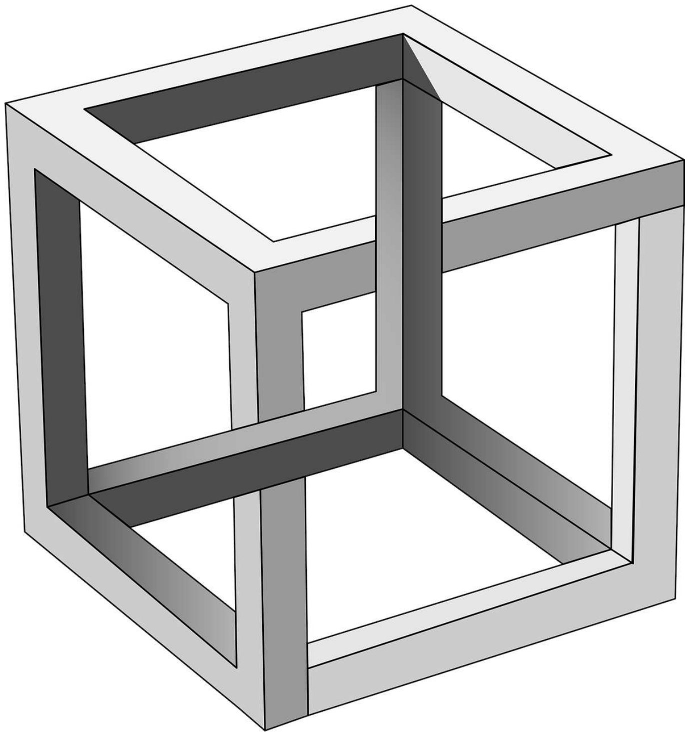
Christian Bär

GEOMETRIE IN POTSDAM

Algebraic Topology

Lecture Notes, Summer Term 2022



Version of August 5, 2022

Universität Potsdam

Picture on title page taken from pixabay.com.

Contents

Preface

These are the lecture notes of an introductory course on algebraic topology which I taught at the University of Potsdam during the summer term 2022. The aim was to introduce the basic tools from homotopy and homology theory. Choices concerning the material had to be made. Since time was too short for a reasonable discussion of cohomology theory after homology had been treated, I decided to skip cohomology altogether and instead included more material from homotopy theory than is often done. In particular, there is a detailed discussion of higher homotopy groups and the long exact sequence for Serre fibrations.
The necessary prerequisites of the students were rather modest. The course contains a quick introduction to set theoretic topology but a certain acquaintance with these concepts was certainly helpful. Familiarity with basic algebraic notions like rings, modules, linear maps etc. was assumed.
These notes are based on the lecture notes of a course I taught back in 2010. I am grateful to Volker Branding who wrote a first draft of those lecture notes and created most of the figures and to Ramona Ziese who improved those notes considerably. Moreover, I would like to thank the participants of the 2022 course, especially for the valuable feedback that they provided.

Potsdam, August 2022

Christian Bär

1. Set Theoretic Topology

1.1. Typical problems in topology

Topology is rough geometry. For example, a sphere is topologically the same as a cube, even though the sphere is smooth and curved while the cube is piecewise flat and has corners. In more precise mathematical terms this means that they are homeomorphic. On the other hand, the sphere is different from a torus even topologically.

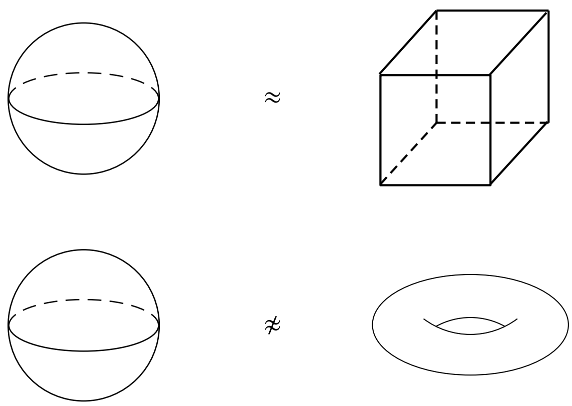


Figure 1. (Non-) homeomorphic spaces

Here are four versions of a typical question that one asks in mathematics. Fix integers $1 \leq n < m$.

1.) Does there exist a linear isomorphism $\varphi : \mathbb{R}^m \to \mathbb{R}^n$?
No, since linear algebra tells us that isomorphic vector spaces have the same dimension.

2.) Does there exist a diffeomorphism $\varphi : \mathbb{R}^m \to \mathbb{R}^n$?
No, otherwise the differential $d\varphi(0) : \mathbb{R}^m \to \mathbb{R}^n$ would be a linear isomorphism.

3.) Does there exist a homeomorphism $\varphi : \mathbb{R}^m \to \mathbb{R}^n$?
We cannot answer that question yet, but we will develop the necessary tools to find the answer.

4.) Does there exist a bijective map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$?
Yes. We will now explicitly construct an example for such a map in the case $n = 1$ and $m = 2$.

Example 1.1. Since the exponential function maps $\mathbb{R}$ bijectively onto $\mathbb{R}_+ = (0, \infty)$ it suffices to construct a bijective map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}_+$.

Given $x > 0$ write x as infinite decimal fraction.

$$x = \dots a_3 a_2 a_1, b_1 b_2 b_3 \dots$$

Here $a_j \in \{0, 1, \dots, 9\}$ and almost all a_j are equal to 0. The b_j are blocks of the form $0 \dots 0 c_j$ with $c_j \in \{1, \dots, 9\}$. This representation of x is unique. Now define $\varphi(x) = (\varphi_1(x), \varphi_2(x))$ where

$$\varphi_1(x) = \dots a_5 a_3 a_1, b_1 b_3 b_5 \dots \qquad \varphi_2(x) = \dots a_6 a_4 a_2, b_2 b_4 b_6 \dots$$

One easily checks that φ maps $\mathbb{R}_+$ bijectively onto $\mathbb{R}_+ \times \mathbb{R}_+$.
Let us evaluate the construction for an example. Let $x = 1987, 30500735 \dots$. Then we can read off the a_j and the b_j as

$$\begin{aligned} a_1 &= 7, a_2 = 8, a_3 = 9, a_4 = 1, a_j = 0 \text{ for } j \geq 5 \\ b_1 &= 3, b_2 = 05, b_3 = 007, b_4 = 3, b_5 = 5, \dots \end{aligned}$$

Hence $\varphi_1(x) = 97, 30075 \dots$ and $\varphi_2(x) = 18, 053 \dots$

Remark 1.2. In the continuous world counter-intuitive things can happen which are not possible in the differentiable world. For example, there exist continuous maps

$$\varphi : [0, 1] \to [0, 1] \times [0, 1]$$

which are surjective. Such a map is called a *plane-filling curve*.

Example 1.3. The first example goes back to Peano [6]. The following example was shortly after given by Hilbert [3] and is known as the *Hilbert curve*. It is defined by

$$\varphi(x) := \lim_{n \to \infty} \varphi_n(x)$$

where the φ_n are defined recursively as indicated in the pictures:

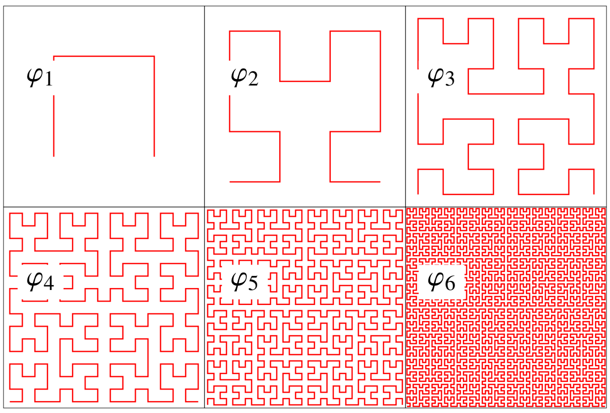


Figure 2. Hilbert's curve[1]

[1] Based on an illustration by Zbigniew Fiedorowicz, see https://commons.wikimedia.org/wiki/File:Hilbert_curve.png

Remark 1.4. This cannot happen for smooth curves due to a Theorem by Sard which states that for smooth $\varphi : [0,1] \to \mathbb{R}^2$ the image $\varphi([0,1]) \subset \mathbb{R}^2$ is a zero set.

Many typical problems in topology are of the following form:

1.) Given two spaces, are they homeomorphic? To show that they are, construct a homeomorphism. To show that they are not, find topological invariants, which are different for given spaces.

2.) Classify all spaces in a certain class up to homeomorphisms.

3.) Fixed point theorems.

Example 1.5 (Classification theorem for surfaces). The *classification theorem for surfaces* states that each orientable compact connected surface is homeomorphic to exactly one in the following infinite list:

S^2 (sphere), genus 0

T^2 (torus), genus 1

F_2 (Pretzel surface), genus 2

F_3 (true pretzel), genus 3

$\vdots$

Figure 3. Surface classification[2]

The genus is the number of "holes" in the surface. We will give a more precise definition later. So the classification theorem says that to each $g \in \mathbb{N}_0$ there exists an orientable compact connected surface F_g with genus g and each orientable compact connected surface is homeomorphic to exactly one F_g.

[2]Images from https://pixabay.com

Example 1.6 (Brouwer fixed point theorem). Any continuous map $f : D^n \to D^n$ with $D^n = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$ has a fixed point, i.e., there exists an $x \in D^n$ such that

$$f(x) = x.$$

We give a proof for $n = 1$. Consider the continuous function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) := f(x) - x$. Now $|f| \leq 1$ implies

$$g(-1) \geq -1 - (-1) = 0, \qquad g(1) \leq 1 - 1 = 0.$$

By the intermediate value theorem we can find an x with $g(x) = 0$ which is equivalent to $f(x) = x$.

1.2. Some basic definitions

First of all let us recall the following

Definition 1.7. A subset $U \subset \mathbb{R}^n$ is called *open*

iff

$$\forall x \in U \; \exists r > 0 : B(x, r) \subset U$$

where $B(x, r) = \{y \in \mathbb{R}^n \mid d(x, y) = \|x - y\| < r\}$.

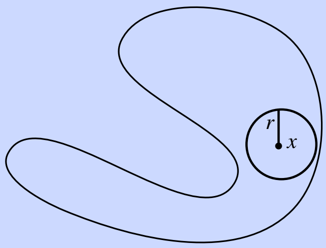


Figure 4. Open subset

The set of open subsets of $\mathbb{R}^n$ is called the *standard topology of* $\mathbb{R}^n$. We write

$$\mathcal{T}_{\mathbb{R}^n} := \{U \subset \mathbb{R}^n \text{ open}\} \subset \mathcal{P}(\mathbb{R}^n).$$

One easily checks

Proposition 1.8. *$\mathcal{T}_{\mathbb{R}^n}$ has the following properties:*

(i) $\emptyset, \mathbb{R}^n \in \mathcal{T}_{\mathbb{R}^n}$;

(ii) $U_j \in \mathcal{T}_{\mathbb{R}^n}, j \in J \Rightarrow \bigcup_{j \in J} U_j \in \mathcal{T}_{\mathbb{R}^n}$;

(iii) $U_1, U_2 \in \mathcal{T}_{\mathbb{R}^n} \Rightarrow U_1 \cap U_2 \in \mathcal{T}_{\mathbb{R}^n}$.

The importance of the concept of open subsets comes from the fact that one can characterize continuous maps entirely in terms of open subsets. Namely, a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff for each $U \in \mathcal{T}_{\mathbb{R}^m}$ the preimage $f^{-1}(U)$ is open, $f^{-1}(U) \in \mathcal{T}_{\mathbb{R}^n}$. This motivates taking open subsets axiomatically as the starting point in topology.

Definition 1.9. A topological space is a pair $(X, \mathcal{T}_X)$ with $\mathcal{T}_X \subset \mathcal{P}(X)$ such that

(i) $\emptyset, X \in \mathcal{T}_X$;

(ii) $U_j \in \mathcal{T}_X, j \in J \Rightarrow \bigcup_{j \in J} U_j \in \mathcal{T}_X$;

(iii) $U_1, U_2 \in \mathcal{T}_X \Rightarrow U_1 \cap U_2 \in \mathcal{T}_X$.

Definition 1.10. Let $(X, \mathcal{T}_X)$ be a topological space. Elements of $\mathcal{T}_X$ are called *open subsets* of X. Subsets of the form $X \setminus U$ with $U \in \mathcal{T}_X$ are called *closed*.

Examples 1.11. 1.) X arbitrary set, $\mathcal{T}_X = \mathcal{P}(X)$ (*discrete topology*). All subsets of X are open and they are also all closed.

2.) X arbitrary set, $\mathcal{T}_X = \{\emptyset, X\}$ (*coarse topology*). Only X and $\emptyset$ are open subsets of X. They are also the only closed subsets.

3.) Let $(X, \mathcal{T}_X)$ be a topological space, let $Y \subset X$ be an arbitrary subset. The *induced topology* or *subspace topology* of Y is defined by $\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}$.

4.) Let (X, d) be a metric space. Then we can imitate the construction of the standard topology on $\mathbb{R}^n$ and define the *induced metric* as the set of all $U \subset X$ such that for each $x \in U$ there exists an $r > 0$ so that $B(x, r) \subset U$. Here $B(x, r) = \{y \in X \mid d(x, y) < r\}$ is the metric ball of radius r centered at x.

Remark 1.12. Let two metrics d_1 and d_2 be given on a set X. If d_1 and d_2 are equivalent, i.e., $\exists\, C \geq 0$ with

$$C^{-1} d_2(x, y) \leq d_1(x, y) \leq C d_2(x, y) \qquad \forall x, y \in X,$$

then d_1 and d_2 induce the same topology.

Remark 1.13. Openness is a relative concept. If $Y \subset X$ be an arbitrary subset of a topological space X, then Y is in general not an open subset of X but it is always open as a subset of itself (w.r.t. the induced topology). The same remark applies to closed subsets.

It is now clear how to define continuous maps between topological spaces.

Definition 1.14. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is called *continuous* iff $\forall U \in \mathcal{T}_Y : f^{-1}(U) \in \mathcal{T}_X$.

Example 1.15. If X carries the discrete topology then every map $f : X \to Y$ is continuous.

Example 1.16. If Y carries the coarse topology then every map $f : X \to Y$ is continuous.

Remark 1.17. In general, a continuous bijective map need not have a continuous inverse. For example, let $\#X > 1$ and consider $f = \mathrm{id}_X : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{coarse}})$.

Remark 1.18. $f : X \to Y$ is continuous iff $f^{-1}(A) \subset X$ is closed for all closed subsets $A \subset Y$.

Definition 1.19. Let X and Y be topological spaces. A bijective continuous map $f : X \to Y$ is called a *homeomorphism* iff $f^{-1} : Y \to X$ is again continuous. If there exists a homeomorphism $f : X \to Y$ then X and Y are called *homeomorphic*. We then write $X \approx Y$.

Remark 1.20. If $f : X \to Y$ and $g : Y \to Z$ are continuous then the composition $g \circ f : X \to Z$ is also continuous.

1.3. Compactness

Definition 1.21. Let X be a topological space. A subset $Y \subset X$ is called *compact* iff for any collection $\{U_i\}_{i \in I}$, $U_i \subset X$ open, with $Y \subset \cup_{i \in I} U_i$ there exist $i_1, \dots, i_n \in I$ such that $Y \subset U_{i_1} \cup \dots \cup U_{i_n}$.

Examples 1.22. 1.) Finite sets are always compact. Namely, et $Y = \{y_1, \dots, y_n\}$ and let $\{U_i\}_{i \in I}$ be an open cover of Y. Then choose i_j such that $y_i \in U_{i_j}$. Then $\{U_{i_1}, \dots, U_{i_n}\}$ still covers Y.

2.) If X carries the discrete topology then a subset $Y \subset X$ is compact if and only if it is finite. We have already seen that finite sets are always compact. Conversely, let Y be a compact subset of X. We cover Y by $\{\{y\} \mid y \in Y\}$. These one-point sets are open because X carries the discrete topology. Since Y is compact this cover must be finite and therefore $\#Y < \infty$.

3.) If X carries the coarse topology then every $Y \subset X$ is compact

4.) A subset $Y \subset \mathbb{R}^n$ is compact iff Y is closed and bounded (Heine-Borel theorem).

Example 1.23. In particular, $\mathbb{R}$ is not compact. We can see this directly by looking at the open cover $\{(x-1, x+1) | x \in \mathbb{R}\}$ of $\mathbb{R}$. Since all intervals in this cover have length 2, any finite subcover can cover only a bounded subset of $\mathbb{R}$.

Proposition 1.24. *Let X be a compact topological space. Let $Y \subset X$ be a closed subset. Then Y is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of Y. Put $U := X \setminus Y \in \mathcal{T}_X$. Then $\{U, U_i\}_{i\in I}$ is an open cover of X. Since X is compact, there exist $i_1, \ldots, i_n$ such that $X \subset U \cup U_{i_1} \cup \cdots \cup U_{i_n}$ and we conclude that $Y \subset U_{i_1} \cup \cdots \cup U_{i_n}$. □

Proposition 1.25. *Let $f : X \to Y$ be continuous. Let $K \subset X$ be compact. Then $f(K) \subset Y$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$, i.e. $f(K) \subset \cup_{i\in I} U_i$. Then we have

$$K \subset f^{-1}\left(f(K)\right) \subset f^{-1}\left(\cup_{i\in I} U_i\right) = \cup_{i\in I} \underbrace{f^{-1}(U_i)}_{\text{open}}.$$

Since K is compact there exist $i_1, \ldots, i_n$ such that for

$$K \subset f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_n}) = f^{-1}(U_{i_1} \cup \cdots \cup U_{i_n})$$

and we conclude that

$$f(K) \subset f(f^{-1}(U_{i_1} \cup \cdots \cup U_{i_n})) \subset U_{i_1} \cup \cdots \cup U_{i_n}.$$ □

1.4. Hausdorff spaces

Definition 1.26. A topological space X is called *Hausdorff* iff $\forall x_1, x_2 \in X$ with $x_1 \neq x_2$ $\exists U_i \subset X$ open with $x_i \in U_i$, such that $U_1 \cap U_2 = \emptyset$.

The Hausdorff property says that any two distinct points can be separated by disjoint open neighborhoods.

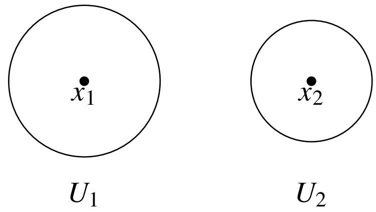


Figure 5. Hausdorff property

Examples 1.27. 1.) Spaces with discrete topology are Hausdorff spaces, simply put $U_i = \{x_i\}$.

2.) Let $\#X \geq 2$. Then X with the coarse topology is not a Hausdorff space.

3.) If the topology of X is induced by a metric d, then X is Hausdorff. Namely, for $x_1 \neq x_2$ put $r := d(x_1, x_2) > 0$. Then the open balls $U_i = B(x_i, r/2)$ separate x_1 and x_2.

4.) Let X be Hausdorff and let $Y \subset X$ any subset. Then Y with its induced topology is again Hausdorff.

Proposition 1.28. *Let X be a Hausdorff space. Let $Y \subset X$ be a compact subset. Then Y is a closed subset.*

Proof. Let $p \in X \setminus Y$. Then for every $y \in Y$ the Hausdorff property tells us that there exist open subsets $U_{y,p}, V_{y,p} \subset X$ such that $y \in U_{y,p}$, $p \in V_{y,p}$ and $U_{y,p} \cap V_{y,p} = \emptyset$. Since Y is compact there exist $y_1, \ldots, y_n \in Y$ such that $Y \subset U_{y_1,p} \cup \cdots \cup U_{y_n,p}$. Now put $V_p := \cap_{j=1}^n V_{y_j,p}$. Since V_p is a finite intersection of open subsets it is open itself. Moreover, $p \in V_p$. Now

$$Y \cap V_p \subset (U_{y_1,p} \cup \cdots \cup U_{y_n,p}) \cap V_p \subset (U_{y_1,p} \cap V_{y_1,p}) \cup \cdots \cup (U_{y_n,p} \cap V_{y_n,p}) = \emptyset,$$

hence $V_p \subset X \setminus Y$. Therefore

$$X \setminus Y = \cup_{p \in X \setminus Y} V_p \subset X \text{ is open.}$$

Thus $Y \subset X$ is closed. □

Corollary 1.29. *Let X be compact, Y Hausdorff. If $f : X \to Y$ is continuous and bijective, then f is a homeomorphism.*

Proof. We have to show that $\forall A \subset X$ closed $f(A) \subset Y$ is closed. Now let $A \subset X$ be closed. Since X is compact, A is compact and also the image $f(A)$ is compact. Since Y is a Hausdorff space we conclude that $f(A) \subset Y$ is closed. □

1.5. Quotient spaces

Let X be a topological space. Let $\sim$ be an equivalence relation on X. For any $x \in X$ let $[x]$ be the equivalence class of x. Denote the set of equivalence classes by

$$X/_{\sim} := \{[x] \mid x \in X\}$$

Let $\pi : X \to X/_{\sim}, x \mapsto [x]$. Define $U \subset X/_{\sim}$ to be open iff $\pi^{-1}(U) \subset X$ is open. One can easily check that this defines a topology on $X/_{\sim}$. Observe that $\pi : X \to X/_{\sim}$ is continuous by definition.
We now state the universal property of the quotient topology: For any topological space Y and for any maps $f : X \to Y$ and $\bar{f} : X/_{\sim} \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ X/_{\sim} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes, f is continuous iff $\bar{f}$ is continuous.

Example 1.30
Let $X = [0,1]$ and let the equivalence relation be given by $x \sim x \quad \forall x \in X$ and $0 \sim 1$.

This equivalence relation identifies the end points of the interval. We expect to obtain a topological space homeomorphic to the circle.

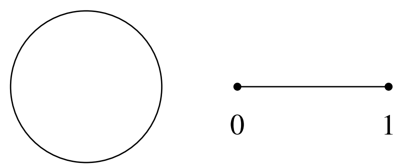


Figure 6. Interval with endpoints identified

Indeed, we can construct such a homeomorphism. Consider $f : X \to S^1$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$. Since $f(0) = f(1)$ there exists a unique $\bar{f} : X/_{\sim} \to S^1$ such that the diagram

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "\pi"] & S^1 \\ {[0,1]/_{\sim}} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes. From the universal property we know that $\bar{f}$ is continuous because f is continuous. Moreover, $\bar{f} : X/_{\sim} \to S^1$ is bijective. Since X is compact (Heine Borel) $\pi(X) = X/_{\sim}$ is compact as well. Since $\mathbb{R}^2$ is Hausdorff, S^1 is also Hausdorff. Hence Corollary 1.29 applies and $\bar{f} : X/_{\sim} \to S^1$ is a homeomorphism.

If X is a topological space and $Y \subset X$ a subset then $X/Y := X/_\sim$ where

$$x_1 \sim x_2 \Leftrightarrow x_1 = x_2 \text{ or } x_1, x_2 \in Y .$$

This equivalence relation identifies all points in Y to one point and performs no further identifications. Example 1.30 is of this form.

Example 1.31. $\mathbb{R}/_{[0,1]}$ is homeomorphic to $\mathbb{R}$.

Example 1.32. $\mathbb{R}/_{(0,\infty)}$ is not a Hausdorff space because the points $[0]$ and $[1]$ cannot be separated.

1.6. Product spaces

Let $X_1, \dots, X_n$ be topological spaces. We set

$$X := X_1 \times \cdots \times X_n$$

Definition 1.33. A subset $U \subset X$ is called *open (for the product topology)* iff for all $p = (p_1, \dots, p_n) \in U$ there exist open subsets $U_i \subset X_i$ with $p_i \in U_i$ and $U_1 \times \cdots \times U_n \subset U$.

It is easy to check that this defines a topology on X.

Examples 1.34. 1.) If all X_i carry the discrete topology then X carries the discrete topology. Namely, the sets $\{(p_1, \dots, p_n)\}$ are open, hence all subsets of the product space are open.

2.) If all X_i carry the coarse topology then X carries the coarse topology.

3.) $\mathbb{R}^n \times \mathbb{R}^m \approx \mathbb{R}^{n+m}$. To see this it is convenient to use the maximum metric on $\mathbb{R}^n$ to characterize the standard topology.

Now we list some important properties of the product topology:

1.) The projection maps $\pi_i : X \to X_i$, $\pi_i(x) = x_i$, are continuous because for any open subset $U_i \subset X_i$

$$\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n$$

is open in X.

2.) Fix $x_i \in X_i$ where $i \in \{1, \dots, i_0 - 1, i_0 + 1, \dots, n\}$. Then the map $\iota : X_{i_0} \to X$ is continuous where $\iota(\xi) = (x_1, \dots, x_{i_0-1}, \xi, x_{i_0+1}, \dots, x_n)$.

3.) Universal property: for all topological spaces Y and for all maps

$$f = (f_1, \dots, f_n) : Y \to X = X_1 \times \cdots \times X_n$$

the map f is continuous if and only if all $f_i : Y \to X_i$ are continuous.

4.) If all X_i are compact then X is also compact.

5.) If all X_i are Hausdorff then X is also Hausdorff.

1.7. Exercises

1.1. Determine all topologies on the set $\{1, 2, 3\}$ and investigate which ones are homeomorphic.

1.2. Let $W^n = [-1, 1] \times \cdots \times [-1, 1] \subset \mathbb{R}^n$ and $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Show that D^n and W^n are homeomorphic.

1.3. Let $\mathcal{T}$ be a topology on $\mathbb{R}$ which contains all half-open intervals $(x, y]$ and $[x, y)$, $x < y$. Show that $\mathcal{T}$ is the discrete topology.

1.4. Show that $(0, 1)$ and $[0, 1]$

a) are not homeomorphic when equipped with the standard topologies induced by $\mathbb{R}$;

b) are homeomorphic when equipped with the discrete topology.

1.5. Let D^n be as in Exercise 1.2. Let $x, y \in \mathring{D}^n$, i.e., $\|x\|, \|y\| < 1$. Construct a homeomorphism $\varphi : D^n \to D^n$ with $\varphi(x) = y$ and $\varphi(z) = z$ for all $z \in \partial D^n$, i.e., $\|z\| = 1$.

1.6. Let X be a topological space, let "$\sim$" be an equivalence relation on X. On page 11 we defined that $U \subset X/\!\sim$ is called open if and only if $\pi^{-1}(U) \subset X$ is open where $\pi : X \to X/\!\sim$ is the standard projection.

a) Show that $X/\!\sim$ equipped with this system of open sets is a topological space.

b) Show that the universal property on page 11 determines the topology on $X/\!\sim$ uniquely, i.e., if $\mathcal{T}$ is a topology on $X/\!\sim$ such that $\pi : (X, \mathcal{T}_X) \to (X/\!\sim, \mathcal{T})$ is continuous and if the universal property holds for $(X/\!\sim, \mathcal{T})$ then $\mathcal{T}$ is the topology defined above.

1.7. Let D^n and W^n be as in Exercise 1.2. Show:

a) $D^n/\partial D^n \approx S^n$;

b) $W^n/\partial W^n \approx S^n$.

1.8. Let $X = \mathbb{R}/\mathbb{Q}$, i.e., the quotient space of $\mathbb{R}$ under the equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Show that X has uncountably many elements but carries the coarse topology.

1.9. Let $X = \mathbb{R}^2/\mathbb{Z}^2$, i.e., the quotient space of $\mathbb{R}^2$ under the equivalence relation $x \sim y$ iff $x - y \in \mathbb{Z}^2$. Let $Y = S^1 \times S^1$ with the product topology. Show that X and Y are homeomorphic.

1.10. Let X be a topological space. One obtains the *cone* CX over X by considering the *cylinder* $X \times [0,1]$ and then passing to the quotient $CX = (X \times [0,1])/\sim$ where the equivalence relation $\sim$ is given by $(x,t) \sim (x',t')$ if and only if $(x,t) = (x',t')$ or $t = t' = 1$. Show:

a) $CS^n \approx D^{n+1}$.

b) If X is compact so is CX.

c) If X is Hausdorff so is CX.

