Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Christmas Sheet 7 **Due on Thursday 01/07/2020 at 10.00 am**

Exercise 1 (4 bonus points)**.** Let S be a primitive substitution over the alphabet $\mathcal{A}$ with $\sharp\mathcal{A} \geq 2$. Prove that $|S^n(a)| \to \infty$ for all $a \in \mathcal{A}$.

Exercise 2 (4 bonus points)**.** Let S be the Fibonacci substitution $S(a) := ab$, $S(b) := a$, over the alphabet $\mathcal{A} := \{a, b\}$. Find periodic $\omega_n \in \mathcal{A}^{\mathbb{Z}}$ such that

- $\lim_{n\to\infty} \omega_n \in \Omega(S)$ and
- $\lim_{n\to\infty} Orb(\omega_n) \neq \Omega(S)$.

Hint: Compute first $W(S)\cap\mathcal{A}^2$ and try to construct a sequence ω_n such that one of the elements of $\mathcal{A}^2 \setminus W(S)$ occurs in ω_n for all $n \in \mathbb{N}$.

Exercise 3 (4 bonus points)**.** Prove that the one-defect $\overline{Orb(\omega)}$ is periodically approximable.

Exercise 4 (4 bonus points)**.** Compute all legal words of the Fibonacci substitution S up to length 4. Specifically, compute $W(S) \cap \mathcal{A}^j$ for $j \in \{1, 2, 3, 4\}$. Justify your claim.

Bonus exercise 1 (1 bonus point)**.** Let $E := \ell^2(\mathbb{N})$, $k \in \mathbb{N}$ and consider the operator $A_k \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by

$$(A_k\psi)(n) := \begin{cases} 0, & k < n, \\ \psi(n), & k \geq n. \end{cases}$$

Prove that $\sigma(A_k) = \{0, 1\}$.

Bonus exercise 2 (1 bonus point)**.** Let E be a Banach space and $A \in \mathcal{L}(E)$. Prove that if p is a polynomial with complex coefficients then

$$\sigma(p(A)) = \{p(\lambda) \,|\, \lambda \in \sigma(A)\}.$$

Bonus exercise 3 (1 bonus point)**.** Let $\lambda \in [0, 1]$. Show that there are $p_n \leq q_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that

- $\frac{p_n}{q_n} \to \lambda$ and $\frac{p_n}{q_n}$ is a reduced fraction,
- $p_n \to \infty$,
- $q_n - p_n \to \infty$.