic media. It is nothing but the Pontryagin dual group of $\mathbb{Z}^d$ (or in our case of $(N+1)\cdot\mathbb{Z}^d$).*

(d) We point out that the concept of Pontryagin duality is only defined for locally compact abelian groups. It is not clear as how to treat periodic Hamiltonians on a non-abelian group. One possible analog of the dual group and its characters f_n is the so-called Gelfand pair and spherical functions.

7.2. Spectrum of periodic Hamiltonians.

Throughout this section, we have a finite alphabet $\mathcal{A}$ and a periodic configuration $\omega \in \mathcal{A}^{\mathbb{Z}^d}$. Invoking Proposition 3.8, there is an $N \in \mathbb{N}_0$ and a $p \in \mathcal{A}^{Q_N^+}$ such that $\omega = \omega_p = p^\infty$.

Lemma 7.3. *Let $M \in \mathcal{L}(\mathcal{H}(N))$ be a linear bounded operator defined by*

$$(Mf)(x) := M(x)f(x), \qquad f \in \mathcal{H}(N),\, x \in \mathbb{T}_N^d,$$

where $M(x) \in \mathcal{L}\left(\bigoplus_{g\in Q_N^+}\mathbb{C}\right)$ *is a hermitian (self-adjoint) matrix such that* $\mathbb{T}_N^d \ni x \mapsto M(x)$ *is continuous in the operator norm. Then* M *is self-adjoint, the eigenvalues of* $M(x)$ *depend continuously on* x *and*

$$\sigma(M) = \overline{\bigcup_{x\in\mathbb{T}^d}\sigma(M(x))}.$$

REMARK. *We point out that the continuity of* $\mathbb{T}_N^d \ni x \mapsto M(x)$ *in the operator norm is equivalent to the continuity of each coefficient of the matrix* $M(x)$.

PROOF. This is an exercise (Sheet 10). One of the inclusion of the spectrum relies on Weyl's criterion for the spectrum (Theorem 5.13). □

Recall that a Hamiltonian H over a dynamical system $(\mathcal{A}^{\mathbb{Z}^d},\mathbb{Z}^d)$ is defined by

$$H_\omega := \sum_{h\in\mathcal{R}}\left(\widehat{t}_h(\omega)R_h + R_{-h}\widehat{t}_h(\omega)^*\right), \qquad \omega\in\mathcal{A}^{\mathbb{Z}^d},$$

where $\mathcal{R}\subseteq\mathbb{Z}^d$ is finite and $t_h : \mathcal{A}^{\mathbb{Z}^d}\to\mathbb{C}, h\in\mathcal{R}$, are continuous.

For any periodic element $\omega_p := p^\infty \in \mathcal{A}^{\mathbb{Z}^d}$, we will show that there is an

$$A(\omega_p,\cdot)\in\mathcal{L}\left(\bigoplus_{k\in Q_N^+} L^2(\mathbb{T}_N^d)\right)$$

such that

- $x\mapsto A(\omega_p,x)$ is continuous,
- $\mathcal{F}_N H_{\omega_p}\varphi = A(\omega_p,\cdot)\mathcal{F}_N\varphi$ for all $\varphi\in\ell^2(\mathbb{Z}^d)$.

THEOREM 7.4. *Let* $(H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ *be a Hamiltonian over* $(\mathcal{A}^{\mathbb{Z}^d},\mathbb{Z}^d)$ *with finite range* $\mathcal{R}\subseteq\mathbb{Z}^d$ *and* $t_h : \mathcal{A}^{\mathbb{Z}^d}\to\mathbb{C}, h\in\mathcal{R}$, *be continuous. Let* $\omega_p = p^\infty\in\mathcal{A}^{\mathbb{Z}^d}$ *be some periodic element for* $p\in\mathcal{A}^{Q_N^+}$ *and* $\Omega_p := Orb(\omega_p)$.
Then there is a self-adjoint $M\times M$*-matrix* $A(\omega_p,x)$ *for each* $x\in\mathbb{T}_N^d$ *with* $M := \sharp Q_N^+$ *such that*

$$\sigma(H_{\Omega_p}) = \sigma(H_{\omega_p}) = \overline{\bigcup_{x\in\mathbb{T}_N^d}\sigma\big(A(\omega_p,x)\big)}.$$

PROOF. The first equation follows from $\Omega_p = Orb(\omega_p)$ and Proposition 6.2. We seek to show that the coefficients (k,l) of the matrix $A(\omega_p,x)$ defined by

$$\langle\delta_k,\mathcal{F}H_{\omega_p}\mathcal{F}^{-1}\delta_l\rangle(x), \qquad x\in\mathbb{T}_N^d,$$

satisfy that $x\mapsto A(\omega_p,x)$ is continuous in the operator norm.

Let $\mathcal{F} := \mathcal{F}_N : \ell^2(\mathbb{Z}^d)\to\mathcal{H}(N)$ be the isomorphism defined in Proposition 7.2. Since $\mathcal{F}$ is unitary ($\langle\mathcal{F}\psi,\mathcal{F}\varphi\rangle = \langle\psi,\varphi\rangle$) and H_{ω_p} is self-adjoint, we conclude

$$\left(\mathcal{F}H_{\omega_p}\mathcal{F}^{-1}\right)^* = \left(\mathcal{F}^{-1}\right)^* H_{\omega_p}^*\mathcal{F}^* = \mathcal{F}H_{\omega_p}\mathcal{F}^{-1}$$

and so $A(\omega_p,x)$ is automatically self-adjoint. Since $\mathcal{F}$ is an isomorphism, we get

$$\sigma(H_{\omega_p}) = \sigma(\mathcal{F}H_{\omega_p}\mathcal{F}^{-1}) = \sigma\big(A(\omega_p,\cdot)\big)$$

by 5.3. Then the second equation $\sigma(H_{\omega_p}) = \overline{\bigcup_{x\in\mathbb{T}^d_N} \sigma\big(A(\omega_p, x)\big)}$ follows by the previous Lemma 7.3.

First note that for each $h \in \mathcal{R} \cup -\mathcal{R}$ and $k \in Q_N^+$, there are unique $M(k,h) \in (N+1)\cdot\mathbb{Z}^d$ and $n(k,h) \in Q_N^+$ such that

$$k + h = M(k,h) + n(k,h).$$

Let $k \in Q_N^+$ and $h \in \mathcal{R} \cup -\mathcal{R}$. Then a short computation leads to

$$\begin{aligned}(\mathcal{F}\ R_h\varphi)_k(x) &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} (R_h\varphi)(n+k)e^{in.x}\\ &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} \varphi(n+k+h)e^{in.x}\\ &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} \varphi(n+M(k,h)+n(k,h))e^{i(n+M(k,h)).x}e^{-iM(k,h).x}\\ &= e^{-iM(k,h).x}\sum_{m\in(N+1)\cdot\mathbb{Z}^d} \varphi(m+n(k,h))e^{im.x}\\ &= e^{-iM(k,h).x}(\mathcal{F}\varphi)_{n(k,h)}(x).\end{aligned}$$

Since $n^{-1}\omega_p = \omega_p$ for all $n \in (N+1)\cdot\mathbb{Z}^d$, we get

$$\begin{aligned}(\mathcal{F}\ \widehat{t}_h(\omega_p)\varphi)_k(x) &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} (\widehat{t}_h(\omega_p)\varphi)(n+k)e^{in.x}\\ &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} t_h\big((n+k)^{-1}\omega_p\big)\varphi(n+k)e^{in.x}\\ &= \sum_{n\in(N+1)\cdot\mathbb{Z}^d} t_h\big(k^{-1}(n^{-1}\omega_p)\big)\varphi(n+k)e^{in.x}\\ &= t_h\big(k^{-1}\omega_p\big)\sum_{n\in(N+1)\cdot\mathbb{Z}^d} \varphi(n+k)e^{in.x}\\ &= t_h\big(k^{-1}\omega_p\big)(\mathcal{F}\varphi)_k(x).\end{aligned}$$

Thus,

$$\begin{aligned}\big(\mathcal{F}\widehat{t}_h(\omega_p)R_h\varphi\big)_k(x) &= t_h\big(k^{-1}\omega_p\big)\ (\mathcal{F}R_h\varphi)_k(x)\\ &= t_h\big(k^{-1}\omega_p\big)\ e^{-iM(k,h).x}\ (\mathcal{F}\varphi)_{n(k,h)}(x)\end{aligned}$$

and

$$\begin{aligned}\big(\mathcal{F}R_{-h}\widehat{t}_h(\omega_p)^*\varphi\big)_k(x) &= e^{-iM(k,-h).x}\ \big(\mathcal{F}\widehat{t}_h(\omega_p)^*\varphi\big)_{n(k,-h)}(x)\\ &= e^{-iM(k,-h).x}\ \overline{t_h\big(n(k,-h)^{-1}\omega_p\big)}\ (\mathcal{F}\varphi)_{n(k,-h)}(x)\end{aligned}$$

follow. Define $a_{(k,h)}, b_{(k,h)} \in C(\mathbb{T}^d_N)$ by

$$\begin{aligned}a_{(k,h)}(x) &:= t_h\big(k^{-1}\omega_p\big)\ e^{-iM(k,h).x} \quad \text{and}\\ b_{(k,h)}(x) &:= e^{-iM(k,-h).x}\ \overline{t_h\big(n(k,-h)^{-1}\omega_p\big)}.\end{aligned}$$

For $k, l \in Q_N^+$ and $x \in \mathbb{T}_N^d$, the previous considerations imply

$$\begin{aligned}\langle \delta_k, \mathcal{F} H_{\omega_p} \mathcal{F}^{-1} \delta_l \rangle(x) &= \sum_{h \in \mathcal{R}} a_{(k,h)}(x) \, \big(\delta_l\big)_{n(k,h)}(x) + b_{(k,h)}(x) \, \big(\delta_l\big)_{n(k,-h)}(x) \\ &= \sum_{h \in \mathcal{R}} a_{(k,h)}(x) \, \delta_l\big(n(k,h)\big) + b_{(k,h)}(x) \, \delta_l\big(n(k,-h)\big),\end{aligned}$$

which are continuous in $x \in \mathbb{T}_N^d$. Thus, if the (k,l) coefficient of the matrix $A(\omega_p, x)$ is definded by $\langle \delta_k, \mathcal{F} H_{\omega_p} \mathcal{F}^{-1} \delta_l \rangle(x)$, then $x \mapsto A(\omega_p, x)$ is continuous in the operator norm. □

Remark. *It is worth pointing out that the dimension of the matrix depends on the size of Q_N^+. Therefore, it is for practical purpose useful to choose this fundamental domain of the periodic element ω_p as small as possible since then the spectrum of the matrix $A(\omega_p, x)$ is easier to compute. This relates to the remark given in the video clip for Section 3.2.*

The previous theorem shows us that we can diagonalize periodic Hamiltonians. Indeed the same strategy would work the same for operator families A_X over a periodic dynamical system $(X, \mathbb{Z}^d)$. However, we have seen that it is tedious to write down the corresponding matrix $A(\omega_p, x)$. In order to simplify and to show as how to use this approach, let us compute this matrix for a specific case.

Let $d = 1$ and $\mathcal{R} = \{0, 1\}$ with $t_1 \equiv 1$ and $t_0 := \frac{1}{2} V$ where $V \in C(\mathcal{A}^{\mathbb{Z}})$ is real-valued. Then the Hamiltonian H over $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ is given by

$$(H_\omega \psi)(n) = \psi(n+1) + \psi(n-1) + V(n^{-1}\omega)\psi(n), \qquad \omega \in \mathcal{A}^{\mathbb{Z}}.$$

Recall that $\mathcal{A}^{Q_N^+} = \mathcal{A}^{N+1}$.

Proposition 7.5. *Let $d = 1$ and $\mathcal{R} = \{0, 1\}$ with $t_1 \equiv 1$ and $t_0 := \frac{1}{2} V$ where $V \in C(\mathcal{A}^{\mathbb{Z}})$ is real-valued. Let $p \in \mathcal{A}^{N+1}$ be a finite word and $\omega_p := p^\infty$. Then*

$$\sigma(H_\Omega) = \sigma(H_{\omega_p}) = \overline{\bigcup_{x \in [0, \frac{2\pi}{N+1})} \sigma\big(A(\omega_p, x)\big)}$$

holds where $A(\omega_p, x)$ is an $(N+1) \times (N+1)$ matrix given by

$$A(\omega_p, x) = \begin{pmatrix} V(\omega_p) & 1 & 0 & \dots & 0 & e^{i(N+1)x} \\ 1 & V(1^{-1}\omega_p) & 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \dots & & & V((N-1)^{-1}\omega_p) & 1 \\ e^{-i(N+1)x} & 0 & \dots & 0 & 1 & V(N^{-1}\omega_p) \end{pmatrix}.$$

Proof. This is an exercise (Sheet 11). □

Week 11

Exercise. *Let $p \in \mathcal{A}^{N+1}$ be a finite word and $\omega_p := p^\infty$. Consider the matrices $A(\omega_p,\theta)$ for $\theta \in [0,2\pi)$ defined by*

$$A(\omega_p,\theta) = \begin{pmatrix} V(\omega_p) & 1 & 0 & \dots & 0 & e^{i\theta} \\ 1 & V(1^{-1}\omega_p) & 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \dots & & & V((N-1)^{-1}\omega_p) & 1 \\ e^{-i\theta} & 0 & \dots & 0 & 1 & V(N^{-1}\omega_p) \end{pmatrix}.$$

Prove that there is a polynomial $P(\lambda) = \sum_{j=0}^{N+1} a_j\lambda^j$ such that each a_j is independent of θ and the following holds.

(a) *The characteristic polynomial $\chi_\theta(\lambda) := det\big(\lambda - A(\omega_p,\theta)\big)$ satisfies*
$$\chi_\theta(\lambda) = P(\lambda) - 2\cos(\theta).$$
(b) *The equality $\sigma(H_{\omega_p}) = \big\{\lambda \in \mathbb{R} \big| |P(\lambda)| \le 2\big\}$ holds.*
(c) *There are $N+1$ intervals $I_j := [a_j, b_j]$, $0 \le j \le N$, such that*
$$\sigma(H_{\omega_p}) = \bigcup_{j=0}^{N} I_j$$
where the interval edges a_j and b_j are elements of $\{\lambda \in \mathbb{R} \,|\, |P(\lambda)| = 2\}$.
(d) *Two consecutive intervals I_j and I_{j+1} can at most touch at their interval edges.*

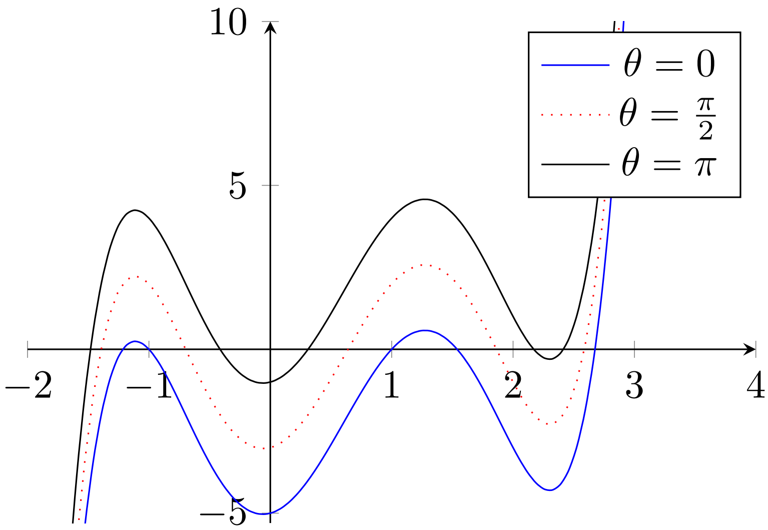

Figure 26. Let $\mathcal{A} := \{a, b\}$. We plotted the characteristic polynomials of $A(\omega_p,\theta)$ for $p = abaab$ with potential V defined by $V(\omega) := 1$ if $\omega(0) = a$ and $V(\omega) := 0$ if $\omega(0) = b$. Exercise: draw the spectral band of this Hamiltonian.

Let $\mathcal{A} := \{0,1\}$ and for $\alpha := \frac{p}{q} \in [0,1] \cap \mathbb{Q}$ (reduced fraction), $\omega_\alpha \in \mathcal{A}^{\mathbb{Z}}$ defined by

$$\omega_\alpha(n) := \chi_{[1-\alpha,1)}(n\alpha \mod 1), \qquad n \in \mathbb{Z},$$

q-periodic (Exercise). Let P_α be the corresponding polynomial of degree q defined in the previous exercise, namely

$$\big(\text{characteristic polynomial of} A(\omega_p,\theta)\big)(\lambda) = P_\alpha(\lambda) - 2\cos(\theta)$$

Then P_α turns out to be a trace of a certain 2×2-matrix defined through the qth time product of so-called transfer matrices.

Let F_n for $n \in \mathbb{N}_0$ be defined by $F_0 := F_1 := 1$ and $F_n := F_{n-1} + F_{n-2}$. These are the so-called Fibonacci numbers. Let $P_n := P_{\alpha_n}$ for $\alpha_n := \frac{F_n}{F_{n+1}}$. These numbers converge to the golden mean $\beta := \frac{1-\sqrt{5}}{2} \in [0,1] \setminus \mathbb{Q}$ and moreover the polynomials P_n satisfy a certain recursion relation, namely

$$P_{n+2} = P_{n+1}P_n - P_{n-1}.$$

One can find similar relations for any rational approximations of an irrational number $\beta \in [0,1] \setminus \mathbb{Q}$. These relations are a powerful tool to study the spectral nature of the Hamiltonians H_{ω_β} as well as to numerical compute these sets.

As a first step one needs that these rational approximations lead to the convergence of the spectra and maybe we can even hope to estimate the Hausdorff distance. In the subsequent chapter, we show that convergence of the associated dynamical systems leads to the convergence of the spectra in general, which is a useful tool to also study higher dimensional systems. It is an open question to find corresponding recursive relations for periodic approximations in higher dimensions ($d \geq 2$).

8. Spectral approximations via the underlying dynamics

In this section we seek to show that the map

$$\Sigma_H : \mathcal{J} \to \mathcal{K}(\mathbb{C}), \quad Y \mapsto \sigma(H_Y),$$

is a Lipschitz continuous function (in the corresponding Hausdorff metric) if the coefficients of the Hamiltonian are certain Lipschitz continuous functions on the underlying space. Here we restrict our considerations to the dynamical system $(\mathcal{A}^{\mathbb{Z}^d}, \mathbb{Z}^d)$ with $\mathcal{A}$ finite. One can prove this statement in a slightly more general setting but this will be discussed at the end of the chapter. The extension to a broader class of dynamical systems (containing e.g. the Harper model) or even more general structures (such as groupoids containing examples like the Penrose and the Octogonal tiling) is the content of current research.

Let (X, d_X) and (Y, d_Y) be two metric spaces. Then a map $f : X \to Y$ is called *Lipschitz continuous* if there is a constant $C_L(f) > 0$ such that

$$d_Y\big(f(x), f(y)\big) \le C_L(f) d_X(x,y), \qquad x, y \in X.$$

We will deal here with various functions on different metric spaces that are Lipschitz continuous. For instance, if $X = \mathcal{A}^{\mathbb{Z}^d}$ and $Y = \mathbb{C}$, then d_Y is the Euclidean distance and $d_X = d$ is given by

$$d(\omega, \rho) := \min\left\{1, \inf\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N} \text{ be such that } \omega|_{Q_n} = \rho|_{Q_n}\right\}\right\}$$

Here Q_n denotes the cube centered at the origin with side length $2n+1$, i.e.,

$$Q_n := \left\{g \in \mathbb{Z}^d \,\middle|\, \|g\|_\infty \le n\right\} \qquad \text{where } \|g\|_\infty := \max_{1 \le i \le d} |g_i|.$$

Definition. *A function $t : \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ is called* (strongly) pattern equivariant with parameter $M(t) \in \mathbb{N}$ *if $t(\omega) = t(\rho)$ holds whenever $\omega|_{Q_{M(t)}} = \rho|_{Q_{M(t)}}$.*

It is immediate to check that pattern equivariant functions are Lipschitz continuous as

$$|t_h(\omega) - t_h(\rho)| \le 2M(t)\|t\|_\infty d(\omega, \rho).$$

We already have seen such functions, confer Chapter 1 (Motivation) that we will discuss here again.

Example. *Let $\mathcal{A} := \{0,1\}$ be our alphabet and consider the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$. For $\alpha \in [0,1]$, define $\omega_\alpha \in \mathcal{A}^{\mathbb{Z}}$ by*

$$\omega_\alpha(n) := \chi_{[1-\alpha,1)}(n\alpha \mod 1), \qquad n \in \mathbb{Z},$$

where $\chi_{[1-\alpha,1)}$ is the characterstic function of the left closed and right open interval $[1-\alpha, 1)$. Define $t_0 : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ by $t_0(\omega) := \frac{1}{2}\omega(0)$, which is clearly a pattern equivariant function with $M_{t_0} = 1$. Then

$$\Big(\big(\widehat{t_0}(\omega) R_0 + R_{-0}\widehat{t_0}(\omega)^*\big)\varphi\Big)(n) = \frac{1}{2}\omega(n)\varphi(n) + \frac{1}{2}\omega(n)\varphi(n) = \omega(n)\varphi(n)$$

holds for all $\varphi \in \ell^2(\mathbb{Z})$ and $n \in \mathbb{Z}$. Set $\mathcal{R} := \{0,1\}$ and $t_1 : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$, $t_1(\omega) := 1$ which is also pattern equivariant with parameter $M_{t_1} = 1$. Then the Hamiltonian $H_{\omega_\alpha} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ satisfies

$$\big(H_{\omega_\alpha}\varphi\big)(n) = \psi(n-1) + \psi(n+1) + \chi_{[1-\alpha,1)}(n\alpha \mod 1)\psi(n), \qquad n \in \mathbb{Z},$$

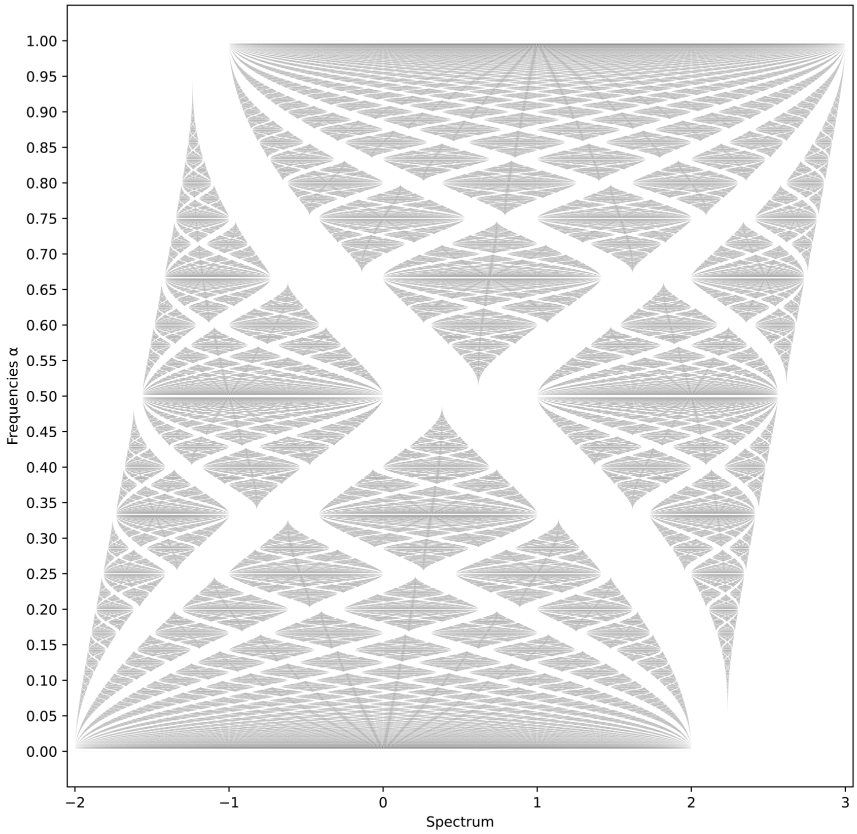

Figure 27. The Kohmoto butterfly plotted by Barak Biber (`https://github.com/DaAnIV/Quasiperiodic`, Technion, Israel). On each horizontal line the spectrum of certain one-dimensional periodic Schrödinger operators H_{ω_α} is plotted for rational $\alpha \in [0,1]$. If these rational α's approach an irrational β, we get a Schrödinger operator for a certain quasicrystal. Then these rational approximations provide us some insight in the spectral nature of them. The spectrum is determined by computing the spectrum of the associated matrices to the periodic Hamiltonians, confer Chapter 7.

which is the family of Schrödinger operators discussed in Chapter 1 (Motivation). For various $\alpha \in [0,1] \cap \mathbb{Q}$, the spectrum of H_{ω_α} (or the operator family $(H_\omega)_{\omega \in \overline{Orb(\omega_\alpha)}}$) is plotted in Figure 27. We are interested in particular in the spectrum of H_{ω_α} when $\alpha \in [0,1]$ is irrational. One of the most studied example is the case where $\alpha := -\frac{1-\sqrt{5}}{2}$ is the golden ratio. The corresponding dynamical system $\overline{Orb(\omega_\alpha)}$ turns out to be equal to the subshift $\Omega(S)$ of the Fibonacci substitution S. The periodic approximations $\omega_n := S^n(a^\infty)$, $n \in \mathbb{N}$, *that we were constructing in Section 4.3 are given by the rational approximations* $\alpha_n := \frac{F_n}{F_{n+1}}$, $n \in \mathbb{N}_0$, *where the* F_n*'s are the Fibonacci numbers defined by* $F_0 := 1, F_1 := 1$ *and* $F_n := F_{n-1} + F_{n-2}$ *for* $n \in \mathbb{N}$. *We will show in this chapter that the convergence of the dynamical systems implies the convergence of the corresponding spectra of Hamiltonians. Moreover, the spectral map* $\Sigma : \mathcal{J} \to \mathcal{K}(\mathbb{R})$, $\Omega \mapsto \sigma(H_\Omega)$, *turns out to be Lipschitz continuous if the coefficients of the Hamiltonian are (strongly) pattern equivariant.*

We will deal with the following class of operators.