1.11. Let X be a topological space. One obtains the *suspension* ΣX of X as the quotient $\Sigma X = (X \times [0,1])/\sim$ where the equivalence relation $\sim$ is given by $(x,t) \sim (x',t')$ if and only if $(x,t) = (x',t')$ or $t = t' = 1$ or $t = t' = 0$.
If furthermore $f : X \to Y$ is a map then $f \times \mathrm{id} : X \times [0,1] \to Y \times [0,1]$ induces a map $\Sigma f : \Sigma X \to \Sigma Y$. Show:

a) If f is continuous so is Σf.

b) $\Sigma D^n \approx D^{n+1}$.

c) $\Sigma S^n \approx S^{n+1}$.

2. Homotopy Theory

2.1. Homotopic maps

Definition 2.1. Let X and Y be topological spaces. Let $A \subset X$ be a subset. Two continuous maps $f_0, f_1 : X \to Y$ are called *homotopic relative to* A iff there exists a continuous map $F : X \times [0,1] \to Y$ such that

(i) $F(x,0) = f_0(x) \qquad \forall x \in X;$

(ii) $F(x,1) = f_1(x) \qquad \forall x \in X;$

(iii) $F(a,t) = f_0(a) \qquad \forall a \in A, \forall t \in [0,1].$

In symbols, $f_0 \simeq_A f_1$. The map F is then called a *homotopy relative to* A.

Remark 2.2. If $f_0 \simeq_A f_1$ then $f_0\big|_A = f_1\big|_A$.

Remark 2.3. If $A = \emptyset$, then we say "f_0 and f_1 are homotopic" instead of "f_0 and f_1 are homotopic relative to $\emptyset$". Similarly, we write "$f_0 \simeq f_1$" instead of "$f_0 \simeq_\emptyset f_1$".

Examples 2.4. 1.) Let $f_0, f_1 : X \to \mathbb{R}^n$ be continuous with $f_0\big|_A = f_1\big|_A$. Put $F : X \times [0,1] \to \mathbb{R}^n$ with $F(x,t) := t f_1(x) + (1-t) f_0(x)$. Then F is a homotopy from f_0 to f_1 relative to A, hence $f_0 \simeq_A f_1$.

2.) Let $f_0 : \mathbb{R}^n \to Y$ be continuous. Put $F(x,t) := f_0((1-t)x)$, then $f_0 \simeq$ const map.

3.) Lef $f = \mathrm{Exp} : \mathbb{R} \to S^1 \in \mathbb{C}$, $\mathrm{Exp}(x) = e^{2\pi i x}$. From the previous example we know $f \simeq$ const map, but we will shortly see that $f \not\simeq_{\mathbb{Z}}$ const map.
Given two topological spaces X, Y we set

$$C(X,Y) := \{f : X \to Y \mid f \text{ is continuous}\}.$$

Lemma 2.5. *Let X, Y be topological spaces, let $A \subset X$ and let $\varphi \in C(A,Y)$. Then "$\simeq_A$" is an equivalence relation on $\{f \in C(X,Y) \mid f|_A = \varphi\}$.*

Proof. a) "$\simeq_A$" is reflexive:
$f \simeq_A f$, because we can put $F(x,t) := f(x)$.

b) "$\simeq_A$" is symmetric:
Let $f \simeq_A g$. We have to show $g \simeq_A f$. Let $F : X \times [0,1] \to X$ be a homotopy relative to A from f to g. Put $G(x,t) := F(x, 1-t)$. This is a homotopy relative to A from g to f, therefore $g \simeq_A f$.

c) "$\simeq_A$" is transitive:
Let $f \simeq_A g$ and $g \simeq_A h$. We have to show $f \simeq_A h$. Let $F : X \times [0,1] \to Y$ be homotopy relative to A from f to g and let $G : Y \times [0,1] \to X$ be homotopy relative to A from g to h. Then put $H : X \times [0,1] \to Y$,

$$H(x,t) := \begin{cases} F(x,2t), & 0 \le t \le 1/2 \\ G(x,2t-1), & 1/2 \le t \le 1 \end{cases}$$

This is a homotopy relative to A from f to h, thus $f \simeq_A h$. □

Lemma 2.6. *Let X, Y, Z be topological spaces. Let $A \subset X, B \subset Y$ be subsets. Let $f_0, f_1 \in C(X,Y)$ be such that $f_0 \simeq_A f_1, f_i(A) \subset B$ and let $g_0, g_1 \in C(Y,Z)$ be such that $g_0 \simeq_B g_1$. Then $g_0 \circ f_0 \simeq_A g_1 \circ f_1$.*

Proof. Let $F : X \times [0,1] \to Y$ be homotopy relative to A from f_0 to f_1 and $G : Y \times [0,1] \to Z$ be homotopy relative to B from g_0 to g_1 Then $H : X \times [0,1] \to Z$ is a homotopy relative to A from $g_0 \circ f_0$ to $g_1 \circ f_1$ where $H(x,t) = G(F(x,t),t)$. □

Definition 2.7. A map $f \in C(X,Y)$ is called a *homotopy equivalence* iff there exists a $g \in C(Y,X)$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ If there exists a homotopy equivalence $f : X \to Y$ then X and Y are *homotopy equivalent*. In symbols, $X \simeq Y$.

Remark 2.8. This defines an equivalence relation on the class of topological spaces.

Remark 2.9. Obviously, homeomorphic implies homotopy equivalent, in short,

$$X \approx Y \Rightarrow X \simeq Y.$$

Example 2.10. Euclidean space is homotopy equivalent to a point, $\mathbb{R}^n \simeq \{0\}$. Namely, put $f : \{0\} \to \mathbb{R}^n$, $f(0) = 0$, and $g : \mathbb{R}^n \to \{0\}$, $g(x) = 0$. Then $g \circ f = \mathrm{id}_{\{0\}}$ and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n}$ by Example 2.4.1. Remark 2.8 implies $\mathbb{R}^n \simeq \mathbb{R}^m$ for all $n, m \in \mathbb{N}$.

Definition 2.11. A topological space is called *contractible* iff it is homotopy equivalent to $\{\text{point}\}$.

Definition 2.12. Let $A \subset X$ and let $\iota : A \longrightarrow X$ be the inclusion map. Then A is called

1.) a *retract* of X iff there exists $r \in C(X, A)$ such that $r\big|_A = \mathrm{id}_A$, i.e. $r \circ \iota = \mathrm{id}_A$. Then r is called a *retraction* from X to A.

2.) a *deformation retract* of X iff there exists a retraction $r : X \longrightarrow A$ such that $\iota \circ r \simeq \mathrm{id}_X$.

3.) a *strong deformation retract* of X iff there exists a retraction $r : X \longrightarrow A$ such that $\iota \circ r \simeq_A \mathrm{id}_X$.

Examples 2.13. 1.) Let X any topological space, let $A = \{x_0\} \subset X$ consist of just one point. Then $r : X \longrightarrow A$, $r(x) = x_0$, is a retraction, hence A is a retract of X. The one-pointed set A is a deformation retract of X iff X is contractible.

2.) Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and $A = S^n$. Consider the map $r : X \longrightarrow A$ with $r(x) = \frac{x}{\|x\|}$. The composition $\iota \circ r : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ then satisfies $\iota \circ r = \frac{x}{\|x\|}$. The map $F \in C(\mathbb{R}^{n+1} \setminus \{0\} \times [0, 1], \mathbb{R}^{n+1} \setminus \{0\})$ given by

$$F(x, t) = tx + (1 - t)\frac{x}{\|x\|}$$

is continuous and satisfies

$$F(x, 0) = (\iota \circ r)(x), \qquad F(x, 1) = \mathrm{id}(x)$$

for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and

$$F(a, t) = a, \qquad a \in S^n$$

We thus conclude that $\iota \circ r \simeq_{S^n} \mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$, hence S^n is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

The difference between a deformation retract and a strong deformation retract is rather subtle.

Example 2.14
We consider the *comb space* given by

$$X = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \text{ and } (x = 0 \text{ or } x = \tfrac{1}{n} \text{ for some } n \in \mathbb{N})\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ and } 0 \le x \le 1\right\}$$

The space X is a bounded and closed subset of $\mathbb{R}^2$, hence compact. Let the set A be given by $A = \{(0, 1)\}$.

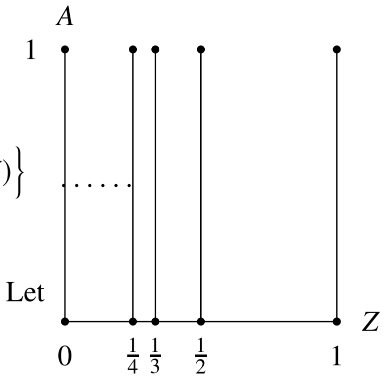


Figure 7. Comb space

First we show that A is a deformation retract of X. The map $F : X \times [0, 1] \to X$ given by $F(x, y, t) := (x, (1 - t)y)$ is continuous and satisfies

$$F(x, y, 0) = (x, y), \qquad F(x, y, 1) = (x, 0) .$$

Therefore $\mathrm{id}_X \simeq \mathrm{Inclusion}_{Z \to X} \circ \pi$ where $\pi : X \to [0, 1] \times \{0\} =: Z$ is the projection $\pi(x, y) = (x, 0)$. Moreover, we have $\pi \circ \mathrm{Inclusion}_{Z \to X} = \mathrm{id}_Z$. This shows that π is a homotopy equivalence between X and Z. Hence $X \simeq Z \approx [0, 1] \simeq \{\mathrm{pt}\}$, which means that X is contractible. Therefore A is a deformation retract of X.

Now we show that A is not a strong deformation retract of X. Suppose it were, then there would exist a continuous map $G : X \times [0, 1] \to X$, such that for all $t \in [0, 1]$ and all $(x, y) \in X$

$$G(x, y, 0) = (x, y), \qquad G(0, 1, t) = (0, 1).$$

Since $X \times [0, 1]$ is compact the map G would be uniformly continuous. Therefore for $\varepsilon = 1/2$ we can find $\delta > 0$ such that

$$\|G(x, y, t) - G(x', y', t')\| < \tfrac{1}{2}$$

whenever $|x - x'| < \delta$, $|y - y'| < \delta$ and $|t - t'| < \delta$.

Now choose n so large that $\frac{1}{n} < \delta$ and consider

$$(x, y) = \left(\tfrac{1}{n}, 1\right), \quad (x', y') = (0, 1), \quad t = t'.$$

Then

$$\left\| G\left(\tfrac{1}{n}, 1, t\right) - G(0, 1, t) \right\| < \tfrac{1}{2}, \qquad \forall t \in [0, 1] .$$

Hence $G(\frac{1}{n}, 1, t) \in B((0, 1), \frac{1}{2})$ for all $t \in [0, 1]$.

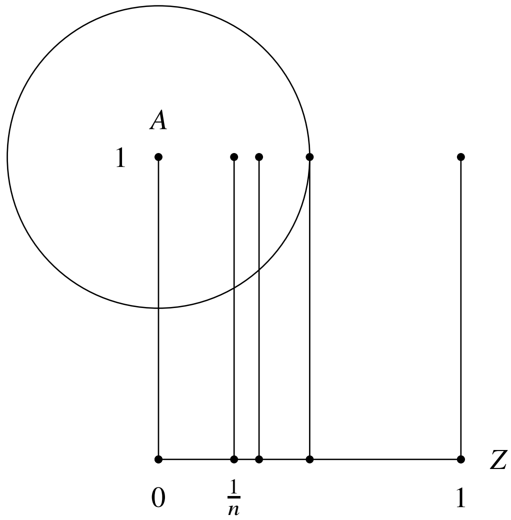


Figure 8. A is not a strong deformation retract of X

On the other hand the mapping $t \mapsto G(\frac{1}{n}, 1, t)$ is a continuous path in X from $(\frac{1}{n}, 1)$ to $(0, 1)$ and must take values in Z for some t.

Remark 2.15. We have the following scheme of implications:

$$\begin{array}{ccc} & A \text{ is a strong deformation retract of } X & \\ & \Downarrow & \\ & A \text{ is a deformation retract of } X & \\ \swarrow & & \searrow \\ A \text{ is a retract of } X & \begin{array}{c}\not\Leftarrow \\ \not\Rightarrow\end{array} & A \simeq X \end{array}$$

That both possible implications in the bottom row do not hold in general can be seen by counterexamples. Let $A = \{x_0\}$ be a point in X and let X be not contractible. Then A is a retract of X but A and X are not homotopically equivalent. This is a counterexample for the implication " $\Rightarrow$ ".

A counterexample for the other direction " $\Leftarrow$ " is given by $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and A = comb space. Then X and A are contractible, hence $X \simeq A$, but one can show that there is no retraction $X \longrightarrow A$.

2.2. The fundamental group

Definition 2.16. Let X be a topological space and let $x_0, x_1, x_2 \in X$. Put

$$\Omega(X; x_0, x_1) := \{\omega \in C([0,1], X) \mid \omega(0) = x_0, \omega(1) = x_1\} \quad \text{and}$$
$$\Omega(X; x_0) := \Omega(X; x_0, x_0)\,.$$

Elements of $\Omega(X; x_0, x_1)$ are called *paths* and elements of $\Omega(X; x_0)$ *loops*.
For $\omega \in \Omega(X; x_0, x_1)$ and $\eta \in \Omega(X; x_1, x_2)$ define $\omega \star \eta \in \Omega(X; x_0, x_2)$ by

$$(\omega \star \eta)(t) = \begin{cases} \omega(2t), & 0 \le t \le 1/2, \\ \eta(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Moreover, we consider $\omega^{-1} \in \Omega(X; x_1, x_0)$ with $\omega^{-1}(t) = \omega(1-t)$ and $\varepsilon_{x_0} \in \Omega(X; x_0)$ where $\varepsilon_{x_0}(t) = x_0$.

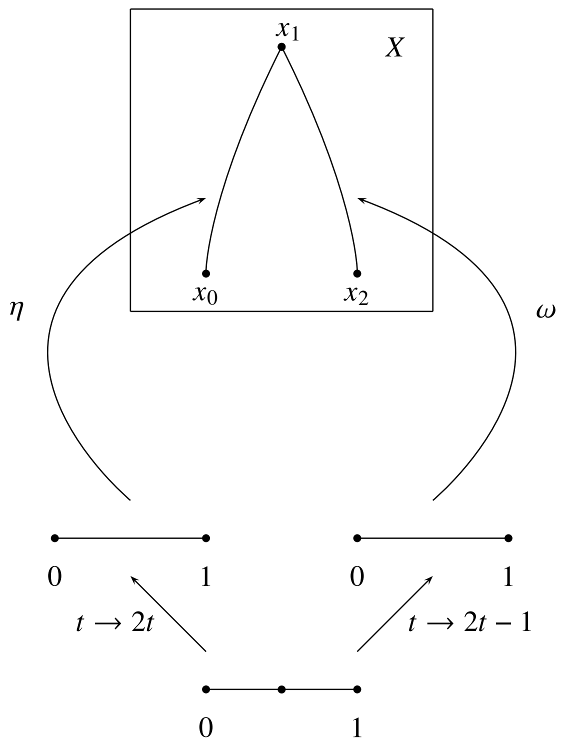


Figure 9. Concatenation of paths

Definition 2.17. For $\omega \in \Omega(X; x_0)$ denote by $[\omega]$ the homotopy class of ω relative to $\{0, 1\}$. Then

$$\pi_1(X; x_0) := \{[\omega] \mid \omega \in \Omega(X; x_0)\}$$

is called the *fundamental group* of (X, x_0).

Lemma 2.18. *For $\omega, \omega', \eta, \eta', \zeta \in \Omega(X, x_0)$ we have*

(i) If $\omega \simeq_{\{0,1\}} \omega'$ and $\eta \simeq_{\{0,1\}} \eta'$, then $\omega \star \eta \simeq_{\{0,1\}} \omega' \star \eta'$;

(ii) $\varepsilon_{x_0} \star \omega \simeq_{\{0,1\}} \omega \simeq_{\{0,1\}} \omega \star \varepsilon_{x_0}$;

(iii) $\omega \star \omega^{-1} \simeq_{\{0,1\}} \varepsilon_{x_0} \simeq_{\{0,1\}} \omega^{-1} \star \omega$;

(iv) $(\omega \star \eta) \star \zeta \simeq_{\{0,1\}} \omega \star (\eta \star \zeta)$.

Proof. The proof will be given graphically. In the following diagrams we draw the domain of the required homotopies. The horizontal axis denotes the loop parameter whereas the vertical axis represents the deformation parameter. The interpolations are piecewise linear. The red area gets mapped to x_0.

(i) We run the loop parameter with twice the speed.

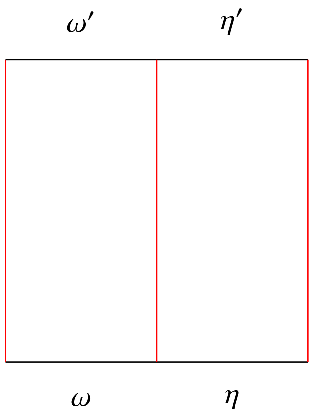


Figure 10. Concatenations of homotopic paths are homotopic.

In formulas, if we denote the homotopy between ω and ω' by F and the one between η and η' by G, then the homotopy $H : [0, 1] \times [0, 1] \to X$ between $\omega \star \eta$ and $\omega' \star \eta'$ is given by

$$H(t, s) = \begin{cases} F(2t, s), & 0 \leq t \leq \frac{1}{2}, \\ G(2t - 1, s), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

(ii) The first diagram in Figure **??** shows $\varepsilon_{x_0} \star \omega \simeq_{\{0,1\}} \omega$, the second proves that $\omega \simeq_{\{0,1\}} \omega \star \varepsilon_{x_0}$.

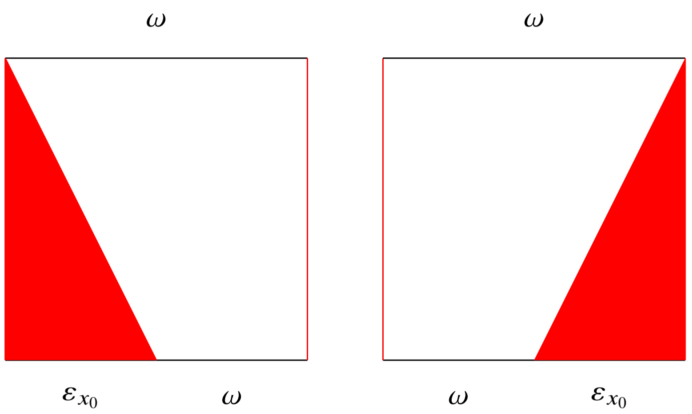


Figure 11. Concatenation with constant path is homotopic to original path

In formulas, the homotopy $H : [0,1] \times [0,1] \longrightarrow X$ between $\varepsilon_{x_0} \star \omega$ and ω is given by

$$H(t,s) = \begin{cases} x_0, & 0 \le t \le \frac{1-s}{2}, \\ \omega\left(\frac{2t}{1+s} - \frac{1-s}{1+s}\right), & \frac{1-s}{2} \le t \le 1, \end{cases}$$

and similarly for $\omega \star \varepsilon_{x_0}$.
(iii) The statement $\omega \star \omega^{-1} \simeq_{\{0,1\}} \varepsilon_{x_0}$ is proven by the following diagram

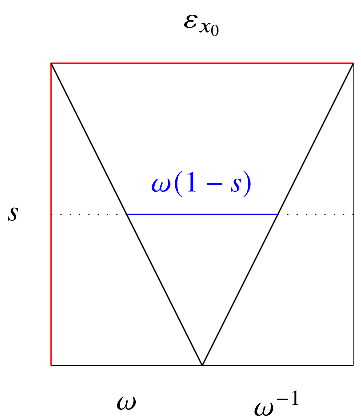


Figure 12. Inverse modulo homotopy

For any $s \in [0,1]$ the corresponding ”blue“ line segment gets mapped to $\omega(1-s)$. In formulas, the homotopy $H : [0,1] \times [0,1] \longrightarrow X$ between $\omega \star \omega^{-1}$ and ε_{x_0} is given by

$$H(t,s) = \begin{cases} \omega(2t), & 0 \le t \le \frac{1-s}{2}, \\ \omega(1-s), & \frac{1-s}{2} \le t \le \frac{1+s}{2}, \\ \omega(2-2t), & \frac{1+s}{2} \le t \le 1. \end{cases}$$

To prove the statement $\varepsilon_{x_0} \simeq_{\{0,1\}} \omega^{-1} \star \omega$ interchange the roles of ω and ω^{-1}.
(iv) The last assertion follows from the homotopy indicated in Figure 13.

□

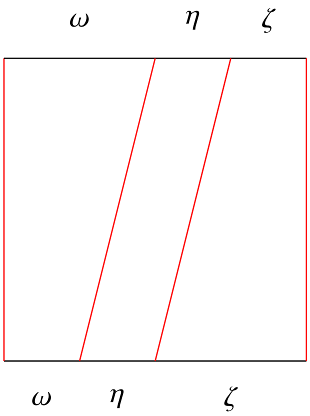


Figure 13. Associativity

Lemma 2.18 has the following implications:

(i) $\Rightarrow [\omega] \cdot [\eta] := [\omega \star \eta]$ is well defined;

(ii) $\Rightarrow [\varepsilon_{x_0}] \cdot [\omega] = [\omega] = [\omega] \cdot [\varepsilon_{x_0}]$;

(iii) $\Rightarrow [\omega] \cdot [\omega^{-1}] = [\varepsilon_{x_0}] = [\omega^{-1}] \cdot [\omega]$;

(iv) $\Rightarrow ([\omega] \cdot [\eta]) \cdot [\zeta] = [\omega] \cdot ([\eta] \cdot [\zeta])$.

Hence $\pi_1(X; x_0)$ together with "$\cdot$" is a group with neutral element $1 := [\varepsilon_{x_0}]$ and inverse element $[\omega]^{-1} = [\omega^{-1}]$.
To any topological space with a preferred point we have associated a group, the fundamental group of the space with that point. Now we consider continuous maps. Let $f \in C(X, Y)$ and $x_0 \in X$. We put $f(x_0) =: y_0 \in Y$. If $\omega \simeq_{\{0,1\}} \omega'$ and H is a homotopy between them relative to $\{0, 1\}$ then $f \circ H$ is a homotopy between $f \circ \omega$ and $f \circ \omega'$ relative to $\{0, 1\}$. Hence $f \circ \omega \simeq_{\{0,1\}} f \circ \omega'$. Therefore we have a well-defined map $f_\# : \pi_1(X; x_0) \to \pi_1(Y; y_0)$, $f_\#([\omega]) = [f \circ \omega]$.

Lemma 2.19. *(i) The map $f_\# : \pi_1(X; x_0) \to \pi_1(Y; y_0)$ is a group homomorphism.*

(ii) It has the functorial properties

a) $(f \circ g)_\# = f_\# \circ g_\#$*;*

b) $(\mathrm{id}_X)_\# = \mathrm{id}_{\pi_1(X; x_0)}$.

(iii) If $f \simeq_{\{x_0\}} f'$ then $f_\# = f'_\#$.

Proof. (i) From the definitions we have $f \circ (\omega \star \eta) = (f \circ \omega) \star (f \circ \eta)$ and hence

$$f_\#([\omega] \cdot [\eta]) = f_\#([\omega \star \eta]) = [f \circ (\omega \star \eta)] = [(f \circ \omega) \star (f \circ \eta)] = f_\#([\omega]) \cdot f_\#([\eta]).$$

(ii) Assertion b) being obvious we compute a):

$$(f \circ g)_\#([\omega]) = [(f \circ g) \circ \omega] = [f \circ (g \circ \omega)] = f_\#([g \circ \omega]) = f_\#(g_\#([\omega])) = (f_\# \circ g_\#)([\omega]) .$$

(iii) By Lemma 2.6, $f \simeq_{\{0,1\}} f'$ implies $f \circ \omega \simeq_{\{0,1\}} f' \circ \omega$. We conclude

$$f_\#([\omega]) = [f \circ \omega] = [f' \circ \omega] = f'_\#([\omega])$$

which proves the statement. □

Corollary 2.20. *If $f : X \to Y$ is a homeomorphism with $f(x_0) = y_0$ then*

$$f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a group isomorphism.

Proof. We only use the functorial properties:

$$(f^{-1})_\# \circ (f_\#) = (f^{-1} \circ f)_\# = (\mathrm{id}_X)_\# = \mathrm{id}_{\pi_1(X;x_0)}$$

and similarly one sees that $(f_\#) \circ (f^{-1})_\# = \mathrm{id}_{\pi_1(Y;y_0)}$. Thus $f_\sharp$ is an isomorphism with $(f_\#)^{-1} = (f^{-1})_\#$. □

Now we want to deal with the question to what extent $\pi_1(X; x_0)$ depends on the choice of x_0.

To study this question let $x_0, x_1 \in X$ and assume there exists a path $\gamma \in \Omega(X; x_0, x_1)$. If such a path does not exist $\pi_1(X; x_0)$ and $\pi_1(X; x_1)$ are not related.

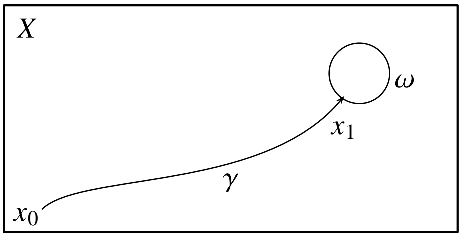


Figure 14. Dependence on base point

Look at the map $\Phi_\gamma : \Omega(X; x_1) \to \Omega(X; x_0)$ where $\omega \mapsto (\gamma \star \omega) \star \gamma^{-1}$. Applying Lemma 2.18 twice we know that if $\omega \simeq_{\{0,1\}} \omega'$ then $\gamma \star \omega \simeq_{\{0,1\}} \gamma \star \omega'$ and hence $(\gamma \star \omega) \star \gamma^{-1} \simeq_{\{0,1\}} (\gamma \star \omega') \star \gamma^{-1}$. Thus the map

$$\begin{aligned} \hat{\Phi}_\gamma : \pi_1(X; x_1) &\to \pi_1(X; x_0), \\ [\omega] &\mapsto [\Phi_\gamma(\omega)] = [(\gamma \star \omega) \star \gamma^{-1}], \end{aligned}$$

is well defined.

Proposition 2.21. *Let X be a topological space, assume $x_0, x_1 \in X$ and $\gamma, \gamma' \in \Omega(X; x_0, x_1)$. Then*

1.) The map $\hat{\Phi}_\gamma : \pi_1(X; x_1) \to \pi_1(X; x_0)$ is a group isomorphism.

2.) If $\gamma \simeq_{\{0,1\}} \gamma'$ then $\hat{\Phi}_\gamma = \hat{\Phi}_{\gamma'}$.

3.) For $\beta \in \Omega(X; x_1, x_2)$ we have $\hat{\Phi}_{\gamma\star\beta} = \hat{\Phi}_\gamma \circ \hat{\Phi}_\beta$.

4.) For the constant path we have $\hat{\Phi}_{\varepsilon_{x_0}} = \mathrm{id}_{\pi_1(X;x_0)}$.

5.) For any $[\omega] \in \pi_1(X; x_1)$ we have $\hat{\Phi}_{\gamma'}([\omega]) = \kappa \cdot \hat{\Phi}_\gamma([\omega]) \cdot \kappa^{-1}$ where $\kappa = [\gamma' \star \gamma^{-1}] \in \pi_1(X; x_0)$.

