Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 9 **Due on Thursday 01/21/2021 at 10.00 am**

Exercise 1. Let $T := \overline{N} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and $H := \ell^2(\mathbb{N})$. Show the following.

(a) There are $A_t \in \mathcal{L}(H)$ self-adjoint for $t \in T$ such that
$$N_p : T \to \mathbb{R}, N_p(t) := \|p(A_t)\|,$$
is continuous for all $p(z) := p_1 z + p_0$ with $p_1, p_0 \in \mathbb{R}$ but $t \mapsto \sigma(A_t)$ is not continuous.

(b) There are $A_t \in \mathcal{L}(h)$ self-adjoint for $t \in T$ such that
$$N_p : T \to \mathbb{R}, N_p(t) := \|p(A_t)\|,$$
is continuous for all $p(z) := p_2 z^2 + p_0$ with $p_2, p_0 \in \mathbb{R}$ but $t \mapsto \sigma(A_t)$ is not continuous.

(c) There are $A_t \in \mathcal{L}(h)$ self-adjoint for $t \in T$ such that
$$N_p : T \to \mathbb{R}, N_p(t) := \|p(A_t)\|,$$
is continuous for all $p(z) := p_2 z^2 + p_1 z$ with $p_2, p_1 \in \mathbb{R}$ but $t \mapsto \sigma(A_t)$ is not continuous.

Exercise 2 (4 points)**.** Let G be countable and $g, h \in G$. Prove that the left-shift
$$L_g : \ell^2(G) \to \ell^2(G), \quad (L_g\psi)(h) := \psi(g^{-1}h),$$
and the right-shift
$$R_g : \ell^2(G) \to \ell^2(G), \quad (R_g\psi)(h) := \psi(hg),$$
are unitary linear bounded operators satisfying

(a) $L_g^* = L_{g^{-1}}$ and $R_g^* = R_{g^{-1}}$,

(b) $\|L_g\| = 1 = \|R_g\|$,

(c) $L_g L_h = L_{gh}$ and $R_g R_h = R_{gh}$,

(d) $L_g R_h = R_h L_g$.

Exercise 3. A function $t : \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ is called *(strongly) pattern equivariant with parameter* $M_t \in \mathbb{N}$ if $t(\omega) = t(\rho)$ holds whenever $\omega|_{Q_{M_t}} = \rho|_{Q_{M_t}}$. Prove that the following statements are equivalent.

(i) $t : \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ is pattern equivariant.

(ii) t is continuous and takes only finitely many values.

Exercise 4 (4 points)**.** Let E be a Banach space and $A \in \mathcal{L}(E)$. Consider a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \rho(A)$ such that $x_n \to x \in \mathbb{C}$. Prove the following statements.

(a) If $\sup_{n\in\mathbb{N}} \|(A - x_n)^{-1}\|$ is finite then $x \in \rho(A)$.

(b) If there are $S_n, T_n \in \mathcal{L}(E),\ n \in \mathbb{N}$, such that
$$(A - x_n)S_n = I + T_n, \qquad \sup_{n\in\mathbb{N}} \|S_n\| < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|T_n\| < 1,$$
then $x \in \rho(A)$.