Definition. *A Hamiltonian* $(H_\omega)_{\omega \in \mathcal{A}^{\mathbb{Z}^d}}$ *of finite range is called* pattern equivariant *if* t_h *is pattern equivariant for all* $h \in \mathcal{R}$.

Our main aim of this chapter is to prove the following statement

Theorem 8.1. *Let* $H = (H_\omega)_{\omega \in \mathcal{A}^{\mathbb{Z}^d}}$ *be a pattern equivariant Hamiltonian of finite range. Then the map*

$$\Sigma_H : \mathcal{J} \to \mathcal{K}(\mathbb{C}), \quad Y \mapsto \sigma(H_Y),$$

is Lipschitz continuous. More precisely, there exists a constant C_d *such that*

$$d_H\big(\sigma(H_\Omega), \sigma(H_\Theta)\big) \le C_d \ M \ \|H\|_S \ \delta_H\big(\Omega, \Theta\big), \qquad \Omega, \Theta \in \mathcal{J},$$

where $\|H\|_S$ *is the Schur norm of* H *and* $M := \sup_{h \in \mathcal{R}} M(t_h)$ *is the maximal parameter of the (strongly) pattern equivariant coefficients of* H.

Remark. *We refer the reader to* [**BBC19**] *for further background where this result was originally proven.*

We remind here the reader that $\sigma(H_\Theta) \in \mathcal{K}(\mathbb{R})$ *since* $H_{\mathcal{A}^{\mathbb{Z}^d}}$ *is self-adjoint and bounded, see Proposition 6.2. Furthermore,* d_H *is the Hausdorff metric on* $\mathcal{K}(\mathbb{R})$ *induced by the Euclidean metric and* δ_H *is the Hausdorff metric on* $\mathcal{K}(\mathcal{A}^{\mathbb{Z}^d})$ *(restricted to* $\mathcal{J}$*) induced by the metric* d *on* $\mathcal{A}^{\mathbb{Z}^d}$.

Recall that there is a choice of a metric on $\mathcal{A}^{\mathbb{Z}^d}$. *However, this specific metric* d *is the correct metric in order to conclude that the regularity of the coefficients of the Hamiltonian pass to the same regularity conditions of the spectrum. More precisely, if all coefficients* t_h *of a Hamiltonian* H *are Lipschitz continuous, then the spectral map* Σ *turns out to be Lipschitz continuous as well.*

8.1. Strategy of the proof. Here we only provide a rough structure of the proof to motivate the various sections that will follow. All details are provided later.

Step 1: By the definition of the metric d on $\mathcal{A}^{\mathbb{Z}^d}$ and its induced Hausdorff metric δ_H on $\mathcal{J}$, there is an $r \in \mathbb{N}$ such that

$$\delta_H(\Omega, \Theta) = \frac{1}{r}.$$

In particular the notion of Hausdorff metric implies that for every $\omega \in \Omega$ there is an $\rho \in \Theta$ such that

$$d(\omega, \rho) \le \frac{1}{r}.$$

Thus, $\omega|_{Q_r} = \rho|_{Q_r}$ follows by the definition of the metric d on $\mathcal{A}^{\mathbb{Z}^d}$.

Step 2: By Step 1, let $r \in \mathbb{N}$ be such that $\delta_H(\Omega, \Theta) = \frac{1}{r}$. Let H be a pattern equivariant Hamiltonian of finite range. We seek to show

$$d_H\big(\sigma(H_\Omega), \sigma(H_\Theta)\big) \leq C_d\ M\ \|H\|_S\ \delta_H\big(\Omega, \Theta\big) = C_d\ M\ \|H\|_S\ \frac{1}{r}$$

for fixed $\Omega, \Theta \in \mathcal{J}$. This estimate can be reduced to the following assertion for $z \in \mathbb{C}$:

$$\operatorname{dist}(z, \sigma(H_\Omega)) \geq C_d\ M\ \|H\|_S\ \frac{1}{r} \quad \Longrightarrow \quad z \in \mathbb{C} \setminus \sigma(H_\Theta).$$

In particular, $z \in \mathbb{C} \setminus \sigma(H_\Theta)$ means that $z \in \rho(H_\rho)$ for all $\rho \in \Theta$. To say the latter in words:

> *If $z \in \mathbb{C}$ is in the resolvent set of H_Ω such that it is far enough away from the spectrum $\sigma(H_\Omega)$ then $H_\rho - z$ is invertible for all $\rho \in \Theta$.*

Step 3: Let $r > M$. We know that $\sigma(H_\Theta)$ and $\sigma(H_\Omega)$ are compact non-empty subsets of $\mathbb{R}$. Thus, if $z \in \mathbb{C} \setminus \mathbb{R}$ then $H_\rho - z$ is invertible for all $\rho \in \mathcal{A}^{\mathbb{Z}^d}$. Let $z \in \mathbb{R}$ be such that

$$\operatorname{dist}(z, \sigma(H_\Omega)) \geq C\ \frac{1}{r - M} \tag{$\blacklozenge$}$$

for a suitable chosen constant $C > 0$ (confer Proposition 8.8) and let $\rho \in \Theta$. Following Step 2, we seek to show that $H_\rho - z$ is invertible. We know that $H_\rho - z_n$ is invertible (as H_ρ is self-adjoint) for $z_n := z + \frac{1}{n}i$. Furthermore, $z_n \to z$ holds. Recall from Sheet 9, Exercise 4, that if there are $S_n, T_n \in \mathcal{L}(\ell^2(G))$, $n \in \mathbb{N}$, such that

$$(H_\rho - z_n)S_n = I + T_n, \qquad \sup_{n \in \mathbb{N}} \|S_n\| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|T_n\| < 1,$$

then $z \in \rho(A)$ follows. We will show that actually for all $z \in \mathbb{C}$ satisfying ($\blacklozenge$), there are $S(z), T(z) \in \mathcal{L}(\ell^2(G))$ such that $(H_\rho - z_n)S(z) = I + T(z)$ and

$$\sup_{z \in \mathbb{C} \text{ satisfying } (\blacklozenge)} \|S(z)\| < \infty \quad \text{and} \quad \sup_{z \in \mathbb{C} \text{ satisfying } (\blacklozenge)} \|T(z)\| \leq \frac{2}{3} < 1,$$

Thus the main task will be to construct these $S(z)$ and $T(z)$. This is done by chopping of the operator into a "sum of finitely supported operators" by using a partition of unity that are supported on $(rg + Q_r)_{g \in \mathbb{Z}^d}$.

Due to Step 1, we find for each $g \in \mathbb{Z}^d$, an $\omega_g \in \Omega$ such that $\rho|_{rg+Q_r}$ is the same patch as $\omega_g|_{Q_r}$. Thus, H_ρ restricted to a cube $rg + Q_r$ acts almost like H_{ω_g} (here we need to work!). In this way we can build $S(z)$ invoking that $z \in \rho(H_{\omega_g})$ for all $g \in \mathbb{Z}^d$. All the error terms are collected in $T(z)$. In order to control this term we need to have further assumptions on the partition of unity (namely it must be Lipschitz continuous in a suitable sense). The existence if a Lipschitz partition of unity might be connected to the amenability of the ambient space.

The explicit details for the partition of unity are discussed in Section 8.2. In Section 8.4, we provide some estimates that will help us at various places to estimate the norm of $S(z)$ and $T(z)$ uniformly in $z \in \mathbb{C}$ satisfying ($\blacklozenge$). Recall that H_ρ is a finite sum of operators of the form $\widehat{t_h}(\rho)R_h$ and their adjoints. The error terms arise by restring the operator to $rg + Q_r$ (with the partition of unity) and the commutator of the operator $\widehat{t_h}(\rho)$ (or R_h for $h \in \mathcal{R}$) with such restrictions. These commutators are estimated in Section 8.3. Finally, we prove the claim in Step 2 in Section 8.5 (we define $S(z)$ and $T(z)$). There we also provide the final proof of Theorem 8.1.

8.2. Lipschitz partition of unity. Recall that the support of a function $\Psi : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\operatorname{supp}(\Psi) := \overline{\left\{x \in \mathbb{R}^d \,|\, \Psi(x) \neq 0\right\}}.$$

Then a countable family $(\Psi_k)_{k\in\mathbb{N}}$ of continuous functions $\Psi_k : \mathbb{R}^d \to [0,1]$ with compact support is called a *partition of unity* of $\mathbb{R}^d$ if

$$\sum_{k\in\mathbb{N}} \Psi_k(x) = 1, \qquad x \in \mathbb{R}^d.$$

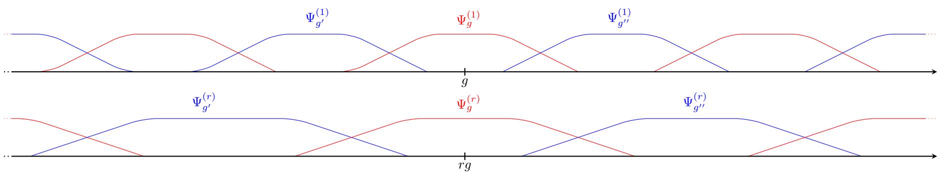

Figure 28. Sketch of the partition of unities $(\Psi_g^{(r)})_{g\in\mathbb{Z}^d}$ for $r \geq 1$.

A function $\Psi : \mathbb{R}^d \to \mathbb{R}$ is called *Lipschitz continuous with constant* $C_L := C_L(\Psi) > 0$, if

$$|\Psi(x) - \Psi(y)| \leq C_L|x-y|, \qquad x, y \in \mathbb{R}^d.$$

Definition. *A family of functions* $(\Psi_k)_{k\in\mathbb{N}}$ *is called* Lipschitz-partition of unity with constants $(C_L, \mathcal{N})$ *if the following holds*

- $(\Psi_k)_{k\in\mathbb{N}}$ *is a partition of unity.*
- $\Psi_k : \mathbb{R}^d \to [0,1]$ *has compact support and is Lipschitz continuous with constant* $C_L(\Psi_k) > 0$ *such that*

$$C_L := \sup_{k\in\mathbb{N}} C_L(\Psi_k) < \infty.$$

- *The set*

$$V_k := \left\{m \in \mathbb{N} \,\middle|\, \operatorname{supp}\left(\Psi_k\right) \cap \operatorname{supp}\left(\Psi_m\right) \neq \varnothing\right\}$$

is finite uniformly in $k \in \mathbb{N}$*, namely,*

$$\mathcal{N} := \sup_{k\in\mathbb{N}} \sharp V_k < \infty.$$

The notion of Lipschitz-partition of unity will play a crucial role in this chapter. It is used to cut our Hamiltonian in small pieces. Therefore, the Lipschitz continuity and the uniform bound on the V_k's plays a crucial role to control the error terms. Such a partition can be chosen to be subordinate to any given covering $(V_k)_{k\in\mathbb{N}}$ of $\mathbb{R}^d$, in general. This means that $\operatorname{supp}(\Psi_k) \subseteq V_k$ for all $k \in \mathbb{N}$. Such a cover is assumed to be uniformly locally finite, which reflects the 3rd constraint of a Lipschitz-partition of unity. Here, a covering $(V_k)_{k\in\mathbb{N}}$ is called *uniformly locally finite* if there is an $\mathcal{N} \in \mathbb{N}$ such that for each $x \in \mathbb{R}^d$ there are at most $\mathscr{N}$ sets V_k containing x. We will identify the index set $\mathbb{N}$ of a Lipschitz partition of unity with $\mathbb{Z}^d$.

We will show that we always have a Lipschitz-partition of unity by using a cover of cubes. For $r > 0$, define the *cube*

$$\widetilde{Q}_r := \left\{ x \in \mathbb{R}^d \,\middle|\, \|x\|_\infty \le r \right\} \qquad \text{where } \|x\|_\infty := \max_{1 \le i \le d} |x_i|.$$

PROPOSITION 8.2 (Lipschitz partition of unity). *There is a Lipschitz continuous function $\Psi : \mathbb{R}^d \to [0,1]$ with constant $C_L > 0$ and $\operatorname{supp}(\Psi) \subseteq \widetilde{Q}_1$ such that for each $g \in \mathbb{Z}^d$ and $r > 0$, the following statements hold.*

(a) *The map*

$$\Psi_g^{(r)} : \mathbb{R}^d \to [0,1], \quad \Psi_g^{(r)}(x) := \Psi\left(\frac{x}{r} - g\right),$$

is Lipschitz continuous with constant $\frac{C_L}{r}$ and $\operatorname{supp}(\Psi_g) \subseteq rg + \widetilde{Q}_r$.

(b) *The family of functions $\left(\Psi_g^{(r)}\right)_{g \in \mathbb{Z}^d}$ defines a Lipschitz partition of unity with constants $\left(\frac{C_L}{r}, \mathscr{N}_r\right)$ satisfying*

$$\sup_{r>0} \mathscr{N}_r \le \sup_{r>0} \sup_{g \in \mathbb{Z}^d} \sharp V_g(r) = \sup_{g \in \mathbb{Z}^d} \sharp V_g(1) =: \mathscr{N} < \infty$$

where $V_g(r) := \left\{ h \in \mathbb{Z}^d \,\middle|\, rg + \widetilde{Q}_r \cap rh + \widetilde{Q}_r \neq \varnothing \right\}$ for $r > 0$ and $g \in \mathbb{Z}^d$.

(c) *The estimate*

$$\left|\Psi_g^{(r)}(x) - \Psi_g^{(r)}(y)\right| \le \frac{|x-y|}{r} C_L \left(\chi_g^{(r)}(x) + \chi_g^{(r)}(y)\right)$$

holds where $\chi_g^{(r)} : \mathbb{R}^d \to \{0,1\}$ is the characteristic function of the set $rg + \widetilde{Q}_r$, namely

$$\chi_g^{(r)}(x) = \begin{cases} 1, & x \in rg + \widetilde{Q}_r, \\ 0, & x \notin rg + \widetilde{Q}_r, \end{cases} \qquad x \in \mathbb{R}^d.$$

PROOF. Let $\psi : \mathbb{R}^d \to [0,1]$ be defined by

$$\psi(x) := \frac{\operatorname{dist}(x,F)}{\operatorname{dist}(x,F) + \operatorname{dist}(x,K)}$$

with $K := \tilde{Q}_{\frac{1}{2}}$ and $F := \overline{\mathbb{R}^d \setminus \tilde{Q}_{\frac{2}{3}}}$. Then $\psi(x) = 1$ if $x \in \widetilde{Q}_{\frac{1}{2}}$ and $\operatorname{supp}(\psi) \subseteq \widetilde{Q}_1 \subseteq \mathbb{R}^d$ holds. Furthermore, ψ is Lipschitz continuous with Lipschitz constant $C_L' := 6 > 0$ since, for $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\left|\psi(x) - \psi(y)\right| =& \frac{\left|\operatorname{dist}(x,F)\operatorname{dist}(y,K) - \operatorname{dist}(y,F)\operatorname{dist}(x,K)\right|}{(\operatorname{dist}(y,F) + \operatorname{dist}(y,K))(\operatorname{dist}(x,F) + \operatorname{dist}(x,K))} \\
\le& \frac{\left|\operatorname{dist}(x,F) - \operatorname{dist}(y,F)\right|\operatorname{dist}(y,K) + \operatorname{dist}(y,F)\left|\operatorname{dist}(y,K) - \operatorname{dist}(x,K)\right|}{(\operatorname{dist}(y,F) + \operatorname{dist}(y,K))(\operatorname{dist}(x,F) + \operatorname{dist}(x,K))} \\
\le& \frac{|x-y|}{\operatorname{dist}(x,F) + \operatorname{dist}(x,K)} \\
\le& 6|x-y|
\end{aligned}$$

using $|\operatorname{dist}(x,F)-\operatorname{dist}(y,F)|\leq|x-y|$, $K\cap F=\varnothing$ and

$$\operatorname{dist}(x,F)+\operatorname{dist}(x,K)\geq\inf_{z_1\in K,z_2\in F}|z_1-z_2|=\frac{1}{6}.$$

Clearly, $\widetilde{Q}_{\frac{2}{3}}\subseteq\widetilde{Q}_1$ and

$$\mathbb{R}^d=\bigcup_{x\in\mathbb{Z}^d}x+\tilde{Q}_{\frac{1}{2}}$$

hold, confer Figure 29. Thus, the constraint $\psi(x)=1$ for $x\in\tilde{Q}_{\frac{1}{2}}$ implies

$$\sum_{g\in\mathbb{Z}^d}\psi(x-g)\geq 1$$

Define the map

$$\Psi:\mathbb{R}^d\to[0,1],\quad \Psi(x):=\frac{\psi(x)}{\sum_{h\in\mathbb{Z}^d}\psi(x-h)}.$$

Since $\operatorname{supp}(\psi)\subseteq\widetilde{Q}_1$, we conclude that $\operatorname{supp}(\Psi)\subseteq\widetilde{Q}_1$, namely Ψ has compact support.

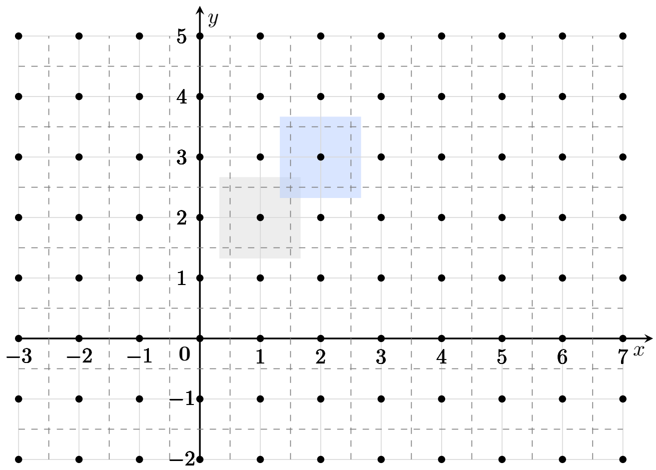

Figure 29. The dashed cubes represent a $x+\tilde{Q}_{\frac{1}{2}}$ for some $x\in\mathbb{Z}^2$. The gray and blue boxes indicate the support $(1,2)+\widetilde{Q}_{\frac{2}{3}}$ of $\Psi^{(1)}_{(1,2)}$ respectively $(2,2)+\widetilde{Q}_{\frac{2}{3}}$ of $\Psi^{(1)}_{(2,2)}$.

Define

$$V_g(r):=\left\{h\in\mathbb{Z}^d\,\middle|\,rg+\widetilde{Q}_r\cap rh+\widetilde{Q}_r\neq\varnothing\right\},\qquad r>0.$$

The family of sets $(x+\widetilde{Q}_1)_{x\in\mathbb{Z}^d}$ satisfies

$$V_g(1)\subseteq\left\{h\in\mathbb{Z}^d\,\middle|\,\|g-h\|_\infty\leq 2\right\},\qquad g\in\mathbb{Z}^d,$$

and so $\mathcal{N}:=\sup_{g\in\mathbb{Z}^d}\sharp V_g(1)<\infty$ (as $\mathbb{Z}^d$ is uniformly discrete). Moreover, for $g,h\in\mathbb{Z}^d$, we have $g+\widetilde{Q}_1\cap h+\widetilde{Q}_1\neq\varnothing$ if and only if $rg+\widetilde{Q}_r\cap rh+\widetilde{Q}_r\neq\varnothing$ for any $r>0$. Hence,

$$\sup_{g\in\mathbb{Z}^d}\sharp V_g(r)=\sup_{g\in\mathbb{Z}^d}\sharp V_g(1)=\mathcal{N}<\infty$$

follows. For $x \in \mathbb{R}^d$, let $I(x) := \{g \in \mathbb{Z}^d : \psi(x-g) \neq 0\}$. Then it is immediate to show $I(x) \subseteq V_g(r)$ for any $g \in \mathbb{Z}^d$ satisfying $\psi(x-g) \neq 0$. Thus, $\sharp\, I(x) \leq \mathscr{N}$ is independent of $x \in \mathbb{R}^d$.

With this at hand, a short computation for $x, y \in \mathbb{R}^d$ leads to

$$\begin{aligned} & |\Psi(x) - \Psi(y)| \\ = & \left| \frac{\psi(x)}{\sum_{h \in \mathbb{Z}^d} \psi(x-h)} - \frac{\psi(y)}{\sum_{h \in \mathbb{Z}^d} \psi(y-h)} \right| \\ \leq & \sum_{h \in I(x) \cup I(y)} \left| \psi(x)\ \psi(y-h) - \psi(y)\ \psi(x-h) \right| \\ \leq & \sum_{h \in I(x) \cup I(y)} \left|\psi(y-h)\right| \left|\psi(x) - \psi(y)\right| + \left|\psi(y)\right| \left|\psi(y-h) - \psi(x-h)\right| \\ \leq & \ 4 \mathscr{N} C_L' \, |x-y| \, . \end{aligned}$$

where we used that $\sum_{g \in \mathbb{Z}^d} \psi(x-g) \geq 1$, $0 \leq \psi \leq 1$ and

$$0 = \psi(y-h)\psi(y) - \psi(y-h)\psi(y).$$

Thus, Ψ is Lipschitz continuous with Lipschitz constant $C_L = 24 \mathscr{N}$.
(a) Let $g \in \mathbb{Z}^d$ and $r > 0$. By the Lipschitz continuity of Ψ, we conclude

$$\left|\Psi_g^{(r)}(x) - \Psi_g^{(r)}(y)\right| \leq C_L \left| \frac{x}{r} - g - \left(\frac{y}{r} - g \right) \right| \leq \frac{C_L}{r} \left|x - y\right|, \qquad x, y \in \mathbb{R}^d.$$

Furthermore, $\Psi_g^{(r)}(x) \neq 0$ holds for $x \in \mathbb{R}^d$ if and only if $\Psi\left(\frac{x}{r} - g\right) \neq 0$. The latter leads to $\frac{x}{r} - g \in \operatorname{supp}(\Psi) \subseteq \widetilde{Q}_1$ proving $x \in rg + \widetilde{Q}_r$.