Proof. a) Assertions 2., 3., and 4. follow directly from Lemma 2.18 and the definitions.

b) The map $\hat{\Phi}_\gamma$ is a group homomorphism because

$$\begin{aligned}
\hat{\Phi}_\gamma\big([\omega] \cdot [\eta]\big) &= \hat{\Phi}_\gamma([\omega \star \eta]) \\
&= \big[(\gamma \star (\omega \star \eta)) \star \gamma^{-1}\big] \\
&= \big[\big(\gamma \star ((\omega \star (\gamma^{-1} \star \gamma)) \star \eta) \star \gamma^{-1}\big)\big] \\
&= \big[\big((\gamma \star \omega) \star \gamma^{-1}\big) \star \big((\gamma \star \eta) \star \gamma^{-1}\big)\big] \\
&= \hat{\Phi}_\gamma([\omega]) \cdot \hat{\Phi}_\gamma([\eta]).
\end{aligned}$$

The map $\hat{\Phi}_\gamma$ is bijective, because

$$\hat{\Phi}_\gamma \circ \hat{\Phi}_{\gamma^{-1}} \overset{3.}{=} \hat{\Phi}_{\gamma\star\gamma^{-1}} \overset{2.}{=} \hat{\Phi}_{\varepsilon_{x_0}} \overset{4.}{=} \mathrm{id}_{\pi_1(X;x_0)} \ .$$

and similarly $\hat{\Phi}_{\gamma^{-1}} \circ \hat{\Phi}_\gamma = \mathrm{id}_{\pi_1(X;x_1)}$. This proves 1.

c) We compute

$$\begin{aligned}
\kappa \cdot \hat{\Phi}_\gamma([\omega]) \cdot \kappa^{-1} &= \big[\gamma' \star \gamma^{-1}\big] \cdot \big[\gamma \star \omega \star \gamma^{-1}\big] \cdot \big[\gamma \star (\gamma')^{-1}\big] \\
&= \big[\gamma' \star \gamma^{-1} \star \gamma \star \omega \star \gamma^{-1} \star \gamma \star (\gamma')^{-1}\big] \\
&= \big[\gamma' \star \omega \star (\gamma')^{-1}\big] \\
&= \hat{\Phi}_{\gamma'}([\omega]).
\end{aligned}$$

□

Proposition 2.22. *Let X, Y be topological spaces and $x_0 \in X$. Let $f, g \in C(X, Y)$ and let $H : X \times [0,1] \to Y$ be a homotopy from f to g. Define $\eta \in \Omega(Y; f(x_0), g(x_0))$ by*

$$\eta(s) := H(x_0, s).$$

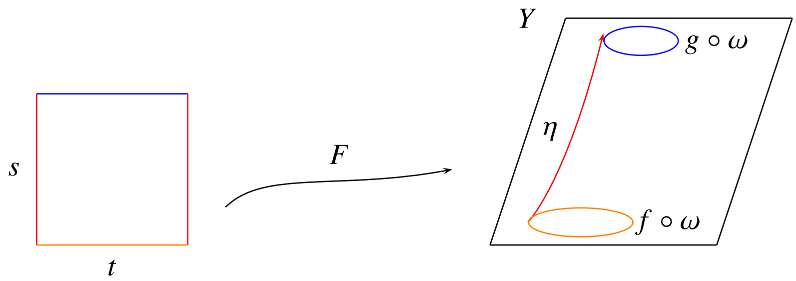


Figure 15. Auxiliary homotopy

Then the following diagram commutes:

$$\begin{tikzcd}
\pi_1(X; x_0) \arrow[d, "g_\#"] \arrow[dr, "f_\#"] & \\
\pi_1(Y; g(x_0)) \arrow[r, "\hat{\Phi}_\eta"'] & \pi_1(Y; f(x_0))
\end{tikzcd}$$

Proof. Let $[\omega] \in \pi_1(X; x_0)$ and define $F : [0,1] \times [0,1] \to Y$ by $F(t,s) := H(\omega(t), s)$.

The deformation indicated in Figure 16 yields a homotopy relative to $\{0,1\}$ from $f \circ \omega$ to $\eta \star (g \circ \omega) \star \eta^{-1}$. We conclude that

$$f_\#[\omega] = [f \circ \omega] = [\eta \star (g \circ \omega) \star \eta^{-1}] = \hat{\Phi}_\eta(g_\#([\omega])) .$$

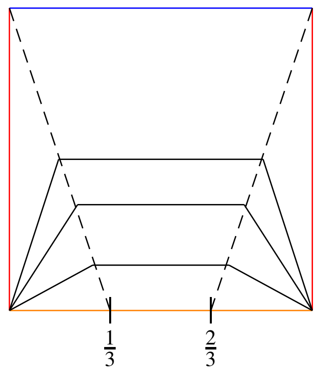


Figure 16. The deformation

$\square$

Now we can improve Corollary 2.20 and show that homotopy equivalent spaces have isomorphic fundamental groups.

Theorem 2.23. *Let $f : X \to Y$ be a homotopy equivalence. Then*

$$f_\# : \pi_1(X; x_0) \to \pi_1(Y; f(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. Let $g : Y \to X$ be a homotopy inverse of f, i.e., $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We know by Proposition 2.22 that for a suitable $\eta \in \Omega(X; x_0, f(g(x_0)))$

$$g_\# \circ f_\# = (g \circ f)_\# = \hat{\Phi}_\eta \circ (\mathrm{id}_X)_\# = \hat{\Phi}_\eta \circ \mathrm{id}_{\pi_1(X;x_0)} = \hat{\Phi}_\eta .$$

Hence $g_\# \circ f_\#$ is an isomorphism. In particular, $f_\#$ is injective. Similarly, we can show that $f_\# \circ g_\#$ is an isomorphism, hence $f_\#$ is surjective. Therefore $f_\#$ is an isomorphism. □

Corollary 2.24. *If $A \subset X$ is a deformation retract then the inclusion $\iota : A \to X$ induces an isomorphism $\iota_\# : \pi_1(A, x_0) \to \pi_1(X; x_0)$ for any $x_0 \in A$.*

Proof. If $A \subset X$ is a deformation retract then the map $\iota : A \to X$ is a homotopy equivalence. Theorem 2.23 yields the claim. □

Example 2.25. If X is contractible then the one-point set $A = \{x_0\} \subset X$ is a deformation retract. Hence $\pi_1(X; x_0) \simeq \pi_1(A; x_0) = \{[\varepsilon_{x_0}]\} = \{1\}$. Thus contactible spaces have trivial fundamental group.

2.3. The fundamental group of the circle

Recall the map $\mathrm{Exp} : \mathbb{R} \to S^1 \subset \mathbb{C}$ where $\mathrm{Exp}(t) = e^{2\pi i t}$.

Lemma 2.26. *Let $t_0 \in \mathbb{R}$ and $z_0 = \mathrm{Exp}(t_0) \in S^1$. Then for all $f \in C(S^1, S^1)$ with $f(1) = z_0$ there exists a unique $\hat{f} \in C(\mathbb{R}, \mathbb{R})$ with $\hat{f}(0) = t_0$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\hat{f}"] \arrow[d, "\mathrm{Exp}"'] & \mathbb{R} \arrow[d, "\mathrm{Exp}"] \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Proof. a) First we show uniqueness of the map $\hat{f}$.

Assume that besides $\hat{f}$ there is a second map $\tilde{f} \in C(\mathbb{R}, \mathbb{R})$ with the same properties as $\hat{f}$. The equality of $\mathrm{Exp}(x) = \mathrm{Exp}(x')$ is equivalent to $x - x' \in \mathbb{Z}$. Since $\mathrm{Exp}(\hat{f}(t)) = f(\mathrm{Exp}(t)) = \mathrm{Exp}(\tilde{f}(t))$ we deduce that $\hat{f}(t) - \tilde{f}(t) \in \mathbb{Z}$ for all $t \in \mathbb{R}$. Since both $\tilde{f}, \hat{f}$ are continuous it follows that $\hat{f} - \tilde{f}$ is constant. Finally, we know that $\hat{f}(0) = t_0 = \tilde{f}(0)$, hence $\tilde{f} = \hat{f}$.

b) Now we show existence of $\hat{f}$.

Since S^1 is compact, f is uniformly continuous. Since Exp is also uniformly continuous, the composition $f \circ \mathrm{Exp} : \mathbb{R} \to S^1$ is uniformly continuous. Hence there exists $\varepsilon > 0$ such that $f(\mathrm{Exp}(I)) \subset S^1$ is contained in a semi-circle for every interval $I \subset \mathbb{R}$ with length $\leq \varepsilon$.

For any closed semi-circle $C \subset S^1$ the pre-image $\mathrm{Exp}^{-1}(C) \subset \mathbb{R}$ is the disjoint union of compact intervals of length $\frac{1}{2}$. More precisely,

$$\mathrm{Exp}^{-1}(C) = \bigcup_{k\in\mathbb{Z}} \left[t_1 + k, t_1 + k + \frac{1}{2}\right].$$

Moreover, the restriction of Exp to each of these intervals is a homeomorphism onto its image,

$$\mathrm{Exp}|_{[t_1+k, t_1+k+\frac{1}{2}]} : \left[t_1 + k, t_1 + k + \frac{1}{2}\right] \xrightarrow{\approx} C.$$

Its inverse can written down explicitly in terms of logarithms but we will not need this.

Now we construct $\hat{f}$ step by step.

Since $f(\mathrm{Exp}([0,\varepsilon]))$ is contained in a closed semi-circle C_0 we can define $\hat{f}$ on $[0,\varepsilon]$ by

$$\hat{f} := (\mathrm{Exp}\,|_{I_0})^{-1} \circ f \circ \mathrm{Exp},$$

where $I_0 \subset \mathbb{R}$ is the compact interval of length $\frac{1}{2}$ with $\mathrm{Exp}(I_0) = C_0$ and $t_0 \in I_0$. This insures that

$$\hat{f}(0) = (\mathrm{Exp}\,|_{I_0})^{-1} \circ f \circ \mathrm{Exp}(0) = (\mathrm{Exp}\,|_{I_0})^{-1} \circ f(1) = (\mathrm{Exp}\,|_{I_0})^{-1}(z_0) = t_0.$$

Put $t_1 := \hat{f}(\varepsilon)$.

Next, $f(\mathrm{Exp}([\varepsilon, 2\varepsilon]))$ is contained in a closed semi-circle C_1 and we define $\hat{f}$ on $[\varepsilon, 2\varepsilon]$ by

$$\hat{f} := (\mathrm{Exp}\,|_{I_1})^{-1} \circ f \circ \mathrm{Exp}$$

where $I_1 \subset \mathbb{R}$ is the compact interval of length $\frac{1}{2}$ with $\mathrm{Exp}(I_1) = C_1$ and $t_1 \in I_1$. This insures that the two definitions of $\hat{f}$ at ε agree so that we obtain a continuous function $\hat{f} : [0, 2\varepsilon] \to \mathbb{R}$.

Repeating this procedure infinitely many times we can extend $\hat{f}$ continuously to $[0,\infty) \to \mathbb{R}$. The extension to the left is done similarly so that we obtain a continuous function $\hat{f} : \mathbb{R} \to \mathbb{R}$. Commutativity of the diagram holds by construction. □

Definition 2.27. For a map $f \in C(S^1, S^1)$ a map $\hat{f} \in C(\mathbb{R},\mathbb{R})$ for which the diagram in Lemma 2.26 commutes is called a *lift* of f.

Example 2.28. Consider the map $f_n : S^1 \to S^1$ with $f_n = z^n, n \in \mathbb{Z}$. Then we have

$$f_n\big(\operatorname{Exp}(t)\big) = \operatorname{Exp}(t)^n = \operatorname{Exp}(nt)\,.$$

Hence the map $\hat{f}_n$ given by $\hat{f}_n(t) = nt$ is a lift of f_n.

Definition 2.29. For $f \in C(S^1, S^1)$ we call

$$\deg(f) := \hat{f}(1) - \hat{f}(0)$$

the *degree* of f, where $\hat{f}$ is a lift of the map f.

Remark 2.30

1.) We have seen that different lifts of f differ by an additive constant in $\mathbb{Z}$. Hence the degree $\deg(f)$ is well defined, independently of the choice of lift $\hat{f}$.

2.) The degree $\deg(f)$ is an integer because

$$\operatorname{Exp}\left(\hat{f}(1)\right) = f\big(\operatorname{Exp}(1)\big) = f(1) = f\big(\operatorname{Exp}(0)\big) = \operatorname{Exp}\left(\hat{f}(0)\right)$$

Hence we have $\deg(f) = \hat{f}(1) - \hat{f}(0) \in \mathbb{Z}$.

3.) The map $t \mapsto \hat{f}(t+1) - \hat{f}(t)$ is continuous and takes values in $\mathbb{Z}$, by the same argument as above. We conclude that $\hat{f}(t+1) - \hat{f}(t) = \deg(f)$ for all $t \in \mathbb{R}$.

4.) For $k \in \mathbb{Z}$ we compute

$$\begin{aligned}
\hat{f}(t_0 + k) - \hat{f}(t_0) \quad &= \quad \hat{f}(t_0 + k) - \hat{f}(t_0 + (k-1)) \\
&\qquad + \hat{f}(t_0 + (k-1)) - \hat{f}(t_0 + (k-2)) \\
&\qquad + \cdots + \\
&\qquad + \hat{f}(t_0 + 1) - \hat{f}(t_0) \\
&\overset{(3)}{=} \quad k \deg(f)\,.
\end{aligned}$$

5.) For $f, g \in C(S^1, S^1)$ and lifts $\hat{f}, \hat{g}$ we compute

$$\operatorname{Exp}\left((\hat{f} + \hat{g})(t)\right) = \operatorname{Exp}\left(\hat{f}(t)\right)\operatorname{Exp}\left(\hat{g}(t)\right) = f\big(\operatorname{Exp}(t)\big)g\big(\operatorname{Exp}(t)\big)\,.$$

Hence $\hat{f} + \hat{g}$ is a lift of fg and we get the following formula for the degree

$$\deg(fg) = \hat{f}(1) + \hat{g}(1) - \left(\hat{f}(0) + \hat{g}(0)\right) = \deg(f) + \deg(g)\,.$$

6.) For $f, g \in C(S^1, S^1)$ and lifts $\hat{f}, \hat{g}$ we compute

$$\operatorname{Exp}\left(\hat{f}(\hat{g}(t))\right) = f\Big(\operatorname{Exp}\left(\hat{g}(t)\right)\Big) = f\Big(g\big(\operatorname{Exp}(t)\big)\Big)$$

and therefore $\hat{f} \circ \hat{g}$ is a lift of $f \circ g$. For the degree of $f \circ g$ this means

$$\deg(f \circ g) = \hat{f}\big(\hat{g}(1)\big) - \hat{f}\big(\hat{g}(0)\big) = \hat{f}\big(\hat{g}(0) + \deg(g)\big) - \hat{f}\big(\hat{g}(0)\big) \overset{(4)}{=} \deg(g)\deg(f)\,,$$

hence $\deg(f \circ g) = \deg(g)\deg(f)$.

7.) Let $f \in C(S^1, S^1)$ with $\deg(f) \neq 0$. We show that f must then be surjective.

Namely, let $\hat{f}$ be a lift of f. Then $\deg(f) = \hat{f}(1) - \hat{f}(0)$ is an integer, not equal to 0. Then, if $\hat{f}(1) > \hat{f}(0)$, the whole interval $I := [\hat{f}(0), \hat{f}(1)]$ must be contained in the image of $\hat{f}$ by the intermediate value theorem. If $\hat{f}(1) < \hat{f}(0)$ this holds for $I := [\hat{f}(1), \hat{f}(0)]$. In either case I is an interval of length at least 1, hence $\operatorname{Exp}(I) = S^1$. We conclude

$$S^1 = \operatorname{Exp}(I) \subset \operatorname{Exp}(\operatorname{im}(\hat{f})) = \operatorname{im}(f).$$

Thus f is onto.

Example 2.31. For the map $f_n : S^1 \to S^1$ with $f_n(z) = z^n$ a lift is given by $\hat{f}(t) = nt$ so that its degree is $\deg(f_n) = \hat{f}_n(1) - \hat{f}_n(0) = n$.

Lemma 2.32. *Let $f, g \in C(S^1, S^1)$. If $f \simeq g$, then we have* $\deg(f) = \deg(g)$.

Proof. Let $F : S^1 \times [0,1] \to S^1$ be a homotopy from f to g. Since $S^1 \times [0,1]$ is compact the map F is uniformly continuous. Hence, there exists a $\delta > 0$ such that

$$|F(z,s) - F(z,s')| < 1$$

whenever $z \in S^1$ and $s, s' \in [0,1]$ with $|s - s'| < \delta$. For such s, s' the map

$$z \mapsto \frac{F(z,s)}{F(z,s')} : S^1 \to S^1$$

is continuous and not surjective because -1 is not contained in the image. Hence, by Remark 2.30, $\deg\left(\frac{F(\cdot,s)}{F(\cdot,s')}\right) = 0$. We now compute using 2.30.5.):

$$\deg(F(\cdot,s)) = \deg\left(\frac{F(\cdot,s)}{F(\cdot,s')} \cdot F(\cdot,s')\right) = \deg\left(\frac{F(\cdot,s)}{F(\cdot,s')}\right) + \deg F(\cdot,s') = \deg F(\cdot,s')\,.$$

We see inductively that

$$\deg(f) = \deg F(\cdot,0) = \deg F(\cdot,s_1) = \cdots = \deg F(\cdot,1) = \deg(g)$$

where $0 = s_0 < s_1 < \cdots < s_r = 1$ is a partition of the unit interval $[0,1]$ satisfying $|s_{i+1} - s_i| < \delta$. □

Corollary 2.33. *Let $f \in C(S^1, S^1)$ be such that $f = g|_{S^1}$ where $g \in C(D^2, S^1)$ then* $\deg(f) = 0$.

Proof. The map $F \in C(S^1 \times [0,1], S^1)$, $F(z,s) := g(sz)$, defines a homotopy from a constant map to f. By Lemma 2.32 we conclude that $\deg(f) = \deg(\mathrm{const}) = 0$. □

Let us give several applications of the concept of the degree.

Theorem 2.34 (Fundamental theorem of algebra). *Every non-constant complex polynomial has a root.*

Proof. Suppose we are given a non-constant polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \qquad a_j \in \mathbb{C},\ a_n \neq 0,\ n \geq 1.$$

Since dividing by a_n does not change the roots, we assume without loss of generality that $a_n = 1$. Now assume that p has no roots. Then the map $f : \mathbb{C} \to S^1$ with $f(z) = \frac{p(z)}{|p(z)|}$ is a well-defined continuous map. To compute $\deg(f|_{S^1})$ consider

$$F(z,s) := \frac{s^n p(z/s)}{|s^n p(z/s)|} = \frac{z^n + s a_{n-1} z^{n-1} + \cdots + s^n a_0}{|z^n + s a_{n-1} z^{n-1} + \cdots + s^n a_0|}.$$

The map $F \in C(S^1 \times [0,1], S^1)$ is a homotopy from $f_n(z) = z^n$ to $f|_{S^1}$. Computing its degree we find $\deg(f|_{S^1}) = \deg(f_n) = n \geq 1$.
On the other hand, f is a continuous map defined on all of $\mathbb{C}$, hence Corollary 2.33 implies $\deg(f|_{S^1}) = 0$. This is a contradiction, thus p must have a root. □

Lemma 2.35. *Suppose the map $f \in C(S^1, S^1)$ satisfies $f(-z) = -f(z)$ for all $z \in S^1$. Then the degree* $\deg(f)$ *is odd.*

Proof. Lef $\hat{f}$ be a lift of the map f. We compute

$$f(-\operatorname{Exp}(t)) = f\left(\operatorname{Exp}\left(\tfrac{1}{2}\right)\operatorname{Exp}(t)\right) = f\left(\operatorname{Exp}\left(t + \tfrac{1}{2}\right)\right) = \operatorname{Exp}\left(\hat{f}\left(t + \tfrac{1}{2}\right)\right).$$

Moreover,

$$-f(\operatorname{Exp}(t)) = -\operatorname{Exp}\left(\hat{f}(t)\right) = \operatorname{Exp}\left(\tfrac{1}{2}\right)\operatorname{Exp}\left(\hat{f}(t)\right) = \operatorname{Exp}\left(\hat{f}(t) + \tfrac{1}{2}\right).$$

From $f(-\operatorname{Exp}(t)) = -f(\operatorname{Exp}(t))$ we conclude $\operatorname{Exp}\left(\hat{f}\left(t+\frac{1}{2}\right)\right) = \operatorname{Exp}\left(\hat{f}(t)+\frac{1}{2}\right)$ and hence

$$\hat{f}(t+\tfrac{1}{2}) - (\hat{f}(t)+\tfrac{1}{2}) =: k(t)$$

is an integer for every t. Due to the continuity of the map, $k(t)$ it is constant, $k(t) = k$. Hence $\hat{f}(t+\frac{1}{2}) - \hat{f}(t) = k + \frac{1}{2}$ for all $t \in \mathbb{R}$. Now we can compute

$$\begin{aligned} \deg(f) &= \hat{f}(1) - \hat{f}(0) = \left(\hat{f}(1) - \hat{f}\left(\tfrac{1}{2}\right)\right) + \left(\hat{f}\left(\tfrac{1}{2}\right) - \hat{f}(0)\right) \\ &= (k+\tfrac{1}{2}) + (k+\tfrac{1}{2}) = 2k+1 \end{aligned}$$

which proves the assertion. □

Theorem 2.36 (Borsuk-Ulam). *Let $f \in C(S^2, \mathbb{R}^2)$ satisfy $f(-x) = -f(x)$ for all $x \in S^2$. Then f has a zero.*

Proof. Assume that the map f has no zero. Then the map $g : S^2 \to S^1$ with $g(x) := \frac{f(x)}{\|f(x)\|}$ is defined and continuous. Moreover, g satisfies $g(-x) = -g(x)$ for all $x \in S^2$. Now define a map $G : D^2 \to S^1$ by $G(y) = g(y, \sqrt{1-\|y\|^2})$. The map $G \in C(D^2, S^1)$ has the property that $G\big|_{S^1} = g|_{S^1}$. By Corollary 2.33 we know that $\deg(g\big|_{S^1}) = 0$. On the other hand we know by Lemma 2.35 that $\deg(g\big|_{S^1})$ is odd, which gives a contradiction. □

Corollary 2.37. *Let $f \in C(S^2, \mathbb{R}^2)$. Then there exists a point $x_0 \in S^2$ with $f(-x_0) = f(x_0)$.*

Proof. Put $g(x) := f(x) - f(-x)$. Then the map $g \in C(S^2, \mathbb{R}^2)$ satisfies $g(-x) = -g(x)$ for all $x \in S^2$. Hence by the Borsuk-Ulam Theorem 2.36 there exists an $x_0 \in S^2$ with $0 = g(x_0) = f(x_0) - f(-x_0)$, which proves the theorem. □

Remark 2.38. In particular, the map $f \in C(S^2, \mathbb{R}^2)$ cannot be injective. Thus the sphere S^2 cannot be homeomorphic to a subset of $\mathbb{R}^2$. This also shows that $\mathbb{R}^3$ cannot be homeomorphic to $\mathbb{R}^2$.

Now suppose you have a sandwich consisting of bread, ham, and cheese. You want to share it evenly with your friend. Can you cut the sandwich into two pieces such that each piece contains the same amount of bread, the same amount of ham and the same amount of cheese? The following theorem tells us that it is possible.

Figure 17. Cutting a sandwich[1]

Theorem 2.39 (Ham-Sandwich-Theorem). *Let $A, B, C \subset \mathbb{R}^3$ be open and bounded. Then there exists an affine hyperplane $H \subset \mathbb{R}^3$ such that each of the sets is divided into pieces of equal volume.*

Proof. The proof consists of four steps.

a) For each $x \in S^2$ and $t \in \mathbb{R}$ we define the affine hyperplane

$$H_{x,t} := \{y \in \mathbb{R}^3 \mid \langle y, x\rangle = t\}\,.$$

It is clear that $H_{-x,-t} = H_{x,t}$. We define the half space $H^+_{x,t}$ by

$$H^+_{x,t} := \{y \in \mathbb{R}^3 \mid \langle y, x\rangle \geq t\}.$$

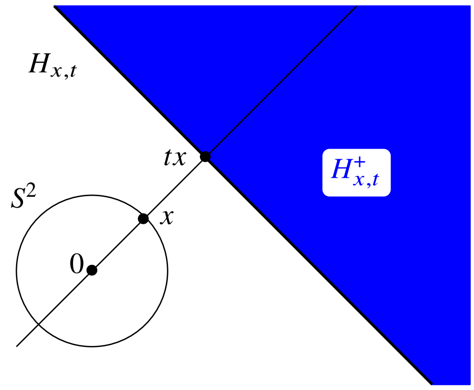


Figure 18. The hyperplane function

b) Now fix $x \in S^2$. Look at the function $a_x : \mathbb{R} \to \mathbb{R}$ defined by

$$a_x(t) := \mathrm{vol}(A \cap H^+_{x,t}).$$

It satisfies $a_{-x}(t) + a_x(-t) = \mathrm{vol}(A)$ and is monotonically decreasing. Since A is bounded there exists $R_A > 0$ such that $A \subset B(0, R_A)$.

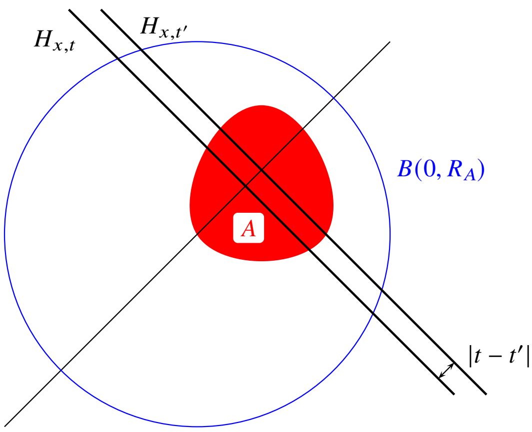


Figure 19. The volume function

[1] Based on public domain images from http://www.sxc.hu

For $t < t' \in \mathbb{R}$ we have

$$|a_x(t') - a_x(t)| = \mathrm{vol}(A \cap (H^+_{x,t} \setminus H^+_{x,t'})) \le \pi R_A^2 \cdot |t - t'|\,.$$

Thus a_x is Lipschitz continuous.

Moreover, $a_x(t) = \mathrm{vol}(A)$ for $t \ll 0$ and $a_x(t) = 0$ for $t \gg 0$.

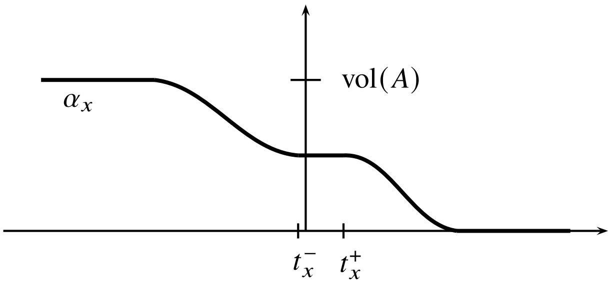


Figure 20. Monotonicity of the volume function

c) By continuity and monotonicity the pre-image of any value under α_x is a closed interval. In particular, $a_x^{-1}(\mathrm{vol}(A)/2) = [t_x^-, t_x^+]$.

Now put $\alpha(x) := \frac{t_x^- + t_x^+}{2}$. Hence $H_{x,\alpha(x)}$ divides A into two pieces of equal volume. Moreover, $\alpha(-x) = -\alpha(x)$ for all $x \in S^2$ and it is not hard to check that α is continuous. Similarly, define the functions β for B and γ for C.

d) Consider the map $f \in C(S^2, \mathbb{R}^2)$ with $f(x) = (\alpha(x) - \beta(x), \alpha(x) - \gamma(x))$. We have

$$\begin{aligned} f(-x) &= (\alpha(-x) - \beta(-x), \alpha(-x) - \gamma(-x)) \\ &= (-\alpha(x) + \beta(x), -\alpha(x) + \gamma(x)) \\ &= -f(x). \end{aligned}$$

Thus the Borsuk-Ulam Theorem 2.36 applies and there exists a point $x_0 \in S^2$ with

$$(0,0) = f(x_0) = (\alpha(x_0) - \beta(x_0), \alpha(x_0) - \gamma(x_0)).$$

Hence $\alpha(x_0) = \beta(x_0) = \gamma(x_0)$. Thus the hyperplance $H_{x_0,\alpha(x_0)}$ does the job. □

Remark 2.40. The ham-sandwich theorem is optimal in the sense that it fails for more than three sets in $\mathbb{R}^3$.

