Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 8 **Due on Thursday 01/14/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let S be a substitution over a finite alphabet $\mathcal{A}$ with $\sharp\mathcal{A} \geq 2$. Define $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$S(\omega) = \quad \ldots S\big(\omega(-3)\big)\, S\big(\omega(-2)\big)\, S\big(\omega(-1)\big) \,\big|\, S\big(\omega(0)\big)\, S\big(\omega(1)\big)\, S\big(\omega(2)\big) \,\ldots .$$

(a) Prove that $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a continuous map.
(b) Suppose S is primitive. Prove that there is always a $k \in \mathbb{N}$ and an $\omega \in \mathcal{A}^{\mathbb{Z}}$ such that $S^k(\omega) = \omega$ and $\omega|_{\{-1,0\}}$ is legal.

Hint: Compare exercise on page 62 in the lecture.

Exercise 2 (4 points)**.** Let $\mathcal{A} := \{a, b, c, d\}$ and consider the Golay-Rudin-Shapiro substitution

$$S(a) := ab, \qquad S(b) := ac, \qquad S(c) := db, \qquad S(d) = dc.$$

- Prove that S is is primitive.
- Prove that $\Omega(S)$ is strongly aperiodic.
- Find a periodic $\omega \in \mathcal{A}^{\mathbb{Z}}$ such that

$$\lim_{n\to\infty} \overline{Orb(S^n(\omega))} = \Omega(S).$$

Exercise 3. Let H be a Hilbert space, T be a topological space and $A_t \in \mathcal{L}(H)$ be self-adjoint for $t \in T$. We say (A_t) is continuous in the strong operator topology if

$$t \mapsto A_t\psi \in H$$

is continuous for all $\psi \in h$. Let (A_t) be continuous in the strong operator topology such that $\sup_{t\in T} \|A_t\| < \infty$. Prove the following assertions.

(a) The map $N : T \to [0, \infty), N(t) := \|A_t\|$ is lower semi-continuous.
(b) The family $(p(A_t))_{t\in T}$ is continuous in the strong operator topology for all polynomials $p(z) := az^2 + bz + c$ with $a, b, c \in \mathbb{C}$.
(c) The map $N_p : T \to [0, \infty), N_p(t) := \|p(A_t)\|$ is lower semi-continuous for all polynomials $p(z) := az^2 + bz + c$ with $a, b, c \in \mathbb{C}$.

Exercise 4 (4 points)**.** Let $\mathcal{A}$ be an finite set (alphabet) and $V : \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ be continuous. For $w \in \mathcal{A}^{\mathbb{Z}}$, define the operator $H_w \in \mathcal{L}\big(\ell^2(\mathbb{Z})\big)$ by

$$(H_\omega\psi)(n) := \psi(n-1) + \psi(n+1) + V(n^{-1}\omega)\psi(n).$$

Prove the following statements.

(a) H_ω is a linear, self-adjoint operator with $\sup_{\omega\in\mathcal{A}^{\mathbb{Z}}} \|H_\omega\| < \infty$.
(b) The map $\omega \mapsto H_\omega$ is strongly continuous, namely

$$\lim_{\rho\to\omega} \|(H_\omega - H_\rho)\psi)\| = 0$$

for all $\psi \in \ell^2(\mathbb{Z})$.

(c) For all $n \in \mathbb{Z}$ and $\omega \in \mathcal{A}^{\mathbb{Z}}$, the equality

$$H_{n\omega} = L_n H_\omega L_{-n}$$

holds where

$$L_n : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad (L_n\psi)(m) := \psi(-n+m),$$

Bonus exercise 1 (1 point)**.** Let (X, G) be a dynamical system and $t : X \to \mathbb{C}$ be continuous. For each $x \in X$, compute the spectrum $\sigma\big(\widehat{t}(x)\big)$ of the operator $\widehat{t}(x) \in \mathcal{L}(\ell^2(G))$ defined by

$$\big(\widehat{t}(x)\psi\big)(g) := t(g^{-1}x)\psi(g)$$

for $\psi \in \ell^2(G)$ and $g \in G$.