(b) Let $x \in \mathbb{R}^d$. Then

$$\begin{aligned} \sum_{g \in \mathbb{Z}^d} \Psi_g^{(r)}(x) = \sum_{g \in \mathbb{Z}^d} \Psi\left(\frac{x}{r} - g\right) &= \sum_{g \in \mathbb{Z}^d} \frac{\psi(\frac{x}{r} - g)}{\sum_{h \in \mathbb{Z}^d} \psi(\frac{x}{r} - g - h)} \\ &= \sum_{g \in \mathbb{Z}^d} \frac{\psi(\frac{x}{r} - g)}{\sum_{h \in \mathbb{Z}^d} \psi(\frac{x}{r} - h)} \\ &= 1 \end{aligned}$$

holds and so $\left(\Psi_g^{(r)}\right)_{g \in \mathbb{Z}^d}$ is a partition of unity.

Due to (a), we know that $\Psi_g^{(r)}$ has compact support and is Lipschitz continuous with constant $C_L(\Psi_g^{(r)}) \leq \frac{C_L}{r}$. Furthermore, $\operatorname{supp}(\Psi_g^{(r)}) \subseteq rg + \widetilde{Q}_r$ holds and the supremum $\mathscr{N} = \sup_{g \in \mathbb{Z}^d} \sharp\, V_g(r)$ is finite and independent of r as we have seen above. Thus, $\left(\Psi_g^{(r)}\right)_{g \in \mathbb{Z}^d}$ is a Lipschitz partition of unity with constants $\left(\frac{C_L}{r}, \mathscr{N}\right)$.

(c) This is trivial as $\Psi_g^{(r)} \leq \chi_g^{(r)}$ for all $g \in \mathbb{Z}^d$ and $r > 0$. □

Remark. *In the subsequent section we fix the Lipschitz partition of unity* $\left(\Psi_g^{(r)}\right)_{g \in \mathbb{Z}^d}$ *with constants* $\left(\frac{C_L}{r}, \mathscr{N}\right)$ *that we constructed in Proposition 8.2.*

8.3. Some estimates of commutators. The aim of this section is to estimate the error terms that appear by commuting an potential term and the kinetic term for the multiplication operator of the quadratic partition $\widehat{\Psi}_g^{(r)}$. As it turns out, there is no error term for potentials arising by strongly pattern equivariant functions. This phenomenon is very particular to the structure of these functions and in general the error term is of the order of regularity of the function to the square root.

LEMMA 8.3. *Let $t : \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ be pattern equivariant with parameter $M(t) \in \mathbb{N}$, $g \in \mathbb{Z}^d$ and $r \geq M(t)$. Then for all $\omega, \rho \in \mathcal{A}^{\mathbb{Z}^d}$ with $d(g^{-1}\omega, g^{-1}\rho) \leq \frac{1}{r}$ we have*

$$t\big(x^{-1}\omega\big) = t\big(x^{-1}\rho\big)$$

for each $x \in g + Q_{r-M(t)}$.

PROOF. Let $x \in g + Q_{r-M(t)}$. We first note that $x + Q_{M(t)} \subseteq g + Q_r$. Since $d(g^{-1}\omega, g^{-1}\rho) \leq \frac{1}{r}$, we conclude $(g^{-1}\omega)|_{Q_r} = (g^{-1}\rho)|_{Q_r}$ or equivalently $\omega|_{g+Q_r} = \rho|_{g+Q_r}$. Thus, $\omega|_{x+Q_{M(t)}} = \rho|_{x+Q_{M(t)}}$ follows implying $(x^{-1}\omega)|_{Q_{M(t)}} = (x^{-1}\rho)|_{Q_{M(t)}}$. Thus, the pattern equivariance of t implies the desired identity $t\big(x^{-1}\omega\big) = t\big(x^{-1}\rho\big)$. □

PROPOSITION 8.4. *Let $t : \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ be pattern equivariant with parameter $M(t) \in \mathbb{N}$ and $\Omega, \Theta \in \mathcal{J}$ be such that $\delta_H(\Omega, \Theta) = \frac{1}{r}$ with $r \geq M(t)$. Then for each $\omega \in \Omega$ and $g \in \mathbb{Z}^d$, there exists an $\rho_g = \rho(\omega, g, M(t)) \in \Theta$ (that only depends on $M(t)$ but not on t itself) such that*

$$\widehat{t}(\omega)\, \widehat{\Psi}_g^{(r-M(t))} = \widehat{\Psi}_g^{(r-M(t))}\, \widehat{t}(\rho_g).$$

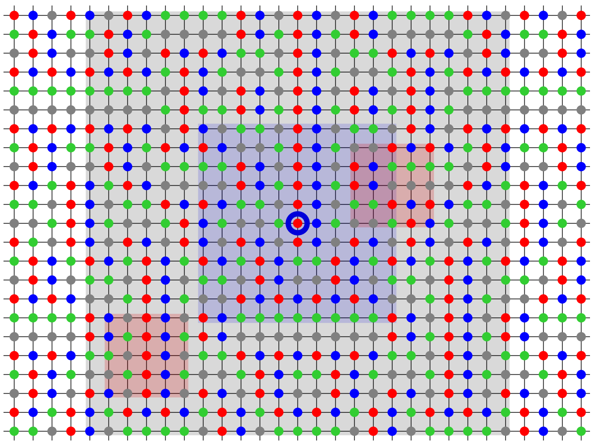

Figure 30. The gray box indicates the cube $g_r + Q_r$. The red cubes represent two different $x + Q_{M(t)}$ and the blue cube is the support $(r - M(t))g + Q_{r-M(t)}$ of $\Psi_g^{(r-M(t))}$.

PROOF. In order to simplify the notation, set $M := M(t)$. Let $g \in \mathbb{Z}^d$ and $r \in \mathbb{N}$ be such that $r \geq M$ and $\delta_H(\Omega, \Theta) = \frac{1}{r}$. We note that we use here the specific structure of the metric on $\mathcal{A}^{\mathbb{Z}^d}$ implying that such a natural number r exists. Since $r \in \mathbb{N}$, we have $g_r := (r - M) \cdot g \in \mathbb{Z}^d$. The invariance of Ω implies $g_r^{-1}\omega \in \Omega$ and so there is an $\tilde{\rho}_g \in \Theta$ such that

$$d\big(g_r^{-1}\omega, \tilde{\rho}_g\big) \le \frac{1}{r} = \delta_H(\Omega, \Theta).$$

Since Θ is invariant, $\rho_g = g_r \tilde{\rho}_g \in \Theta$. Then

$$d\big(g_r^{-1}\omega, g_r^{-1}\rho_g\big) = d\big(g_r^{-1}\omega, \tilde{\rho}_g\big) \le \frac{1}{r}.$$

Consider some $\varphi \in \ell^2(G)$ and $x \in \mathbb{Z}^d$. Then a short computation leads to

$$\begin{aligned}
&\Big(\widehat{t}(\omega)\, \widehat{\Psi}_g^{(r-M)}\varphi - \widehat{\Psi}_g^{(r-M)}\, \widehat{t}(\rho_g)\varphi\Big)(x)\\
&= t(x^{-1}\omega)\big(\widehat{\Psi}_g^{(r-M)}\varphi\big)(x) - \Psi^{(r-M)}(x)\, \big(\widehat{t}(\rho_g)\varphi\big)(x)\\
&= t(x^{-1}\omega)\, \Psi^{(r-M)}(x)\, \varphi(x) - \Psi^{(r-M)}(x)\, t(x^{-1}\rho_g)\, \varphi(x)\\
&= \Psi_g^{(r-M)}(x)\, \Big(t\big(x^{-1}\omega\big) - t\big(x^{-1}\rho_g\big)\Big)\, \varphi(x)\,.
\end{aligned}$$

Clearly, the term vanishes if $x \notin \operatorname{supp}\big(\Psi_g^{(r-M)}\big) \subseteq g_r + Q_{r-M}$ where the inclusion follows from Proposition 8.2. If $x \in g_r + Q_{r-M}$, then the previous Lemma 8.3 implies $t\big(x^{-1}\omega\big) = t\big(x^{-1}\rho_g\big)$ since $d\big(g_r^{-1}\omega, g_r^{-1}\rho_g\big) \le \frac{1}{r}$. $\square$

Week 12

For two operators $A, B \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$, their (additive) commutator is defined by $[A, B] := AB - BA$. The following estimate is using Proposition 8.2 (c). Recall that for $h \in \mathbb{Z}^d$, the right shift $R_h \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ is given by

$$(R_h\varphi)(x) = \varphi(x + h)$$

Lemma 8.5. *For each $g,\ h \in \mathbb{Z}^d$ and $r \ge 1$, the estimate*

$$\Big|\Big(\Big[R_h\,, \widehat{\Psi}_g^{(r)}\Big]\varphi\Big)\,(x)\Big| \le \frac{|h|}{r}\; C_L\, \Big(\chi_g^{(r)}(x + h) + \chi_g^{(r)}(x)\Big)\; \big|\big(R_h\varphi\big)(x)\big|$$

holds for all $\varphi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$.

Proof. Let $\varphi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. A short computation leads to

$$\Big(\Big[R_h\,, \widehat{\Psi}_g^{(r)}\Big]\varphi\Big)\,(x)\; = \;\Big(\Psi_g^{(r)}(x + h) - \Psi_g^{(r)}(x)\Big)\; \big(R_h\varphi\big)(x)\,.$$

In addition, Proposition 8.2 asserts

$$\Big|\Psi_g^{(r)}(x + h) - \Psi_g^{(r)}(x)\Big| \le \frac{|h|}{r}\; C_L\, \Big(\chi_g^{(r)}(x + h) + \chi_g^{(r)}(x)\Big).$$

$\square$

8.4. Lipschitz maps on $\ell^2(\mathbb{Z}^d)$. Here we provide an estimate of a map $\mathfrak{D}_B(A) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ that will help us to estimate $S(z)$ and the error term $T(z)$.

Recall that $\chi_g^{(r)} : \mathbb{R}^d \to \{0, 1\}$ denotes the characteristic function of the support $rg + \widetilde{Q}_r$. The corresponding multiplication operator by the function $\chi_g^{(r)}$ on $\ell^2(\mathbb{Z}^d)$ is denoted by the symbol $\widehat{\chi}_g^{(r)}$, with $\|\widehat{\chi}_g^{(r)}\| = 1$ for each $g \in \mathbb{Z}^d$ and $r > 0$. In the following $C_c(\mathbb{Z}^d)$ denotes the set of functions $\varphi : \mathbb{Z}^d \to \mathbb{C}$ with finite support in $\mathbb{Z}^d$.

We will prove some estimates that will be useful later. Therefore, we need the concept of a positivity preserving operator. An operator $B \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ is called *positivity preserving* if

$$Bf \geq 0 \qquad \text{for all } f \in \ell^2(\mathbb{Z}^d) \text{ with } f \geq 0.$$

LEMMA 8.6. *Let $B \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ be a positivity preserving and $(A_g)_{g\in\mathbb{Z}^d} \subseteq \mathcal{L}(\ell^2(\mathbb{Z}^d))$ be an operator family satisfying $\|A\| := \sup_{g\in\mathbb{Z}^d} \|A_g\| < \infty$. Then the map $\mathfrak{O}_B(A) : C_c(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ defined by*

$$\big(\mathfrak{O}_B(A)\varphi\big)(x) \;:=\; \sum_{g\in\mathbb{Z}^d} \Big(\widehat{\chi}_g^{(r)}\, B \left|A_g\, \widehat{\chi}_g^{(r)}\varphi\right|\Big)(x)\,, \qquad \varphi \in C_c(\mathbb{Z}^d)\,,\, x \in \mathbb{Z}^d,$$

satisfies

$$\left\|\mathfrak{O}_B(A)\varphi - \mathfrak{O}_B(A)\psi\right\|_2 \leq \mathscr{N}\, \|A\|\, \|B\|\, \|\varphi-\psi\|_2\,, \qquad \varphi,\psi \in C_c(\mathbb{Z}^d)\,.$$

Furthermore, $\mathfrak{O}_B(A)$ extends to a continuous bounded map on $\ell^2(\mathbb{Z}^d)$ to $\ell^2(\mathbb{Z}^d)$ such that

$$\sup_{\|\varphi\|_2\leq 1} \|\mathfrak{O}_B(A)\varphi\|_2 \;\leq\; \mathscr{N}\, \|A\|\, \|B\|\,.$$

PROOF. Recall that $\left|A_g\, \widehat{\chi}_g^{(r)}\varphi\right|$ is an element of $\ell^2(\mathbb{Z}^d)$ and so $B\left|A_g\, \widehat{\chi}_g^{(r)}\varphi\right|$ is well-defined.

Let

$V_g(r) := \left\{h\in\mathbb{Z}^d \,\middle|\, rg+\widetilde{Q}_r \cap rh+\widetilde{Q}_r \neq \varnothing\right\} \supseteq \left\{h\in\mathbb{Z}^d \,\middle|\, \operatorname{supp}(\Psi_g^{(r)}) \cap \operatorname{supp}(\Psi_h^{(r)}) \neq \varnothing\right\}$

which satisfies $\sharp\, V_g(r) \leq \mathscr{N}$, confer Proposition 8.2 (b).

Let $\varphi \in C_c(\mathbb{Z}^d)$. First note that

$$\varphi_g := B\left|A_g\, \widehat{\chi}_g^{(r)}\varphi\right|, \qquad g\in\mathbb{Z}^d,$$

defines an element in $\ell^2(\mathbb{Z}^d)$. Thus, the Cauchy-Schwarz inequality on $\ell^2(\mathbb{Z}^d)$ yields

$$\left|\left\langle \widehat{\chi}_g^{(r)}\,\varphi_g\,,\, \widehat{\chi}_h^{(r)}\,\varphi_h\right\rangle\right| \leq \left\|\widehat{\chi}_g^{(r)}\,\varphi_g\right\|_2 \left\|\widehat{\chi}_h^{(r)}\,\varphi_h\right\|_2 \leq \|\varphi_g\|_2\, \|\varphi_h\|_2\,.$$

Note that the latter inner product vanishes if $g\in\mathbb{Z}^d$ and $h\notin V_g(r)$. Furthermore, we have

$$\|\varphi_g\|_2 \leq \|B\|\, \left\|A_g\widehat{\chi}_g^{(r)}\varphi\right\|_2 \leq \|B\|\, \|A\|\|\widehat{\chi}_g^{(r)}\varphi\|_2\,.$$

Since $2ab \leq a^2+b^2$ for $a,b\geq 0$, the previous considerations lead to

$$\begin{aligned} \|\mathfrak{O}_B(A)\varphi\|_2^2 &\leq \sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)} \left|\left\langle \widehat{\chi}_g^{(r)}\,\varphi_g\,,\, \widehat{\chi}_h^{(r)}\,\varphi_h\right\rangle\right| \\ &\leq \frac{\|B\|^2\, \|A\|^2}{2} \sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)} \left(\|\widehat{\chi}_g^{(r)}\varphi\|_2^2 + \|\widehat{\chi}_h^{(r)}\varphi\|_2^2\right). \end{aligned}$$

For $x\in\mathbb{R}^d$, the number of $h\in\mathbb{Z}^d$ with $\chi_h^{(r)}(x)\neq 0$ is bounded by $\mathscr{N}$. For each such h, the number of $g\in\mathbb{Z}^d$ with $h\in V_g(r)$ is also bounded $\mathscr{N}$. Hence,

$$\sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)} \chi_h^{(r)}(x) \;\leq \mathscr{N}\; \max_{g\in\mathbb{Z}^d} \sum_{h\in V_g^{(r)}} \chi_h^{(r)}(x) \leq\; \mathscr{N}^2$$

follows. Hence, the estimate

$$\sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)}\|\widehat{\chi}_h^{(r)}\varphi\|_2^2 = \sum_{x\in\mathbb{Z}^d}|\varphi(x)|^2\left(\sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)}\chi_h^{(r)}(x)\right) \leq \|\varphi\|_2^2\,\mathcal{N}^2$$

is proven. Similarly, we can show

$$\sum_{g\in\mathbb{Z}^d}\sum_{h\in V_g(r)}\|\widehat{\chi}_g^{(r)}\varphi\|_2^2 = \sum_{x\in\mathbb{Z}^d}|\varphi(x)|^2\,\mathcal{N}\,\sharp\left\{g\in\mathbb{Z}^d\,\middle|\,\chi_g^{(r)}(x)\neq 0\right\} \leq \|\varphi\|_2^2\,\mathcal{N}^2.$$

Combining this with the previous estimates, we get

$$\|\mathfrak{O}_B(A)\varphi\|_2^2 \leq \mathcal{N}^2\ \|A\|^2\ \|B\|^2\ \|\varphi\|_2^2$$

for all $\varphi\in C_c(G)$.

Let $\varphi,\psi\in C_c(\mathbb{Z}^d)$. Since B is a positivity preserving linear operator we have

$$0\leq B\big(|\varphi-\psi|-|\varphi|+|\psi|\big) = B|\varphi-\psi|-\big(B|\varphi|-B|\psi|\big).$$

Thus,

$$\Big|B\big|A_g\,\widehat{\chi}_g^{(r)}\varphi\big|-B\big|A_g\,\widehat{\chi}_g^{(r)}\psi\big|\Big| \leq B\big|A_g\,\widehat{\chi}_g^{(r)}(\varphi-\psi)\big|$$

follows implying

$$\big|\mathfrak{O}_B(A)\varphi-\mathfrak{O}_B(A)\psi\big| \leq \sum_{g\in\mathbb{Z}^d}\widehat{\chi}_g^{(r)}\,B\,\big|A_g\,\widehat{\chi}_g^{(r)}(\varphi-\psi)\big| = \mathfrak{O}_B(A)(\varphi-\psi).$$

Together with the first step, we conclude

$$\big\|\mathfrak{O}_B(A)\varphi-\mathfrak{O}_B(A)\psi\big\|_2 \leq \big\|\mathfrak{O}_B(\varphi-\psi)\big\|_2 \leq \mathcal{N}\,\|B\|\,\|A\|\,\|\varphi-\psi\|_2,$$

namely $\mathfrak{O}_B(A): C_c(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ is Lipschitz continuous and defined on a dense set with $\|\mathfrak{O}_B(A)\varphi\|_2\leq\mathcal{N}\ \|B\|\ \|A\|\ \|\varphi\|_2$. Thus, we can extend $\mathfrak{O}_B(A)$ continuosly onto $\ell^2(\mathbb{Z}^d)$ such that

$$\sup_{\|\varphi\|_2\leq 1}\|\mathfrak{O}_B(A)\varphi\|_2\leq\mathcal{N}\ \|A\|\ \|B\|.$$

□

8.5. Lipschitz continuity of the spectra. Now we have all tools together to prove Theorem 8.1.

LEMMA 8.7. *Let $H=(H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ be a pattern equivariant Hamiltonian of finite range. Consider an $r>0$, $\Theta\in\mathcal{J}$ and for each $g\in\mathbb{Z}^d$, let $\rho_g\in\Theta$ be arbitrary. Then for every $z\in\rho(H_\Theta)=\mathbb{C}\setminus\sigma(H_\Theta)$, the operator $S(z)\in\mathcal{L}(\ell^2(G))$ given by*

$$S(z) := \sum_{g\in\mathbb{Z}^d}\widehat{\Psi}_g^{(r)}\,(H_{\rho_g}-z)^{-1}\,\widehat{\chi}_g^{(r)}.$$

is well-defined and its operator norm is bounded by $\frac{\mathcal{N}}{\operatorname{dist}(z,\sigma(H_\Theta))}$.

PROOF. Note that $H_\rho - z$ is invertible for each $\rho \in \Theta$ as $z \in \rho(H_\Theta)$. Consider the operator family $A_g := \widehat{\Psi}_g^{(r)}\,(H_{\rho_g} - z)^{-1}$ for $g \in \mathbb{Z}^d$. Its operator norm is bounded by

$$\|A\| = \sup_{g\in\mathbb{Z}^d} \|A_g\| \le \sup_{\rho\in\Theta} \|(H_\rho - z)^{-1}\| = \sup_{\rho\in\Theta} \frac{1}{\operatorname{dist}\big(z, \sigma(H_\rho)\big)} = \frac{1}{\operatorname{dist}\big(z, \sigma(H_\Theta)\big)}$$

using Proposition 5.5 and Theorem 5.11 (Spectral radius and norm for normal elements). Let $\varphi \in C_c(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. Since $\widehat{\Psi}_g^{(r)} = \widehat{\chi}_g^{(r)}\,\widehat{\Psi}_g^{(r)}$, the estimate

$$\big|\big(S(z)\varphi\big)(x)\big| = \left|\sum_{g\in\mathbb{Z}^d} \Big(\widehat{\chi}_g^{(r)}\, A_g\, \widehat{\chi}_g^{(r)}\varphi\Big)(x)\right| \le \big(\mathfrak{O}_I(A)\varphi\big)(x)$$

follows. Note that the identity operator $I \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ is positivity preserving and $\|I\| = 1$. Hence, Lemma 8.6 implies $\|\big(S(z)\big)\| \le \mathscr{N}\,\|A\|$ which coupled with the estimate on $\|A\|$ finnishes the proof. □

Recall that a pattern equivariant Hamiltonian $H = (H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ of finite range is of the form

$$H_\omega := \sum_{h\in\mathcal{R}} \Big(\widehat{t}_h(\omega) R_h + R_{-h}\widehat{t}_h(\omega)^*\Big)$$

where $\mathcal{R} \subseteq \mathbb{Z}^d$ is finite and each $t_h : \mathcal{A}^{\mathbb{Z}} \to \mathbb{C}$ is a (strongly) pattern equivariant function with parameter $M(t_h) \in \mathbb{N}$.
Now we have all tools at hand to prove the heart of Theorem 8.1.

PROPOSITION 8.8. *Let $H = (H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ be a pattern equivariant Hamiltonian of finite range. Let $\Omega, \Theta \in \mathcal{J}$ be such that*

$$r := \delta_H(\Omega, \Theta)^{-1} > M := \max_{h\in\mathcal{R}} M(t_h)$$

If $z \in \rho(H_\Theta)$ satisfies

$$\operatorname{dist}\big(z, \sigma(H_\Theta)\big) > \frac{3\,\mathscr{N}\,\max\{C_L\,, 1\}}{(r - M)}\;\|H\|_S\,, \tag{1}$$

we have that $z \in \rho(H_\Omega)$.

PROOF. Let $z \in \rho(H_\Theta)$ obeying (1). In particular, $z \in \rho(H_\rho)$ for all $\rho \in \Theta$ and $\operatorname{dist}\big(z, \sigma(H_\rho)\big) \ge \operatorname{dist}\big(z, \sigma(H_\Theta)\big)$. It suffices to prove that there exists $\delta > 0$ independent of $\omega \in \Omega$ such that $\operatorname{dist}\big(z, \sigma(H_\omega)\big) > \delta$. This would imply that $z \in \rho(H_\Omega) = \mathbb{C} \smallsetminus \sigma(H_\Omega)$.

In light of this, let $\omega \in \Omega$ be fixed and we will proceed as follows:

(i) An operator $S(z)$ is constructed such that $(H_\omega - z)S(z) = \mathrm{I} + T(z)$ where the error term $T(z)$ comes from the kinetic term.
(ii) It is shown that $\|T(z)\| \le \frac{2}{3}$.
(iii) Using (i) and (ii), $z \in \rho(H_\omega)$ is verified and moreover $\operatorname{dist}(z, \sigma(H_\Omega)) \ge \delta$ is proven for a suitable $\delta > 0$.