Example 2.41. A ball $B(x,r) \subset \mathbb{R}^3$ with center x is cut into two halves of equal volume exactly by those planes that contain x. If you choose four balls in $\mathbb{R}^3$ in such a way that their centers are not contained in one plane, then no plane will cut them all into halves of equal volume.

In the following we want to use the concept of degree to determine $\pi_1(S^1; 1)$.

a) For $\omega \in \Omega(S^1;1)$ and $I = [0,1]$ consider the following diagram:

$$\begin{tikzcd}
S^1 \arrow[rr, bend left, "f_\omega"] & I \arrow[l, "\mathrm{Exp}"'] \arrow[r, "\omega"] \arrow[d] & S^1 \\
& I/\partial I \arrow[ul, "\overline{\mathrm{Exp}}", "\approx"'] \arrow[ur, "\overline{\omega}"'] &
\end{tikzcd}$$

Here $\overline{\omega}$ and $\overline{\mathrm{Exp}}$ are the continuous maps induced on the quotient space. They exist because $\omega(0) = \omega(1)$ and $\mathrm{Exp}(0) = \mathrm{Exp}(1)$, compare Section 1.5. By Example 1.30 we know that $\overline{\mathrm{Exp}} : I/\partial I \to S^1$ is a homeomorphism. Now put

$$f_\omega := \overline{\omega} \circ (\overline{\mathrm{Exp}})^{-1} \in C\big(S^1, S^1\big)$$

and define $\deg(\omega) := \deg(f_\omega)$. We have obtained a map

$$\deg : \Omega(S^1;1) \to \mathbb{Z}.$$

b) Now suppose $\omega \simeq_{\{0,1\}} \omega'$. We choose a homotopy $F : I \times I \to S^1$ from ω to ω' relativ to $\{0,1\}$. Then the map $G : S^1 \times I \to S^1$ defined by $G(z,s) := F((\overline{\mathrm{Exp}})^{-1}(z), s)$ is a homotopy from f_ω to $f_{\omega'}$, hence $f_\omega \simeq f_{\omega'}$. Therefore

$$\deg(\omega) = \deg(f_\omega) = \deg(f_{\omega'}) = \deg(\omega') \,.$$

Hence we get a well-defined map

$$\deg : \pi_1(S^1;1) \to \mathbb{Z} \quad \text{where} \quad [\omega] \mapsto \deg(\omega).$$

c) This map is surjective because for $n \in \mathbb{Z}$ we can consider the map $\omega(t) = \mathrm{Exp}(nt)$. The commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r, "z \mapsto z^n = f_n(z)"] & S^1 \\
& I \arrow[ul, "t \mapsto \mathrm{Exp}(t)"] \arrow[u, "t \mapsto \omega(t) = \mathrm{Exp}(nt) = \mathrm{Exp}(t)^n"']
\end{tikzcd}$$

shows $f_\omega(z) = z^n = f_n(z)$ and we get $\deg(\omega) = \deg(f_n) = n$.

d) Now let $\omega, \omega' \in \Omega(S^1;1)$ and consider the map

$$\begin{aligned}
f_{\omega \star \omega'}(\mathrm{Exp}(t)) &= \omega \star \omega'(t) \\
&= \begin{cases} \omega(2t), & 0 \le t \le 1/2 \\ \omega'(2t-1), & 1/2 \le t \le 1 \end{cases} \\
&= \begin{cases} f_\omega(\mathrm{Exp}(2t)), & 0 \le t \le 1/2 \\ f_{\omega'}(\mathrm{Exp}(2t-1)), & 1/2 \le t \le 1 \end{cases}
\end{aligned}$$

Let $\hat{f}_\omega, \hat{f}_{\omega'}$ be lifts of $f_\omega, f_{\omega'}$ with $\hat{f}_{\omega'}(0) = \hat{f}_\omega(1)$. Then we have

$$\begin{aligned} f_{\omega\star\omega'}(\operatorname{Exp}(t)) &= \begin{cases} \operatorname{Exp}(\hat{f}_\omega(2t)), & 0 \le t \le 1/2 \\ \operatorname{Exp}(\hat{f}_{\omega'}(2t-1)), & 1/2 \le t \le 1 \end{cases} \\ &= \operatorname{Exp}\left(\underbrace{\begin{cases} \hat{f}_\omega(2t), & 0 \le t \le 1/2 \\ \hat{f}_{\omega'}(2t-1), & 1/2 \le t \le 1 \end{cases}}_{=:g(t)} \right) \end{aligned}$$

Note that the map $g(t)$ is continuous because of $\hat{f}_{\omega'}(0) = \hat{f}_\omega(1)$. Thus $g(t)$ is a lift of $f_{\omega\star\omega'}$. Now we compute the degree of $\omega \star \omega'$.

$$\begin{aligned} \deg(\omega \star \omega') &= g(1) - g(0) \\ &= \hat{f}_{\omega'}(1) - \hat{f}_\omega(0) \\ &= \hat{f}_{\omega'}(1) - \hat{f}_{\omega'}(0) + \hat{f}_\omega(1) - \hat{f}_\omega(0) \\ &= \deg(\omega') + \deg(\omega) \end{aligned}$$

Hence the map $\deg : \pi_1(S^1; 1) \longrightarrow (\mathbb{Z}, +)$ is a group homomorphism.

e) Finally we compute its kernel. Let $\omega \in \Omega(S^1; 1)$ with $\deg(\omega) = 0$. Let $\hat{f}_\omega$ be the lift of f_ω with $\hat{f}_\omega(0) = 0$. Since $0 = \deg(\omega) = \hat{f}_\omega(1) - \hat{f}_\omega(0)$ we have $\hat{f}_\omega(1) = \hat{f}_\omega(0) = 0$. Next consider the continuous map $F : I \times I \longrightarrow S^1$ with $F(t,s) := \operatorname{Exp}(s\hat{f}_\omega(t))$. It satisfies:

$$\begin{aligned} &F(t,0) = 1 \ = \ \varepsilon_1(t)\,, \\ &F(t,1) = f_\omega(\operatorname{Exp}(t)) = \omega(t)\,, \\ &F(0,s) = \operatorname{Exp}(s \cdot 0) = 1\,, \\ &F(1,s) = \operatorname{Exp}(s \cdot \hat{f}_\omega(1)) = \operatorname{Exp}(s\, 0) = 1\,. \end{aligned}$$

We conclude that $\omega \simeq_{\{0,1\}} \varepsilon_1$, hence $[\omega] = [\varepsilon_1] = 1 \in \pi_1(S^1; 1)$. Therefore the kernel is trivial and the map $\deg : \pi_1(S^1; 1) \longrightarrow \mathbb{Z}$ is injective.

We summarize the discussion in the following

Theorem 2.42. *The map* $\deg : \pi_1(S^1; 1) \longrightarrow \mathbb{Z}$ *is a group isomorphism.*

Example 2.43. We already know that S^1 is a strong deformation retract of $\mathbb{C} \setminus \{0\}$. Hence the inclusion $\iota : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ induces an isomorphism

$$\iota_\# : \pi_1(S^1; 1) \longrightarrow \pi_1(\mathbb{C} \setminus \{0\}; 1)$$

and therefore $\pi_1(\mathbb{C} \setminus \{0\}; 1) \cong \mathbb{Z}$.

Example 2.44
We show that $S^1 = \partial D^2$ is not a retract of D^2. Suppose the map $r : D^2 \to S^1$ is a retraction and denote the inclusion by $\iota : S^1 \to D^2$.

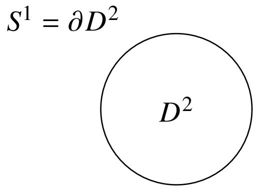


Figure 21. The disk and its boundary

Then we would have $r \circ \iota = \mathrm{id}_{S^1}$, hence $r_\# \circ \iota_\# = (r \circ \iota)_\# = (\mathrm{id}_{S^1})_\# = \mathrm{id}_{\pi_1(S^1;1)}$. We then get a contradiction because of the following diagram

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(S^1;1) \arrow[r, "\iota_\#"] \arrow[rr, bend right, "\mathrm{id}_{\pi_1(S^1;1)}"'] & \pi_1(D^2;1) \cong \{1\} \arrow[r, "r_\#"] & \pi_1(S^1;1) \cong \mathbb{Z} \end{tikzcd}$$

As a corollary we get a proof of Brouwer's fixed point theorem in dimension two. See page 93 for the theorem in general dimensions.

Theorem 2.45 (Brouwer's fixed point theorem in 2 dimensions). *Let $f : D^2 \to D^2$ be a continuous map. Then f has a fixed point, i.e., there exists an $x \in D^2$ such that $f(x) = x$.*

Proof. Assume that $f \in C(D^2, D^2)$ has no fixed point. Then $f(x) \neq x$ for all $x \in D^2$ so that we can consider the half line emanating from $f(x)$ through x. We let $r(x)$ be its intersection point with $\partial D^2 = S^1$ as indicated in the picture.

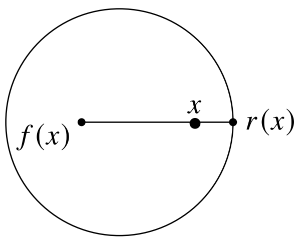


Figure 22. Constructing a retraction

This yields a retraction $r : D^2 \to S^1$ and we get a contradiction. □

Example 2.46. We show that the system of equations

$$\begin{aligned} 1 + \tfrac{1}{2}\sin(x)y - 2x &= 0, \\ \tfrac{1}{2}\cos(x)y + \tfrac{x^2}{2} - 2y &= 0, \end{aligned} \tag{2.1}$$

has a solution. We rewrite this as a fixed point equation and apply the Brouwer fixed point theorem. To do this we put

$$f(x,y) := \left(\frac{1}{2} + \frac{\sin(x)y}{4}, \frac{\cos(x)y}{4} + \frac{x^2}{4}\right)$$

Fixed points of $f(x,y)$ are then the same as solutions to the above system of equations (2.1). To apply Brouwer's fixed point theorem we have to show that $f(D^2) \subset D^2$. Let $x, y \in D^2$. Then

$$\begin{aligned}
\|f(x,y)\|^2 &= \left(\frac{1}{2} + \frac{\sin(x)y}{4}\right)^2 + \left(\frac{\cos(x)y}{4} + \frac{x^2}{4}\right)^2 \\
&= \frac{1}{4} + \frac{\sin(x)y}{4} + \frac{\sin(x)^2y^2}{16} + \frac{\cos(x)^2y^2}{16} + \frac{\cos(x)yx^2}{8} + \frac{x^4}{16} \\
&= \frac{1}{4} + \frac{\sin(x)y}{4} + \frac{y^2}{16} + \frac{\cos(x)yx^2}{8} + \frac{x^4}{16} \\
&\leq \frac{1}{4} + \frac{|y|}{4} + \frac{y^2}{16} + \frac{|y|x^2}{8} + \frac{x^4}{16} \\
&\leq \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} \\
&= \frac{3}{4} \\
&\leq 1.
\end{aligned}$$

Example 2.47. Consider $f_n \in C(S^1, S^1)$ with $f_n(z) = z^n$. We check that the diagram

$$\begin{tikzcd}
\pi_1(S^1;1) \arrow[r, "\deg", "\cong"'] \arrow[d, "(f_n)_\#"'] & \mathbb{Z} \arrow[d, "n\cdot"] \\
\pi_1(S^1;1) \arrow[r, "\deg", "\cong"'] & \mathbb{Z}
\end{tikzcd}$$

commutes. Let $[\omega] \in \pi_1(S^1;1)$. We compute

$$\begin{aligned}
\deg\big((f_n)_\#([\omega])\big) = \deg\big([f_n \circ \omega]\big) &= \deg(f_n \circ \omega) \\
= \deg(f_n \circ f_\omega) &= \deg(f_n)\deg(f_\omega) = n \deg\big([\omega]\big).
\end{aligned}$$

Remark 2.48. From Exercise 2.5 we know that for X_1, X_2 and $x_1 \in X_1, x_2 \in X_2$ we have

$$\pi_1(X_1 \times X_2; (x_1, x_2)) \cong \pi_1(X_1; x_1) \times \pi_1(X_2; x_2).$$

For the two-dimensional torus $T^2 = S^1 \times S^1$ we get $\pi_1(T^2;x) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. More generally, we get inductively for the n-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ that $\pi_1(T^n;x) \cong \mathbb{Z}^n$. In particular, $T^n \not\simeq T^m$ for $n \neq m$.

Definition 2.49. Let X be a topological space.

1.) We call X *connected* iff X and $\emptyset$ are the only subsets of X which are both open and closed.

2.) We call X *path-connected* iff for all $x_1, x_2 \in X$ there exists a path $\omega \in \Omega(X; x_1, x_2)$.

3.) Let X be path-connected. Then X is called *simply-connected* (or *1-connected*) iff $\pi_1(X; x_0) = \{1\}$ for some (and hence all) $x_0 \in X$.

Example 2.50. The interval $[0, 1]$ is connected. To see this let $I \subset [0, 1]$ be open and closed and assume that I is neither empty nor all of $[0, 1]$. Then there exists $t_0 \in I$ and $t_1 \in [0, 1] \setminus I$. W.l.o.g. let $t_0 < t_1$, the other case being analogous. We put $T := \sup(I \cap [0, t_1))$. Then $0 \leq t_0 \leq T \leq t_1 \leq 1$. Since I is closed $T \in I$.
If $T = 1$ then $T = t_1$ which contradicts $t_1 \notin I$. If $T < 1$ then there exists $\varepsilon > 0$ such that $[T, T + \varepsilon) \subset I$ because I is open. This contradicts the maximality of T.

Remark 2.51. If X is 1-connected then it is path-connected by definition but the converse is not true. For example, S^1 is path-connected but not 1-connected.

Remark 2.52. If X is path-connected then X is connected.

Proof. Let X be path-connected and let $U \subset X$ be open and closed, $U \neq \emptyset$. We show $U = X$. Since U is non-empty we can find $x_1 \in U$. Let $x_2 \in X$ be any point.
Since X is path-connected there is a path $\omega \in \Omega(X; x_1, x_2)$. Then $I := \omega^{-1}(U)$ is an open and closed subset of $[0, 1]$. Since $\omega(0) = x_1 \in U$ we have $0 \in I$ and hence I is non-empty. Thus $I = [0, 1]$ because $[0, 1]$ is connected. We conclude $1 \in I$ and hence $x_2 = \omega(1) \in U$. This shows that U contains all points of X. □

Again, the converse implication does not hold in general. Consider for example the space

$$X := \{(t, \sin(1/t)) \mid t > 0\} \cup \{(0, s) \mid -1 \leq s \leq 1\},$$

see Figure 23. Then X is connected but not path-connected.

Remark 2.53. Let X be path-connected. Then the following are equivalent (see Exercise 2.2):

(i) X is simply connected;

(ii) Every $\omega \in C(S^1, X)$ is homotopic to a constant map;

(iii) Every $\omega \in C(S^1, X)$ has a continuous extension to a map $D^2 \to X$.

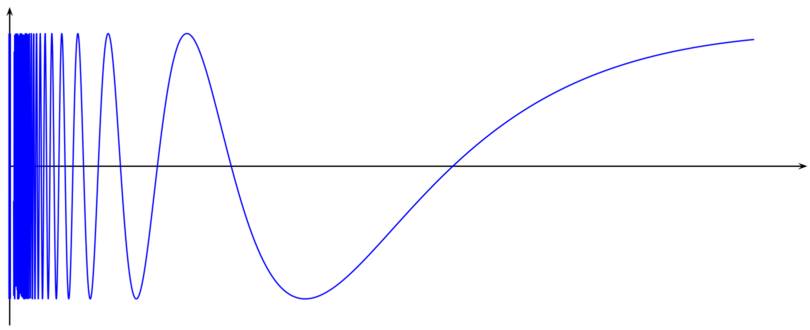


Figure 23. Connected but not path-connected

2.4. The Seifert-van Kampen theorem

The Seifert-van Kampen theorem will allow us to compute the fundamental group of spaces which are built out of simpler spaces whose fundamental groups we already know. We start with an excursion to group theory.

Definition 2.54. Let G be a group. A subgroup $H \subset G$ is called *normal* iff

$$g \cdot H = H \cdot g \quad \text{for all } g \in G .$$

The condition on a subgroup of being normal can be reformulated in various ways. It is equivalent to any of the following:

(i) $g \cdot H \cdot g^{-1} = H$ for all $g \in G$;

(ii) $g \cdot H \cdot g^{-1} \subset H$ for all $g \in G$;

(iii) $g \cdot h \cdot g^{-1} \in H$ for all $g \in G$ and $h \in H$.

Example 2.55. Consider the cartesian product of two groups $G = G_1 \times G_2 = \{(g_1, g_2) \mid g_j \in G_j\}$ with componentwise multiplication. Then $\{(g_1, 1) \mid g_1 \in G_1\} \cong G_1$ is a normal subgroup of G because

$$(g_1, g_2) \cdot (\tilde{g}_1, 1) \cdot (g_1, g_2)^{-1} = \big(g_1\tilde{g}_1 g_1^{-1}, g_2 1 g_2^{-1}\big) = \big(g_1\tilde{g}_1 g_1^{-1}, 1\big) .$$

Example 2.56. Let $\varphi : G \to K$ be a group homomorphism. Then $H = \ker(\varphi)$ is a normal subgroup because for $h \in \ker(\varphi)$ and $g \in G$ we have

$$\varphi(g \cdot h \cdot g^{-1}) = \varphi(g) \cdot \underbrace{\varphi(h)}_{1} \cdot \varphi(g)^{-1} = 1 ,$$

hence $g \cdot h \cdot g^{-1} \in \ker(\varphi)$.

Remark 2.57. If $H \subset G$ is a normal subgroup then G/H is again a group via

$$(g \cdot H) \cdot (\tilde{g} \cdot H) = (g\tilde{g}) \cdot H.$$

Normality of H ensures that this multiplication is well defined. The group H is then the kernel of $G \to G/H$ with $g \mapsto g \cdot H$. Thus the normal subgroups are exactly those which arise as kernels of group homomorphisms.

Now let $S \subset G$ be any subset. Then

$$\mathcal{N}(S) := \bigcap_{\substack{H \subset G \text{ normal subgroup,} \\ H \supset S}} H$$

is the smallest normal subgroup containing S. We call $\mathcal{N}(S)$ the *normal subgroup generated by* S.

Example 2.58. $\mathcal{N}(\emptyset) = \{1\}$.

Definition 2.59
Let G_1 and G_2 be groups. A group G is called *free product* of G_1 and G_2 iff there exist homomorphisms $i_j : G_j \to G$ such that for all groups H and for all homomorphisms $\varphi_j : G_j \to H$ there exists a unique homomorphism

$$\varphi_1 \star \varphi_2 : G \to H$$

such that the diagram to the right commutes.

$$\begin{tikzcd}
& G_1 \arrow[dl, "i_1"'] \arrow[dr, "\varphi_1"] & \\
G \arrow[rr, dotted, "\varphi_1 \star \varphi_2"] & & H \\
& G_2 \arrow[ul, "i_2"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Remark 2.60. Here we have characterized free products by their *universal property*. This universal property implies for example that the maps $i_j : G_j \to G$ are injective.
Namely, choose $H = G_1$, $\varphi_1 = \mathrm{id}_{G_1}$ and $\varphi_2(g_2) = 1$ for all $g_2 \in G_2$. The diagram now tells us that the map i_1 must be injective because the identity is injective. Similarly, we see that i_2 is injective.

$$\begin{tikzcd}
& G_1 \arrow[dl, "i_1"'] \arrow[dr, equal, "\mathrm{id}"] & \\
G \arrow[rr, "\mathrm{id} \star \varphi_2"] & & G_1 \\
& G_2 \arrow[ul, "i_2"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Remark 2.61. The free product of G_1 and G_2 is unique up to isomorphism.

Namely, let G' be another free product of G_1 and G_2 with $i'_j : G_j \to G'$ the corresponding homomorphisms. By the universal property of G with $H = G'$ and $\varphi_j = i'_j$ we get the following commutative diagram:

$$\begin{tikzcd}
& G_1 \arrow[dl, "i_1"'] \arrow[dr, "i'_1"] & \\
G \arrow[rr, "i'_1 \star i'_2"] & & G' \\
& G_2 \arrow[ul, "i_2"] \arrow[ur, "i'_2"'] &
\end{tikzcd}$$

Interchanging the roles of G and G' we get another commutative diagram:

$$\begin{tikzcd}
& G_1 \arrow[dl, "i'_1"'] \arrow[dr, "i_1"] & \\
G' \arrow[rr, "i_1 \star i_2"] & & G \\
& G_2 \arrow[ul, "i'_2"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

Combining both diagrams we obtain

$$\begin{tikzcd}
& G_1 \arrow[dl, "i_1"'] \arrow[d, "i'_1"] \arrow[dr, "i_1"] & \\
G \arrow[r, "i'_1 \star i'_2"] & G' \arrow[r, "i_1 \star i_2"] & G \\
& G_2 \arrow[ul, "i_2"] \arrow[u, "i'_2"'] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

On the other hand, this diagram commute as well.

$$\begin{tikzcd}
& G_1 \arrow[dl, "i_1"'] \arrow[dr, "i_1"] & \\
G \arrow[rr, "\mathrm{id}"] & & G \\
& G_1 \arrow[ul, "i_2"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

By the uniqueness of the induced homomorphisms we have

$$(i_1 \star i_2) \circ \left(i'_1 \star i'_2\right) = \mathrm{id}_G$$

and similarly

$$\left(i'_1 \star i'_2\right) \circ (i_1 \star i_2) = \mathrm{id}_{G'} .$$

Hence the map $i_1 \star i_2 : G \to G'$ is a group isomorphism with inverse $i'_1 \star i'_2$.

Next we show the existence of the free product of two groups by a direct construction. For this purpose let G_1 and G_2 be groups. For formal reasons we assume without loss of generality that

$G_1 \cap G_2 = \emptyset$.[2] We define

$$G_1 \star G_2 := \big\{(x_1, \dots x_n) \mid n \in \mathbb{N}_0,\ x_j \in \big(G_1 \setminus \{1_{G_1})\} \cup (G_2 \setminus \{1_{G_2}\}\big) \text{ such that} \\ \text{if } x_i \in G_1 \text{ then } x_{i+1} \in G_2 \text{ or conversely}\big\}.$$

The group multiplication in $G_1 \star G_2$ is then inductively defined as

$$(x_1, \dots, x_n) \cdot (x_{n+1}, \dots, x_{n+m}) \\ := \begin{cases} (x_1, \dots, x_{n-1}, x_n \cdot x_{n+1}, x_{n+2}, \dots, x_{n+m}) & \text{if } (x_n, x_{n+1} \in G_1 \text{ or } x_n, x_{n+1} \in G_2) \\ & \text{and } x_n \cdot x_{n+1} \neq 1, \\ (x_1, \dots, x_{n-1}) \cdot (x_{n+2}, \dots, x_{n+m}) & \text{if } (x_n, x_{n+1} \in G_1 \text{ or } x_n, x_{n+1} \in G_2) \\ & \text{and } x_n \cdot x_{n+1} = 1, \\ (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) & \text{otherwise.} \end{cases}$$

This turns $G_1 \star G_2$ into a group with neutral element the empty sequence $()$. The inverse element for $(x_1, \dots x_n)$ is given by $(x_n^{-1}, \dots, x_1^{-1})$ because of

$$(x_1, \dots, x_n) \cdot \big(x_n^{-1}, \dots, x_1^{-1}\big) = (x_1, \dots, x_{n-1}) \cdot \big(x_{n-1}^{-1}, \dots, x_1^{-1}\big) = \dots = (x_1)(x_1^{-1}) = ().$$

Now consider the map $i_j : G_j \to G_1 \star G_2$ given by

$$i_j(x) = \begin{cases} (x), & x \neq 1 \\ (), & x = 1 \end{cases}$$

For $\varphi_j : G_j \to H$ homomorphisms we put

$$(\varphi_1 \star \varphi_2)(x_1, \dots, x_n) := \varphi_{i_1}(x_1) \cdot \varphi_{i_2}(x_2) \cdot \ldots \cdot \varphi_{i_n}(x_n)$$

where i_j is chosen such that $x_j \in G_{i_j}$.

Remark 2.62
1.) The subset $i_j(G_j) \subset G_1 \star G_2$ is a subgroup isomorphic to G_j. The intersection $i_1(G_1) \cap i_2(G_2) = \{1\}$ is trivial. The union $i_1(G_1) \cup i_2(G_2)$ generates $i_1(G_1) \star i_2(G_2)$ as a group.

2.) If $G_1 = \{1\}$ then $G_1 \star G_2 = i_2(G_2) \cong G_2$. Similarly, if $G_2 = \{1\}$ then $G_1 \star G_2 \cong G_1$.

3.) If $G_1 \neq \{1\}$ and $G_2 \neq \{1\}$ then we may choose $x \in G_1 \setminus \{1\}$ and $y \in G_2 \setminus \{1\}$. Then

$$(x), (x, y), (x, y, x), (x, y, x, y), \dots$$

yields infinitely many pairwise different elements in $G_1 \star G_2$, hence $|G_1 \star G_2| = \infty$ (even if $|G_j| < \infty$). In addition, we have

$$(x) \cdot (y) = (x, y) \neq (y, x) = (y) \cdot (x)\,,$$

hence the group $G_1 \star G_2$ is not abelian, even if G_1 and G_2 are.

[2] Otherwise replace G_2 by an isomorphic group which is disjoint to G_1.

Remark 2.63. Usually one identifies G_1 with $i_1(G_1)$ and G_2 with $i_2(G_2)$ and writes

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \text{ instead of } (x_1, x_2, \ldots, x_n).$$

Example 2.64. Consider $G_1 = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$ and $G_2 = \mathbb{Z}/2\mathbb{Z} = \{1', -1'\}$. Elements of $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ are for example

$$\begin{aligned} a &= (-1) \cdot (-1') \cdot (-1) \cdot (-1') \\ b &= (-1') \cdot (-1) \cdot (-1') \cdot (-1) \cdot (-1')\,. \end{aligned}$$

Now we calculate

$$\begin{aligned} a \cdot b &= (-1) \cdot (-1') \cdot (-1) \cdot \underbrace{(-1') \cdot (-1')}_{1'} \cdot (-1) \cdot (-1') \cdot (-1) \cdot (-1') \\ &= (-1) \cdot (-1') \cdot \underbrace{(-1) \cdot (-1)}_{1} \cdot (-1') \cdot (-1) \cdot (-1') \\ &= (-1) \cdot \underbrace{(-1') \cdot (-1')}_{1'} \cdot (-1) \cdot (-1') \\ &= \underbrace{(-1) \cdot (-1)}_{1} \cdot (-1') \\ &= (-1')\,. \end{aligned}$$

Now we are ready to return to topology.

Proposition 2.65. *Let X be a topological space and let $U, V \subset X$ be open such that $U \cup V = X$. Let $x_0 \in U \cap V$. Furthermore, assume that U, V and $U \cap V$ are path-connected. Then X is path-connected and the map*

$$i_\# \star j_\# : \pi_1(U; x_0) \star \pi_1(V; x_0) \to \pi_1(X; x_0)$$

is onto where $i : U \to X$ and $j : V \to X$ are the corresponding inclusion maps.

Proof. First of all we note that the space X is path-connected because each point in X lies in U or in V and can therefore be connected to x_0 by a path.
The statement of the proposition is equivalent to saying that

$$i_\#\big(\pi_1(U; x_0)\big) \cup j_\#\big(\pi_1(V; x_0)\big)$$

generates $\pi_1(X; x_0)$ as a group. Now let $[\omega] \in \pi_1(X; x_0)$ and subdivide the unit interval $I = [0, 1]$ by $0 = t_0 \leq t_1 \cdots < t_n = 1$ such that $\omega([t_i, t_{i+1}]) \subset U$ or $\subset V$.

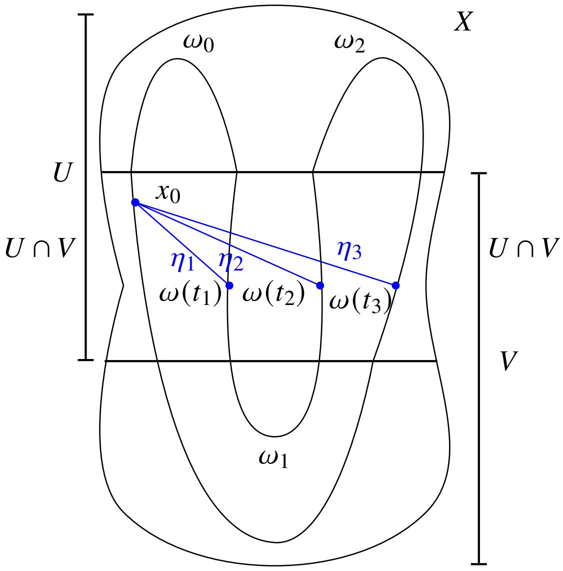


Figure 24. Subdividing ω

By removing subdivision points if necessary we can assume that if $\omega([t_{i-1}, t_i]) \subset U$ then $\omega([t_i, t_{i+1}]) \subset V$ or conversely. Reparametrize $\omega_i(t) = \omega((1-t)t_i + tt_{i+1})$. Then $\omega_i \in \Omega(U \text{ or } V; \omega(t_i), \omega(t_{i+1}))$ and in particular $\omega(t_i) \in U \cap V$. Since by assumption $U \cap V$ is path-connected there exist $\eta_j \in \Omega(U \cap V; x_0, \omega(t_j))$. Then

$$\omega_0 \star \eta_1^{-1}, \eta_1 \star \omega_1 \star \eta_2^{-1}, \eta_2 \star \omega_2 \star \eta_3^{-1}, \ldots, \eta_{n-1} \star \omega_{n-1}$$

are loops with base point x_0 contained entirely in U or V. We then calculate

$$\left(\omega_0 \star \eta_1^{-1}\right) \star \left(\eta_1 \star \omega_1 \star \eta_2^{-1}\right) \star \left(\eta_2 \star \omega_2 \star \eta_3^{-1}\right) \star \cdots \star \left(\eta_{n-1} \star \omega_{n-1}\right) \simeq_{\{0,1\}} \omega_0 \star \omega_1 \star \cdots \star \omega_{n-1} \simeq_{\{0,1\}} \omega$$

in X. Denoting the homotopy class of a loop ω in X, U, V, or $U \cap V$ by $[\omega]_X$, $[\omega]_U$, $[\omega]_V$, and $[\omega]_{U\cap V}$ respectively, we have

$$[\omega]_X = \left[\omega_0 \star \eta_1^{-1}\right]_X \cdot \left[\eta_1 \star \omega_1 \star \eta_2^{-1}\right]_X \cdot \ldots \cdot \left[\eta_{n-1} \star \omega_{n-1}\right]_X$$

and it follows that

$$\begin{aligned} [\omega]_X &= i_\#\left(\left[\omega_0 \star \eta_1^{-1}\right]_U\right) \cdot j_\#\left(\left[\eta_1 \star \omega_1 \star \eta_2^{-1}\right]_V\right) \cdot i_\#\left(\left[\eta_2 \star \omega_2 \star \eta_3^{-1}\right]_U\right) \ldots \\ &= (i_\# \star j_\#)\Big(\underbrace{\left[\omega_0 \star \eta_1^{-1}\right]_U \cdot \left[\eta_1 \star \omega_1 \star \eta_2^{-1}\right]_V \cdot \left[\eta_2 \star \omega_2 \star \eta_3^{-1}\right]_U \cdots}_{\in \pi_1(U;x_0) \star \pi_1(V;x_0)}\Big) \end{aligned}$$

which proves the assertion. □

Corollary 2.66. *Let X is a topological space and let $U, V \subset X$ be open such that $U \cup V = X$ and $U \cap V \neq \emptyset$. If U and V are 1-connected and $U \cap V$ is path-connected, then the space X is 1-connected.*

Proof. Since

$$\{1\} = \{1\} \star \{1\} = \pi_1(U; x_0) \star \pi_1(V; x_0) \longrightarrow \pi_1(X; x_0)$$

is onto we get that $\pi_1(X; x_0) = \{1\}$. □

Example 2.67. Consider $X = S^n$ and put $U = S^n \setminus \{e_1\}$. The stereographic projection yields a homeomorphism $U \to \mathbb{R}^n$. Hence U is 1-connected. Similarly, $V = S^n \setminus \{-e_1\}$ is also 1-connected. Now

$$U \cap V = S^n \setminus \{e_1, -e_1\} \approx \mathbb{R}^n \setminus \{0\}.$$

For $n \geq 2$ the space $U \cap V$ is path-connected. Corollary 2.66 shows that S^n is simply connected for $n \geq 2$.
Recall that we know already that this is not true for $n = 1$ because $\pi_1(S^1; 1) \cong \mathbb{Z}$.

To determine $\pi_1(X; x_0)$ more precisely we compute the kernel of the homomorphism

$$i_\# \star j_\# : \pi_1(U; x_0) \star \pi_1(V; x_0) \longrightarrow \pi_1(X; x_0).$$

Consider the inclusion maps $i' : U \cap V \to U$ and $j' : U \cap V \to V$. Clearly the diagram on the right commutes. This implies

$$\begin{tikzcd} U \arrow[r, "i"] & X \\ U \cap V \arrow[u, "i'"] \arrow[r, "j'"'] & V \arrow[u, "j"'] \end{tikzcd}$$

$$j_\# \circ j'_\# = i_\# \circ i'_\#.$$

For $\alpha \in \pi_1(U \cap V; x_0)$ we calculate

$$1 = i_\#\big(i'_\#(\alpha)\big) \cdot \Big(j_\#\big(j'_\#(\alpha)\big)\Big)^{-1} = i_\#\big(i'_\#(\alpha)\big) \cdot j_\#\big(j'_\#(\alpha)^{-1}\big) = (i_\# \star j_\#)\ \underbrace{\big(i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1}\big)}_{\in \pi_1(U; x_0) \star \pi_1(V; x_0)}.$$

Hence $i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1} \in \ker(i_\# \star j_\#)$ and it follows that

$$\mathcal{N}\big(\{i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V; x_0)\}\big) \subset \ker(i_\# \star j_\#).$$

Theorem 2.68 (Seifert-van Kampen). *Let X be a topological space and let $U, V \subset X$ be open subsets such that $U \cup V = X$ and $x_0 \in U \cap V$. Let U, V and $U \cap V$ be path connected. Then the map $i_\# \star j_\#$ induces an isomorphism*

$$\frac{\pi_1(U; x_0) \star \pi_1(V; x_0)}{\mathcal{N}\big(\{i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V; x_0)\}\big)} \cong \pi_1(X; x_0).$$

Proof. It remains to show that

$$\ker(i_\# \star j_\#) \subset \mathcal{N}\big(\{i'_\#(\alpha)\cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U\cap V;x_0)\}\big).$$

Let $\omega_1, \omega_3, \dots \in \Omega(U;x_0)$ and $\omega_2, \omega_4, \dots \in \Omega(V;x_0)$ be such that

$$\begin{aligned} 1 &= i_\# \star j_\#\big([\omega_1]_U \cdot [\omega_2]_V \cdot [\omega_3]_U \cdot \dots\big) \\ &= i_\#([\omega_1]_U)\cdot j_\#([\omega_2]_V)\cdot i_\#([\omega_3]_U)\cdot \dots \\ &= [\omega_1]_X \cdot [\omega_2]_X \cdot [\omega_3]_X \cdot \dots \\ &= [\omega_1 \star \omega_2 \star \omega_3 \star \dots]_X \end{aligned}$$

Then there exists a homotopy $H : [0,1]\times[0,1] \to X$ such that

$$\begin{aligned} H(t,0) &= (\omega_1 \star \omega_2 \star \omega_3 \star \dots)(t), \\ H(t,1) &= x_0, \\ H(0,s) &= H(1,s) = x_0. \end{aligned}$$

In Figure 25 the red area gets mapped to x_0.

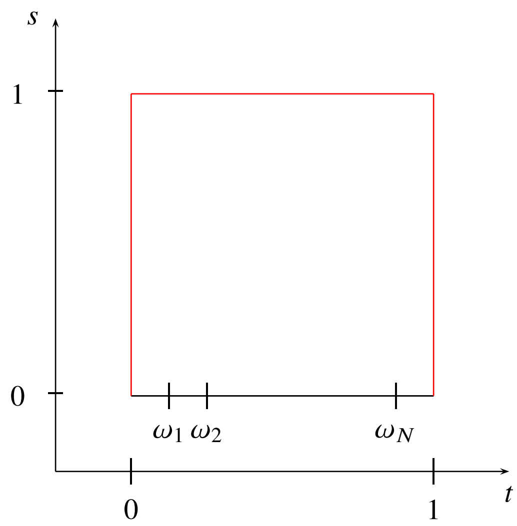


Figure 25. The homotopy to start with

Now subdivide $[0,1]\times[0,1]$ further such that H maps each closed subsquare entirely to U or to V.

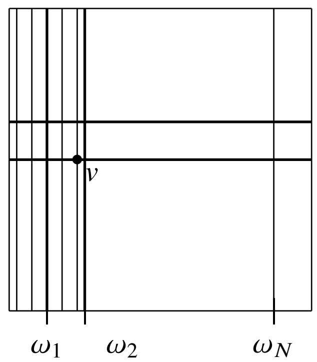


Figure 26. Subdividing the homotopy

Considering the edges of the subsquares we get homotopies

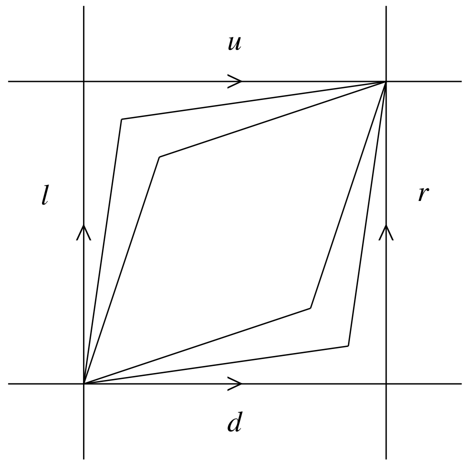


Figure 27. Deforming the homotopy in each square

and hence the relation

$$d \star r \simeq_{\{0,1\}} l \star u \tag{2.2}$$

resulting in

$$(H \circ d) \star (H \circ r) \simeq_{\{0,1\}} (H \circ l) \star (H \circ u) \qquad \text{in } U \text{ or in } V\,.$$

For each vertex v in this subdivision of $[0,1] \times [0,1]$ choose $\eta_v \in \Omega(X; x_0, H(v))$ in such a way that $\eta_v \in \Omega(W; x_0, H(v))$ if $H(v) \in W$ where $W = U, V$ or $U \cap V$. This is possible because $x_0 \in W$ and W is path-connected by assumption. If $H(v) = x_0$ choose $\eta_v = \varepsilon_{x_0}$. For each edge with endpoints v_0 and v_1 we obtain a loop in U or in V by

$$\eta_{v_0} \star (H \circ c) \star \eta_{v_1}^{-1} \in \Omega(U \text{ or } V; x_0)\,.$$

Now look at one row of the subdivision, see Figure 28. We find that

$$D_i := \eta_{d_i(0)} \star (H \circ d_i) \star \eta_{d_i(1)}^{-1} \in \Omega(U \text{ or } V; x_0).$$

Similarly we define $L_i, R_i, U_i \in \Omega(U \text{ or } V; x_0)$ and we conclude that by (2.2)

$$[D_i]_{W_i} \cdot [R_i]_{W_i} = [L_i]_{W_i} \cdot [U_i]_{W_i} \in \pi_1(W_i; x_0) \tag{2.3}$$

where $W_i = U$or V. We now compute in $\pi_1(U; x_0) \star \pi_1(V; x_0)$ modulo $\mathcal{N}(\{i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V; x_0)\})$:

$$\begin{aligned}
[D_1]_{W_1} \cdot [D_2]_{W_2} \cdot \ldots \cdot [D_N]_{W_N} &= [D_1]_{W_1} \cdot [D_2]_{W_2} \cdot \ldots \cdot [D_N]_{W_N} \cdot [R_N]_{W_N} \\
&= [D_1]_{W_1} \cdot \ldots \cdot [D_{N-1}]_{W_{N-1}} \cdot [L_N]_{W_N} \cdot [U_N]_{W_N} \\
&= [D_1]_{W_1} \cdot \ldots \cdot [D_{N-1}]_{W_{N-1}} \cdot [R_{N-1}]_{W_N} \cdot [U_N]_{W_N}\,.
\end{aligned}$$

If now $W_{N-1} = W_N$ then we can apply (2.3) once more and get

$$[D_1]_{W_1} \cdot [D_2]_{W_2} \cdot \ldots \cdot [D_N]_{W_N} = [D_1]_{W_1} \cdot \ldots \cdot [D_{N-2}]_{W_{N-2}} \cdot [L_{N-1}]_{W_{N-1}} \cdot [U_{N-1}]_{W_{N-1}} \cdot [U_N]_{W_N} \tag{2.4}$$

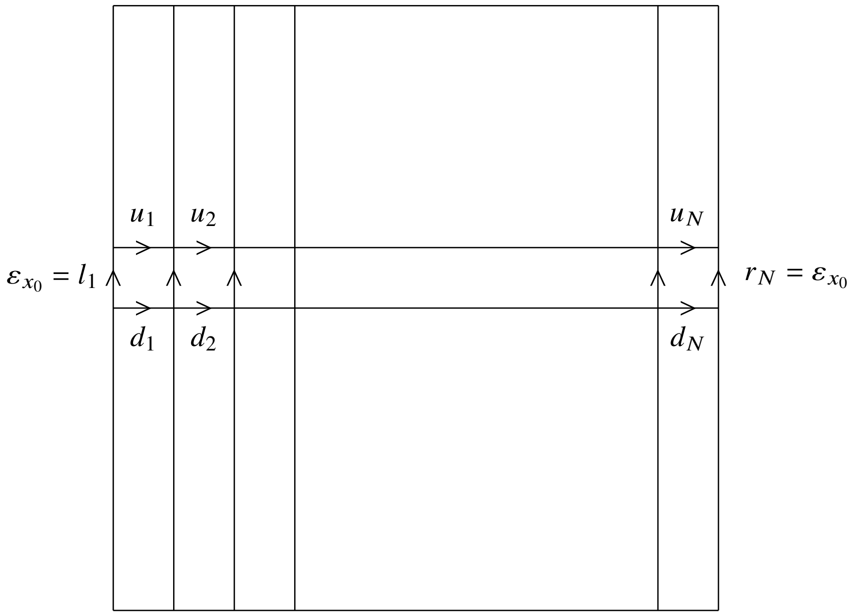


Figure 28. Deforming along one row

In case $W_{N-1} \neq W_N$, say $W_{N-1} = U$ and $W_N = V$, then $L_N = R_{N-1} \in \Omega(U \cap V; x_0)$. Hence

$$[R_{N-1}]_V = j'_\#([R_{N-1}]_{U\cap V}) \overset{\mathrm{mod}\ \mathcal{N}(\dots)}{=} i'_\#([R_{N-1}]_{U\cap V}) = [R_{N-1}]_U\,.$$

In

$$G := \frac{\pi_1(U;x_0) \star \pi_1(V;x_0)}{\mathcal{N}(\{i'_\#(\alpha)\cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V; x_0)\})}$$

the computation (2.4) is still possible. By induction on all squares in the row from right to the left we find that

$$[D_1]_{W_1} \cdot \ldots \cdot [D_N]_{W_N} = \overbrace{[L_1]_{W_1}}^{=1} \cdot [D_1]_{W_1} \cdot \ldots \cdot [D_N]_{W_N} = [U_1]_{W_1} \cdot \ldots \cdot [U_N]_{W_N}\,.$$

A second induction on all rows from bottom to top yields in G

$$[\omega_1]_U \cdot [\omega_2]_V \cdot [\omega_3]_U \cdot \ldots \overset{\mathrm{mod}\ \mathcal{N}(\dots)}{=} [\varepsilon_{x_0}]_{W_1} \cdot [\varepsilon_{x_0}]_{W_2} \cdot [\varepsilon_{x_0}]_{W_3} \cdot \ldots = 1\,.$$

We have shown that in $\pi_1(U;x_0) \star \pi_1(V;x_0)$

$$[\omega_1]_U \cdot [\omega_2]_V \cdot [\omega_3]_U \cdot \ldots \in \mathcal{N}(\{i'_\#(\alpha)\cdot j'_\#(\alpha)^{-1} \,|\, \alpha \in \pi_1(U \cap V; x_0)\})$$

and hence

$$\ker(i_\# \star j_\#) \subset \mathcal{N}\big(\{i'_\#(\alpha)\cdot j'_\#(\alpha)^{-1} \,|\, \alpha \in \pi_1(U \cap V; x_0)\}\big)\,. \qquad \square$$

Corollary 2.69. *Let X be a topological space. Let $U, V \subset X$ be open subsets such that $U \cup V = X$ and let $x_0 \in U \cap V$. Assume that U and V are path connected and that $U \cap V$ is 1-connected. Then*

$$\pi_1(X; x_0) \cong \pi_1(U; x_0) \star \pi_1(V; x_0)$$

where the isomorphism is induced by the inclusion maps.

Proof. By assumption $\pi_1(U \cap V, x_0) = \{1\}$ and hence

$$\mathcal{N}(\{i'_\#(\alpha) \cdot j'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V; x_0)\}) = \{1\}\,.$$

The assertion then follows from Theorem 2.68. □

Example 2.70. Consider the figure 8 space and the two subsets U and V as indicated in the picture:

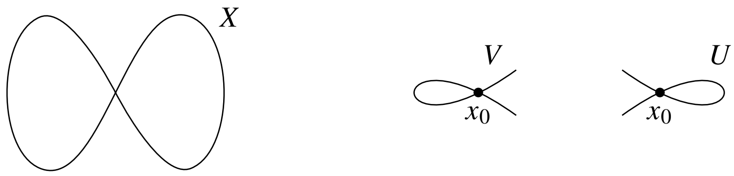


Figure 29. Covering the figure 8

It is easy to see that $U \simeq S^1$ and also $V \simeq S^1$. Moreover, the intersection

$$U \cap V \simeq \text{point}$$

is 1-connected. Hence we have

$$\pi_1(X; x_0) \cong \pi_1(S^1; x_0) \star \pi_1(S^1; x_0) \cong \mathbb{Z} \star \mathbb{Z}\,.$$

Example 2.71. Let X be a connected n-dimensional manifold with $n \geq 3$.

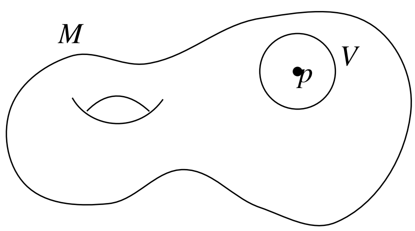


Figure 30. Punctured manifold

Let $p \in X$ and $U := X \setminus \{p\}$. Now let V be an open neighborhood of p homeomorphic to $\mathbb{R}^n$, which is contractible and hence 1-connected. Then the space

$$U \cap V \approx \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$$

is 1-connected. By Corollary 2.69 we then deduce

$$\pi_1(X;x_0) \cong \pi_1(U;x_0) \star \pi_1(V;x_0) = \pi_1(X \setminus \{p\};x_0)\,.$$

Removing a point from a manifold of dimension at least 3 does not change its fundamental group.

Example 2.72. Let M and N be two connected manifold of dimension $n \geq 3$. Let $X = M\#N$.

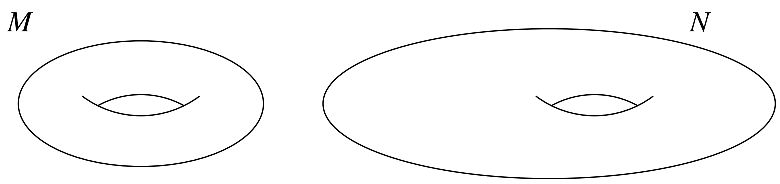


Figure 31. Start with two manifolds...

Now choose U and V as in Figure 32. Then we have that $U \approx M \setminus \{p\}$ and $V \approx N \setminus \{q\}$.

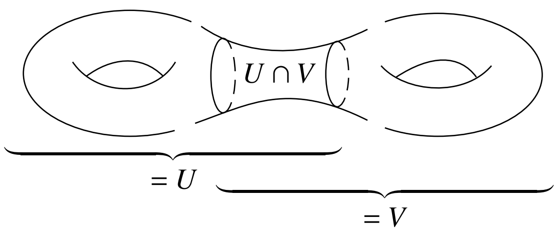


Figure 32. ... and consider their connected sum.

Since

$$U \cap V \approx S^{n-1} \times (0,1) \simeq S^{n-1}$$

is 1-connected we find by Corollary 2.69 once again that

$$\pi_1(M\#N) \cong \pi_1(U) \star \pi_1(V) \cong \pi_1(M) \star \pi_1(N)\,.$$

For example, if $M = N = T^3$ then $\pi_1(T^3\#T^3) = (\mathbb{Z}^3) \star (\mathbb{Z}^3)$.

Remark 2.73. We now see easily that the torus T^n with $n \geq 3$ cannot be homotopy equivalent to the connected sum $T^n \simeq M\#N$ of two non-1-connected manifolds M and N.[3] If it were possible then $\pi_1(T^n) \cong \pi_1(M) \star \pi_1(N)$ would not be abelian but we know that $\pi_1(T^n) \cong \mathbb{Z}^n$, a contradiction.

2.5. The fundamental group of surfaces

Our aim is to prove that orientable compact connected surfaces of different genus (as depicted by the pastries in Example 1.5) are not homotopy equivalent and therefore not homeomorphic.

[3]If one allows simply connected summands then it is of course possible, $T^n \approx T^n\#S^n$.

Definition 2.74. We call $F_g := \underbrace{T^2\#\cdots\#T^2}_{g-\text{times}}$ a *surface of genus* $g \geq 1$.

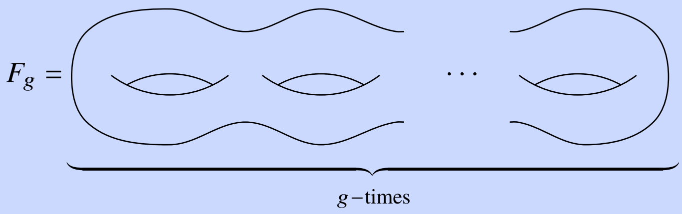


Figure 33. Surface of genus g

Remark 2.75. We also put $F_0 := S^2$.

We now want to compute $\pi_1(F_g)$ and show that the fundamental groups for surfaces of different genus are not isomorphic. This then shows in particular that they are not homotopy equivalent.

Proposition 2.76. *For any* $g \geq 1$

$$F_g = \underbrace{T^2\#\cdots\#T^2}_{g-\text{times}} \approx D^2/\sim$$

where $x \sim y$ *iff* $x = y$ *or* $x, y \in \partial D^2$ *and are identified according to the following scheme:*

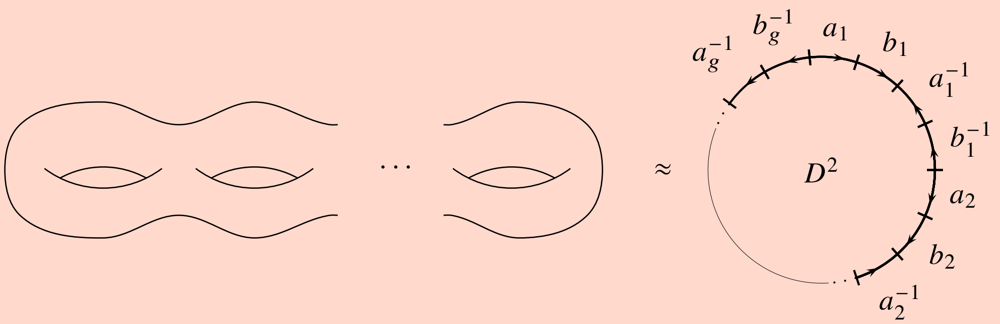


Figure 34. Building a surface from the disk

This identification is to be understood as follows: Each line segment labelled by a_i *(or* b_j *respectively) is identified with with* a_i^{-1} *(or* b_j^{-1}*) with respect to the direction indicated by the arrow. Note that there are* $2g$ *labels* $a_i, b_i, i = 1, .., g$.

Proof. We do an induction on g.

Induction basis for $g = 1$**:** We see that the labelled disk $D^2/\sim$ is homeomorphic to the labelled $W^2/\sim$ which is homeomorphic to the two-dimensional torus, i.e. to F_1.

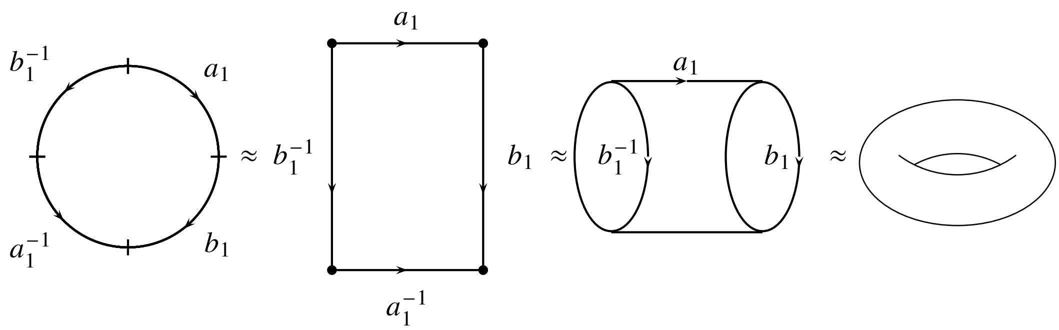


Figure 35. Starting the induction with the torus

Inductive step, $g - 1 \Rightarrow g$**:** We perform a cut along the line c.

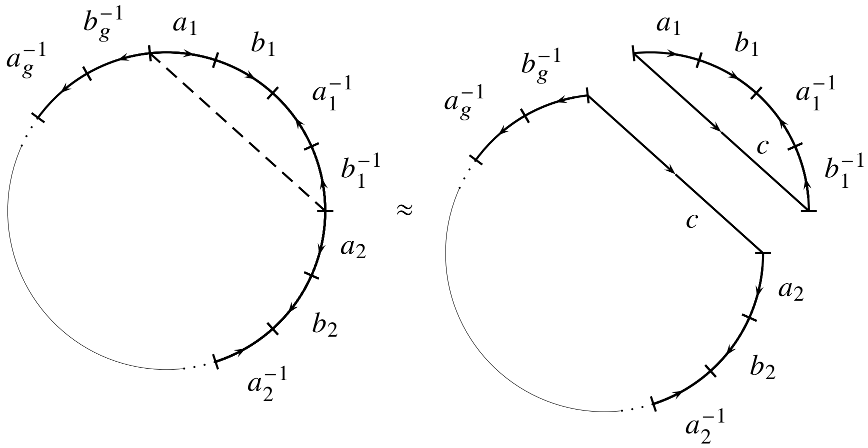


Figure 36. Induction step by cutting off a segment

The endpoints of c in the two pieces are identified. This yields a homeomorphism where the interior of the now closed loop c is cut out of the two remaining disks.