(i): By definition each t_h is a (strongly) pattern equivariant function with parameter M. Proposition 8.4 asserts that there is for each $g \in \mathbb{Z}^d$, an $\rho_g \in \Theta$ (depending on M but not on t_h) such that

$$\widehat{t}_h(\omega)\,\widehat{\Psi}_g^{(r-M)} = \widehat{\Psi}_g^{(r-M)}\,\widehat{t}_h(\rho_g) \quad \text{and} \quad \widehat{t}_h(\omega)^*\,\widehat{\Psi}_g^{(r-M)} = \widehat{\Psi}_g^{(r-M)}\,\widehat{t}_h(\rho_g)^* \qquad (\clubsuit)$$

for all $h \in \mathcal{R}$ using that $\widehat{\Psi}_g^{(r-M)}$ is self-adjoint.

With this chosen $\rho_g \in \Theta$ for $g \in \mathbb{Z}^d$, define $S(z) \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ by

$$S(z) := \sum_{g \in \mathbb{Z}^d} \widehat{\Psi}_g^{(r-M)}\,(H_{\rho_g} - z)^{-1}\,\widehat{\chi}_g^{(r-M)},$$

which is a bounded linear operator by Lemma 8.7.

Define the operators $T(z) \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ by $T(z) := (H_\omega - z)S(z) - I$. Furthermore, consider the operator $A_g(z) := (H_{\rho_g} - z)^{-1} \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$ for $g \in \mathbb{Z}^d$. Then the operator norm of this operator family $(A_g(z))_{g \in \mathbb{Z}^d}$ satisfies

$$\begin{aligned} \|A(z)\| := \sup_{g \in \mathbb{Z}^d} \|A_g(z)\| \le \sup_{g \in \mathbb{Z}^d} \|(H_{\rho_g} - z)^{-1}\| &= \sup_{g \in \mathbb{Z}^d} \frac{1}{\operatorname{dist}\big(z, \sigma(H_{\rho_g})\big)} \\ &\le \frac{1}{\operatorname{dist}\big(z, \sigma(H_\Theta)\big)}\,. \end{aligned}$$

In order to determine $T(z)$, we investigate the operator product $H_\omega\,S(z)$. For $g \in \mathbb{Z}^d$, a short computation gives

$$\begin{aligned} H_\omega\,\widehat{\Psi}_g^{(r-M)} &= \sum_{h \in \mathcal{R}} \widehat{t}_h(\omega) R_h \widehat{\Psi}_g^{(r-M)} + R_{-h}\widehat{t}_h(\omega)^* \widehat{\Psi}_g^{(r-M)} \\ &\overset{(\clubsuit)}{=} \sum_{h \in \mathcal{R}} \widehat{t}_h(\omega)\widehat{\Psi}_g^{(r-M)} R_h + R_{-h}\widehat{\Psi}_g^{(r-M)}\widehat{t}_h(\rho_g)^* + \widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] \\ &\overset{(\clubsuit)}{=} \sum_{h \in \mathcal{R}} \widehat{\Psi}_g^{(r-M)}\widehat{t}_h(\rho_g) R_h + \widehat{\Psi}_g^{(r-M)} R_{-h}\widehat{t}_h(\rho_g)^* \\ &\qquad + \widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] + \widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big] \\ &= \widehat{\Psi}_g^{(r-M)} H_{\rho_g} + \sum_{h \in \mathcal{R}} \widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] + \widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big]. \end{aligned}$$

Since $\big(\Psi_g^{(r-M)}\big)_{g \in \mathbb{Z}^d}$ is a Lipschitz-partition of unity and the identity $\widehat{\Psi}_g^{(r-M)} = \widehat{\Psi}_g^{(r-M)}\widehat{\chi}_g^{(r-M)}$ holds, we conclude

$$\sum_{g \in \mathbb{Z}^d} \widehat{\Psi}_g^{(r-M)}\,\widehat{\chi}_g^{(r-M)} = I.$$

With this at hand, we conclude

$$
\begin{aligned}
&(H_\omega - z)S(z)\\
&= \sum_{g\in\mathbb{Z}^d} (H_\omega - z)\widehat{\Psi}_g^{(r-M)}\,(H_{\rho_g} - z)^{-1}\,\widehat{\chi}_g^{(r-M)}\\
&= \sum_{g\in\mathbb{Z}^d} \widehat{\Psi}_g^{(r-M)}\,(H_{\rho_g} - z)(H_{\rho_g} - z)^{-1}\,\widehat{\chi}_g^{(r-M)}\\
&\quad+ \sum_{g\in\mathbb{Z}^d}\sum_{h\in\mathcal{R}} \Big(\widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] + \widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big]\Big)\,A_g(z)\,\widehat{\chi}_g^{(r-M)}\\
&= \mathrm{I} + \sum_{g\in\mathbb{Z}^d}\sum_{h\in\mathcal{R}} \Big(\widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] + \widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big]\Big)\,A_g(z)\,\widehat{\chi}_g^{(r-M)}.
\end{aligned}
$$

Hence,

$$
T(z) = \sum_{g\in\mathbb{Z}^d}\sum_{h\in\mathcal{R}} \Big(\widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big] + \widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big]\Big)\,A_g(z)\,\widehat{\chi}_g^{(r-M)}
$$

is derived.

(ii): We will estimate the operator norm of $T(z)$. Therefore, define the Hamiltonian $H^+ = (H^+_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}}}$ with the same range $\mathcal{R}$ as H and coefficients $t_h^+ : \mathcal{A}^{\mathbb{Z}} \to [0,\infty)$ defined by $t_h^+(\omega) := |h|\,\|t_h\|_\infty$. Since t_h^+ is just a constant, it is clearly pattern equivariant and

$$
\begin{aligned}
\|H^+\| = \sup_{\omega\in\mathcal{A}^{\mathbb{Z}}}\|H^+_\omega\| \le \sup_{\omega\in\mathcal{A}^{\mathbb{Z}}}\sum_{h\in\mathcal{R}} \|\widehat{t}_h^+(\omega)\|\|R_h\| + \|\widehat{t}_h^+(\omega)^*\|\|R_{-h}\| &= 2\sum_{h\in\mathcal{R}}|h|\|t_h\|_\infty\\
&\le \|H\|_S
\end{aligned}
$$

where $\|H\|_S$ denotes the Schur norm of our Hamiltonian H, confer Section 6.1. Since each t_h^+, $h\in\mathcal{R}$, is non-negative, it is straight forward to show that H^+ is positivity preserving.

We claim that the commutator estimates in Lemma 8.5 of the shift operator with the Lipschitz partition of unity lead to

$$
\|T(z)\| \le \frac{C_L}{r-M}\left(\|\mathfrak{D}_{H^+}(A(z))\| + \|H^+\|\;\|\mathfrak{D}_I(A(z)\|\right)
$$

where the operator family $(A_g(z))_{g\in\mathbb{Z}^d}$ was defined by $A_g(z) := (H_{\rho_g} - z)^{-1} \in \mathcal{L}(\ell^2(\mathbb{Z}^d))$. Before proving this claim let us show that this implies $\|T(z)\| \le \frac{2}{3}$. Since $\|A_g(z)\| \le \frac{1}{\mathrm{dist}(z,\sigma(H_\Theta))}$, Lemma 8.6 and our assumption (1) imply

$$
\|T(z)\| \le \frac{2C_L}{r-M}\mathcal{N}\;\|H^+\|\sup_{g\in G}\|A_g(z)\| \le \frac{2C_L\;\mathcal{N}}{r-M}\|H\|_S\frac{1}{\mathrm{dist}(z,\sigma(H_\Theta))} \overset{(1)}{\le} \frac{2}{3}.
$$

Now let us show the desired estimate for $\|T(z)\|$. Therefore, let $\varphi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. For $h \in \mathcal{R}$ and $g \in G$, set $\varphi_g := A_g(z)\,\widehat{\chi}_g^{(r-M)}\varphi$, Lemma 8.5 yields

$$\begin{aligned}
&\left|\left(\widehat{t}_h(\omega)\big[R_h, \widehat{\Psi}_g^{(r-M)}\big]\varphi_g\right)(x)\right| \\
&\leq \|t_h\|_\infty \left|\big[R_h, \widehat{\Psi}_g^{(r-M)}\big]\varphi_g(x)\right| \\
&\leq \frac{C_L}{r-M}|h|\ \|t_h\|_\infty \left(\chi_g^{(r-M)}(x+h)\left|\big(R_h\varphi_g\big)(x)\right| + \chi_g^{(r-M)}(x)\left|\big(R_h\varphi_g\big)(x)\right|\right) \\
&\leq \frac{C_L}{r-M}|h|\ \|t_h\|_\infty \left(\left(R_h\left|\widehat{\chi}_g^{(r-M)}\varphi_g\right|\right)(x) + \left(\widehat{\chi}_g^{(r-M)}\left|R_h\varphi_g\right|\right)(x)\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left|\left(\widehat{t}_h(\rho_g)^*\big[R_{-h}, \widehat{\Psi}_g^{(r-M)}\big]\varphi_g\right)(x)\right| \\
&\leq \frac{C_L}{r-M}|h|\ \|t_h\|_\infty \left(\left(R_{-h}\left|\widehat{\chi}_g^{(r-M)}\varphi_g\right|\right)(x) + \left(\widehat{\chi}_g^{(r-M)}\left|R_{-h}\varphi_g\right|\right)(x)\right).
\end{aligned}$$

With this at hand, the estimate

$$\begin{aligned}
|T(z)\varphi(x)| \leq &\frac{C_L}{r-M}\sum_{g\in\mathbb{Z}^d}\left(\widehat{\chi}_g^{(r-M)}\left(\sum_{h\in\mathcal{R}}|h|\|t_h\|_\infty\big(R_h+R_{-h}\big)\right)|\varphi_g|\right)(x) \\
&+\frac{C_L}{r-M}\left(\sum_{h\in\mathcal{R}}|h|\|t_h\|_\infty\big(R_h+R_{-h}\big)\right)\left(\sum_{g\in\mathbb{Z}^d}\widehat{\chi}_g^{(r-M)}|\varphi_g|\right)(x) \\
= &\frac{C_L}{r-M}\left(\left|\big(\mathfrak{D}_{H^+}(A(z))\varphi\big)(x)\right| + \left|\big(H^+\mathfrak{D}_I(A(z))\varphi\big)(x)\right|\right)
\end{aligned}$$

is concluded proving the claim.

(iii): According to Step (ii), the estimate $\|T(z)\| \leq \frac{2}{3}$ holds uniformly in $\omega \in \Omega$, for all $z \in \rho(H_\Theta)$ satisfying (1). Note that if $z \notin \mathbb{R}$, we clearly have $z \notin \sigma(H_\Omega)$ as H is self-adjoint. Let $z \in \mathbb{R}$ be such that it satisfies (1). Then $z_n := z + i\frac{1}{n}$ satisfies (1) as well and $H_\omega - z_n$ is invertible because H_ω is self-adjoint. Since, furthermore,

$$(H_\omega - z_n)^{-1} = S(z_n)\left(\mathrm{I}+T(z_n)\right)^{-1}$$

and $\sup_{n\in\mathbb{N}}\|S(z_n)\| < \infty$, we conclude $z \in \rho(H_\omega)$ by Sheet 9, Exercise 4. It is left to show that there is a $\delta > 0$ (independent of ω) such that $\operatorname{dist}(z, \sigma(H_\omega)) \geq \delta$. The operator $I + T(z)$ is invertible by the Neumann series and moreover,

$$\|(I+T(z))^{-1}\| = \left\|\sum_{k=0}^{\infty}(-T(z))^n\right\| \leq \sum_{k=0}^{\infty}\|T(z)\|^n \leq \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$$

holds since $\|T(z)\| \leq \frac{2}{3}$. Thus, Lemma 8.7 leads to

$$\frac{1}{\operatorname{dist}(z_n, \sigma(H_\omega))} = \|(H_\omega - z_n)^{-1}\| \leq \|S(z_n)\|\ \|(\mathrm{I}+T(z_n))^{-1}\| \leq \frac{3\,\mathscr{N}}{\operatorname{dist}(z_n, \sigma(H_\Theta))}$$

uniformly both in ω. Thus, $\operatorname{dist}(z, \sigma(H_\omega)) \geq \delta$ follows by sending $n \to \infty$ where $\delta := \frac{\operatorname{dist}(z,\sigma(H_\Theta))}{3\,\mathscr{N}}$. Hence, $z \notin \sigma(H_\Omega)$ is concluded. $\square$

Proof of Theorem 8.1. Recall the notation $M := \max_{h\in\mathcal{R}} M(t_h)$ and define the constant

$$C_d := 6\,\mathscr{N}\ \max\{C_L\,, 1\}.$$

Recall that $\mathscr{N}$ and C_L are constants depending essentially on the dimension d, confer Section 8.2. If $\Omega = \Theta$, then $\sigma(H_\Omega) = \sigma(H_\Theta)$. Now suppose $\Omega \neq \Theta$, namely $\delta_H(\Omega, \Theta) > 0$. Set $r := \delta_H(\Omega, \Theta)^{-1}$. We analyze two cases: (i) $1 \leq r \leq \mathscr{N} M$ and (ii) $\mathscr{N} M < r$.

(i): From $1 \leq r \leq \mathscr{N}\ M$, we infer $1 \leq \mathscr{N} \frac{M}{r}$. Due to Proposition 6.4 (Basic properties of a Hamiltonian), we have $\|H_\omega\| \leq \|H\|_S$ for $\omega \in \mathcal{A}^{\mathbb{Z}^d}$. Then Proposition 5.15 (Spectrum is norm continuous) implies

$$\begin{aligned} d_H\big(\sigma(H_\omega), \sigma(H_\rho)\big) \leq 2\ \|H\|_S \leq 2\mathscr{N}\ \|H\|_S \frac{M}{r} &\leq 2\mathscr{N}\ M\ \|H\|_S\ \delta_H(\Omega, \Theta) \\ &\leq C_d\ M\ \|H\|_S\ \delta_H(\Omega, \Theta) \end{aligned}$$

for all $\omega \in \Omega$ and $\rho \in \Theta$.

(ii): Suppose $r > \mathscr{N} M \geq 2M$ (using that $\mathscr{N} \geq 2$ holds always). For $z \in \sigma(H_\Theta)$, Proposition 8.8 leads to

$$\operatorname{dist}\big(z, \sigma(H_\Omega)\big) \ \leq\ \frac{3\mathscr{N}\ \max\{C_L\,,1\}}{(r - M)}\ \|H\|_S\,.$$

By interchanging the role of Ω and Θ we obtain:

$$d_H\big(\sigma(H_\Omega), \sigma(H_\Theta)\big) \ \leq\ \frac{C_d}{2}\ \|H\|_S\ \frac{1}{(r - M)}.$$

In addition, the constraint $r > 2\ M$ implies

$$r - M = r\left(1 - \frac{M}{r}\right) > \frac{r}{2}.$$

Hence, the desired estimate

$$\begin{aligned} d_H\big(\sigma(H_\Omega), \sigma(H_\Theta)\big) \ \leq\ \frac{C_d}{2}\ \|H\|_S\ \frac{1}{(r - M)} &\leq C_d\ \|H\|_S\ \delta_H(\Omega, \Theta) \\ &\leq C_d\ M\ \|H\|_S\ \delta_H(\Omega, \Theta). \end{aligned}$$

follows since $\delta_H(\Omega, \Theta) = \frac{1}{r}$. □

8.6. Discussion and extensions. Theorem 8.1 can be extended to the setting where the alphabet $\mathcal{A}$ is replaced by a compact metric space $(\mathcal{A}, d_{\mathcal{A}})$ and $\mathbb{Z}^d$ is replaced by the countable group $G := M\mathbb{Z}^d$, where $M \in M_d(\mathbb{R})$ is an invertible matrix. Then it gets natural to treat Hamiltonians where the coefficients are β-Hölder continuous in $\mathcal{A}$. As long as the the coefficients are still in a suitable sense (strongly) pattern equivariant, this yields the β-Hölder continuity of the spectrum.

Furthermore, one can deduce estimates for Hamiltonians with infinite range. In this case the decay of the off-diagonal terms influences the regularity of the spectral map Σ, see Beckus/Bellissard/Cornean - *Hölder continuity of the spectra for aperiodic Hamiltonians*, 2019. Specifically, one can prove the following.

THEOREM 8.9. *Let $H = (H_\omega)_{\omega \in \mathcal{A}^{\mathbb{Z}^d}}$ be a pattern equivariant Hamiltonian (possible infinite range). If the radius of influence of H has a linear growth with constant $1 \leq C_H < \infty$, then*

$$\max_{\|h\|_\infty \leq r} M(t_h) \leq Cr$$

(linear growth of the parameter of the (strongly) pattern equivariant coefficients), then

$$d_H\big(\sigma(H_\Omega),\sigma(H_\Theta)\big) \;\leq C_d\; \|H\|_S\; C_H\; \delta_H\big(\Xi,\Theta\big), \qquad \Omega,\Theta\in\mathcal{J},$$

where $C_H > 0$ is constant depending on C and H.

Another consequence of Theorem 8.1 is the following.

Corollary 8.10. *Let $H = (H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ be a pattern equivariant Hamiltonian of finite range. Then*

$$\Sigma : \mathcal{J}\to\mathcal{K}(\mathbb{R}), \qquad \Omega\mapsto\sigma(H_\Omega),$$

is continuous in the corresponding Hausdorff topologies.

This continuity holds for a much larger class of Hamiltonians. First of all it can be easily extended to all Hamiltonians of finite range using that the (strongly) pattern equivariant functions are dense in $C(\mathcal{A}^{\mathbb{Z}^d})$ with respect to the uniform norm (use Proposition 5.15). Using again Proposition 5.15, the same statement holds for all Hamiltonians with infinite range that can be approximated in norm $\|H\| := \sup_{x\in X}\|H_x\|$ by Hamiltonians of finite range. Recall from Theorem 2.19 that the set of invariant probability measures of a dynamical system only behave semi-continuous. Here we have seen that the spectrum behaves actually continuous.

Finally let us remark that the continuity result holds in a much larger generality of groupoid C^*-algebras, confer Beckus/Bellissard/De Nittis - *Spectral continuity for aperiodic quantum systems I. General theory*, 2018.

8.7. Applications. We discuss here shortly as how to apply the later theorem and to connect to previous considerations. Therefore, recall the notion of substitutions and the various periodic approximations that we defined in Section 4.3.

Corollary 8.11. *Let $H = (H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}^d}}$ be a pattern equivariant Hamiltonian of finite range. Then there exists a constant C_d such that*

$$d_H\big(\sigma(H_\omega),\sigma(H_\rho)\big) \leq C_d\; M\; \|H\|_S\; \delta_H\big(\overline{Orb(\omega)},\overline{Orb(\rho)}\big), \qquad \omega,\rho\in\mathcal{A}^{\mathbb{Z}^d},$$

where $\|H\|_S$ is the Schur norm of H and $M := \sup_{h\in\mathcal{R}} M(t_h)$ is the maximal parameter of the (strongly) pattern equivariant coefficients of H.

Proof. This follows directly from Theorem 8.1 and Proposition 6.2 stating that $\sigma(H_{\overline{Orb(\omega)}}) = \sigma(H_\omega)$. □

Let $G := \mathbb{Z}$, $\mathcal{A}\subseteq\mathbb{R}$ be a finite set and consider for $w\in\mathcal{A}^{\mathbb{Z}}$, the Schrödinger operator $H_w\in\mathcal{L}\big(\ell^2(\mathbb{Z})\big)$ defined by

$$(H_\omega\psi)(n) := \psi(n-1)+\psi(n+1)+\omega(n)\psi(n).$$

Then $V:\mathcal{A}^{\mathbb{Z}}\to\mathbb{R}, V(\omega):=\omega(0)$ is (strongly) pattern equivariant with parameter $M=1$ and $\omega(n) := V(n^{-1}\omega)$. Note that $t_1\equiv 1$ which are also (strongly) pattern equivariant with parameter $M=1$. Specifically, $(H_\omega)_{\omega\in\mathcal{A}^{\mathbb{Z}}}$ is a self-adjoint operator family over a dynamical system $(\mathcal{A}^{\mathbb{Z}},\mathbb{Z})$ by Sheet 8. In particular, we can represent the operator H_ω by

$$H_w = R_1 + R_{-1} + \widehat{V}(w)$$

where $(\widehat{V}(w)\psi)(n) = V(n^{-1}w)\psi(n)$ is a multiplication operator. Then the Schur norm of H is

$$\|H\|_S = 2\left(\|t_0\|_\infty + \|t_1\|_\infty\sqrt{1+\frac{1}{2}}\right) = 2\max_{a\in\mathcal{A}}|a| + \sqrt{6}.$$

Furthermore, we can compute that for $d = 1$, we have $\mathscr{N} = 3$, $C_L = 72$ and so $C_d = 1296$.

Let S be one of the primitive substitution (Fibonacci, Thue-Morse, Period Doubling and Golay-Rudin-Shapiro sequence) over the alphabet $\mathcal{A}$. We also have seen that there is an $x \in \mathcal{A}$ such that $\omega_n := \big(S^n(x)\big)^\infty$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} Orb(\omega_n) = \Omega(S).$$

Let $\omega \in \Omega(S)$, then $\overline{Orb(\omega)} = \Omega(S)$ holds as $\Omega(S)$ is minimal, see Theorem 4.5. Then the previous considerations lead to

$$d_H\big(\sigma(H_\omega), \sigma(H_{\omega_n})\big) \le 1296\left(2\max_{a\in\mathcal{A}}|a| + \sqrt{6}\right)\delta_H\big(\overline{Orb(\omega)}, Orb(\omega_n)\big),$$

where the left hand side goes to zero if $n \to \infty$. Recall that ω_n is periodic for each $n \in \mathbb{N}$ and so we can compute explicitly the spectrum $\sigma(H_{\omega_n})$, confer Section 7.2.

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APPENDIX A

A primer on topology

Open and closed sets

We refer to the book *Mengentheoretische Topologie* by B. von Querenburg, the book *Topology* by M. Manetti or any other topology book for further background.

Definition. *Let X be a set. Then $\mathscr{O} \subseteq \mathcal{P}(X)$ of subsets of X is called a* topology *of X if*

(1) $X, \varnothing \in \mathscr{O}$,
(2) any union of sets in $\mathscr{O}$ belongs to $\mathscr{O}$:

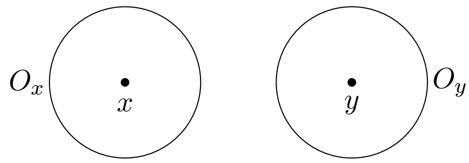
$$O_i \in \mathscr{O},\, i \in I \quad \Longrightarrow \quad \bigcup_{i \in I} O_i \in \mathscr{O}\,,$$

(3) every intersection of finitely many elements of $\mathscr{O}$ belongs to $\mathscr{O}$:

$$O_1, \ldots, O_n \in \mathscr{O} \quad \Longrightarrow \quad \bigcap_{i=1}^{n} O_i \in \mathscr{O}\,.$$

The tuple $(X, \mathscr{O})$ is called a topological space *if X is a set and $\mathscr{O}$ is a topology on X. The subsets of X belonging to $\mathscr{O}$ are called* open. *A subset $F \subseteq X$ is called closed if $X \setminus F$ is open. The family of all closed subsets is denoted by $\mathscr{F}$. Furthermore, $A \subseteq X$ is called* clopen *if A is an open and a closed subset (examples are $\varnothing$ and X).*

There are various seperation properties for topologies. For the purpose of this lecture we will mainly deal with the Hausdorff property: A topological space $(X, \mathscr{O})$ is called *Hausdorff* if for each $x, y \in X$, there $O_x, O_y \in \mathscr{O}$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \varnothing$.