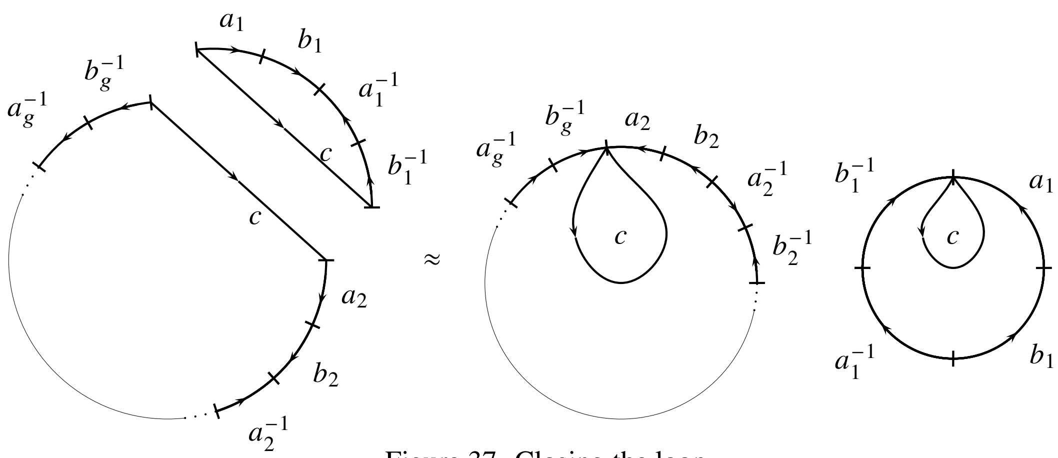


Figure 37. Closing the loop c

By **induction hypothesis** the left-hand disc is homeomorphic to a surface of genus $g-1$ with a disc removed that is bounded by c. The right-hand disc becomes a torus, also with a disc removed that is bounded by c.
Gluing these two spaces together along c, we obtain a space which is homeomorphic to a surface of genus g:

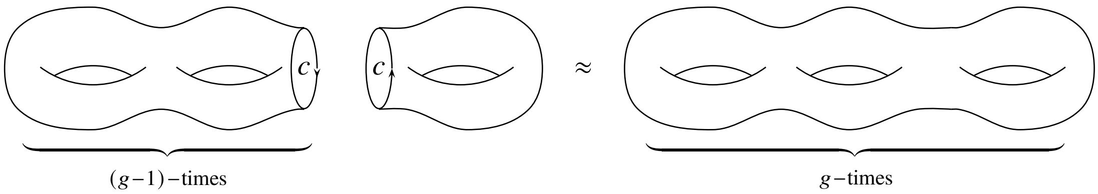


Figure 38. Completing the induction step

□

Remark 2.77. We note that $F_0 \approx D^2/\partial D^2$.

Remark 2.78. Let G be a group. Assume G is generated by $g_1, ..., g_n \in G$ as a group. We have group homomorphisms $\varphi_i : \mathbb{Z} \to G$, $\varphi_i(k) = g_i^k$. Repeated application of the universal property of free products of groups yields the group homomorphism

$$(\dots(\varphi_1 \star \varphi_2) \star \varphi_3) \star \cdots \star \varphi_n =: \varphi_1 \star \cdots \star \varphi_n : (\dots(\mathbb{Z} \star \mathbb{Z}) \star \mathbb{Z}) \star \cdots \star \mathbb{Z} =: \mathbb{Z} \star \cdots \star \mathbb{Z} \to G$$

with

$$(\varphi_1 \star \cdots \star \varphi_n)((k_1, i_1), ..., (k_r, i_r)) = g_{i_1}^{k_1} \cdots g_{i_r}^{k_r},$$

where $i_j \in \{1, ..., n\}$ is an index used to make the $\mathbb{Z}$-factors formally disjoint.

The fact that $g_1, ..., g_n$ generate G is equivalent to the fact that $\varphi_1 \star \cdots \star \varphi_n : \mathbb{Z} \star \cdots \star \mathbb{Z} \to G$ is onto. It follows that

$$G \cong \frac{\mathbb{Z} \star \cdots \star \mathbb{Z}}{\ker(\varphi_1 \star \cdots \star \varphi_n)}$$

If the normal subgroup $\ker(\varphi_1 \star \cdots \star \varphi_n)$ is also finitely generated as a group, with generators $x_1, ..., x_m$, then we call G *finitely presentable* and

$$\langle g_1, ..., g_n \mid x_1, ..., x_m \rangle$$

a *presentation* of G.

Example 2.79. For the cartesian product $\mathbb{Z}^2$ of two copies of $\mathbb{Z}$ we have the presentation

$$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

The cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order 2 is presentable as

$$\mathbb{Z}/2\mathbb{Z} \cong \langle a \mid a^2 \rangle.$$

Remark 2.80. A word of caution: isomorphic groups may have several, quite different presentations. For example

$$\langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle \cong \langle x, y, w, z \mid xyxy^{-1}x^{-1}y^{-1},\ xyxw^{-1},\ zy^{-1}x^{-1} \rangle$$

because the generators $xyxw^{-1}$ and $zy^{-1}x^{-1}$ on the right-hand side can be used to eleminate w and z. It is therefore often not obvious whether two presentations give rise to isomorphic groups.

Theorem 2.81. *For any $g \in \mathbb{N}$ we have*

$$\pi_1(F_g) \cong \langle a_1, b_1, ..., a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Proof. Recall that $F_g \approx D^2/\sim$ as in Proposition 2.76. To apply the Seifert-van Kampen theorem 2.68 we put $U := \mathring{D}^2$ and $V := (D^2/\sim) \setminus D^2(\frac{1}{2})$. We have $F_g = U \cup V$.

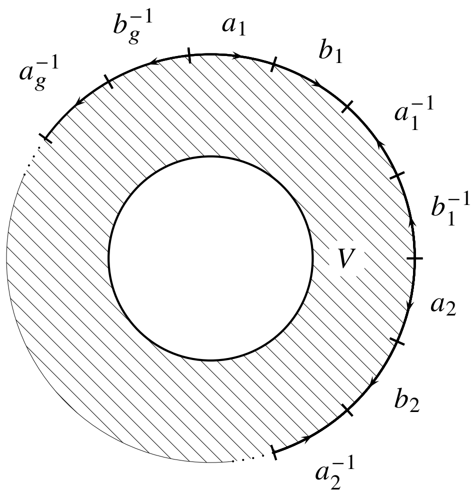


Figure 39. Applying the Seifert-van Kampen theorem to the disk representation

The subset U is contractible and therefore $\pi_1(U) = \{1\}$. The subset V is homotopy equivalent to the boundary ∂D subject to the identifications of the equivalence relation, $V \simeq \partial D/\sim$. The identifications induced by $\sim$ generate a bouquet of $2g$ circles, one for each relation a_i and b_i. The bouquet is denoted by $S^1 \vee \cdots \vee S^1$, where the *wedge sum* "$\vee$" of two topological spaces X and Y is defined to be the disjoint union of X and Y with identification of two base points $x_0 \in X$, $y_0 \in Y$ such that $X \vee Y := X \cup Y/\{x_0 \sim y_0\}$. For the bouquet of circles all S^1's are joined at the same base point. Graphically we can depict the bouquet of circles as follows

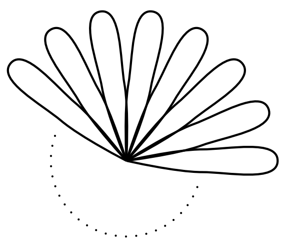
$S^1 \vee \cdots \vee S^1 =$ 

Figure 40. Bouquet of circles

So we have

$$V \simeq \partial D^2/\!\sim\, \approx \underbrace{S^1 \vee \cdots \vee S^1}_{2g \text{ times}}.$$

The fundamental group of V follows immediately from Example 2.70 by induction:

$$\pi_1(V) \cong \underbrace{\mathbb{Z} \star \mathbb{Z} \star \cdots \star \mathbb{Z}}_{2g \text{ times}}$$

with generators again denoted by $a_1, b_1, a_2, b_2, ..., a_g, b_g$. For the intersection of U and V we have $U \cap V = \mathring{D}^2 \setminus \bar{D}^2(\frac{1}{2}) \simeq S^1$ and thus by Theorem 2.42 we have $\pi_1(U \cap V) \cong \mathbb{Z}$. A generator of $\pi_1(U \cap V)$ is given by a loop c of degree 1.

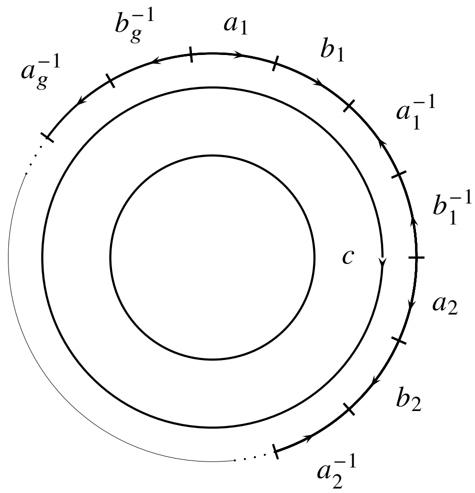


Figure 41. Finding the generator

The inclusion map $i' : U \cap V \longrightarrow U$ induces the trivial homomorphism $i'_\#$ because $\pi_1(U)$ is trivial. For the inclusion map $j' : U \cap V \longrightarrow V$ we note that the induced isomorphism maps c onto $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.
By the Seifert-van Kampen theorem 2.68 we find

$$\begin{aligned}
\pi_1(F_g) &\cong \frac{\pi_1(U) \star \pi_1(V)}{\mathcal{N}(j'_\#(\alpha) \cdot i'_\#(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V))} \\
&= \frac{\pi_1(V)}{\mathcal{N}(j'_\#(\alpha) \mid \alpha \in \pi_1(U \cap V))} \\
&= \frac{\mathbb{Z} \star \cdots \star \mathbb{Z}}{\mathcal{N}(j'_\#(c))} \\
&= \langle a_1, b_1, ..., a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.
\end{aligned}$$

□

Example 2.82. For the two-dimensional torus T^2 we find with Example 2.79

$$\pi_1(T^2) = \pi_1(F_1) = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2,$$

in agreement with Remark 2.48.

Corollary 2.83. *For $g, g' \in \mathbb{N}$, $g \neq g'$ we have $F_g \not\simeq F_{g'}$ and hence $F_g \not\approx F_{g'}$.*

Proof. The statement follows once we see that $\pi_1(F_g) \ncong \pi_1(F_{g'})$. Attention here: as noted in 2.80 different presentations can yield isomorphic groups.
For any group G let $[G, G]$ be the normal subgroup generated by all commutators $aba^{-1}b^{-1}$, $a, b \in G$. The abelian factor group $G/[G, G]$ is called the *abelianization of* G. We now calculate the abelianization of $\pi_1(F_g)$.

$$\begin{aligned}
\frac{\pi_1(F_g)}{[\pi_1(F_g), \pi_1(F_g)]} &= \langle a_1, b_1, ..., a_g, b_g \mid a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}, a_1b_1a_1^{-1}b_1^{-1}, ... \\
&\qquad ..., a_1a_2a_1^{-1}a_2^{-1}, ...\rangle \\
&= \langle a_1, b_1, ..., a_g, b_g \mid a_1b_1a_1^{-1}b_1^{-1}, ..., a_1a_2a_1^{-1}a_2^{-1}, ...\rangle \\
&\cong \mathbb{Z}^{2g},
\end{aligned}$$

where the second equality follows because the simple commutators $a_ib_ia_i^{-1}b_i^{-1}$, $i = 1, ..., g$ generate the relation $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$.
Hence if $F_g \simeq F_{g'}$ then $\pi_1(F_g) \cong \pi_1(F_{g'})$ and thus

$$\pi_1(F_g)/[\pi_1(F_g), \pi_1(F_g)] \cong \pi_1(F_g')/[\pi_1(F_g'), \pi_1(F_g')].$$

Thus $\mathbb{Z}^{2g} \cong \mathbb{Z}^{2g'}$ and therefore $g = g'$. □

Remark 2.84. This proves the uniqueness part of the classification for surfaces, see Example 1.5.

2.6. Higher homotopy groups

We now generalize the definition of $\pi_1(X; x_0)$.

Definition 2.85. Let $W^n = \underbrace{[0, 1] \times \ldots \times [0, 1]}_{n \text{ times}}$ be the *n-cube*. Let X be a topological space and $x_0 \in X$. Then

$$\pi_n(X; x_0) := \{[\sigma]_{\partial W^n} \mid \sigma \in C(W^n, X),\ \sigma(\partial W^n) = \{x_0\}\}$$

is called the *n-th homotopy group* of X with base point x_0. Here $[\sigma]_{\partial W^n}$ denotes the homotopy class of σ relative to ∂W^n.

The group structure on $\pi_n(X; x_0)$ is obtained as follows: For $\sigma, \tau \in C(W^n, X)$ with $\sigma(\partial W^n) = \tau(\partial W^n) = \{x_0\}$ define $\sigma \star \tau$ by

$$(\sigma \star \tau)(t_1, \ldots, t_n) = \begin{cases} \sigma(2t_1, t_2, \ldots, t_n), & 0 \leq t_1 \leq 1/2, \\ \tau(2t_1 - 1, t_2, \ldots, t_n), & 1/2 \leq t_1 \leq 1. \end{cases}$$

Then $\sigma \star \tau \in C(W^n, X)$ with $(\sigma \star \tau)(\partial W^n) = \{x_0\}$.

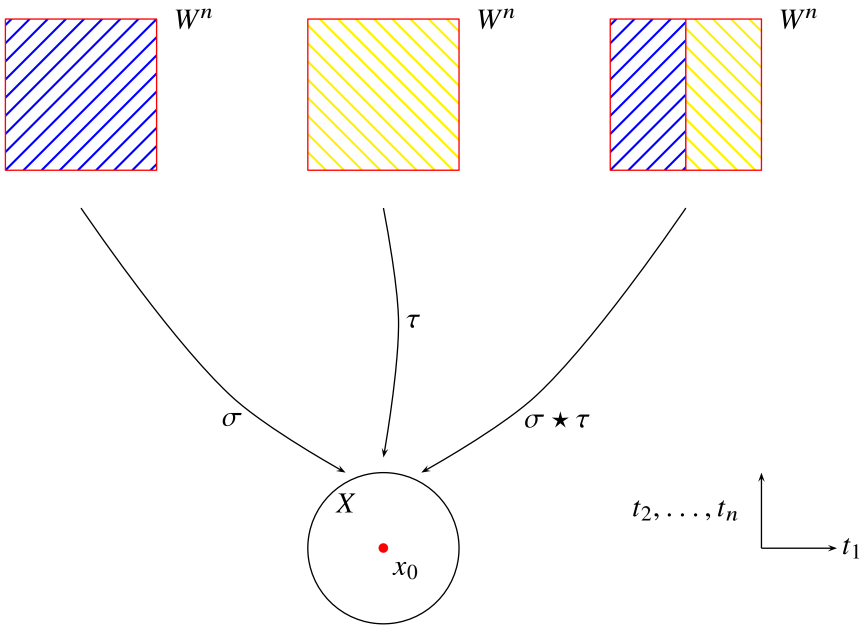


Figure 42. Concatenation

Now put $[\sigma]_{\partial W^n} \cdot [\tau]_{\partial W^n} := [\sigma \star \tau]_{\partial W^n}$. The proof that this yields a well-defined group multiplication on $\pi_n(X; x_0)$ for $n \geq 2$ is literally the same as in the case for $n = 1$. The neutral element is represented by the constant map $\varepsilon^n_{x_0} : W^n \to X$, $\varepsilon^n_{x_0}(t_1, \ldots, t_n) = x_0$, and $[\sigma]^{-1}_{\partial W^n}$ is represented by $\sigma(1 - t_1, t_2, \ldots, t_n)$.

Unlike for the case $n = 1$ the higher homotopy groups are abelian:

Proposition 2.86. *Let X be a topological space and $x_0 \in X$. Then for $n \geq 2$ the group $\pi_n(X; x_0)$ is abelian.*

Proof. For $\sigma, \tau \in C(W^n, X)$ with $\sigma(\partial W^n) = \tau(\partial W^n) = \{x_0\}$ we need to show that $\sigma \star \tau$ and $\tau \star \sigma$ are homotopic relative to ∂W^n. The homotopy is obtained by precomposing with the homotopy of the n-cube indicated in the following picture (which illustrates the case $n = 2$):

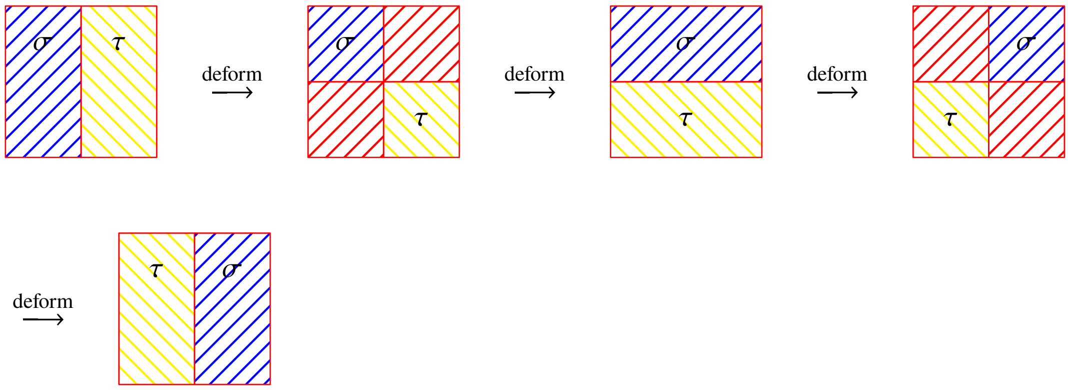


Figure 43. Commutativity of higher homotopy groups

Remark 2.87. This proof also shows that replacing t_1 by any other variable in the definition of $\sigma \star \tau$ gives the same group multiplication on $\pi_n(X; x_0)$.

For $f \in C(X, Y)$ with $f(x_0) = y_0$ we get a group homomorphism

$$f_\# : \pi_n(X; x_0) \longrightarrow \pi_n(Y; y_0)$$

defined by $[\sigma]_{\partial W^n} \mapsto [f \circ \sigma]_{\partial W^n}$.

Remark 2.88. Lemma 2.18, 2.19, Corollary 2.20, Propositions 2.21, 2.22, Theorem 2.23 and Corollary 2.24 also hold for $\pi_n(X; x_0)$. In particular, if $f : X \longrightarrow Y$ is a homotopy equivalence then the map $f_\# : \pi_n(X; x_0) \longrightarrow \pi_n(Y; f(x_0))$ is an isomorphism. For $\gamma \in \Omega(X; x_0, x_1)$ there is an isomorphism

$$\Phi_\gamma : \pi_n(X; x_1) \longrightarrow \pi_n(X; x_0)$$

given by $[\sigma]_{\partial W^n} \mapsto [\sigma']_{\partial W^n}$ where

$$\sigma'(t) = \begin{cases} \sigma(2t), & \|t\|_{\max} \leq \frac{1}{2}, \\ \gamma(2 - 2\|t\|_{\max}), & \frac{1}{2} \leq \|t\|_{\max} \leq 1. \end{cases}$$

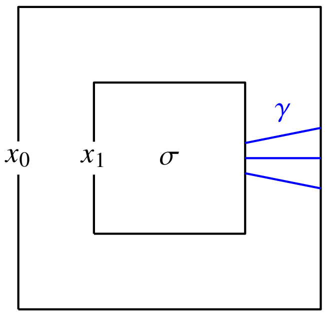


Figure 44. “Independence” of base point

Remark 2.89. By Exercise 1.7 we have the maps

$$W^n \xrightarrow{\pi} W^n/_{\partial W^n} \xrightarrow[\approx]{\varphi} S^n$$

and we see that $\sigma \in C(W^n, X)$ with $\sigma(\partial W^n) = \{x_0\}$ corresponds uniquely to $f_\sigma \in C(S^n, X)$ with $f_\sigma(s_0) = x_0$, where $s_0 = \varphi(\partial W^n)$ such that $\sigma = f_\sigma \circ \varphi \circ \pi$. Thus there is a canonical bijection

$$\pi_n(X; x_0) \overset{1:1}{\longleftrightarrow} \{[f]_{\{s_0\}} \mid f \in C(S^n, X),\ f(s_0) = x_0\}\,.$$

Remark 2.90. The Seifert-van-Kampen theorem for π_n works only under very restrictive assumptions. For this reason the computation of π_n for explicit examples can be very difficult. We will be able to compute $\pi_n(S^m)$ for $n \leq m$ but many of the $\pi_n(S^m)$ for $n > m$ are actually still unknown.

Remark 2.91. The definition of $\pi_n(X; x_0)$ also works for $n = 0$. A map $\sigma \in C(W^0, X)$ corresponds to the point $\sigma(W^0) \in X$. Two such maps are homotopic iff the corresponding points can be joined by a path. Hence

$$\pi_0(X; x_0) = \{\text{path components of } X\}.$$

But there is no (natural) group structure on $\pi_0(X; x_0)$. More precisely, $\pi_0(X; x_0)$ is a *pointed set*, i.e., a set with a distinguished point, namely the path component containing x_0. This corresponds to the neutral element in $\pi_n(X; x_0)$ for $n \geq 1$.

Definition 2.92. Let W, E and B be topological spaces and let $p \in C(E, B)$. We say that p has the *homotopy lifting property* (*HLP* for short) for W iff for every $f \in C(W, E)$ and every $h \in C(W \times [0,1], B)$ with $h(w, 0) = p(f(w))$ for all $w \in W$ there exists $H \in C(W \times [0,1], E)$ such that

$$H(w, 0) = f(w) \quad \forall w \in W \qquad \text{and} \qquad h = p \circ H\,.$$

In other words, there exists an $H \in C(W \times [0,1], E)$ such that the diagram

$$\begin{tikzcd} w \arrow[d, mapsto] & \in & W \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ (w,0) & \in & W \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

commutes.

Example 2.93. Consider the spaces $E = \{\text{point}\}$, $B = \mathbb{R}$, $W = W^0$ and the map given by $p(e) = 0$. No $h : W \times [0,1] \longrightarrow B$ except the constant path ε_0 can be lifted because it leaves the image of p. Here the problem is the lack of surjectivity of p.

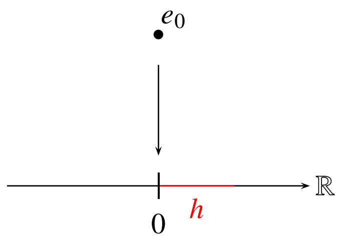


Figure 45. Failure of HLP due to lack of surjectivity

Example 2.94. Now consider $E = [0,\infty) \times \{0\} \cup (-\infty, 0] \times \{1\} \subset \mathbb{R}^2$, $B = \mathbb{R}$ and let p be the projection onto the first factor, $p(t,s) = t$.

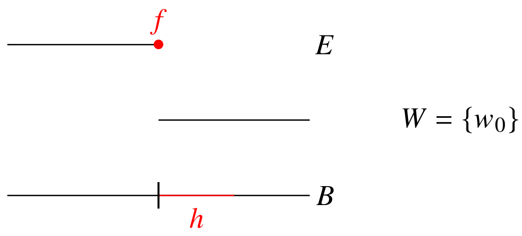


Figure 46. Surjective but HLP still fails

The map p is surjective but still does not have the HLP for $W = W^0$. For example, choose $h(w_0, t) = t$, $f(w_0) = (0, 1)$.

Definition 2.95. A map $p \in C(E,B)$ is called a *Serre fibration* or *weak fibration* iff it has the HLP for all W^n, $n \geq 0$. The space E is called the *total space* and B is called the *base space* of the fibration. For $b_0 \in B$ we call $p^{-1}(b_0)$ the *fiber* over b_0.

Example 2.96. For topological spaces F and B put $E := B \times F$ and $p = pr_1$, the projection on the B-factor. Then p has the HLP for all W. In particular, p is a Serre fibration. Namely, let $f \in C(W, B \times F)$ and $h \in C(W \times [0,1], B)$ be given such that $p(f(w)) = h(w,0)$ for all $w \in W$. Now write $f(w) = (\beta(w), \varphi(w))$ with $\beta \in C(W,B)$ and $\varphi \in C(W,F)$. Hence

$$h(w,0) = p(f(w)) = \beta(w).$$

Now put $H(w,t) := (h(w,t), \varphi(w))$. Then $H \in C(W \times [0,1], B \times F)$ and

$$\begin{aligned} p(H(w,t)) &= pr_1(h(w,t), \varphi(w)) = h(w,t)\,, \\ H(w,0) &= (h(w,0), \varphi(w)) = (\beta(w), \varphi(w)) = f(w)\,, \end{aligned}$$

as required. Hence p has the HLP for any W.

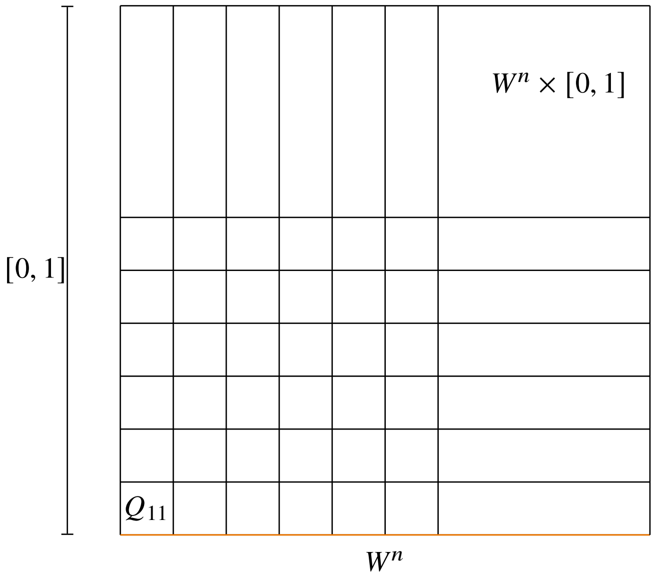


Figure 47. Subdivision for which the fiber bundle is trivial over each subcube

Definition 2.97. A map $p \in C(E, B)$ is called *fiber bundle* with fiber F iff for each $b \in B$ there exists an open subset $U \subset B$ with $b \in U$ and a homeomorphism $\Phi : p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} U \times F \arrow[d, "pr_1"'] & p^{-1}(U) \arrow[l, "\Phi"', "\approx"] \arrow[dl, "p|_{p^{-1}(U)}"] \\ U & \end{tikzcd}$$

commutes.

Lemma 2.98. *Every fiber bundle is a Serre fibration.*

Proof. Let $f \in C(W^n, E)$ and $h \in C(W^n \times [0,1], B)$ such that $h(w, 0) = p(f(w))$ for all $w \in W$. Now subdivide $W^n \times [0,1]$ into small subcubes such that h maps each subcube entirely into an open subset U as in the definition of the fiber bundle, see Figure 47.

By Example 2.96, products have the HLP, hence we can extend the map f to a continuous map $H_{11} : (W^n \times \{0\}) \cup Q_{11} \to E$ such that $p \circ H_{11} = h$.

Next we want to extend the lift to Q_{12}, see Figure 48. Now there seems to be a problem because the required lift H_{12} need not only coincide with f along the edge $Q_{12} \cap (W^n \times \{0\})$ but also with H_{11} along $Q_{11} \cap Q_{12}$.

But there are homeomorphisms of a cube onto itself mapping two edges onto one as indicated in Figure 49.