A family $\mathscr{B}$ of open sets of a topological space $(X, \mathscr{O})$ is called *base*, if each open set of $(X, \mathscr{O})$ is represented as the union of elements of $\mathscr{B}$. Thus, for each $x \in O \in \mathscr{O}$ there is an $B \in \mathscr{B}$ such that $x \in B \subseteq O$. Furthermore, $\mathscr{G} \subseteq \mathscr{O}$ is called *generator of the topology $\mathscr{O}$* if each element in $\mathscr{O}$ can be represented by finite intersections and/or arbitrary unions of elements of $\mathscr{G}$.

Remark. *Let $\mathscr{B}$ be a family of subsets of the set X. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are two topologies on X with base $\mathscr{B}$ then $\mathscr{O}_1 = \mathscr{O}_2$. Thus, in order to define a topology it suffices to give a base.*

Let X be a set and $\mathscr{B}_1$ and $\mathscr{B}_2$ are two bases for a topology $\mathscr{O}_1$ respectively $\mathscr{O}_2$. Then $\mathscr{O}_1 = \mathscr{O}_2$ holds if

- for any $B_1 \in \mathscr{B}_1$ and $x \in B_1$, there is an $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subseteq B_1$,
- for any $B_2 \in \mathscr{B}_2$ and $x \in B_2$, there is an $B_1 \in \mathscr{B}_1$ such that $x \in B_1 \subseteq B_2$.

Example. Let X be a set and $\mathscr{O} := \{\varnothing, X\}$ which is called the *trivial topology* or *indiscrete topology*. If $\sharp X \geq 2$ then X is not Hausdorff. □

Example. Let X be a set and $\mathscr{O}$ is the the power set $\mathcal{P}(X)$ of X, which is called the *discrete topology*. A base for the topology is given by $\mathscr{B} := \big\{\{x\} : x \in X\big\}$. Note that the discrete topology is always Hausdorff. □

Example. Let $X := \mathbb{R}$ and $\mathscr{O}$ be the collection those sets that can be represented by unions of open intervals (a,b) for $a,b \in \mathbb{R}$. This defines the *Euclidean topology* or *natural topology*. A base $\mathscr{B}$ for the Euclidean topology on $\mathbb{R}$ is given by the collection of all open balls with rational center and rational radius. Furthermore, the Euclidean topology is Hausdorff. □

Let $(X, \mathscr{O})$ be a topological space and $x \in X$ then $U \subseteq X$ is called *neighborhood of* x, if there is an open $O \in \mathscr{O}$ such that $x \in O \subseteq U$. If U itself is open, we call it an open neighborhood. The family of all neighborhoods of x is called *neighborhood system of* x, which is denoted by

$$\mathscr{O}(x) := \big\{U \subseteq X : U \text{ is a neighborhood of } x\big\}.$$

A subfamily $\mathscr{B}(x)$ of $\mathscr{O}(x)$ is called a *neighborhood base of* x if for each neighborhood U of x there is an $B \in \mathscr{B}(x)$ such that $B \subseteq U$. A point $x \in X$ of a topological space X is called *isolated* if the singleton $\{x\}$ is a neighborhood of x. A point x is called *cluster point* of a sequence $(x_n)_{n\in\mathbb{N}}$ if each neighborhood of x contains infinitely many members of (x_n). In general, not every sequence admits a cluster point (e.g. $X := \mathbb{N}$ and $x_n := n$). Later we will get to know properties on X so that each sequence has at least one cluster point.

A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called *convergent to* $x \in X$, if for every neighborhood U of X there is an $n_U \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_U$. In this case, x is called the limit of the sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$. Note that a limit (if it exists) is unique if $(X, \mathscr{O})$ is Hausdorff. Otherwise, this is not the case. Furthermore, note that a convergent sequence has only one cluster point if X is Hausdorff.

Definition. *A topological space* $(X, \mathscr{O})$ *is called*

(a) first countable *if each* $x \in X$*, admits a countable neighborhood base* $\mathscr{B}(x)$.
(b) second countable *if* $\mathscr{O}$ *admits a countable base* $\mathscr{B}$.

Remark. *Each second countable space is also first countable.*
Later we get to know properties like continuity of a function. This can be characterized by sequential continuity if X *is first countable (otherwise not).*

Example. *Let* X *be an uncountable space (e.g.* $X = \mathbb{R}$*) equipped with the discrete topology. Then* $\{x\}$ *defines a neighborhood base for each* $x \in X$*.*

Thus, X is first countable. However it is not second countable as X is uncountable.

Example. Let $\mathbb{R}^d$ be equipped with the Euclidean topology (unions of open balls). Let $\mathscr{B}$ be the set of all open balls with rational center and rational radius. Clearly, $\mathscr{B}$ is countable and a base for the Euclidean topology. □

If $\mathscr{O}, \mathscr{O}'$ are two topologies on X, then $\mathscr{O}$ is called *finer* than $\mathscr{O}'$ (and $\mathscr{O}'$ coarser than $\mathscr{O}$) if $\mathscr{O}' \subseteq \mathscr{O}$. In particular, $\mathscr{O}$ has more open sets than $\mathscr{O}'$. For instance, the discrete topology on $\mathbb{R}$ is finer than the Euclidean topology on $\mathbb{R}$. Clearly, convergence of a sequence in the topology $\mathscr{O}$ implies convergence in the topology $\mathscr{O}'$ if $\mathscr{O}$ is finer than $\mathscr{O}'$.

Specific topologies are given by a metric:

Definition. *A map $d : X \times X \to [0, \infty)$ is called a* metric *if*

(a) $d(x,y) = 0 \Leftrightarrow x = y$ ***(identity of indiscernibles)***
(b) $d(x,y) = d(y,x)$ *for all* $x, y \in X$ ***(symmetry)***
(c) $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x, y, z \in X$ ***(triangle inequality)***

Then the tuple (X, d) is called metric space. Furthermore, d is called an ultra metric *if (c) is replaced by*

(c') $d(x,z) \leq \max\{d(x,y),\ d(y,z)\}$ *for all* $x, y, z \in X$.

Remark. *A metric space (X, d) is called* complete *if every Cauchy-sequence (i.e. (x_n) Cauchy sequence if for all $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for $n, m \geq n_0$) admits a limit point in X. This is often implicitly assumed in the literature.*

Let (X, d) be a metric space. Then

$$B_r(x) := \{y \in X : d(x,y) < r\}$$

for $r > 0$ denotes the open ball around x with radius r. In a metric space, a set $O \subseteq X$ is open if for each $x \in O$, there is a radius $r > 0$ such that $B_r(x) \subseteq O$. The collection of all open sets defines a topology $\mathscr{O}_d$ induced by the metric d. This topology is generated (finite intersections and arbitrary unions) by the family of all open balls in X. Due to the constraint (a), the topology $\mathscr{O}_d$ is always Hausdorff.

A topological space $(X, \mathscr{O})$ is called metrizable, if there exists a metric d on X such that $\mathscr{O}_d = \mathscr{O}$.

A family $(U_i)_{i \in I}$ of subsets of X is called *cover of* $A \subseteq X$ if $A \subseteq \bigcup_{i \in I} U_i$. Let $(V_j)_{j \in J}$ and $(U_i)_{i \in I}$ be a cover of A. Then $(V_j)_{j \in J}$ is called a *subcover* $(U_i)_{i \in I}$, if for each $j \in J$ there is an $i \in I$ such that $V_j = U_i$.

Definition. *Let X be a topological space. A set $K \subseteq X$ is called* compact *if for every family $(U_i)_{i \in I}$ of open sets with $K \subseteq \bigcup_{i \in I} U_i$ there are finitely many $i_1, \ldots i_n \in I$ with $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$. ("Every open cover has a finite subcover.") The topological space X is called compact if the set X is compact. By an abuse of notation the family $U_{i_1}, \ldots, U_{i_n}$ will usually be denoted simply by $U_1, \ldots U_n$.*

Remark. *A set* $K \subseteq X$ *of a topological space* X *is called* sequentially compact *if every sequence* $(x_n)_{n\in\mathbb{N}} \subseteq K$ *has a convergent subsequence in* K*. Whenever* X *is metrizable, sequential compactness and compactness are equivalent. However, for general topological spaces there is no relation.*

Proposition A.1. *Let* X *be a second countable topological space. Then* X *is sequentially compact if and only if it is compact.*

Proof. See *Mengentheoretische Topologie* by B. von Querenburg, Kapitel 8. □

Let (X, d) be a metric space. Then X is called *totally bounded (präkompakt)*, if for every $\varepsilon > 0$, there are $x_1, \dots, x_{n(\varepsilon)} \in X$ such that

$$X \subseteq \bigcup_{j=1}^{n(\varepsilon)} B_\varepsilon(x_j),$$

confer Satz 13.11 in *Mengentheoretische Topologie* by B. von Querenburg.

Proposition A.2. *Let* (X, d) *be a complete metric space. Then* X *is compact if and only if* X *is totally bounded.*

Proof. See Korollar 13.3 in *Mengentheoretische Topologie* by B. von Querenburg. □

A topological space X is called *locally compact* if each $x \in X$ has a compact neighborhood, namely there is a compact $K \subseteq X$ and an open $O \in \mathcal{B}(x)$ such that $O \subseteq K$.

Lemma A.3. *Let* X *be a topological space,* $K \subseteq X$ *be compact and* $A \subseteq K$ *be closed. Then,* A *is compact.*

Proof. Let $(U_i)_{i\in I}$ be an open covering of A. Then, we have $K \subseteq (X \setminus A) \cup \bigcup_{i\in I} U_i$. And, as the set $X \setminus A$ is open by closedness of A, we have an open cover of K. Thus, there is a finite subcover $U_1, \dots, U_n, X \setminus A$ of K. But then, $U_1, \dots, U_n$ is an open cover of A. This concludes the proof. □

Lemma A.4. *Let* X *be a Hausdorff space. Then, every compact* $K \subseteq X$ *is closed.*

Proof. We show that $X \setminus K$ is open. Let $x \in X \setminus K$ be arbitrary. As X is a Hausdorff space, for every $y \in K$, there are open neighborhoods U_x, V_y of x and y such that $U_x \cap V_y = \varnothing$. Then the sets $(V_y)_{y\in K}$ form an open cover of K. Since K is compact, there are $y_1, \dots, y_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} V_{y_i}.$$

Define $U(x) \coloneqq \bigcap_{i=1}^n U_x^i$, where U_x^i is the corresponding open neighborhood of x for the point y_i, namely $U_x^i \cap V_{y_i} = \varnothing$. Then, $U(x)$ is an open neighborhood of X. Furthermore, for every $y \in K$, we have $y \in U_{y_i}$ for some y_i and, hence, $y \notin U_x^i$. This yields $y \notin U(x)$. Therefore, we infer $U(x) \cap K = \varnothing$. Thus, for each $x \in X \setminus K$ there is an open neighborhood $U(x)$ with $U(x) \cap K = \varnothing$ implying $X \setminus K = \bigcup_{x\in X\setminus K} U(x)$ is open. Thus, K is closed. □

LEMMA A.5. *Let X be a compact Hausdorff space and let $(K_n)_{n\in\mathbb{N}}$ be a decreasing sequence of non-empty, compact subsets of X. Then the set $\bigcap_{n\in\mathbb{N}} K_n$ is non-empty and compact.*

PROOF. By Lemma A.4 every K_n is closed. Hence, the intersection $\bigcap_{n\in\mathbb{N}} K_n$ is closed and, obviously, a subset of the compact space X. Thus, Lemma A.3 yields compactness of $\bigcap_{n\in\mathbb{N}} K_n$. It is left to show $\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$. Suppose the opposite is true. Then, we have

$$X = X \setminus \bigcap_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} X \setminus K_n.$$

As every $X \setminus K_n$ is open, they form an open cover of X. Thus, there is a finite subcover $X \setminus K_{i_1}, \dots, X \setminus K_{i_n}$. Without loss of generality we can assume $i_1 \leq \dots \leq i_n$. Thus, we have $X = \bigcup_{k=1}^n X \setminus K_{i_k}$ and, therefore,

$$\varnothing = X \setminus \left(\bigcup_{k=1}^n X \setminus K_{i_k} \right) = \bigcap_{k=1}^n K_{i_k}.$$

As the i_j's are ordered by size, and since the sequence (K_n) is decreasing, we have

$$\bigcap_{k=1}^n K_{i_k} = K_{i_n} \neq \varnothing.$$

This is a contradiction. □

A topological space X is called *regular*, if $x \in X$ and $F \subseteq X$ is closed such that $x \notin F$, then there are open subsets $U, V \subseteq X$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.

THEOREM A.6 (Urysohn's metrization theorem). *Every Hausdorff second-countable regular space is metrizable.*

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be topological spaces and $f : X \to Y$. Then f is called *continuous* if the preimage $f^{-1}(O) \in \mathscr{O}_X$ for all $O \in \mathscr{O}_Y$. The set of all continuous functions on $f : X \to \mathbb{C}$ is denotes by $C(X)$.

We finish the section with Urysohn's lemma and Tietze's extension Theorem. For $K \subseteq X$ compact and $f \in C_c(X)$ with $0 \leq f \leq 1$, we write $K \prec f$ if $f(x) = 1$ for all $x \in K$. For $V \subseteq X$ open and $f \in C_c(X)$ with $0 \leq f \leq 1$, we write $f \prec V$ if $\operatorname{supp}(f) \subseteq \overline{V}$.

LEMMA A.7 (Urysohn's lemma). *Let X be a compact metric space, $V \subseteq X$ open and $K \subseteq X$ compact such that $K \subseteq V$. Then there exists an $f \in C_c(X)$ such that*

$$K \prec f \prec V.$$

PROOF. Let $F := X \setminus V$. Then $F \cap K = \varnothing$. Define

$$f(x) := \frac{dist(x, F)}{dist(x, K) + dist(x, F)}$$

where

$$dist(x, F) := \inf\{d(x, y) \, : \, y \in F\}.$$

It is an easy exercise to show that $dist(\cdot, F)$ is a continuous function and by construction, we have $supp(f) \subseteq V$. Furthermore, $f(x) = 1$ whenever $x \in K$. □

THEOREM A.8 (Tietze's extension theorem). *Let X be a normal topological space and $f : A \to \mathbb{R}$ be continuous on the closed set $A \subseteq X$. Then there exists a continuous map $F : X \to \mathbb{R}$ with $f(x) = F(x)$ for all $x \in A$. Moreover, F can be chosen such that*

$$\sup\left\{|f(x)| \,\middle|\, x \in A\right\} = \sup\left\{|F(y)| \,\middle|\, y \in X\right\}.$$

APPENDIX B

A primer on measure theory

Let X be a non-empty set. A measure is a map that assigns to subsets a non-negative number which can be thought as the volume (weight, energy,...). In order to guarantee nice properties one has to make restrictions on the set of measurable sets.

A family $\mathcal{A}$ of subsets of X is called *σ-algebra* if

- $\varnothing \in \mathcal{A}$,
- if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$,
- if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

A set $A \in \mathcal{A}$ is called *measurable.*

EXAMPLE. *In the following few examples for σ-algebras are given.*

(a) *The trivial system $\mathcal{A} := \{\varnothing, X\}$.*
(b) *The power set $\mathcal{A} := \mathcal{P}(X)$.*
(c) *Let X be a topological space. Then, the smallest (with respect to set inclusion) σ-algebra that contains all open sets of X is called the* Borel-*σ-algebra on X. It is denoted by $\mathscr{B}(X)$.*

We call a map $\mu : \mathcal{A} \to [0, \infty]$ a *measure*, if

- $\mu(\varnothing) = 0$,
- $\mu\big(\bigcup_n A_n\big) = \sum_n \mu(A_n)$ if $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint elements of the σ-algebra $\mathcal{A}$.

A triple $(X, \mathcal{A}, \mu)$ is called a *measure space* if $\mathcal{A}$ is a σ-algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. A measurable space $(X, \mathcal{A}, \mu)$ is called *finite* if $\mu(X) < \infty$. The measure μ is called a *probability measure* if $\mu(X) = 1$.

We say a property holds *μ-almost everywhere(μ-a.e.)* if there is a measurable set $X_0 \subseteq X$ with $\mu(X_0) = 0$ such that the property holds for all $x \in X \setminus X_0$.

EXAMPLE. (a) *Let X be a countable set equipped with the σ-algebra $\mathcal{P}(X)$. Then, every function $\mu : X \to [0, \infty]$ defines a measure via $\mu(A) := \sum_{x\in A} \mu(x)$.*
(b) *The Lebesgue-measure λ on $\mathscr{B}(\mathbb{R}^d)$. It is the unique translation invariant measure λ (i.e. $\lambda(A) = \lambda(\{x + y : y \in A\})$ for each $x \in A$) that satisfies*

$$\lambda([a_1, b_1) \times \ldots \times [a_d, b_d)) = (b_1 - a_1) \cdot \ldots \cdot (b_d - a_d).$$

LEMMA B.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, the following holds.*

(a) *If $A_j \in \mathcal{A}$ with $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1) < \infty$, then $\mu(\bigcap_{j\in\mathbb{N}} A_j) = \lim_{n\to\infty} \mu(A_n)$.*
(b) *If $A_j \in \mathcal{A}$ with $A_1 \subset A_2 \subset \ldots$, then $\mu(\bigcup_{j\in\mathbb{N}} A_j) = \lim_{n\to\infty} \mu(A_n)$.*

Let $(X_1, \mathcal{A}_1, \mu_1)$, $(X_2, \mathcal{A}_2, \mu_2)$ be measure spaces. A function

$$f : X_1 \to X_2$$

is called *measurable* if $f^{-1}(A) \in \mathcal{A}_1$ holds for all $A \in \mathcal{A}_2$.

If X is a topological space then the σ-algebra $\mathscr{B}(X)$ generated by the open sets in X is called the Borel-σ-algebra. Then it follows that continuous functions $f : X \to Y$ on topological spaces X and Y are measurable in the corresponding Borel-σ-algebras.

Recall that $f + g$, λf and $f \cdot g$ are measurable if f and g are measurable functions. In addition, the minimum, maximum, supremum or infimum of measurable functions are measurable.

Let $(X, \mathcal{A}, \mu)$ be a measure space. A function $\varphi : X \to \mathbb{R}$ is called *simple* if there are $a_1, \ldots a_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{A}$ such that $\varphi = \sum_{i=1}^n a_i 1_{A_i}$. We define the integral of a simple function by

$$\int_X \varphi \, d\mu := \int_X \varphi(x) \, d\mu(x) := \sum_{i=1}^n a_i \mu(A_i).$$

It can be shown that this is independent of the representation of φ.

For a measurable function $f : X \to [0, \infty)$ (the latter equipped with the Borel-σ-algebra) we define the integral

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \sup \left\{ \int_X \varphi \, d\mu \, : \, 0 \le \varphi \le f, \, \varphi \text{ simple} \right\}.$$

A measurable function $f : X \to \mathbb{C}$ is called *integrable* if $\int_X |f| \, d\mu < \infty$. If$f : X \to \mathbb{R}$ is integrable, the integral of f is defined as

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \int_X f_+ \, d\mu - \int_X f_- \, d\mu,$$

where $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$. Similarly, let $f : X \to \mathbb{C}$ be integrable, then the integral of f is defined by

$$\int_X f \, d\mu := \int_X f(x) \, d\mu(x) := \int_X \Re(f) \, d\mu + \imath \int_X \Im(f) \, d\mu,$$

For $p \in [1, \infty)$ we define

$$\mathcal{L}^p(X) = \{ f : X \to \mathbb{R} : f \text{ measurable}, \, \|f\|_p < \infty \}, \qquad \|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Note, that $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(X)$ but generally not a norm. However, it is a norm on the space $L^p(X) = \mathcal{L}^p(X)/\sim$, where $\sim$ is the equivalence relation "equality μ-almost everywhere". The pair $(L^p(X), \|\cdot\|_p)$ is a Banach space.

PROPOSITION B.2 (Hölder's inequality). *Let $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(X, \mathcal{A}, \mu)$ be a measurable space. If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, then $fg \in L^1(X, \mu)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

hold.

PROPOSITION B.3. *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space. Then*

$$\|f\|_p \le \mu(X)^{\frac{q-p}{qp}} \|f\|_q, \qquad f \in L^q(X,\mu),$$

and $L^q(X,\mu) \subseteq L^p(X,\mu)$ *hold for all* $1 \le p < q \le \infty$

Next, we will shortly recall some theorems on integrals and limits.

THEOREM B.4 (Fatou). *Let* $f_n : X \to [0,\infty]$ *be measurable functions. If* $\liminf_{n\to\infty} f_n$ *is measurable, then*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

THEOREM B.5 (Beppo Levi). *Let* $f_n : X \to [0,\infty)$ *be measurable functions with* $f_n \le f_{n+1}$ *for every* n μ*-almost everywhere. Denote the (*μ*-almost everywhere) limit of* $\{f_n\}_{n\in\mathbb{N}}$ *by* f*. Then,* f *is measurable and*

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

THEOREM B.6 (Lebesgue's dominated convergence theorem). *Let* $f_n : X \to \mathbb{C}$*,* $n \in \mathbb{N}$*, be measurable functions with* $|f_n| \le g$ μ*-almost everywhere for an integrable function* $g : X \to [0,\infty]$*. Denote the (*μ*-almost everywhere) limit of* $\{f_n\}_{n\in\mathbb{N}}$ *by* f*. Then,* f *is integrable and satisfies*

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

APPENDIX C

A primer on complex analysis

Here we collect the facts from complex analysis needed for the lecture.

A function $f : U \longrightarrow \mathbb{C}$ from an open subset $U \subseteq \mathbb{C}$ is called *analytic* or *holomorphic* if it is complex differentiable, i.e., there exists a continuous function $f' : U \longrightarrow \mathbb{C}$ such that for all $z_0 \in U$

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \eta(z - z_0)$$

where $\eta(w) - |w| \to 0$ as $w \to 0$ (where the choice of η also depends on z_0). It is remarkable that being analytic is already equivalent to f being arbitrarily often complex differentiable and, moreover, f equals its Taylor expansion about every z_0 in some neighborhood $V \subseteq U$ of z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{\infty} a_n z^n$$

for $z \in V$ and a suitable sequence (a_n) in $\mathbb{C}$ (which can be directly calculated from the complex derivatives $f^{(n)}(z_0)$).

For an integrable function $f : U \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ and a (piecewise) continuously differentiable curve $\gamma : I \longrightarrow \mathbb{C}$ from an interval I we denote

$$\int_\gamma f(z)dz := \int_I f(\gamma(t))\gamma'(t)dt.$$

A first theorem says that the integral of a function over a simply closed curve in a domain of analyticity of the function vanishes. A curve is called simply closed if it is homotopically equivalent to a trivial curve.

THEOREM C.1. *Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function on an open set $U \subseteq \mathbb{C}$. For any simply closed curve γ in U we have*

$$\int_\gamma f(z)dz = 0.$$

An even more fundamental result is Cauchy's integral formula. Recall that $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $rA = \{ra \mid a \in A\}$ for any $A \subseteq \mathbb{C}$ and $r \in \mathbb{C}$. By the integral $\int_{\partial r\mathbb{D}} \ldots dz$ we mean the integral $\int_\gamma \ldots dz$ along the simply closed curve γ which goes counterclockwise along the boundary of $r\mathbb{D}$.