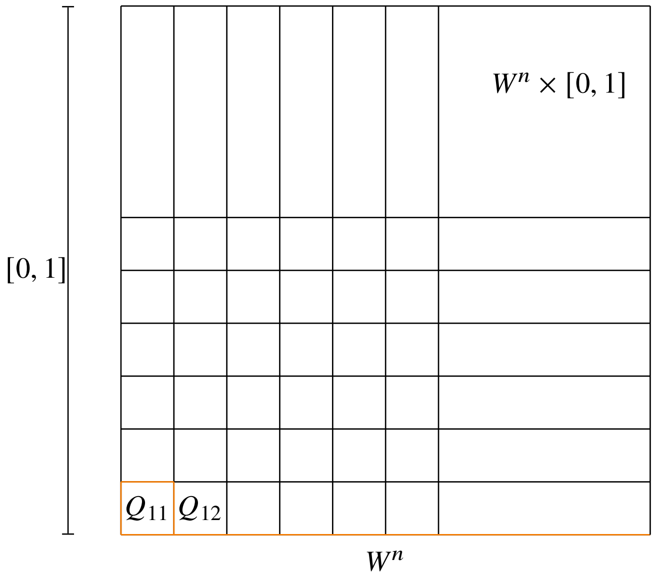


Figure 48. Lift over second cube

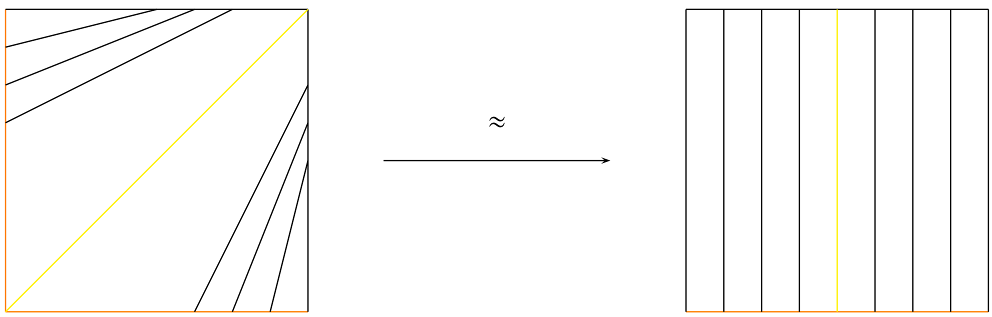


Figure 49. Homeomorphism mapping two faces to one

Thus we can apply the HLP and extend the lift to $(W^n \times \{0\}) \cup Q_{11} \cup Q_{12}$. Iteration of this procedure proves the assertion. □

Example 2.99. Let G be a Lie group, e.g. a closed subgroup of $\mathrm{GL}(n;K)$ with $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $H \subset G$ be a closed subgroup. We equip the space $B = G/H = \{g \cdot H \mid g \in G\}$ with the quotient topology. Such a space is called a *homogeneous space*. Then $G \to G/H$ with $g \mapsto g \cdot H$ is a fiber bundle with fiber H. The proof of this fact requires some technical work, see [8, p. 120 ff].

Example 2.100. Let $G = \mathrm{SO}(n+1)$ and

$$H = \left\{ \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array} \right) \middle| B \in \mathrm{SO}(n) \right\}.$$

Then H is a closed subgroup of G isomorphic to $\mathrm{SO}(n)$. We now show that $G/H \approx S^n$. Consider the map $f : G \to S^n$ with $A \mapsto A \cdot e_{n+1}$ where e_{n+1} the $(n+1)$-st unit vector of the canonical basis. The map f is continuous and surjective. We observe

$$\begin{aligned} f(A) = f(\tilde{A}) \quad &\iff \quad A \cdot e_{n+1} = \tilde{A} \cdot e_{n+1} \\ &\iff \quad \tilde{A}^{-1} \cdot A \cdot e_{n+1} = e_{n+1} \\ &\iff \quad \tilde{A}^{-1} \cdot A = \left(\begin{array}{c|c} \star & 0 \\ \hline \star & 1 \end{array} \right) \\ &\iff \quad \tilde{A}^{-1} \cdot A \in H \\ &\iff \quad \pi(A) = \pi(\tilde{A}) \end{aligned}$$

where the map $\pi : G \to G/H$ is the canonical projection. We conclude that the map f descends to a bijective map $\bar{f} : G/H \to S^n$. By the universal property of the quotient topology the map $\bar{f} : G/H \to S^n$ is continuous. Since G is compact the space G/H is also compact. Moreover, the sphere S^n is a Hausdorff space, hence the map $\bar{f}$ is a homeomorphism. Thus we obtain a fiber bundle $\mathrm{SO}(n+1) \to S^n$ with fiber $\mathrm{SO}(n)$.

Let $p : E \to B$ be any Serre fibration and fix $e_0 \in E$. Put $b_0 := p(e_0) \in B$ and let $F := p^{-1}(b_0)$. Then $e_0 \in F$. Let $\iota : F \to E$ be the inclusion map. We obtain the following two homomorphisms:

$$\begin{aligned} \iota_\# &: \pi_n(F; e_0) \to \pi_n(E; e_0), \\ p_\# &: \pi_n(E; e_0) \to \pi_n(B; b_0). \end{aligned}$$

Now we construct a map $\partial : \pi_n(B; b_0) \to \pi_{n-1}(F; e_0)$. We define

$$\mathrm{Box}_0 := \{1\} \quad \text{and} \quad \mathrm{Box}_k := (W^k \times \{1\}) \cup (\partial W^k \times [0,1]) \text{ for } k \geq 1.$$

Then we have

$$\begin{aligned} \partial W^n &= (W^{n-1} \times \{0\}) \cup \mathrm{Box}_{n-1}, \\ (W^{n-1} \times \{0\}) \cap \mathrm{Box}_{n-1} &= \partial W^{n-1} \times \{0\}. \end{aligned}$$

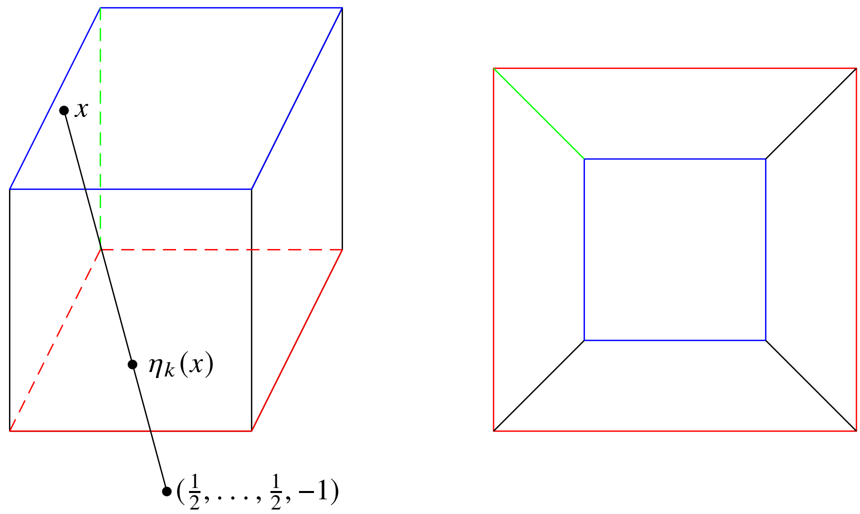


Figure 50. Mapping the box to the (bottom) cube

Consider the homeomorphism $\eta_k : \mathrm{Box}_k \to W^k$ obtained by the central projection from $(\frac{1}{2}, \dots, \frac{1}{2}, -1)$. This homeomorphism maps the faces out of which Box_k is built onto the regions depicted in Figure 50.
In order to define $\partial : \pi_n(B; b_0) \to \pi_{n-1}(F; e_0)$ let $\sigma \in C(W^n, B)$ with $\sigma(\partial W^n) = \{b_0\}$. Since $(t_1, \dots, t_{n-1}) \mapsto \sigma(t_1, \dots, t_{n-1}, 0) = b_0$ is constant we can lift it to the constant map $(t_1, \dots, t_{n-1}) \mapsto e_0$, see Figure 51. Now the HLP of p for W^{n-1} yields a continuous map $\Sigma : W^n \to E$ with $\Sigma(t_1, \dots, t_{n-1}, 0) = e_0$ and $p \circ \Sigma = \sigma$. From $\sigma(\partial W^n) = \{b_0\}$ we have $\Sigma(\partial W^n) \subset F$. We put $\tilde{\sigma} := \Sigma \circ \eta_{n-1}^{-1} : W^{n-1} \to F$. We want to define the map $\partial([\sigma]_{\partial W^n}) := [\tilde{\sigma}]_{\partial W^{n-1}}$. We have to check well-definedness of this map:

a) We have to show that $[\tilde{\sigma}]_{\partial W^{n-1}}$ does not depend on the particular choice of the lift Σ.

Let $\Sigma' \in C(W^n, E)$ be another lift of σ with $\Sigma'(t_1, \dots, t_{n-1}, 0) = e_0$. Then $\Sigma^{-1} \bullet \Sigma'$ is a lift of $\sigma^{-1} \bullet \sigma$. Here $\bullet$ denotes the concatenation with respect to the variable t_n, $\Sigma^{-1}(t_1, \dots, t_n) = \Sigma(t_1, \dots, t_{n-1}, 1 - t_n)$ and similarly for σ^{-1}. Since $\sigma^{-1} \bullet \sigma \simeq_{\partial W^n} \varepsilon^n_{b_0}$ we can find a homotopy $h : W^{n+1} \to B$ relative to ∂W^n with

$$h(t_1, \dots, t_n, 0) = (\sigma^{-1} \bullet \sigma)(t_1, \dots, t_n) \quad \text{and} \quad h(t_1, \dots, t_n, 1) = b_0.$$

Then $h(\mathrm{Box}_n) = \{b_0\}$. We apply the HLP of p for W^{n+1} to get a lift $H \in C(W^{n+1}, E)$ of h with

$$H(t_1, \dots, t_n, 0) = \Sigma^{-1} \bullet \Sigma'(t_1, \dots, t_n).$$

From $h(\mathrm{Box}_n) = \{b_0\}$ we have $H(\mathrm{Box}_n) \subset F$. Then we get a homotopy in F relative to ∂W^{n-1} from $\tilde{\sigma} = \Sigma \circ \eta_{n-1}^{-1}$ to $\tilde{\sigma}' = \Sigma' \circ \eta_{n-1}^{-1}$ as shown in Figure 52.

b) We also have to show that $\sigma \simeq_{\partial W^n} \sigma'$ in B implies $\tilde{\sigma} \simeq_{\partial W^{n-1}} \tilde{\sigma}'$ in F.

Let $h : W^n \times [0, 1] \to B$ be a homotopy in B from σ to σ' relative to ∂W^n. The HLP for W^n

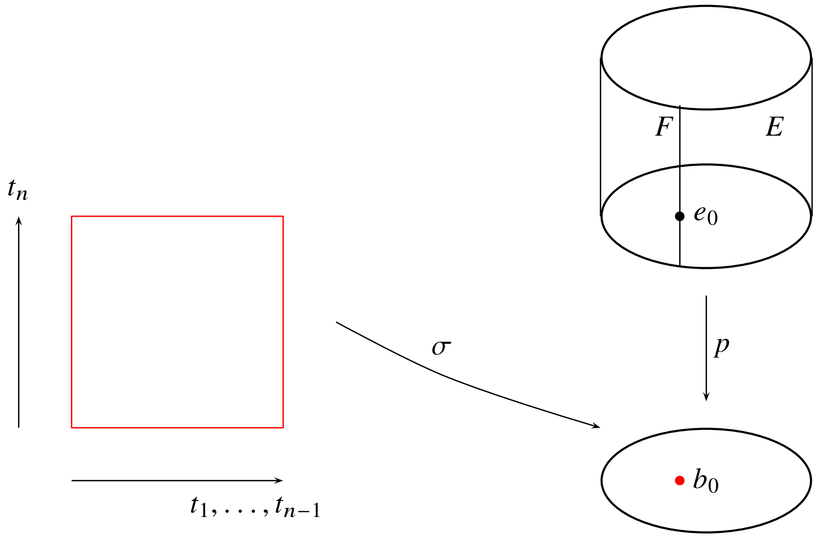


Figure 51. Lift σ

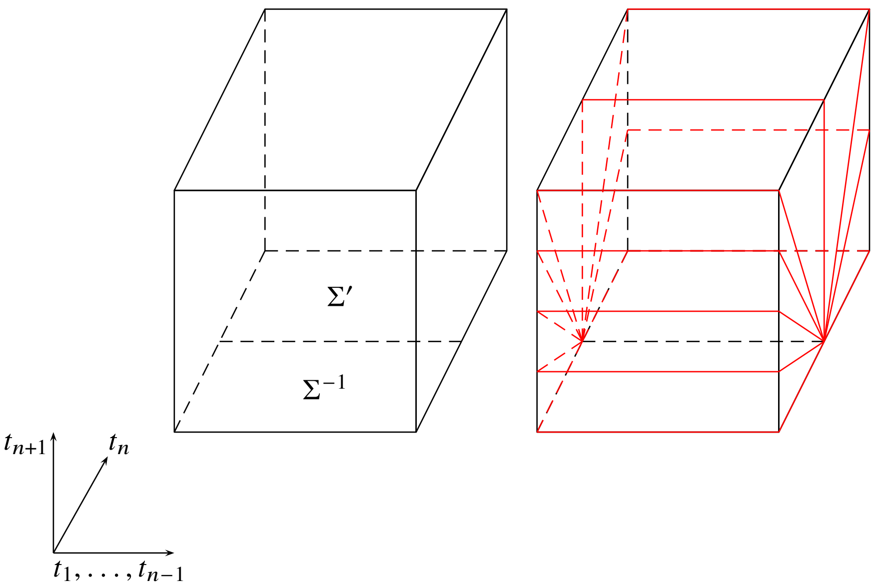


Figure 52. Homotopy between projected lifts

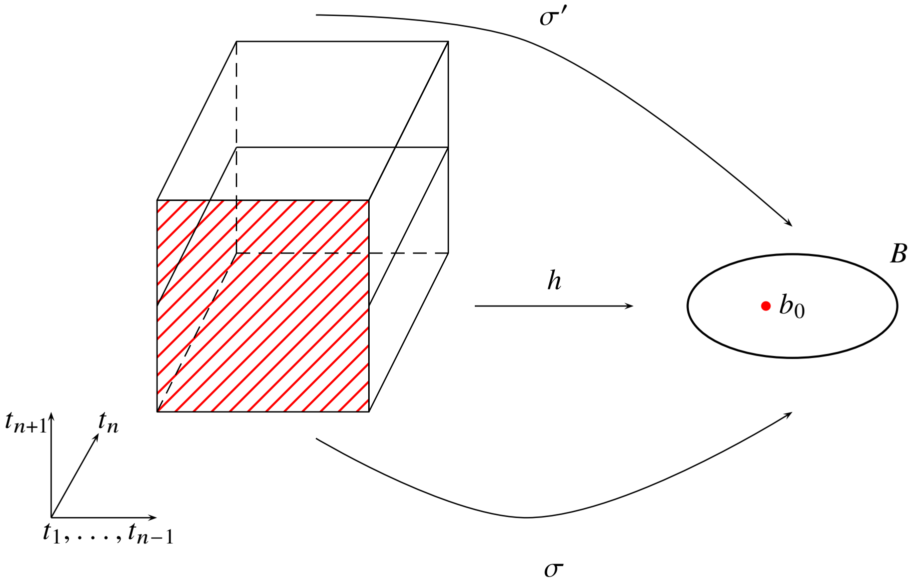


Figure 53. Homotopy invariance

yields a lift $H : W^{n+1} \to E$ of h with $H(t_1, \ldots, t_{n-1}, 0, t_{n+1}) = e_0$, see Figure 53. We thus obtain a homotopy

$$\hat{H}(t_1, \ldots, t_{n-1}, s) = H(\eta_{n-1}^{-1}(t_1, \ldots, t_{n-1}), s)$$

in F from $\tilde{\sigma}$ to $\tilde{\sigma}'$ relative to ∂W^{n-1} and hence

$$[\tilde{\sigma}]_{\partial W^{n-1}} = [\tilde{\sigma}']_{\partial W^{n-1}} .$$

We have shown that the map $\partial : \pi_n(B; b_0) \to \pi_{n-1}(F, e_0)$ is well defined.

Lemma 2.101. *For $n \geq 2$ the map $\partial : \pi_n(B; b_0) \to \pi_{n-1}(F, e_0)$ is a group homomorphism.*

Proof. Let $\sigma, \tau \in C(W^n, B)$ such that $\sigma(\partial W^n) = \tau(\partial W^n) = \{b_0\}$. Choose a lift $H \in C(W^n, E)$ of $\sigma \star \tau$ with $H(t_1, \ldots, t_{n-1}, 0) = e_0$. Restriction yields lifts Σ of σ and T of τ up to stretching in the t_1-direction. Moreover, we have

$$(\Sigma \circ \eta_{n-1}^{-1}) \star (T \circ \eta_{n-1}^{-1}) \simeq_{\partial W^{n-1}} H \circ \eta_{n-1}^{-1}.$$

The homotopy is given by shrinking the marked region in the t_1-direction, see Figure 54. We conclude that

$$\begin{aligned}
\partial([\sigma]_{\partial W^n}) \cdot \partial([\tau]_{\partial W^n}) &= [\Sigma \circ \eta_{n-1}^{-1}]_{\partial W^{n-1}} \cdot [T \circ \eta_{n-1}^{-1}]_{\partial W^{n-1}} \\
&= [(\Sigma \circ \eta_{n-1}^{-1}) \star (T \circ \eta_{n-1}^{-1})]_{\partial W^{n-1}} \\
&= [H \circ \eta_{n-1}^{-1}]_{\partial W^{n-1}} \\
&= \partial([\sigma \star \tau]_{\partial W^n}) \\
&= \partial([\sigma]_{\partial W^n} \cdot [\tau]_{\partial W^n}).
\end{aligned}$$

□

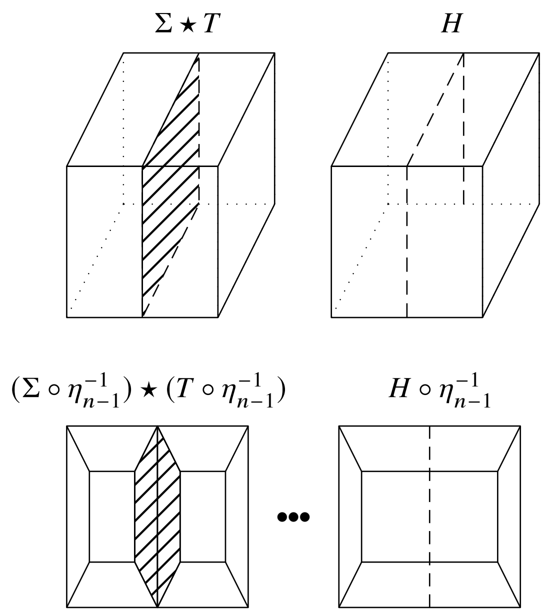


Figure 54. Boundary map is a group homomorphism

Hence $\partial : \pi_n(B; b_0) \to \pi_{n-1}(F, e_0)$ is a homomorphism if $n \geq 2$. For $n = 1$ this statement does not make sense because $\pi_0(F, e_0)$ is not a group. But ∂ still maps the neutral element of $\pi_1(B, b_0)$ to the distinguished element of $\pi_0(F, e_0)$.

Theorem 2.102 (Long exact homotopy sequence of a Serre fibration). *Let $p : E \to B$ be a Serre fibration, $e_0 \in E$, $b_0 = p(e_0) \in B$ and $F = p^{-1}(b_0)$. Let $\iota : F \to E$ be the inclusion map. Then the following sequence is exact:*

$$\dots \xrightarrow{\partial} \pi_n(F; e_o) \xrightarrow{\iota_\#} \pi_n(E; e_0) \xrightarrow{p_\#} \pi_n(B; b_0) \xrightarrow{\partial} \pi_{n-1}(F; e_0) \xrightarrow{\iota_\#} \dots$$

$$\dots \xrightarrow{\partial} \pi_0(F; e_0) \xrightarrow{\iota_\#} \pi_0(E; e_0) \xrightarrow{p_\#} \pi_0(B; b_0)$$

Remark 2.103. Exactness means that the image of the incoming map equals the kernel of the outgoing map. The question arises what this means on the π_0-level where we do not have homomorphisms. The image is defined for an arbitrary map. For the kernel we recall that π_0 is a set together with a distinguished element which corresponds to the neutral element of a group. It is therefore natural to define the kernel of a map to be the set of all elements in the domain of the map which are mapped to the distinguished element. Having clearified this, exactness of the above sequence also makes sense on the π_0-level.

Proof. a) Exactness at $\pi_n(E;e_0)$ for $n \geq 0$:

i) $\operatorname{im}(\iota_\#) \subset \ker(p_\#)$:
Since $p \circ \iota$ is the constant map, we have for any $[\sigma]_{\partial W^n} \in \pi_n(F;e_0)$:

$$p_\#(\iota_\#([\sigma]_{\partial W^n})) = (p \circ \iota)_\#([\sigma]_{\partial W^n}) = [p \circ \iota \circ \sigma]_{\partial W^n} = [\varepsilon^n_{b_0}]_{\partial W^n} = 0.$$

ii) $\ker(p_\#) \subset \operatorname{im}(\iota_\#)$:
Let $[\tau]_{\partial W^n} \in \pi_n(E;e_0)$ with $p_\#([\tau]_{\partial W^n}) = [p \circ \tau]_{\partial W^n} = 0$. Hence $p \circ \tau \simeq_{\partial W^n} \varepsilon^n_{b_0}$. Let $h : W^n \times [0,1] \to B$ be a homotopy in B relative to ∂W^n from $p \circ \tau$ to $\varepsilon^n_{b_0}$. Lift the map h to a homotopy $H : W^n \times [0,1] \to E$ with initial conditions τ, i.e. $H(\cdot, 0) = \tau$. The red area in the diagram gets mapped to F by H because it gets mapped to b_0 by h.

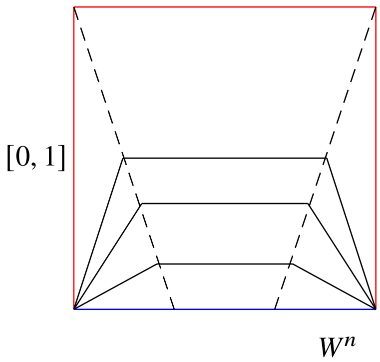


Figure 55. Bottom-to-box homotopy

We obtain a homotopy in E relative to ∂W^n from τ to $H \circ \eta_n^{-1}$. Hence $\tau \simeq_{\partial W^n} H \circ \eta_n^{-1}$ and we conclude that

$$[\tau]_{E,\partial W^n} = [H \circ \eta_n^{-1}]_{E,\partial W^n} = \iota_\#([H \circ \eta_n^{-1}]_{F,\partial W^n}) \in \operatorname{im} \iota_\#.$$

b) Exactness at $\pi_n(F;e_0)$ for $n \geq 0$:

i) $\operatorname{im} \partial \subset \ker \iota_\#$:

Let $[\sigma]_{\partial W^{n+1}} \in \pi_{n+1}(B;b_0)$. Then we have $\partial([\sigma]_{\partial W^{n+1}}) = [\Sigma \circ \eta_n^{-1}]_{F,\partial W^n}$ where Σ is a lift of σ with initial conditions $\varepsilon^n_{e_0}$. The map Σ yields a homotopy relative to ∂W^n in E from $\varepsilon^n_{e_0}$ to $\Sigma \circ \eta_n^{-1}$, see Figure 56. Hence

$$\iota_\#(\partial([\sigma]_{\partial W^{n+1}})) = \iota_\#([\Sigma \circ \eta_n^{-1}]_{F,\partial W^n}) = [\Sigma \circ \eta_n^{-1}]_{E,\partial W^n} = [\varepsilon^n_{e_0}]_{E,\partial W^n} = 0.$$

ii) $\ker \iota_\# \subset \operatorname{im} \partial$:
Let $[\tau]_{\partial W^n} \in \pi_n(F;e_0)$ with $0 = \iota_\#([\tau]_{\partial W^n})$. Hence $\tau \simeq_{\partial W^n} \varepsilon^n_{e_0}$ in E. Let H be a homotopy in E relative to ∂W^n from $\varepsilon^n_{e_0}$ to τ, see Figure 57. Then H is a lift of $h := p \circ H$. Since H maps ∂W^{n+1} to F, the map h maps ∂W^{n+1} to b_0. Thus h represents an element in $\pi_{n+1}(B;b_0)$. By definition of ∂, we have $[H \circ \eta_n^{-1}]_{\partial W^n} = \partial([h]_{\partial W^{n+1}})$.

In the image of Box_n under η_n we let the interior cube grow and thereby obtain a homotopy in F relative to ∂W^n from $H \circ \eta_n^{-1}$ to τ, see Figure 58. Therefore $[\tau]_{\partial W^n} = [H \circ \eta_{n-1}^{-1}]_{\partial W^n} = \partial[h]_{\partial W^{n+1}} \in \operatorname{im}(\partial)$.

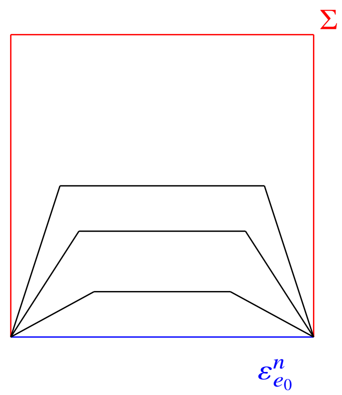


Figure 56. Bottom-to-box homotopy, again

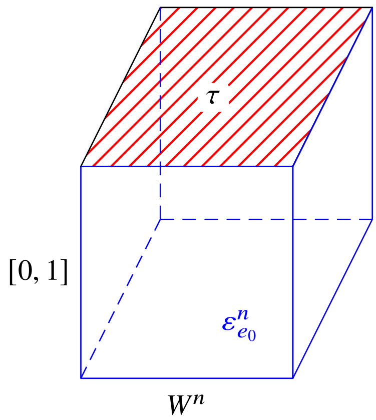


Figure 57. Homotopy between τ and constant map

c) Exactness at $\pi_n(B;b_0)$ for $n \geq 1$:

i) $\operatorname{im} p_\# \subset \ker \partial$:
Let $[\tau]_{\partial W^n} \in \pi_n(E;e_0)$. Then τ is a lift of $p \circ \tau$ with initial conditions $\varepsilon^{n-1}_{e_0}$. Since τ maps the boundary of W^n to e_0 we have $\tau \circ \eta^{-1}_{n-1} = \varepsilon^{n-1}_{e_0}$. Thus

$$\partial(p_\#([\tau]_{\partial W^n}) = \partial([p \circ \tau]_{\partial W^n})) = [\tau \circ \eta^{-1}_{n-1}]_{\partial W^{n-1}} = [\varepsilon^{n-1}_{e_0}]_{\partial W^{n-1}} = 0.$$

ii) $\ker \partial \subset \operatorname{im} p_\#$:
Let $[\sigma]_{\partial W^n} \in \pi_n(B;b_0)$ with $\partial([\sigma]_{\partial W^n}) = 0$. Let Σ be a lift of σ with initial condition $\varepsilon^{n-1}_{e_0}$. Then $\Sigma \circ \eta^{-1}_{n-1}$ represents $\partial([\sigma]_{\partial W^n}) = 0$. Hence we have in F

$$\Sigma \circ \eta^{-1}_{n-1} \simeq_{\partial W^{n-1}} \varepsilon^{n-1}_{e_0}$$

Now choose a homotopy in F relative to ∂W^{n-1} from $\Sigma \circ \eta^{-1}_{n-1}$ to $\varepsilon^{n-1}_{e_0}$. Use this homotopy to continuously extend Σ to the larger cube with boundary values e_0, see Figure 59. Call this extension τ. We then have

$$[\tau]_{\partial W^n} \in \pi_n(E;e_0) \qquad \text{and} \qquad p_\#([\tau]_{E,\partial W^n}) = [p \circ \tau]_{\partial W^n}.$$

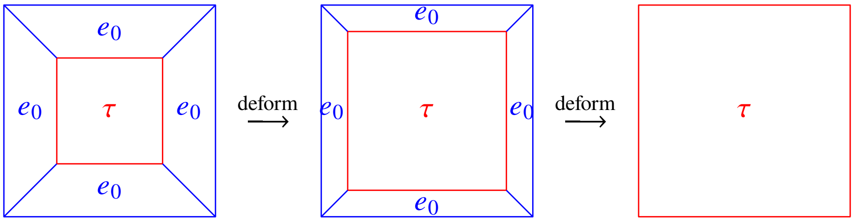


Figure 58. Shrink "constant region" to boundary

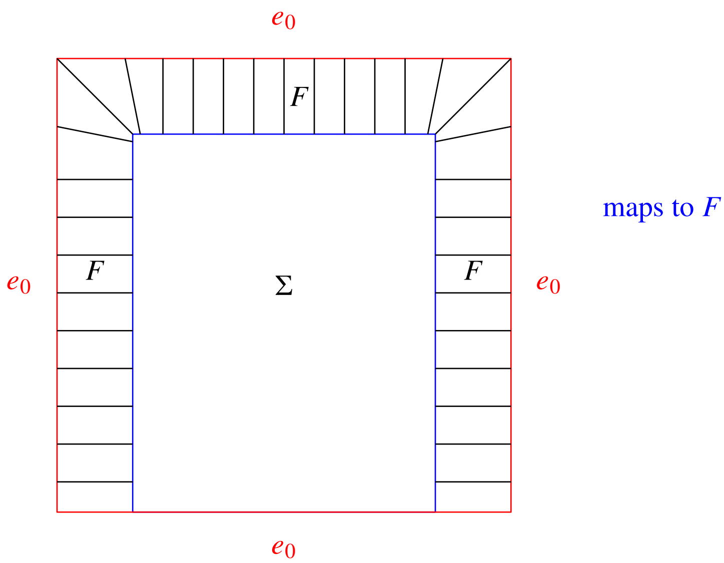


Figure 59. Extension of Σ

Now $p \circ \tau$ is homotopic relative to ∂W^n to σ as shown in Figure 60. Thus $[\sigma]_{\partial W^n} = [p \circ \tau]_{\partial W^n} \in \operatorname{im} p_\#$. □

Definition 2.104. A fiber bundle with discrete fiber is called a *covering*.