THEOREM C.2 (Cauchy's integral formula). *Let $U \subseteq \mathbb{C}$ be open such that $r\overline{\mathbb{D}} \subseteq U$ for some $r > 0$. Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function. Then, for all $z_0 \in r\mathbb{D}$*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial r\mathbb{D}} \frac{f(z)}{z - z_0} dz.$$

Example. *For example we get with the function* $f \equiv 1$, $z_0 = 0$ *and* $n \geq 0$ *for any* $r > 0$

$$\int_{\partial r\mathbb{D}} z^{n-1} dz = 2\pi i \delta_{n,0}.$$

A simple yet important consequence is Liouville's theorem. It says that every non-constant entire function has to be unbounded. An *entire* function is an analytic function $f : \mathbb{C} \longrightarrow \mathbb{C}$. We provide a short proof here as it follows immediately from Cauchy's integral formula

Theorem C.3 (Liouville's theorem). *Every bounded entire function is constant.*

Proof. Let $C \geq 0$ such that $|f| \leq C$. We calculate using Cauchy's integral formula

$$|f'(z_0)| = \frac{1}{2\pi} \left| \int_{\partial r\mathbb{D}} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{C}{r^2} \to 0$$

as $r \to \infty$. This finishes the proof. □

Another remarkable theorem is that if two holomorphic functions agree on a large enough set, then they agree everywhere. Indeed, the coincidence set can be chosen rather small.

Theorem C.4 (Identity theorem). *If two holomorphic functions* f *and* g, *defined on an open and connected* $U \subseteq \mathbb{C}$, *agree on a set* $S \subseteq U$ *which has an accumulation point in* U, *then* $f = g$ *on* U.

APPENDIX D

A primer on functional analysis

1. The Hahn-Banach theorem

The theorem of Hahn-Banach is one of the most fundamental and most useful theorems in functional analysis.

Similar to Tietze's continuation theorem for continuous functions it allows to find a proper extension of a continuous function. However, the category here are topological vector spaces, so one does not only look for a continuous extension but for an extension which is additionally linear.

This theorem is proven in every course on functional analysis and its proof is a fairly simple application of Zorn's lemma.

Theorem D.1 (Hahn-Banach theorem). *Let $(E, \|\cdot\|)$ be a normed space and $F \subseteq E$ be a linear subspace. Let $\varphi : F \longrightarrow \mathbb{C}$ be a linear functional. Then, there exist a linear functional $\psi : E \longrightarrow \mathbb{C}$ such that*

$$\psi|_F = \varphi \qquad \text{and} \qquad \|\psi\| = \|\varphi\|.$$

There is an important corollary of the Hahn-Banach theorem.

Corollary D.2 (Realizing the norm by functionals). *Let X be a normed space and $X' := \mathcal{L}(X, \mathbb{C})$ be the dual space of continuous linear functionals on X. Then, for all $x \in X$*

$$\|x\| = \sup_{\varphi \in X',\ \|\varphi\|=1} |\varphi(x)| = \sup_{\varphi \in X',\ \|\varphi\|\leq 1} |\varphi(x)|.$$

Proof. Without loss of generality we can suppose that $x \neq 0$. Let $\varphi' \in X'$ be arbitrary with $\|\varphi'\| \leq 1$. Then

$$|\varphi'(x)| = \frac{|\varphi'(x)|}{\|x\|}\|x\| \leq \|\varphi'\|\|x\| \leq \|x\|$$

implies

$$\sup_{\varphi' \in X', \|\varphi'\|=1} |\varphi'(x)| \leq \sup_{\varphi' \in X', \|\varphi'\|\leq 1} |\varphi'(x)| \leq \|x\|$$

If we show that there is a $\varphi \in X'$ with $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$ then

$$\sup_{\varphi' \in X', \|\varphi'\|=1} |\varphi'(x)| \geq |\varphi(x)| = \|x\|$$

follows proving the desired statement. Such a φ can be found by using the Hahn-Banach Theorem. Let $x \in X$ and define $\psi : \operatorname{span}(x) \to \mathbb{C}$ by $\psi(\lambda x) := \lambda \|x\|$. Then

$$\|\psi\| = \sup_{y \in \operatorname{span}(x),\, \|y\|\leq 1} \|\psi(y)\| = \sup_{|\lambda| \leq \frac{1}{\|x\|}} \|\psi(\lambda x)\| = 1.$$

Due to the Hahn-Banach theorem, there is a $\varphi \in X'$ such that $\varphi|_{\mathrm{span}(x)} = \psi$ and $\|\varphi\| = \|\psi\| = 1$. Furthermore, $\varphi(x) = \psi(x) = \|x\|$ holds finishing the proof. □

Another look at this corollarys yield the following corollary.

COROLLARY D.3 (Separating points by functionals). *Let X be a normed space and $x \in X$. If $\varphi(x) = 0$ for all continuous linear functionals φ on X, then $x = 0$.*

PROOF. This follows immediately from the previous corollary as

$$\|x\| = \sup_{\varphi \in X',\ \|\varphi\|=1} |\varphi(x)| = 0$$

yields $x = 0$. □

2. The Stone-Weierstraß theorem

Consider the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A $\mathbb{K}$-vector space A is called $\mathbb{K}$-*algebra* if there is a bilinear, associative map $\cdot : A \times A \to A$, i.e., for all $a, b, c \in A$ and $\lambda \in \mathbb{K}$, we have

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- $(a + b) \cdot c = a \cdot c + b \cdot c$,
- $a \cdot (b + c) = a \cdot b + a \cdot c$,
- $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$.

A linear subspace $B \subseteq A$ of an algebra A is called *subalgebra* if $a, b \in B$ implies $a \cdot b \in B$. If A is equipped with a norm $\|\cdot\|$, then $(A, \cdot, \|\cdot\|)$ is called *Banachalgebra* if

- $(A, +, \|\cdot\|)$ is a Banach space,
- $(A, \cdot)$ is a $\mathbb{K}$-algebra,
- $\|a \cdot b\| \leq \|a\|\,\|b\|$ holds for all $a, b \in A$. (submultiplicative)

EXAMPLE. *Let X be a locally compact space and $C_0(X)$, the set of continuous functions $f : X \to \mathbb{C}$ that vanish at infinity, namely for all $\varepsilon > 0$ there exists an $K_\varepsilon \subseteq X$ compact such that*

$$|f(x)| < \varepsilon \qquad \text{for all } x \in X \setminus K_\varepsilon.$$

Then $C_0(X)$ is a Banach algebra if equipped with the uniform norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|, \qquad f \in C_0(X)$$

and pointwise multiplication

$$(f \cdot g)(x) := f(x)g(x), \qquad f, g \in C_0(X),\ x \in X.$$

Let B be a subalgebra of $C_0(X)$. We say that B *separates the points* from X if for any $x, y \in X$ with $x \neq y$ there is $f \in B$ with $f(x) \neq f(y)$. Moreover, it is said that B does *not vanish anywhere* if for every $x \in X$ there is a $f \in B$ with $f(x) \neq 0$. Finally, B is called *self-adjoint* if $f \in B$ implies $\overline{f} \in B$.

THEOREM D.4 (Stone-Weierstrass theorem). *Let X be locally compact and let B be subalgebra of $C_0(X)$ with the following properties:*

- *B separates the points,*
- *B does not vanish anywhere,*

- *B is self-adjoint.*

Then, B is dense in $C_0(X)$ with respect to $\|\cdot\|_\infty$.

3. The Riesz-Markov theorem

Next, we discuss the Riesz-Markov theorem which relates the positive functionals with Radon measures on locally compact spaces.

THEOREM D.5 (Riesz-Markov representation theorem). *Let X be locally compact. Then, for any positive linear functional φ on $C_0(X)$ there exists a unique finite regular measure μ on X such that*

$$\varphi(f) = \int f d\mu$$

for all $f \in C_0(X)$ and $\|\varphi\| = \mu(X)$.

APPENDIX E

Spectrum of bounded linear operators

We introduce the notions of spectrum, resolvent set of bounded linear operators on Banach spaces and prove basic properties of these objects. Therefore, recall what we discussed in Section 2.3 "Linear bounded maps". Afterwards we will focus on linear bounded operators on a Hilbert space H and in particular on self-adjoint operators. We discuss consequences on the convergence of the spectra w.r.t. the Hausdorff metric d_H on $\mathbb{C}$ if $\mathcal{L}(H)$ is equipped with various topologies.

1. Spectrum and resolvent set

Let E be a Banach space. Let $I := I_E \in \mathcal{L}(E)$ be the identity operator on E, namely, $Ix := x$. In the following we will write $AB := A \circ B$ for $A \in \mathcal{L}(E_1, E_2)$ and $B \in \mathcal{L}(E_2, E_3)$. Recall that $\mathcal{L}(E) := \mathcal{L}(E, E)$.

Definition. *An operator $A \in \mathcal{L}(E,F)$ is called* invertible *if there is a $B \in \mathcal{L}(F,E)$ such that $AB = I_F$ and $BA = I_E$. Then B is called the* inverse *of A that we denote by A^{-1}.*

Proposition E.1 (Basic properties of invertible elements). *Let E be a Banach space and $A, B, C \in \mathcal{L}(E)$. Then the following holds.*

(a) *If $AB = I$ and $CA = I$, then $B = C$ and A is invertible with inverse $B = C$. In particular, the inverse of an operator is unique.*
(b) *If A and B are invertible then AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.*
(c) *If AB and BA are invertible, then A and B are invertible.*
(d) *If A and AB are invertible, then BA and B are invertible.*
(e) *For $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, the operator λI is invertible with inverse $\lambda^{-1} I$.*

Proof. (a) This follows from

$$B = IB = (CA)B = C(AB) = CI = C$$

(b) Clearly $B^{-1}A^{-1} \in \mathcal{L}(H)$. Furthermore,

$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$$

and similarly $(B^{-1}A^{-1})(AB) = I$.

(c) Clearly it suffices to show that A is invertible. Set $C := (BA)^{-1}B$ and $D := B(AB)^{-1}$. Then a short computation leads to

$$CA = (BA)^{-1}(BA) = I \qquad \text{and} \qquad AD = (AB)(AB)^{-1} = I.$$

Hence A is invertible with inverse $C = D$ by (a).

(d) Since A and AB are invertible,

$$BA = A^{-1}(AB)A$$

is invertible as composition of invertible elements by (b). Thus, AB and BA are invertible and so B is invertible by (c).

(e) This is trivial. □

EXAMPLE. *If $E := \mathbb{C}^d$ for some $d \in \mathbb{N}$, then $\mathcal{L}(E)$ is nothing but than all $d \times d$-matrices. Then $A \in \mathcal{L}(E)$ is invertible if the matrix is invertible.*

EXAMPLE. *Let $E := \ell^2(\mathbb{N})$, $k \in \mathbb{N}$ and consider the operator $A_k \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by $A_k := \widehat{\chi_{[0,k]}}$, namely the multiplication operator where $\chi_{[0,k]} : \mathbb{N} \to \{0,1\}$ is the characteristic function of the interval $[0,k]$. More precisely, we have*

$$(A_k\psi)(n) = \begin{cases} 0, & n > k, \\ \psi(n), & n \le k. \end{cases}$$

Then $A_k\delta_{k+1} = 0$ and $A_k\delta_{k+2} = 0$ while $\delta_{k+1} \neq \delta_{k+2}$. Thus, A_k is not injective and so it is in particularly not invertible. Another way to see that A_k is not invertible is by taking any linear bounded operator $B \in \mathcal{L}(\ell^2(\mathbb{N}))$. Then $B0 = 0$ and so $BA_k\delta_{k+1} = 0 \neq \delta_{k+1}$ follows. Thus, A_k does not admit an inverse in $\mathcal{L}\big(\ell^1(\mathbb{N})\big)$. Note that A_k is actually a projection, i.e. $A_k^2 = A_kA_k = A_k$.

The operator $A_c \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by

$$(A_c\psi)(n) = c^n\psi(n)$$

is invertible if and only if $c \neq 0$. Furthermore, its inverse is given by $A_{c^{-1}}$.

DEFINITION. *Let E be a Banach space. For $A \in \mathcal{L}(E)$, define the* resolvent set *by*

$$\rho(A) := \{z \in \mathbb{C} \,|\, A - zI \text{ bijective with continuous inverse}\}$$

and its spectrum

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

The map

$$R_A : \rho(A) \to \mathcal{L}(E), \quad R_A(z) := (A - zI)^{-1},$$

is called resolvent.

REMARK. *(a) In classical mechanics, the possible results of measurements are given precisely by the possible values of the observable functions. In quantum mechanics, the spectrum of an operator (i.e., the generalized eigenvalues) appears as the possible results of measurements of the associated observables. This corresponds to the transition from the spectrum in commutative algebras to the spectrum in non-commutative algebras.*

(b) About the term spectrum*: In optics, acoustics and harmonic analysis it is common practice to decompose objects in eigenfunctions (i.e., waves) to eigenvalues (i.e., frequencies). This is known as the spectral decomposition or as frequency analysis.*

(c) About the term resolvent*: For a bounded operator $A \in \mathcal{L}(E)$ on a Banach space, one has $\lambda \in \rho(A)$ if and only if the equation $(A-\lambda I)x = y$ has a unique solution for all y, i.e., the equation can be uniquely* resolved *with regard to*

x. In light of this we seek to figure out when we can solve such "generalized equation systems".

(d) In the lecture "Functional Analysis" you will get to know the bounded inverse theorem (also called inverse mapping theorem). This states that every bijective $A \in \mathcal{L}(E)$ admits an inverse $A^{-1} \in \mathcal{L}(E)$ which is bounded where E is some Banach space. Note that this result is wrong if E is just a normed space.

(e) With a slight abuse of notation, we will use $\lambda := \lambda I$ for all $\lambda \in \mathbb{C}$.

EXAMPLE (Operators on a finite-dimensional space). *For a matrix $A \in \mathbb{C}^{n\times n}$, we have*

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(A - \lambda) = 0\} = \{\textit{eigenvalues of } A\}.$$

EXAMPLE. *Let $E := \ell^2(\mathbb{N})$, $k \in \mathbb{N}$ and consider the operator $A_k \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by $A_k := \widehat{\chi_{[0,k]}}$ as in the previous example. Then $\sigma(A_k) = \{0, 1\}$ for all $k \in \mathbb{N}$ (**Exercise**).*

PROPOSITION E.2 (Neumann series). *Let E be a Banach space and $A \in \mathcal{L}(E)$ with $\limsup_{n\to\infty} \|A^n\|^{1/n} < 1$ be given. Then $(I - A)$ is invertible and*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

In particular, $(I - A)$ is invertible if $\|A\| < 1$.

PROOF. The assumption $\limsup_{n\to\infty} \|A^n\|^{1/n} < 1$ yields the existence of $0 < \theta < 1$ with $\|A^n\| \le \theta^n$ for large n. We know that $\sum_{n\ge 0} \theta^n$ converges (geometric series). Hence, $\sum_{n\ge m} \theta^n$ gets arbitrarily small if m gets large. Thus, we conclude that $S_k := \sum_{n=1}^{k} A^n \in \mathcal{L}(E)$, $k \in \mathbb{N}$, is a Cauchy-sequence invoking the triangle inequality. Therefore, the sum $\sum_{n\ge 0} A^n$ converges absolutely and exists due to completeness of $\mathcal{L}(E)$, namely $\sum_{n\ge 0} A^n \in \mathcal{L}(E)$. Moreover,

$$(I - A)\sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = A^0 = I$$

and analogously

$$\left(\sum_{n=0}^{\infty} A^n\right)(I - A) = \sum_{n=0}^{\infty} A^n - \sum_{n=1}^{\infty} A^n = A^0 = I.$$

The last statement follows from $\|A^n\| \le \|A\|^n$. □

REMARK. *(a) The formula should not come as a surprise. It is well known that the geometric series is an inverse of $1 - q$ for $|q| < 1$. As in analysis the geometric series, the Neumann series will play a crucial role in spectral theory.*

(b) If $\limsup_{n\to\infty} \|A^n\|^{1/n} < 1$, then $(I + A)$ is also invertible with inverse

$$(I + A)^{-1} = \sum_{k=0}^{\infty} (-A)^n.$$

(c) The identity operator $I \in \mathcal{L}(E)$ is always invertible. In this sense the lemma is a stability result.

PROPOSITION E.3. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then the resolvent set $\rho(A) \subseteq \mathbb{C}$ is open and the resolvent R_A is analytic (i.e. it can be locally developed into a norm-convergent power series). In particular, R_A is continuous.*

PROOF. *$\rho(A)$ is open:* We prove that for every $z_0 \in \rho(A)$, there is an $\varepsilon > 0$ such that the open ball $B_\varepsilon(z_0) \subseteq \rho(A)$.

Let $z_0 \in \rho(A)$. Then $(A - z_0)$ is invertible and $A - z_0 \in \mathcal{L}(E)$. Then we get

$$A - z = (A - z_0) + (z_0 - z) = \big(I + (z_0 - z)(A - z_0)^{-1}\big)(A - z_0).$$

Since $A - z_0$ is invertible, we get that $A - z$ is invertible whenever $I + (z_0 - z)(A - z_0)^{-1}$ is invertible by Proposition E.1 (Basic properties of invertible elements). Invoking the previous Proposition E.2 (Neumann series), we get invertibility whenever

$$|z - z_0| \, \|(A - z_0)^{-1}\| = \|(z - z_0)(A - z_0)^{-1}\| < 1$$

Since $A - z_0$ is invertible, $\|(A - z_0)^{-1}\|$ is finite. Set $\varepsilon := \frac{1}{2\|(A-z_0)^{-1}\|}$. Then for all $z \in \mathbb{C}$ with $|z - z_0| < \varepsilon$, we conclude

$$\|(z - z_0)(A - z_0)^{-1}\| \le |z - z_0| \, \|(A - z_0)^{-1}\| \le \frac{1}{2} < 1.$$

Hence, $A - z$ is invertible with bounded inverse for all $z \in \mathbb{C}$ with $|z - z_0| < \varepsilon$, namely $\rho(A)$ is open.

R_A is analytic: We will show that R_A can be developed around each $z_0 \in \rho(A)$ into a norm-convergent power series. Let $z_0 \in \rho(A)$ and $\varepsilon := \frac{1}{2\|(A-z_0)^{-1}\|}$ as before. Consider some $z \in \mathbb{C}$ with $|z - z_0| < \varepsilon$. Then

$$(I - (z_0 - z)(A - z_0)^{-1})^{-1} = \sum_{n=0}^{\infty} (z_0 - z)^n (A - z_0)^{-n}$$

follows by the Proposition E.2 (Neumann series). Thus,

$$\begin{aligned}
(A - z)^{-1} &= \Big(\big(I + (z_0 - z)(A - z_0)^{-1}\big)(A - z_0)\Big)^{-1} \\
&= (A - z_0)^{-1} \left(\sum_{n=0}^{\infty} (z_0 - z)^n (A - z_0)^{-n} \right) \\
&= \sum_{n=0}^{\infty} (z - z_0)^n (A - z_0)^{-n-1}
\end{aligned}$$

follows using Proposition E.1 (Basic properties of invertible elements). Thus $(A - z)^{-1}$ is a norm convergent power series locally around z_0. Hence, the resolvent is analytic on $\rho(A)$.

R_A is continuous: The continuity follows, since the resolvent is analytic. Here are the details. By the previous considerations,

$$(A - z)^{-1} - (A - z_0)^{-1} = \sum_{n=1}^{\infty} (z - z_0)^n (A - z_0)^{-n-1}$$

holds and the power series is norm convergent. Thus, there is a constant $C > 0$ such that the norm of this difference can be estimated by

$$\| \dots \| \le |z - z_0| \|(A - z_0)^{-2}\| \sum_{k=0}^{\infty} \|(z - z_0)(A - z_0)^{-1}\|^k \le |z - z_0| C < \delta C$$

for $z \in \mathbb{C}$ with $|z - z_0| < \delta$ and $\delta < \varepsilon$. This proves the continuity of the resolvent. $\square$

With this at hand, we can prove fundamental properties of the spectrum. For the proof, we need the Hahn-Banach theorem and Liouville's theorem. (See Appendix C and Appendix D for the statements and a short discussion.)

THEOREM E.4 (Basic properties of the spectrum). *Let $A \in \mathcal{L}(E)$. Then the spectrum $\sigma(A) \subseteq \mathbb{C}$ is compact and non-empty. In particular, $\sigma(A) \subseteq B_{\|A\|}(0)$ holds.*

PROOF. $\sigma(A)$ *compact:* Since $\sigma(A)$ is closed (as the resolvent set $\rho(A)$ is open by the previous Proposition E.3), it is sufficient for the compactness to show that $\sigma(A) \subseteq B_{\|A\|}(0)$. To this end, we consider $A - \lambda$ for $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$. Then

$$A - \lambda = (-\lambda)\left(I - \frac{1}{\lambda}A\right).$$

According to the Neumann series (Proposition E.2) $I - \frac{1}{\lambda}A$ is invertible since $|\lambda| > \|A\|$. Thus, $A - \lambda$ is invertible if $|\lambda| > \|A\|$ proving $\sigma(A) \subset B_{\|A\|}(0)$. In particular, $\sigma(A)$ is a compact subset of $\mathbb{C}$.

$\sigma(A)$ *non-empty:* Assume by contradiction that $\sigma(A) = \varnothing$ and so $\rho(A) = \mathbb{C}$. Then, $\lambda \longmapsto (A - \lambda)^{-1}$ is analytic on $\mathbb{C}$ and, thus, an entire function. This means that the map

$$\mathbb{C} \longrightarrow \mathbb{C}, \qquad \lambda \longmapsto \varphi((A - \lambda)^{-1}),$$

is an entire function for every continuous linear functional φ on $\mathcal{L}(E)$. More precisely, $\varphi((A - \lambda)^{-1})$ can be expanded locally to a absolute convergent power series by linearity of φ and since $R_A(\lambda) = (\lambda - A)^{-1}$ can be locally expanded to an norm convergent power series by Proposition E.3.

Using Proposition E.2 (Neumann series), we have

$$(A - \lambda)^{-1} = \frac{1}{-\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} = \frac{1}{-\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}A\right)^{n} \qquad (*)$$

for all $|\lambda| > \|A\|$. Hence,

$$\varphi\left((A - \lambda)^{-1}\right) = \varphi\left(\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{1}{\lambda}A\right)^{n}\right) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\varphi\left(A^n\right)$$

is concluded. With $|\varphi(A^n)| \le \|\varphi\|\|A\|^n$, it follows by the geometric series that

$$|\varphi((\lambda - A)^{-1})| \le \frac{1}{|\lambda|}\|\varphi\|\left(\frac{1}{1 - \frac{\|A\|}{|\lambda|}}\right) \to 0 \qquad \text{if} \quad |\lambda| \to \infty.$$

Hence, the map $\mathbb{C} \to \mathbb{C}$, $\lambda \longmapsto \varphi((A-\lambda)^{-1})$ is a bounded entire function (the resolvent R_A vanishes at infinity and it is continuous by Proposition E.3 and so it is bounded on any compact set). With Liouville's theorem, we conclude that $\varphi\left((A - \lambda)^{-1}\right)$ is constant and, therefore, for all $\lambda \in \mathbb{C}$

$$\varphi((A - \lambda)^{-1}) = 0$$

since it must vanish already at infinity. Since φ was an arbitrary continuous functional on $\mathcal{L}(H)$, the Corollary D.3 (corollary of Hahn-Banach theorem) implies

$$(A-\lambda)^{-1} = 0$$

for every $\lambda \in \mathbb{C}$. However, this stands in contradiction to the basic fact

$$0 = (A-\lambda)0 = (A-\lambda)(A-\lambda)^{-1} = I \neq 0.$$

Thus, $\sigma(A) \neq \varnothing$ is concluded. □

Remark. *This is indeed a result "over $\mathbb{C}$" and which is not a coincidence: On $\mathbb{R}$, a corresponding result is already wrong for matrices (e.g. the rotation on $\mathbb{R}^2$ has no real eigenvalues). The proof makes extensive use of complex analysis in the form of Liousville's theorem. This also does not come as a surprise. Already for matrices one uses complex analysis, in terms of Liouville's theorem or Rousseau's theorem, to prove the fundamental theorem of algebra (every non-constant polynomial with complex coefficients has at least one complex root) which yields the existence of eigenvalues (in $\mathbb{C}$).*

Proposition E.5 (Resolvent identities). *Let E be a Banach space and $A, B \in \mathcal{L}(E)$. Then the following assertions hold.*

(a) *If $\lambda \in \rho(A) \cap \rho(B)$, then*

$$(A-\lambda)^{-1} - (B-\lambda)^{-1} = (A-\lambda)^{-1}(B-A)(B-\lambda)^{-1} = (B-\lambda)^{-1}(B-A)(A-\lambda)^{-1}.$$

(b) *If $\lambda, \mu \in \rho(A)$, then*

$$(A-\lambda)^{-1} - (A-\mu)^{-1} = (\lambda-\mu)(A-\lambda)^{-1}(A-\mu)^{-1} = (\mu-\lambda)(A-\mu)^{-1}(A-\lambda)^{-1}.$$

Proof. (a) Since $\lambda \in \rho(A) \cap \rho(B)$, the elements $(A-\lambda)$ and $(B-\lambda)$ are invertible. Then a short computation leads to

$$\begin{aligned}(A-\lambda)^{-1} - (B-\lambda)^{-1} &= (A-\lambda)^{-1}(B-\lambda)(B-\lambda)^{-1} - (A-\lambda)^{-1}(A-\lambda)(B-\lambda)^{-1}\\ &= (A-\lambda)^{-1}\big(B-\lambda-(A-\lambda)\big)(B-\lambda)^{-1}\\ &= (A-\lambda)^{-1}(B-A)(B-\lambda)^{-1}.\end{aligned}$$

Similarly, we derive

$$\begin{aligned}(A-\lambda)^{-1} - (B-\lambda)^{-1} &= (B-\lambda)^{-1}(B-\lambda)(A-\lambda)^{-1} - (B-\lambda)^{-1}(A-\lambda)(A-\lambda)^{-1}\\ &= (B-\lambda)^{-1}\big(B-\lambda-(A-\lambda)\big)(A-\lambda)^{-1}\\ &= (B-\lambda)^{-1}(B-A)(A-\lambda)^{-1}.\end{aligned}$$

(b) Let $\lambda, \mu \in \rho(A)$. Thus, $(A-\lambda)$ and $(A-\mu)$ are invertible. A similar computation as before leads to

$$\begin{aligned}(A-\lambda)^{-1} - (A-\mu)^{-1} &= (A-\lambda)^{-1}(A-\mu)(A-\mu)^{-1} - (A-\lambda)^{-1}(A-\lambda)(A-\mu)^{-1}\\ &= (A-\lambda)^{-1}\big(A-\mu-(A-\lambda)\big)(A-\mu)^{-1}\\ &= (\lambda-\mu)(A-\lambda)^{-1}(A-\mu)^{-1}.\end{aligned}$$

From this identity we deduce

$$(A-\lambda)^{-1}(A-\mu)^{-1} = \frac{\big((A-\lambda)^{-1}-(A-\mu)^{-1}\big)}{\lambda-\mu}$$
$$= \frac{(A-\mu)^{-1}-(A-\lambda)^{-1}}{\mu-\lambda} = (A-\mu)^{-1}(A-\lambda)^{-1}.$$

□

PROPOSITION E.6 (Transformation and the spectrum). *Let $A \in \mathcal{L}(E)$ and $U \in \mathcal{L}(E)$ be such that U is invertible. Then*

$$\sigma(A) = \sigma(UAU^{-1}) \quad \text{and} \quad \rho(A) = \rho(UAU^{-1}).$$

PROOF. Clearly $\sigma(A) = \sigma(UAU^{-1})$ is equivalent to $\rho(A) = \rho(UAU^{-1})$. Thus, we will only show $\rho(A) = \rho(UAU^{-1})$. First note that $\lambda \in \mathbb{C}$ is an element of $\rho(A)$ if and only if $A-\lambda$ is invertible in $\mathcal{L}(E)$. Since U is invertible, this is equivalent to $U(A-\lambda)U^{-1}$ is invertible by Proposition E.1 (Basic properties of invertible elements). Then the identity

$$U(A-\lambda)U^{-1} = UAU^{-1} - \lambda UU^{-1} = UAU^{-1} - \lambda$$

finishes the proof. □

EXERCISE. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Prove that if p is a polynomial with complex coefficients then*

$$\sigma(p(A)) = \{p(\lambda) \mid \lambda \in \sigma(A)\}.$$

2. Spectral radius

We can say more about the location of the spectrum in terms of the operator norm. But we need some preparation.

LEMMA E.7 (Fekete's lemma). *Let $\alpha : \mathbb{N} \longrightarrow [0,\infty)$ be submultiplicative i.e. $\alpha_{n+m} \le \alpha_n \alpha_m$ for all $n, m \in \mathbb{N}$. Then,*

$$\lim_{n\to\infty} \alpha_n^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \alpha_n^{\frac{1}{n}}.$$

In particular the limit $\lim_{n\to\infty} \alpha_n^{\frac{1}{n}}$ exists.

PROOF. If $\alpha_k = 0$ for some $k \in \mathbb{N}$, then $0 \le \alpha_{k+n} \le \alpha_k \alpha_n = 0$ for all $n \ge 0$. This yields the statement.

We now consider the case $\alpha_k > 0$ for all $k \in \mathbb{N}$. Apparently,

$$\inf_{n\in\mathbb{N}} \alpha_n^{\frac{1}{n}} \le \liminf_{n\to\infty} \alpha_n^{\frac{1}{n}} \le \limsup_{n\to\infty} \alpha_n^{\frac{1}{n}}.$$

So it is enough to show for every $\alpha > \inf_n \alpha_n^{\frac{1}{n}}$

$$\limsup_{n\to\infty} \alpha_n^{\frac{1}{n}} \le \alpha$$

Choose such an α. Then, there exists a $m \in \mathbb{N}$ with

$$\alpha_m^{\frac{1}{m}} < \alpha.$$

Now, let $n \in \mathbb{N}$ be large. Then, there are $k \in \mathbb{N}$ and $r \in \mathbb{N}$ with $0 \le r < m$ such that

$$n = km + r.$$

Using the submultiplicativity, we obtain $\alpha_n \le \alpha_m^k \alpha_r$ and so

$$\alpha_n^{\frac{1}{n}} \le \alpha_m^{\frac{k}{n}} \alpha_r^{\frac{1}{n}}.$$

We have $\alpha_r^{\frac{1}{n}} \to 1, n \to \infty$ (since there are only finitely many $0 \le r < m$ and $\alpha_r > 0$) and $\alpha_m^{\frac{k}{n}} \to \alpha_m^{\frac{1}{m}}$ (because $\frac{k}{n} = \frac{k}{km+r} \to \frac{1}{m}$). This implies

$$\limsup_{n\to\infty} \alpha_n^{\frac{1}{n}} \le \alpha_m^{\frac{1}{m}} < \alpha. \qquad \square$$

Remark (Submultiplicative vs. subadditive). *The statement of the proposition can be reformulated as a statement about subadditive functions. For some subadditive function $\beta : \mathbb{N} \longrightarrow \mathbb{R}$ (i.e. $\beta_{n+m} \le \beta_n + \beta_m$), we conclude the existence of*

$$\lim_{n\to\infty} \frac{1}{n}\beta_n = \inf_{n\in\mathbb{N}} \frac{1}{n}\beta_n,$$

by setting $\alpha_n := e^{\beta_n}$ and applying the proposition. Reversely, given the result on subadditve functions, one can set $\beta_n := \log \alpha_n$ and recover statement for submultiplicative α_n.

Corollary E.8. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then the limit $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists.*

Proof. The sequence $\alpha_n := \|A^n\|$, $n \in \mathbb{N}$, is submultiplicative as

$$\|A^{n+m}\| = \|A^n A^m\| \le \|A^n\| \; \|A^m\|.$$

So, Fekete's lemma (Lemma E.7) leads to the desired statement. $\square$

Definition (Spectral radius). *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then*

$$r(A) := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

is called the spectral radius *of A.*

We will often make use of the fact that continuity of multiplication in $\mathcal{L}(E)$ yields that $r(A)$ is in general smaller than $\|A\|$. Later we will see that $\|A\| = r(A)$ if $A \in \mathcal{L}(H)$ is a so-called normal operator on a Hilbert space H.

Lemma E.9. *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then,*

$$r(A) \le \|A\|.$$

Proof. We have $\|AB\| \le \|A\| \; \|B\|$, $A, B \in \mathcal{L}(E)$. This yields $\|A^n\| \le \|A\|^n$ and the result follows. $\square$

We now come to a theorem about the location of the spectrum which goes back to Beurling. The proof requires a fundamental theorem from complex analysis which is known as Cauchy's integral formula (see Appendix C).

The theorem of Beurling is remarkable because it connects algebra and topology. Algebra enters via invertibility in form of the spectrum and topology enters in form of the spectral radius.

Theorem E.10 (Location of Spectrum – Beurling's Theorem). *Let E be a Banach space and $A \in \mathcal{L}(E)$. Then,*

$$\max_{\lambda \in \sigma(A)} |\lambda| = r(A).$$

Remark. *The spectral radius $r(A)$ is the smallest number r such that $\sigma(A)$ is contained in $B_r(0)$. Hence, the name spectral radius.*

Proof. The spectrum is compact and the absolute value is continuous. Therefore, the maximum on the left hand side exists.

Every $\lambda \in \mathbb{C}$ with $|\lambda| > r(A)$ belongs to the resolvent: This follows directly from the Proposition E.2 (Neumann series) by writing

$$A - \lambda = (-\lambda)\left(I - \frac{1}{\lambda}A\right)$$

and observing

$$\lim_{n\to\infty} \left\| \left(\frac{1}{\lambda}A\right)^n \right\|^{\frac{1}{n}} = \frac{1}{|\lambda|} r(A) < 1.$$

Hence, $\max_{\lambda \in \sigma(A)} |\lambda| \le r(A)$ follows.

Every $r > \max_{\lambda\in\sigma(A)} |\lambda|$ satisfies $r \ge r(A)$: Let φ be a linear continuous functional on $\mathcal{L}(E)$. Then, the map

$$\lambda \longmapsto \varphi\left(\lambda^n (A - \lambda)^{-1}\right)$$

is analytic outside of $\sigma(A)$ by Proposition E.3. If $|\lambda| > \|A\|$, then the Neumann series (Proposition E.2) even implies

$$\lambda^n (A-\lambda)^{-1} = -\lambda^{n-1}\left(I - \frac{1}{\lambda}A\right)^{-1} = -\sum_{k=0}^{\infty} \lambda^{n-k-1} A^k.$$

Therefore, the integral

$$\int_{|\lambda|=r} \varphi\big(\lambda^n (A-\lambda)^{-1}\big) d\lambda$$

exists. Furthermore, we also have that all $\lambda \in \mathbb{C}$ with $|\lambda| = \|A\| + 1 > r(a)$ belong to the resolvent set by the first step. Thus, we get by complex analysis (see Appendix C) and analyticity of the map on the resolvent set

$$\begin{aligned}
\int_{|\lambda|=r} \varphi\left(\lambda^n (A-\lambda)^{-1}\right) d\lambda &= \int_{|\lambda|=\|A\|+1} \varphi(\lambda^n (A-\lambda)^{-1}) d\lambda \\
\text{(Neumann series)} \quad &= \int_{|\lambda|=\|A\|+1} \varphi\left(-\sum_{k=0}^{\infty} \lambda^{n-k-1} A^k\right) d\lambda \\
\text{(absolute convergence)} \quad &= -\sum_{k=0}^{\infty} \varphi(A^k) \int_{|\lambda|=\|A\|+1} \lambda^{n-k-1} d\lambda \\
\text{(Cauchy's integral formula)} \quad &= -\sum_{k=0}^{\infty} 2\pi i\ \delta_{n-k,0}\ \varphi(A^k) \\
&= -2\pi i \varphi(A^n).
\end{aligned}$$

By the virtue of the theorem of Hahn-Banach (Corollary D.2), we conclude

$$\begin{aligned}
\|A^n\| &= \sup_{\|\varphi\|=1} |\varphi(A^n)| \\
&= \frac{1}{2\pi} \sup_{\|\varphi\|=1} \left| \int_{|\lambda|=r} \varphi(\lambda^n (A-\lambda)^{-1}) d\lambda \right| \\
&\leq \frac{1}{2\pi} \sup_{\|\varphi\|=1} \|\varphi\| \int_{|\lambda|=r} |\lambda^n| \|(A-\lambda)^{-1}\| d\lambda \\
&\leq r r^n \sup_{|\lambda|=r} \|(A-\lambda)^{-1}\|.
\end{aligned}$$

Since $r > \max_{\lambda\in\sigma(A)} |\lambda|$ and the resolvent $\lambda \mapsto R_A(\lambda) = (A-\lambda)^{-1}$ is continuous (Proposition E.3), the supremum is finite. Thus, we infer

$$\|A^n\|^{1/n} \leq r r^{1/n} C^{1/n} \longrightarrow r$$

as $n \to \infty$ which yields the desired statement $r(A) \leq r$. □

Remark (Holomorphic functional calculus). *The formula*

$$\int_{|\lambda|=r} \varphi(\lambda^n (A-\lambda)^{-1}) d\lambda = -2\pi i \varphi(A^n).$$

is a weak version of what is known as Cauchy's integral formula. "Weak" means that it holds after application a continuous functional. This can be generalized to the so called holomorphic functional calculus which is however not a part of this lecture.

Recall that $R_A(z) = (A-z)^{-1}$ denotes the resolvent of an operator $A \in \mathcal{L}(E)$ and $z \in \rho(A)$.

Proposition E.11. *Let E be a Banach space, $A \in \mathcal{L}(E)$ and $z \in \rho(A)$. Then*

$$\sigma\big(R_A(z)\big) = \big(\sigma(A) - z\big)^{-1} := \left\{ \frac{1}{\lambda - z} \,\middle|\, \lambda \in \sigma(A) \right\} \tag{a}$$

and

$$r\big(R_A(z)\big) = \frac{1}{\operatorname{dist}(z, \sigma(A))} \tag{b}$$

hold.

Proof. We will first show that (a) implies (b): By Beurling's Theorem (Theorem E.10), we have

$$r\big(R_A(z)\big) = \max_{\lambda\in\sigma(R_A(z))} |z|.$$

Furthermore, $\big(\sigma(A) - z\big)^{-1}$ is nonempty and compact as $\sigma(A)$ is nonempty and compact by Theorem E.4 (Basic properties of the spectrum). Thus,

$$\frac{1}{\operatorname{dist}(z, \sigma(A))} = \frac{1}{\min_{\lambda\in\sigma(A)} |\lambda - z|} = \max_{\lambda\in\sigma(A)} \left| \frac{1}{\lambda - z} \right| = r\big(R_A(z)\big)$$

follows using (a).

In order to prove (a), let $\lambda \in \mathbb{C}$. There is no loss of generality in assuming that $\lambda \neq 0$ since $A - z \in \mathcal{L}(E)$ is the inverse of $R_A(z)$. Then a short computation leads to

$$R_A(z) - \lambda = R_A(z) - \lambda(A-z)R_A(z) = \big((1+\lambda z)I - \lambda A\big)R_A(z)$$
$$= \left(\left(z + \frac{1}{\lambda}\right)I - A\right)\lambda R_A(z).$$

By assumption $\lambda R_A(z)$ is invertible. Using Proposition E.1 (Basic properties of invertible elements) $R_A(z) - \lambda$ is invertible if and only if $\left(z + \frac{1}{\lambda}\right)I - A$ is invertible, if and only if $z_0 := z + \frac{1}{\lambda} \in \rho(A)$. This is equivalent to $\lambda = \frac{1}{z_0 - z}$ for some $z_0 \in \rho(A)$. Hence, $\lambda \in \rho(R_A(z))$ if and only if $\lambda = \frac{1}{z_0 - z}$ for some $z_0 \in \rho(A)$. We claim that from this, the desired identity follows. Here are the details:

$\underline{\sigma\big(R_A(z)\big) \supseteq \big(\sigma(A) - z\big)^{-1}}$: We will work with the complements of these sets. Let $\lambda \notin \sigma\big(R_A(z)\big)$, then $\lambda \in \rho\big(R_A(z)\big)$. Since $\lambda \neq 0$, we conclude that $\lambda = \frac{1}{z_0 - z}$ for some $z_0 \in \rho(A)$ by the previous considerations. Since $\rho(A) \cap \sigma(A) = \varnothing$, we deduce that $\lambda \notin \big(\sigma(A) - z\big)^{-1}$.

$\underline{\sigma\big(R_A(z)\big) \subseteq \big(\sigma(A) - z\big)^{-1}}$: We will work with the complements of these sets. Let $\lambda \notin \big(\sigma(A) - z\big)^{-1}$. If $\lambda = 0$, then clearly $\lambda \notin \sigma\big(R_A(z)\big)$. Thus suppose $\lambda \neq 0$. Then $\frac{1}{\lambda} \notin \sigma(A) - z$ follows or equivalently $z + \frac{1}{\lambda} \in \rho(A)$. Using again the previous considerations, we derive that $\lambda \in \rho(R_A(z))$, namely $\lambda \notin \sigma\big(R_A(z)\big)$. □

3. A short reminder on Hilbert spaces

We seek to study a specific class of operators that are called self-adjoint. In order to do so, we need to restrict to operators defined on Hilbert spaces. Therefore, a short introduction/reminder on Hilbert spaces and inner products is provided.

Let H be vector space over $\mathbb{C}$. A map $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ is an *inner product* if

- it is *complex linear (antilinear)* in the first and linear in the second component, namely
$$\langle \lambda x + y, z\rangle = \overline{\lambda}\langle x, z\rangle + \langle y, z\rangle\,, \qquad \langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle\,,$$
hold for all $x, y, z \in H$ and $\lambda \in \mathbb{C}$,
- it is *symmetric*, i.e., $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in H$,
- it is *positive-definite*, i.e. if $x \neq 0$ then $\langle x, x\rangle > 0$.

Exercise. *The map $\|\cdot\| : H \to [0,\infty)$, $\|x\| := \sqrt{\langle x, x\rangle}$, defines a norm on H, which we call the induced norm by the inner product.*

Then $(H, \langle\cdot,\cdot\rangle)$ is called *Hilbert space* if $(H, \|\cdot\|)$ is a Banach space.

Let H be a Hilbert space. Then $x, y \in H$ are called *orthogonal ($x \perp y$)* if $\langle x, y\rangle = 0$.

PROPOSITION E.12. *Let H be an Hilbert space and $\|\cdot\|$ the induced norm by the inner product.*

(a) $\|\lambda x+y\|^2 = |\lambda|^2\|x\|^2 + 2\Re(\lambda\langle x,y\rangle) + \|y\|^2$ *holds for $x,y \in H$,*
(b) $|\langle x,y\rangle| \le \|x\|\|y\|$ *holds for $x,y\in H$,* ***(Cauchy-Schwarz)***
(c) *The parallelogram law holds, i.e.*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2\,, \qquad x,y\in H\,.$$

(d) *Let $x,y\in H$. Then $x\perp y$ if and only if $\|y\|\le\|\lambda x+y\|$ for each $\lambda\in\mathbb{C}$.*
(e) *If $x\perp y$ then $\|x\|^2+\|y\|^2 = \|x+y\|^2$.* ***(Pythagoras)***

PROOF. (a) This follows by a short computation:

$$\begin{aligned}\|\lambda x+y\|^2 &= \langle\lambda x+y,\lambda x+y\rangle = <\lambda x,\lambda x> + \langle\lambda x,y\rangle + \overline{\langle\lambda x,y\rangle} + \langle y,y\rangle\\ &= |\lambda|^2\|x\|^2 + 2\Re\langle\lambda x,y\rangle + \|y\|^2\end{aligned}$$

as $z+\overline{z} = 2\Re(z)$.

(b) If $x=0$, the statement is obvious. Suppose $x\ne 0$. Set $\alpha := \langle x,y\rangle$ and $\lambda := -\frac{\overline{\alpha}}{\|x\|^2}$. Then (a) yields

$$0\le\|\lambda x+y\|^2 = \frac{|\alpha|^2}{\|x\|^2} - 2\frac{|\alpha|^2}{\|x\|^2} + \|y\|^2 = \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2}$$

implying

$$|\alpha|^2 \le \|x\|^2\|y\|^2$$

and so the desired estimate follows by taking the square root.

(c) This follows just by direct computations (**Exercise**).

(d) By (a) we have

$$0\le\|\lambda x+y\|^2 = |\lambda|^2\|x\|^2 + 2\Re(\lambda\langle x,y\rangle) + \|y\|^2.$$

If $\langle x,y\rangle = 0$ then the desired norm estimate follows immediately as $|\lambda|^2\|x\|^2\ge 0$. Conversely suppose $\|y\|\le\|\lambda x+y\|$ holds for each $\lambda\in\mathbb{C}$. Set $\alpha := \langle x,y\rangle$ and $\lambda := -\frac{\overline{\alpha}}{\|x\|^2}\in\mathbb{C}$. As we have seen in the proof of (b), (a) leads to

$$\|\lambda x+y\|^2 \overset{(a)}{=} \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2} \le \|\lambda x+y\|^2 - \frac{|\alpha|^2}{\|x\|^2}.$$

Thus $\alpha = 0$ follows, namely $x\perp y$.

(e) This follows immediately from (a) for $\lambda = 1$ since $2\Re(\langle x,y\rangle) = 0$ holds if $x\perp y$. □

REMARK. *Note that the Cauchy-Schwarz inequality yields the continuity of the inner product in both components.*

EXAMPLE. *For $d\in\mathbb{N}$, the set $H := \mathbb{C}^d$ equipped with inner product*

$$\langle x,y\rangle := \sum_{j=1}^{d}\overline{x_j}y_j$$

defines a (finite dimensional) Hilbert space.

Example. *Let X be a countable discrete set. Then*

$$\ell^2(X) := \left\{ \psi : X \to \mathbb{C} \;\middle|\; \sum_{x \in X} |\psi(x)|^2 < \infty \right\}$$

is a Hilbert space with inner product

$$\langle \psi, \phi \rangle = \sum_{x \in X} \overline{\psi(x)} \phi(x)$$

and induced norm $\|\psi\|_2 = \sqrt{\sum_{x \in X} |\psi(x)|^2}$. We leave the details as an exercise.

A subset $E \subseteq F$ of a vector space is called *convex* if for all $x, y \in E$ and $0 \le \lambda \le 1$, then $\lambda x + (1 - \lambda) y \in E$.

Theorem E.13 (Hilbert projection theorem). *Every nonempty closed convex set $E \subseteq H$ contains a unique x of minimal norm, i.e. $\|x\| = \inf\{\|y\| \mid y \in E\}$.*

Remark. *Note that we do not assume that E is compact.*

Proof. Define $d := \inf\{\|y\| \mid y \in E\}$. Choose $x_n \in E$ such that $\|x_n\| \to d$ and $\|x_n\| \ge d$. Note that $z := \frac{1}{2}(x_n + x_m) \in E$ (by convexity). Then the parallelogram identity (Proposition E.12 (c) for $x, y = z$) leads to

$$\|x_n + x_m\|^2 = 2\|z\|^2 + 2\|z\|^2 \ge 4d^2.$$

Furthermore, using the parallelogram identity again, we conclude

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 \to 4d^2.$$

This together with the previous estimate yields

$$\begin{aligned} \|x_n - x_m\|^2 =& 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \le 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \\ \le& 2\left|\|x_n\|^2 - d^2\right| + 2\left|\|x_m\|^2 - d^2\right|. \end{aligned}$$

As the right hand side gets arbitrary small if n, m are large enough since $\|x_n\| \to d$, we derive that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in H. Hence, $x_n \to x$ for some $x \in E$ (as it is closed) and $\|x\| = d$ as $\|x_n\| \to d$.

If $y \in E$ is another element satisfying $\|y\| = d$, then the sequence

$$\{x, y, x, y, x, y, x, \dots\} \subset E$$

must converge (as we just observed before). Hence, $x = y$ implying the uniqueness. □

We say E is *orthogonal* to F ($E \perp F$) for $E, F \subseteq H$ if $x \perp y$ for all $x \in E$ and $y \in F$. For $E \subseteq H$, define its *orthogonal complement*

$$E^\perp := \{y \in H \mid \langle x, y \rangle = 0 \text{ for all } x \in E\}.$$

Let $M \subseteq H$ be a closed subspace. If there is a closed subspace $N \subseteq H$ such that $H = M + N$ and $M \cap N = \{0\}$ then N is called the *complement of M* and we write $H = M \oplus N$.

Proposition E.14. *Let H be an Hilbert space and $\|\cdot\|$ the induced norm by the inner product.*

(a) *If $E \subseteq H$, then $E^\perp$ is a closed subspace and $E \subseteq (E^\perp)^\perp$,*
(b) *If $E \subseteq H$ then $E^\perp = \left(\overline{\mathrm{Lin}}(E)\right)^\perp$*

(c) *If E is a closed subspace, then $H = E \oplus E^\perp$,*
(d) *If E is a closed subspace, then $E = (E^\perp)^\perp$.*
(e) *If $E \subseteq H$, then $\overline{\mathrm{Lin}}(E) = (E^\perp)^\perp$.*

Proof. (a) If $x, y \in E^\perp$ then

$$\langle z, \lambda x + y\rangle = \lambda\langle z, x\rangle + \langle z, y\rangle = 0\,, \qquad z \in E,$$

implying $\lambda x + y \in E^\perp$. Let $x_n \to x$ with $x_n \in E^\perp$ and $z \in E$ be arbitrary. Then the Cauchy-Schwarz inequality (Proposition E.12 (b)) yields

$$|\langle x, z\rangle| = |\langle x - x_n, z\rangle| \le \|x - x_n\|\,\|z\| \to 0$$

and so $\langle x, z\rangle = 0$. Thus, $E^\perp$ is a closed subspace. The inclusion $E \subseteq (E^\perp)^\perp$ as $\langle z, x\rangle = 0$ for $z \in E$ and $x \in E^\perp$.

(b) The inclusion $E^\perp \supseteq \big(\overline{\mathrm{Lin}}(E)\big)^\perp$ is trivial as $E \subseteq \overline{\mathrm{Lin}}(E)$. For the converse inclusion, let $z \in E^\perp$. If $x, y \in E$ and $\lambda \in \mathbb{K}$ then

$$\langle x + \lambda y, z\rangle = \langle x, z\rangle + \overline{\lambda}\langle y, z\rangle = 0 + 0 = 0.$$

Thus, for each $x' \in \mathrm{Lin}(E)$, we have $x' \perp z$. If $x \in \overline{\mathrm{Lin}}(E)$, then there is a sequence $x_n \in \mathrm{Lin}(E)$ converging to x and $x_n \perp z$ for all $n \in \mathbb{N}$ by the previous consideration. Due to the Cauchy-Schwarz inequality, the inner product is continuous and so

$$\langle x, z\rangle = \lim_{n\to\infty}\langle x_n, z\rangle = 0.$$

Consequently, $z \in \big(\overline{\mathrm{Lin}}(E)\big)^\perp$ and so $E^\perp \subseteq \big(\overline{\mathrm{Lin}}(E)\big)^\perp$ follows.

(c) By (a) $E^\perp$ is a closed subspace of H. If $x \in E \cap E^\perp$, then $\langle x, x\rangle = 0$ and so $x = 0$ follows (requirement of inner product is that the inner product with itself is strictly positive if $x \neq 0$). Thus, $E \cap E^\perp = \{0\}$.

Let $x \in H$. Then $x - E \coloneqq \{x - y \,|\, y \in E\}$ is closed and convex since E is a closed subspace and

$$\begin{aligned}\lambda(x - y) + (1 - \lambda)(x - z) &= \big(\lambda + (1 - \lambda)\big)x - \lambda y + (1 - \lambda)(-z)\\ &= x - \big(\lambda y - (1 - \lambda)(-z)\big)\end{aligned}$$

for $0 \le \lambda \le 1$ and $y, z \in E$. Thus, there is a unique $x_1 \in E$ that minimizes $\|x - x_1\|$ by Theorem E.13 (Hilbert projection theorem). Set $x_2 \coloneqq x - x_1$. Then

$$\|x_2\| \le \|x_2 + y\|, \qquad y \in E,$$

holds since $x_2 + y = x - x_1 + y$ is an arbitrary element in $x - E$ (using that E is a subspace). Consequently, $x_2 \perp E^\perp$ follows by Proposition E.12 (d) as E is a subspace. Since $x = x_1 + x_2 \in E + E^\perp$, the desired result is proven.

(d) According to (a), we have $E \subset (E^\perp)^\perp$. Since $E \oplus E^\perp = H = E^\perp \oplus (E^\perp)^\perp$ using (a) and (c), E cannot be a proper subspace of $(E^\perp)^\perp$.

(e) According to (b), we have $\overline{\mathrm{Lin}}(E)^\perp = E^\perp$. Thus, $\big(\overline{\mathrm{Lin}}(E)^\perp\big)^\perp = (E^\perp)^\perp$ follows. Since $\overline{\mathrm{Lin}}(E)$ is a closed subspace, (d) implies

$$\overline{\mathrm{Lin}}(E) = \big(\overline{\mathrm{Lin}}(E)^\perp\big)^\perp = (E^\perp)^\perp$$

finishing the proof. ∎

Recall that elements of $H' := \mathcal{L}(H, \mathbb{C})$ are called linear bounded functionals, see Section 2.3 and Appendix D.

THEOREM E.15 (Riesz-Fréchet representation theorem). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then for every $y \in H$, the map*

$$F_y : H \longrightarrow \mathbb{C}, \quad x \mapsto \langle y, x\rangle$$

defines a continuous linear functional on H with $\|F_y\| = \|y\|$. Furthermore, the map

$$H \longrightarrow H', y \mapsto F_y,$$

is complex linear (i.e. $F_{\lambda x + \mu z} = \overline{\lambda} F_x + \overline{\mu} F_z$) and bijectiv. In particular, for every linear bounded functional φ on H, there is a unique $y \in H$ such that $\varphi = F_y$.

REMARK. *The important statement is the surjectivity, namely that each linear bounded functional is represented by an F_y.*

PROOF. Clearly F_y is a linear functional for each $y \in Y$ with

$$|F_y(x)| = |\langle y, x\rangle| \le \|y\|\|x\|.$$

Thus, F_y is bounded (and hence continuous!) with $\|F_y\| \le \|y\|$. Since $F_y(y) = \|y\|^2$ holds, we conclude $\|F_y\| = \|y\|$. The complex linearity follows directly from the complex linearity of the inner product. Hence $y \mapsto F_y$ is isometric and complex linear, and so it is injective (**Exercise**).

It is left to show that $y \mapsto F_y$ is surjective: Let $\varphi \in H'$. If $\varphi \equiv 0$, then clearly, we have to choose $y = 0$ to get $\varphi = F_y$.

Thus, we can assume without loss of generality that $\varphi \not\equiv 0$. Then

$$N := \ker(\varphi) = \{z \in H \,|\, \varphi(z) = 0\}$$

is a proper closed subspace of H, namely $H \neq N$, confer Proposition 2.8. By Proposition E.14 (c) we conclude $N^\perp \neq \{0\}$. (Otherwise: $N = N^{\perp\perp} = H$, namely $\varphi = 0$, a contradiction). Let $z \in N^\perp$ be such that $z \neq 0$. Then

$$\varphi(x)z - \varphi(z)x$$

belongs to $\ker(\varphi)$ (why?) for all $x \in H$. Since $z \in N^\perp$, we conclude

$$0 = \langle z, \varphi(x)z - \varphi(z)x\rangle = \varphi(x)\langle z, z\rangle - \varphi(z)\langle z, x\rangle.$$

Hence,

$$\varphi(x) = \frac{\varphi(z)}{\|z\|^2}\langle z, x\rangle = \left\langle \overline{\frac{\varphi(z)}{\|z\|^2}} z, x \right\rangle$$

follows. This leads to the desired claim with $y = \overline{\frac{\varphi(z)}{\|z\|^2}} z$. □

REMARK. *In the first glance, it is surprising that any choice of $z \in N^\perp$ does the job. However, it comes out of the proof that $N = \{y\}^\perp = \{z\}^\perp$ implying that*

$$N^\perp = (Lin\{z\})^{\perp\perp} = Lin\{z\}$$

is one-dimensional.

4. Self-adjoint operators

From now on, we will focus on linear bounded operators over a Hilbert space H. We will introduce the concept of adjoint operator and study particularly operators where the adjoint operator of A coincides with A. These operators are called self-adjoint. Let us begin with a short exercise.

EXERCISE. *Let H, K be Hilbert spaces and $A \in \mathcal{L}(H,K)$. Show that*

$$\|A\| = \sup\left\{|\langle x, Ay\rangle| \,\middle|\, \|x\| \le 1, \|y\| \le 1\right\}.$$

holds.

Using the Riesz-Fréchet representation theorem (previous Theorem E.15) we define the concept of an adjoint operator.

PROPOSITION E.16 (Adjoint operator). *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H,K)$. Then there is a unique $A^* : K \to H$ satisfying*

$$\langle y, Ax\rangle_K = \langle A^* y, x\rangle_H$$

for all $x \in H$ and $y \in K$. Furthermore, we have

$$\|A^*\| = \|A\|.$$

PROOF. *Uniqueness:* The uniqueness is clear as all inner products of A^*y are fixed. Specifically, let $A_1, A_2 : K \to H$ be such that $\langle y, AX\rangle_K = \langle A_i y, x\rangle_H$ for all $x \in H$ and $y \in K$. Then the previous exercise leads to

$$\|A_1 - A_2\| \sup\left\{|\langle A_1 x, y\rangle - \langle A_2 x, y\rangle| \,\middle|\, \|x\| \le 1, \|y\| \le 1\right\}$$

Existence: Let $y \in K$ and consider the map

$$H \to \mathbb{C}, \quad x \mapsto \langle y, Ax\rangle_K.$$

This map is clearly linear and continuous since

$$|\langle y, Ax\rangle| \le \|y\|\ \|A\|\ \|x\|.$$

Using Riesz-Fréchet representation theorem (Theorem E.15, there is a unique $y' \in H$ with

$$\langle y, Ax\rangle_K = \langle y', x\rangle_H$$

for all $x \in H$. Define $A^* y := y'$. Then, the map $A^* : K \to H, y \mapsto y'$, is linear as

$$\langle A^*(g_1 + \lambda g_2), x\rangle = \langle g_1, Ax\rangle + \overline{\lambda}\langle g_2, Ax\rangle = \langle A^* g_1 + \lambda A^* g_2, x\rangle$$

for all $x \in H$.

Finally, the norm identity is derived from

$$\begin{aligned}\|A\| &= \sup\left\{|\langle y, Ax\rangle| \,\middle|\, \|x\| \le 1, \|y\| \le 1\right\}\\ &= \sup\left\{|\langle A^* y, x\rangle| \,\middle|\, \|x\| \le 1, \|y\| \le 1\right\}\\ &= \|A^*\|\end{aligned}$$

invoking the previous exercise. □

DEFINITION. *Let H, K be Hilbert spaces and $A \in \mathcal{L}(H,K)$. Then we call A^* (defined in the previous proposition) the* adjoint operator *of A.*

PROPOSITION E.17 ($*$ is an involution). *Let H and K be Hilbert spaces, $\lambda \in \mathbb{C}$ and $A, B \in \mathcal{L}(H,K)$. Then the following statements hold.*

(a) $(A+\lambda B)^* = A^* + \overline{\lambda} B^*$.
(b) $(AB)^* = B^* A^*$.
(c) $(A^*)^* = A$.
(d) *If* $I \in \mathcal{L}(H)$ *is the identity, then* $I^* = I$.
(e) $(A^*)^{-1} = (A^{-1})^*$ *if* A *is invertible.*

PROOF. These statements follow by direct computations which are left as an exercise (Sheet 9). □

PROPOSITION E.18 (C^*-property). *Let* H *and* K *be Hilbert spaces and* $A, B \in \mathcal{L}(H,K)$. *Then*

$$\|A\|^2 = \|A^* A\| = \|AA^*\|.$$

PROOF. Since $\|A\| = \|A^*\|$ and $(A^*)^* = A$, it suffices to show the first identity. Invoking the submultiplicativity of the norm, we conclude

$$\|A^* A\| \le \|A^*\| \; \|A\| = \|A\|^2.$$

Furthermore, we have

$$\begin{aligned} \|A^* A\| &= \sup\left\{ \|A^* A x\| \,\middle|\, \|x\| \le 1 \right\} \\ \text{Cauchy-Schwarz} \quad &\ge \sup\left\{ |\langle A^* A x, x\rangle| \,\middle|\, \|x\| \le 1 \right\} \\ &= \sup\left\{ |\langle Ax, Ax\rangle| \,\middle|\, \|x\| \le 1 \right\} \\ &= \sup\left\{ \|Ax\|^2 \,\middle|\, \|x\| \le 1 \right\} \\ &= \|A\|^2 \end{aligned}$$

finishing the proof. □

REMARK. *The previous identity has a great structural impact. If one considers the space* $\mathcal{L}(H)$ *of all continuous linear operators on a Hilbert space, then*

- $\mathcal{L}(H)$ *is an algebra with norm and an involution* $*$.
- *The norm* $\|\cdot\|$ *is submultiplicative and* $\mathcal{L}(H)$ *is complete.*
- *We have* $\|A\|^2 = \|A^* A\| = \|AA^*\|$.

A normed algebra with these properties is called a C^**-algebra and so* $\mathcal{L}(H)$ *is a* C^**-algebra. Indeed one can show that every* C^**-algebra is a subalgebra of* $\mathcal{L}(H)$ *(where* H *might be a huge Hilbert sapce) via the GNS-construction.* C^**-algebras play a crucial role in physics.*

With the notion of adjoint operator at hand, we can define the following.

DEFINITION. *Let* H *be a Hilbert space and* $A \in \mathcal{L}(H)$. *Then*

- A *is called* self-adjoint *if* $A^* = A$.
- A *is called* normal *if* $A^* A = AA^*$.
- A *is called* unitary *if* $A^* = A^{-1}$.

EXAMPLE. *Let* $H := \ell^2(\mathbb{Z})$ *be the Hilbert and* $m \in \mathbb{Z}$. *Define*

$$L_m : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (L_m \psi)(n) = \psi(-m+n).$$

Clearly, L_m *is a linear operator such that*

$$\|L_m \psi\|^2 = \sum_{n\in\mathbb{Z}} |L_m \psi(n)|^2 = \sum_{n\in\mathbb{Z}} |\psi(-m+n)|^2 = \sum_{k\in\mathbb{Z}} |\psi(k)|^2 = \|\psi\|^2.$$

Thus, $L_m \in \mathcal{L}(\ell^2(\mathbb{Z}))$ *with* $\|L_m\| = 1$. *A short computation leads to*

$$\begin{aligned} \langle \psi, L_m \phi \rangle &= \sum_{n\in\mathbb{Z}} \overline{\psi(m-m+n)}\phi(-m+n) \\ &= \sum_{k\in\mathbb{Z}} \overline{\psi(m+k)}\phi(k) \\ &= \sum_{k\in\mathbb{Z}} \overline{(L_{-m}\psi)(k)}\phi(k) \\ &= \langle L_{-m}\psi, \phi\rangle \end{aligned}$$

Thus, $L_m^*\psi = L_{-m}\psi$ *follows as the adjoint operator is uniquely determined by this identity by Proposition E.16. Furthermore*

$$(L_m L_m^* \psi)(n) = (L_m^*\psi)(-m+n) = \psi(m-m+n) = \psi(n)$$

holds and similarly $L_m^* L_m \psi = \psi$. *Thus,* L_m *is unitary and so it is also normal.*

Example. *Let*

$$H := L^2(\mathbb{R}) = \Big\{\psi : \mathbb{R} \to \mathbb{C} \,\Big|\, \psi \text{ measurable}, \int_{\mathbb{R}} |\psi(x)|^2 \, dx < \infty\Big\}$$

be the Hilbert space of square integrable functions with inner product

$$\langle \psi, \phi \rangle := \int_{\mathbb{R}} \overline{\psi(x)}\phi(x) \, dx.$$

Let $t : \mathbb{R} \to \mathbb{C}$ *be a continuous bounded function and define* $\widehat{t} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *by*

$$(\widehat{t}\psi)(x) := t(x)\psi(x).$$

Then $\widehat{t}$ *is a linear bounded operator satisfying*

$$\|\widehat{t}\| \le \|t\|_\infty := \sup_{x\in\mathbb{R}} |t(x)| < \infty.$$

Furthermore, a short computation leads to

$$\begin{aligned} \langle \psi, \widehat{t}\phi \rangle &= \int_{\mathbb{R}} \overline{\psi(x)} t(x)\phi(x) \, dx \\ &= \int_{\mathbb{R}} \overline{\overline{t(x)}\psi(x)}\phi(x) \, dx \\ &= \langle \widehat{\overline{t}}\psi, \phi\rangle. \end{aligned}$$

Hence, $\widehat{t}$ *is self-adjoint if and only if* $t(x) = \overline{t(x)}$ *for Lebesgue almost-every* $x \in \mathbb{R}$, *or equivalently if* t *is real-valued for Lebesgue almost-every* $x \in \mathbb{R}$. *It is elementary to check that* $\widehat{t}$ *is normal.*

Our main focus will lie on self-adjoint bounded operators. Their spectrum is always contained in the real line. In order to prove this, we will need the following.

For $A \in \mathcal{L}(H, K)$, the *kernel* of A is defined by

$$\ker(A) := \{x \in H \,|\, Ax = 0\}$$

and the *range* of A is defined by

$$\operatorname{ran}(A) := \{Ax \,|\, x \in H\}.$$

It is straightforward to check that $\operatorname{ran}(A) \subseteq K$ and $\ker(A) \subseteq H$ are subspaces and that $\ker(A)$ is closed, see Proposition 2.8.

PROPOSITION E.19. *Let $A \in \mathcal{L}(H)$. Then*

$$\ker(A) = \operatorname{ran}(A^*)^\perp \quad \text{and} \quad \ker(A^*) = \operatorname{ran}(A)^\perp.$$

PROOF. Since $(A^*)^* = A$, it suffices to prove the first identity. A direct computation leads to

$$\begin{aligned}\ker(A) &= \{x \in H \mid Ax = 0\} \\ &= \{x \in H \mid \langle Ax, y\rangle = 0 \text{ for all } y \in K\} \\ &= \{x \in H \mid \langle x, A^*y\rangle = 0 \text{ for all } y \in K\} \\ &= \{A^*y \mid y \in K\}^\perp.\end{aligned}$$

□

LEMMA E.20. *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then*

$$\|(A - zI)x\|^2 = \|(A - \Re(z))x\|^2 + |\Im(z)|^2\|x\|^2$$

for all $z \in \mathbb{C}$ and $x \in H$. In particular,

$$\|(A - zI)x\| \ge |\Im(z)|\ \|x\|$$

for all $z \in \mathbb{C}$ and $x \in H$.

PROOF. Set $z := \alpha + i\beta$ with $\alpha := \Re(z) \in \mathbb{R}$ and $\beta := \Im(z) \in \mathbb{R}$. Thus Proposition E.12 (a) leads to

$$\|(A - zI)x\|^2 = \|(A - \alpha)x\|^2 + 2\Re\big(\langle Ax - \alpha x, -i\beta x\rangle\big) + |\beta|^2\|x\|^2.$$

Since A is self-adjoint, we have $\langle Ax, x\rangle \in \mathbb{R}$. Thus

$$\langle Ax - \alpha x, -i\beta x\rangle = -i\beta\langle Ax, x\rangle - i\alpha\beta\|x\|^2$$

has vanishing real part and so the desired identity is proven. Then the desired inequality follows immediately. □

COROLLARY E.21 (Spectrum of self-adjoint operators). *Let $A \in \mathcal{L}(H)$ be self-adjoint. Then $A - zI$ for each $z \in \mathbb{C} \setminus \mathbb{R}$ is bijective and the inverse $(A - zI)^{-1} \in \mathcal{L}(H)$ is bounded by $\frac{1}{|\Im(z)|}$. In particular, $\sigma(A) \subseteq \mathbb{R}$.*

REMARK. *The inclusion $\sigma(A) \subseteq \mathbb{R}$ allows us in quantum mechanics to interpret the spectrum of a self-adjoint operator as possible measurements.*

PROOF. We will use the estimate from the previous lemma

$$\|(A - zI)x\| \ge |\Im(z)|\ \|x\| \tag{$\star$}$$

for all $z \in \mathbb{C}$ and $x \in H$. This implies that $A - zI : H \to H$ is injective.

The image of $A - zI$ is closed: Let $y_n := (A - zI)x_n$ be a sequence in the image that converge to y. Then $(y_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. Thus, $(\star)$ yields that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence and so it converges in H. Denote its limit point by $x \in H$. Then

$$(A - zI)x = \lim_{n\to\infty}(A - zI)x_n = \lim_{n\to\infty} y_n = y$$

follows proving that $ran(A - zI)$ is closed. Furthermore, $(\star)$ and the self-adjointness of A imply that $\ker(A^* - \overline{z}) = \ker(A - \overline{z}) = \{0\}$. Then, the previous Proposition E.19 leads to

$$\operatorname{ran}(A - z)^\perp = \ker(A^* - \overline{z}) = \{0\}.$$

Thus, $\operatorname{ran}(A-z) \subseteq H$ is dense and closed and so $\operatorname{ran}(A-z) = H$ follows. Hence, $A - zI$ is bijective.

Set $x := (A - zI)^{-1}y$ for some $y \in H$. Then $(\star)$ leads to

$$\|y\| = \|(A - zI)x\| \geq |\Im(z)| \, \|x\|$$

and so

$$\|(A - zI)^{-1}y\| = \|x\| \leq \frac{1}{|\Im(z)|}\|y\|.$$

Hence, the operator norm of $(A - zI)^{-1}$ is bounded by $\frac{1}{|\Im(z)|}$. Thus, $A - zI$ is invertible whenever $\Im(z) \neq 0$, namely $\sigma(A) \subseteq \mathbb{R}$. □

Remark. *The reverse of the previous statement is not true in general as can be easily seen from the example of suitable matrices (Exercise).*

Theorem E.22 (Spectral radius and norm for normal elements). *Let $A \in \mathcal{L}(H)$ be normal. Then,*

$$\|A\| = r(A) = \max\left\{|\lambda| \,\middle|\, \lambda \in \sigma(A)\right\}.$$

Proof. According to Beurling's theorem (Theorem E.10, Location of the spectrum), we have

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \max\left\{|\lambda| \,\middle|\, \lambda \in \sigma(A)\right\}.$$

We show that normality and the C^*-algebra property yield

$$\|A^{2^n}\| = \|A\|^{2^n} \tag{$\star$}$$

for every $n \in \mathbb{N}$. This then yields the statement.

The proof of $(\star)$ is carried out by induction: We will use multiple times that for normal elements A and $k \in \mathbb{N}$

$$A^k(A^k)^* = A^k(A^*)^k = (AA^*)^k.$$

$n = 0$: This is clear.

$n \implies n+1$: We calculate the square of the norm we are looking for by using the C^*-property

$$\begin{aligned}
\|A^{(2^{n+1})}\|^2 &= \|A^{2^{n+1}}\left(A^{2^{n+1}}\right)^*\| \\
(A \text{ normal and } (A^k)^* = (A^*)^k) \quad &= \|(AA^*)^{2^n}(AA^*)^{2^n}\| \\
(C^*\text{-property}) \quad &= \|(AA^*)^{2^n}\|^2 \\
(A \text{ normal}) \quad &= \|A^{2^n}(A^*)^{2^n}\|^2 \\
(C^*\text{-property}) \quad &= \|A^{2^n}\|^4 \\
(\text{Induction hypothesis}) \quad &= \|A\|^{2^{n+2}}.
\end{aligned}$$

This finishes the proof. □