Corollary 2.105. *If $p : E \longrightarrow B$ is a covering with $e_0 \in F$, $b_0 = p(e_0) \in B$, then the map*

$$p_\# : \pi_n(E, e_0) \longrightarrow \pi_n(B; b_0)$$

is an isomorphism for all $n \geq 2$.

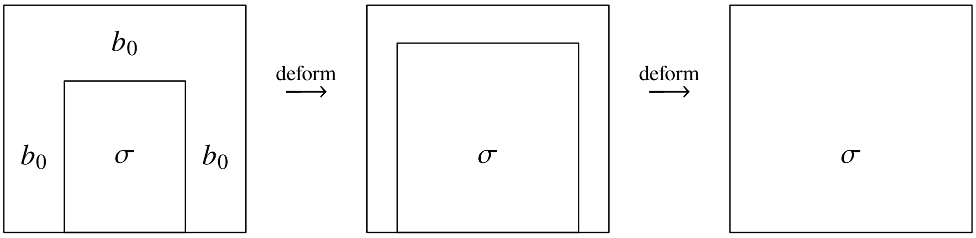


Figure 60. Homotopy between $p \circ \tau$ and σ

Proof. The assertion follows from the long exact sequence:

$$\{0\} = \pi_n(F; e_o) \xrightarrow{\iota_\#} \pi_n(E; e_0) \xrightarrow{p_\#} \pi_n(B; b_0) \xrightarrow{\partial} \pi_{n-1}(F; e_0) = \{0\}. \qquad \square$$

Example 2.106. The map Exp : $\mathbb{R} \to S^1$ with $t \mapsto e^{2\pi i t}$ is a covering with fiber $\mathbb{Z}$. Hence

$$\pi_k(S^1; 1) \cong \pi_k(\mathbb{R}, 0) = \{0\}$$

for all $k \geq 2$. More generally, Exp : $\mathbb{R}^n \to T^n = S^1 \times \ldots \times S^1$ with

$$(t_1, \ldots, t_n) \mapsto (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$$

is a covering with fiber $\mathbb{Z}^n$. Hence $\pi_k(T^n) \cong \pi_k(\mathbb{R}^n) = \{0\}$ for all $k \geq 2$.

Example 2.107. Consider the real projective space, defined by $\mathbb{RP}^n := S^n/\sim$, where $x \sim y \iff x = y$ or $x = -y$. Then the map $p : S^n \to \mathbb{RP}^n$ with $x \mapsto [x]_\sim$ is a covering with fiber $\mathbb{Z}/2\mathbb{Z}$. Hence $\pi_k(\mathbb{RP}^n) \cong \pi_k(S^n)$ for all $k \geq 2$. For $n \geq 2$ we investigate the sequence:

$$\{0\} = \pi_1(S^n) \longrightarrow \pi_1(\mathbb{RP}^n) \longrightarrow \pi_0(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \pi_0(S^n) = \{0\}$$

We deduce that $\pi_1(\mathbb{RP}^n) \cong \pi_0(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ as sets. But then $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ also as groups because there is only one group of order 2 (up to isomorphism).

Example 2.108. We know that $\mathrm{SO}(n+1)/\mathrm{SO}(n) \approx S^n$ from Example 2.99. In the case of $n = 2$ this means that $\mathrm{SO}(3)/\mathrm{SO}(2) \approx S^2$. We also know that $\mathrm{SO}(2) \approx S^1$. For $k \geq 3$ we consider the long exact sequence

$$\cdots \longrightarrow \pi_k(\mathrm{SO}(2)) \longrightarrow \pi_k(\mathrm{SO}(3)) \longrightarrow \pi_k(S^2) \longrightarrow \pi_{k-1}(\mathrm{SO}(2))$$

We know that $\pi_k(S^1) = \{0\}$ and $\pi_{k-1}(S^1) = \{0\}$. Hence we find

$$\pi_k(\mathrm{SO}(3)) \cong \pi_k(S^2) \text{ for all } k \geq 3.$$

The map $f : S^3 \to \mathrm{SO}(3)$ given by

$$f(x,y,u,v) = \begin{pmatrix} x^2+y^2-u^2-v^2 & 2(yu-xv) & 2(yv-xu) \\ 2(yu+xv) & x^2-y^2+u^2-v^2 & 2(uv-xy) \\ 2(yv-xu) & 2(uv+xy) & x^2-y^2-u^2+v^2 \end{pmatrix}$$

satisfies $f(-x,-y,-z,-v) = f(x,y,z,v)$ and therefore induces a continuous map $\bar{f} : \mathbb{RP}^3 \to \mathrm{SO}(3)$. This map is bijective and thus a homeomorphism. Hence we have $\mathrm{SO}(3) \approx \mathbb{RP}^3$. It follows that

$$\pi_k(S^2) \cong \pi_k(\mathrm{SO}(3)) \cong \pi_k(\mathbb{RP}^3) \cong \pi_k(S^3)$$

for all $k \geq 3$. Later we will see (Example 3.121 on page 151) that $\pi_3(S^3) \cong \mathbb{Z}$ and consequently $\pi_3(S^2) \cong \mathbb{Z}$.

Example 2.109. By the same proof as for $\mathrm{SO}(n+1)/\mathrm{SO}(n) \approx S^n$ we get that

$$\mathrm{SU}(n+1)/\mathrm{SU}(n) \approx S^{2n+1}.$$

Therefore there is a fiber bundle $\mathrm{SU}(n+1) \to S^{2n+1}$ with fiber $\mathrm{SU}(n)$. For $n \geq 1$ we consider the following part of the long exact homotopy sequence:

$$\pi_0(\mathrm{SU}(n)) \xrightarrow{\iota_\#} \pi_0(\mathrm{SU}(n+1)) \xrightarrow{p_\#} \pi_0(S^{2n+1}) = \{0\}.$$

Hence the map $\iota_\# : \pi_0(\mathrm{SU}(n)) \to \pi_0(\mathrm{SU}(n+1))$ is onto. Since $\mathrm{SU}(1) = \{1\}$ we have $\pi_0(\mathrm{SU}(1)) = \{0\}$ and thus $\pi_0(\mathrm{SU}(n)) = \{0\}$ by induction on n. Thus $\mathrm{SU}(n)$ is path-connected for all $n \geq 1$.
Now let us analyze $\pi_1(\mathrm{SU}(n))$. Consider

$$\pi_1(\mathrm{SU}(n)) \xrightarrow{\iota_\#} \pi_1(\mathrm{SU}(n+1)) \xrightarrow{p_\#} \pi_1(S^{2n+1}) = \{0\}.$$

Again, we conclude that the map $\iota_\# : \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{SU}(n+1))$ is onto. By the same induction as before we find $\pi_1(\mathrm{SU}(n)) = \{0\}$ for all $n \geq 1$. Thus $\mathrm{SU}(n)$ is simply connected for all $n \geq 1$.

2.7. Exercises

2.1. Let X be a set and let $x_0 \in X$. Determine $\pi_1(X; x_0)$ where

a) X carries the discrete topology;

b) X carries the coarse topology.

2.2. Let X be a topological space and let $\omega : S^1 \to X$ be continuous. Show that the following are equivalent:

(i) ω is homotopic to a constant map.

(ii) ω has a continuous extension $D^2 \to X$.

2.3. Let X be a topological space and let $f, g : X \to S^n$ be continuous. Assume $f(x) \neq -g(x)$ for all $x \in X$. Show that f and g are homotopic.

2.4. Let X and Y be topological spaces. Show that $X \times Y$ is contractible if and only if X and Y are contractible.

2.5. Let X_1, X_2 be topological spaces, $x_i \in X_i$. Put $X := X_1 \times X_2$ and $x := (x_1, x_2) \in X$. Let $p_i : X \to X_i$ be the canonical projections. Show that

$$(p_{1\#}, p_{2\#}) : \pi_1(X; x) \to \pi_1(X; x_1) \times \pi_1(X_2; x_2)$$

is a group isomorphism.

2.6. Let $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and let $A \subset X$ be the comb space. Show that there is no retraction $X \to A$.

2.7. For $f \in C(S^1, \mathbb{R}^2)$ and $p \in \mathbb{R}^2 \setminus f(S^1)$ consider $f_p \in C(S^1, S^1)$ given by

$$f_p(z) = \frac{f(z) - p}{|f(z) - p|}.$$

Then

$$U(f, p) := \deg(f_p)$$

is called the *winding number* of f around p.

a) Show that for $p, q \in \mathbb{R}^2 \setminus f(S^1)$ which can be joined by a continuous path in $\mathbb{R}^2 \setminus f(S^1)$ we have $U(f, p) = U(f, q)$.

b) Compute $U(f_n, p)$ for all $p \in \mathbb{R}^2 \setminus f(S^1)$ and all $n \in \mathbb{Z}$ where $f_n(z) = z^n$.

2.8. Show that the system of equations

$$\cos\left(1 + x^2y^3 + \sin(xy^2)\right) - x^2 = 0,$$
$$y + \frac{1}{\cosh(x + y + 10)} = 0,$$

has a solution $(x, y) \in \mathbb{R}^2$.

2.9. a) Compute $\pi_1(D^2 \setminus \{x_0\}; x_1)$ for all $x_0 \neq x_1 \in D^2$.

b) Show that each homeomorphism $f : D^2 \to D^2$ maps the boundary onto itself, $f(\partial D^2) = \partial D^2$.

2.10. On $[0,1] \times [-1,1]$ consider the equivalence relation $\sim$ given by $(t,s) \sim (t',s')$ iff $(t,s) = (t',s')$ or $|t-t'| = 1$ and $s' = -s$. The quotient space $M := [0,1] \times [-1,1]/\sim$ is called the *Möbius strip*. The image S of $[0,1] \times \{0\}$ is called the *chord* of M, that of $[0,1] \times \{-1,1\}$ is the boundary ∂M of M.

a) Show that ∂M is homeomorphic to S^1.

b) Show that S is a strong deformation retract of M.

c) Determine $\pi_1(M;x_0)$ for some $x_0 \in \partial M$ and the subgroup $\iota_\#(\pi_1(\partial M;x_0)) \subset \pi_1(M;x_0)$ where $\iota : \partial M \hookrightarrow M$ is the inclusion map.

d) Show that ∂M is not a retract of M.

2.11. Let $X = \{(t, \frac{t}{n}) \mid 0 \le t \le 1, n \in \mathbb{N}\} \cup \{(s,0) \mid \frac{1}{2} \le s \le 1\} \subset \mathbb{R}^2$ equipped with the induced topology. Show that X is connected but not path-connected.

2.12. Let G_1 and G_2 be groups. Show that $G_1 * G_2 \cong G_2 * G_1$

a) using the construction of the free product;

b) using the universal property.

2.13. Let G_1 and G_2 be groups. Show:

a) If $g \in G_1 * G_2$ has finite order then g is conjugate to an element in $i_1(G_1)$ or in $i_2(G_2)$.

b) If G_1 and G_2 are nontrivial then $G_1 * G_2$ contains elements of infinite order.

2.14. Provide a presentation for the following group G:

a) $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$;

b) $\mathbb{Z} * \mathbb{Z}$;

c) $G = (\mathbb{Z} * \mathbb{Z}) \times (\mathbb{Z} * \mathbb{Z})$;

d) $G = \mathbb{Z} * \mathbb{Z}/[\mathbb{Z} * \mathbb{Z}, \mathbb{Z} * \mathbb{Z}]$.

2.15. Decide whether or not the groups G and H are isomorphic where

a) $G = \langle a, b \mid a^2b^2 \rangle$ and $H = \langle x, y, z \mid xy^2, xz^2 \rangle$;

b) $G = \langle a, b \mid ab \rangle$ and $H = \langle x, y \mid x^2 \rangle$.

2.16. Let X be a path-connected topological space. Show that the suspension ΣX (see Exercise 1.11) is also path-connected.

2.17. Show that the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$, has the homotopy lifting property for W^0 but not for W^1.

2.18. Let $X = \mathbb{R}^3 \setminus (S^1 \times \{0\}) = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ z = 0\}$. Show:

$$\pi_1(X, \{0\}) \cong \mathbb{Z}$$

and draw a generator of the fundamental group.

2.19. On S^n consider the equivalence relation $\sim$ given by $x \sim y \Leftrightarrow x = y$ or $x = -y$. The quotient $\mathbb{R}\mathrm{P}^n := S^n/\sim$ is called the n-dimensional *real projective space*.
Show inductively using the Seifert-van Kampen theorem that for $n \geq 2$

$$\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2\mathbb{Z}.$$

Hint: Exercise 2.10 may be helpful for the induction base $n = 2$.

2.20. Decide by proof or counter-example whether or not the following assertion holds true: The Seifert-van Kampen theorem also holds if one only assumes that $U \cap V$ is connected rather than path-connected.

2.21. Let $E = B \times F$ and $p = \mathrm{pr}_1 : E \longrightarrow B$ be the product fibration. Show that the boundary map $\partial : \pi_{n+1}(B; b_0) \longrightarrow \pi_n(F; e_0)$ is trivial in this case.

2.22. Show that the inclusion $\mathrm{SU}(n) \hookrightarrow \mathrm{U}(n)$ induces an isomorphism

$$\pi_k\left(\mathrm{SU}(n)\right) \cong \pi_k\left(\mathrm{U}(n)\right)$$

for all $k \geq 2$.

2.23. The *complex projective space* is defined as $\mathbb{C}P^n := S^{2n+1}/\sim$ where $z \sim w$ iff there exists $u \in S^1 \subset \mathbb{C}$ such that $z = u \cdot w$. Here we have regarded $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$ as a subset of $\mathbb{C}^{n+1}$. The map $p : S^{2n+1} \longrightarrow \mathbb{C}P^n$, $z \mapsto [z]_\sim$, is called the *Hopf fibration*.
Compute

$$\pi_k(\mathbb{C}P^n)$$

for all $k \leq 2n$.
Hint: You can use $\pi_k(S^m) = \{0\}$ for all $1 \leq k < m$ and $m \geq 2$.

2.24. Let $p : X \longrightarrow Y$ and $q : Y \longrightarrow Z$ be Serre fibrations. Prove that $q \circ p : X \longrightarrow Z$ is a Serre fibration as well.

2.25. Let $p : E \longrightarrow B$ be a Serre fibration, $e_0 \in E$, $b_0 = p(e_0)$ and $F = p^{-1}(b_0)$. Show

a) If F is simply connected then $\pi_1(E; e_0) \cong \pi_1(B; b_0)$.

b) If E is contractible then $\pi_{n+1}(B;b_0) \cong \pi_n(F;e_0)$ for all $n \geq 1$.

2.26. Let $0 \to A \xrightarrow{i} A' \xrightarrow{p} A'' \to 0$ be an exact sequence of abelian groups. Show that the following three conditions are equivalent:

(i) There exists an isomorphism $\Psi : A' \to A \times A''$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "="] & A' \arrow[r, "p"] \arrow[d, "\Psi"] & A'' \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \times A'' \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

where the arrows $A \to A \times A''$ and $A \times A'' \to A''$ are given by the canonical maps $a \mapsto (a, 0)$ and $(a, a'') \mapsto a''$, respectively.

(ii) There exists a homomorphism $p' : A'' \to A'$ such that $p \circ p' = \mathrm{id}_{A''}$.

(iii) There exists a homomorphism $r : A' \to A$ such that $r \circ i = \mathrm{id}_A$.

If these conditions hold then we say that the exact sequence is *split*.

2.27. a) Show that every exact sequence $0 \to A \xrightarrow{i} A' \xrightarrow{p} A'' \to 0$ of vector spaces (i.e. all spaces are vector spaces over some fixed field and all homomorphisms are linear maps) is split.

b) Show that the exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ is not split.

3. Homology Theory

Homotopy groups are in general hard to compute. For instance, for spheres not all homotopy groups are known even now. In this chapter we introduce rougher invariants which are much easier to determine, the homology groups.

3.1. Singular homology

We will use the notation

$$\begin{aligned} e_0 &= (0, \dots, 0) \in \mathbb{R}^n \\ e_1 &= (1, \dots, 0) \in \mathbb{R}^n \\ &\vdots \\ e_n &= (0, \dots, 1) \in \mathbb{R}^n \end{aligned}$$

Now we define

$$\Delta^n := \text{convex hull of } e_0, \dots, e_n = \left\{ \sum_{i=0}^n t_i e_i \;\middle|\; t_i \geq 0, \ \sum_{i=0}^n t_i \leq 1 \right\}.$$

Then Δ^n is called n-dimensional *standard simplex*.

Example 3.1. For $n = 0, 1, 2$, and 3 the standard simplices are familiar, compare Figure 61:

$$\begin{aligned} n = 0: &\qquad \Delta^0 = \{e_0\} = \text{point} \\ n = 1: &\qquad \Delta^1 = [0, 1] = \text{line segment} \\ n = 2: &\qquad \Delta^2 = \text{triangle} \\ n = 3: &\qquad \Delta^3 = \text{tetrahedron} \end{aligned}$$

Definition 3.2. Let X be a topological space. A *singular n-simplex* in X is a continuous map $\Delta^n \to X$.

Now we fix a commutative ring R with 1. The most important examples will be $R = \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$.

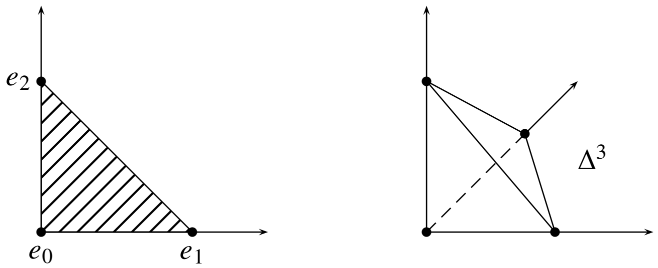


Figure 61. 2 and 3-dimensional standard simplices

Definition 3.3. We define the set of *singular n-chains* $S_n(X;R)$ as the free R-module generated by $C(\Delta^n, X)$.

Then $S_n(X;R)$ is an R-module. Elements of $S_n(X;R)$ are formal linear combinations $\sum_{i=1}^m \alpha_i\sigma_i$ where $\alpha_i \in R$ and $\sigma_i \in C(\Delta^n, X)$. See Appendix A.1 for more on free modules generated by sets.
For $n \geq 0$ we consider the affine linear map $F^i_{n+1} : \Delta^n \longrightarrow \Delta^{n+1}$ given by

$$F^i_{n+1}(e_j) = \begin{cases} e_j, & j < i \\ e_{j+1}, & j \geq i \end{cases} \tag{3.1}$$

Note that F^i_n maps Δ^n to the face of Δ^{n+1} opposite to e_i.

Example 3.4. Consider the special case F^1_2:

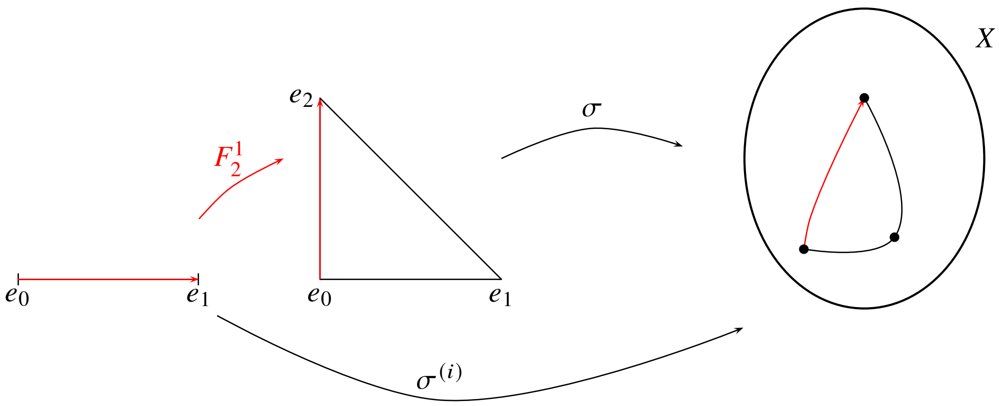


Figure 62. Face maps

Definition 3.5. If σ is an $(n+1)$-dimensional singular simplex in X then $\sigma^{(i)} := \sigma \circ F^i_n$ is called the *i-th face* of σ.

The *boundary* of a singular n-simplex in X is given by:

$$\partial\sigma := \sum_{i=0}^{n}(-1)^i\sigma^{(i)}.$$

We see that the boundary of a singular n-simplex is a singular $(n-1)$-chain. We extend ∂ to chains by linearity. The boundary of a singular n-chain in X is thus given by

$$\partial\left(\sum_{j=0}^{m}\alpha_j\sigma_j\right) = \sum_{j=0}^{m}\alpha_j\partial\sigma_j\,.$$

Hence we obtain a linear map $\partial : S_n(X;R) \to S_{n-1}(X;R)$ and we set $\partial(0\text{-chain}) := 0$.

Lemma 3.6. $\partial \circ \partial = 0$.

Proof. It suffices to prove $\partial\partial\sigma = 0$ for all n-simplices σ. For $j < i$ we have

$$F_n^i \circ F_{n-1}^j = F_n^j \circ F_{n-1}^{i-1}. \tag{3.2}$$

We compute

$$\begin{aligned}
\partial\partial\sigma &= \partial\left(\sum_{i=0}^{n}(-1)^i\sigma\circ F_n^i\right)\\
&= \sum_{i=0}^{n}(-1)^i\partial\left(\sigma\circ F_n^i\right)\\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j\sigma\circ F_n^i\circ F_{n-1}^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ F_n^i\circ F_{n-1}^j + \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ F_n^i\circ F_{n-1}^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ F_n^j\circ F_{n-1}^{i-1} + \sum_{0\le j'<i'\le n}(-1)^{j'+i'-1}\sigma\circ F_n^{j'}\circ F_{n-1}^{i'-1}\\
&= 0.
\end{aligned}$$

In the last step we used (3.2) for the first sum and changed the summation indices from $i \to j'$ and $j \to i'-1$ in the second sum. □

Now we define the set of *singular n-cycles* by

$$\begin{aligned}
Z_n(X;R) :=& \ker(\partial : S_n(X;R) \to S_{n-1}(X;R))\\
=& \{c \in S_n(X;R) \mid \partial c = 0\}
\end{aligned}$$

and the set of *singular n-boundaries* by

$$\begin{aligned} B_n(X;R) &:= \operatorname{im}(\partial : S_{n+1}(X;R) \to S_n(X;R)) \\ &= \{c \in S_n(X;R) \mid \exists\, b \in S_{n+1}(X;R) \text{ such that } c = \partial b\}\,. \end{aligned}$$

Lemma 3.6 says $B_n(X;R) \subset Z_n(X;R)$.

Definition 3.7. The quotient

$$H_n(X;R) := \frac{Z_n(X;R)}{B_n(X;R)}$$

is called *n-th singular homology* of X with coefficients in R.

Remark 3.8. The n-th homology $H_n(X;R)$ is an R-module.

Example 3.9. Assume that $X = \{\text{point}\}$. Then there is only one singular n-simplex, namely the constant map $\sigma_n : \Delta^n \to X$. In other words, $S_n(X;R) = R \cdot \sigma_n$. Consequently, $\sigma_n^{(i)} = \sigma_n \circ F_n^i = \sigma_{n-1}$ and

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0, & n \text{ odd}, \\ \sigma_{n-1}, & n \text{ even}, n \neq 0, \\ 0, & n = 0. \end{cases}$$

This implies

$$Z_n(X;R) = \begin{cases} R \cdot \sigma_n, & n \text{ odd or } n = 0, \\ 0, & n \text{ even and } n \neq 0, \end{cases}$$

and

$$B_n(X;R) = \begin{cases} R \cdot \sigma_n, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

We conclude that

$$H_n(X;R) \cong \begin{cases} R, & \text{if } n = 0, \\ 0, & \text{otherwise}. \end{cases}$$

As in homotopy theory we not only associate groups to spaces but also homomorphisms to maps. Let $f : X \to Y$ be a continuous map. Then we obtain an R-module homomorphism $S_n(f) : S_n(X;R) \to S_n(Y;R)$ by setting

$$S_n(f)\Big(\sum_{i=1}^{m} \alpha_i \cdot \sigma_i\Big) := \sum_{i=1}^{m} \alpha_i \cdot (f \circ \sigma_i).$$

Lemma 3.10. *The diagram*

$$\begin{tikzcd} S_n(X;R) \arrow[r, "S_n(f)"] \arrow[d, "\partial"'] & S_n(Y;R) \arrow[d, "\partial"'] \\ S_{n-1}(X;R) \arrow[r, "S_{n-1}(f)"] & S_{n-1}(Y;R) \end{tikzcd}$$

commutes for all $n \geq 0$.

Proof. We compute

$$\begin{aligned} \partial(S_n(f)(\sigma)) &= \partial(f \circ \sigma) \\ &= \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ F_n^i \\ &= \sum_{i=0}^{n} (-1)^i f \circ (\sigma \circ F_n^i) \\ &= S_{n-1}(f)\Big(\sum_{i=0}^{n} (-1)^i \sigma \circ F_n^i\Big) \\ &= S_{n-1}(f)(\partial\sigma). \end{aligned}$$ □

This lemma implies $S_n(f)(Z_n(X;R)) \subset Z_n(Y;R)$ and $S_n(f)(B_n(X;R)) \subset B_n(Y;R)$. Hence we obtain a well-defined R-module homomorphism $H_n(f) : H_n(X;R) \to H_n(Y;R)$ where $H_n(f)([x]) := [S_n(f)(x)]$. Here the square brackets denote the homology classes of the n-cycles. One sees directly from the definition that $H_n(\cdot)$ has the functorial properties

(i) $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X;R)}$,

(ii) $H_n(f \circ g) = H_n(f) \circ H_n(g)$.

Exactly as for homotopy groups these functorial properties imply that a homeomorphism $f : X \to Y$ induces an isomorphism $H_n(f) : H_n(X;R) \to H_n(Y;R)$. Homeomorphic spaces have isomorphic homology groups.

3.2. Relative homology

For a topological space X and $A \subset X$ we call (X,A) a *pair of spaces* and set

$$C((X,A),(Y,B)) := \{f \in C(X,Y) \mid f(A) \subset B\}.$$

We abbreviate $S_n(X) = S_n(X;R)$ if R is understood. We observe that $S_n(A) \subset S_n(X)$ and $\partial(S_n(A)) \subset S_{n-1}(A)$. Writing $S_n(X,A) = S_n(X,A;R) := S_n(X;R)/S_n(A;R)$, the map ∂

induces a well-defined homomorphism $\bar{\partial}$ such that

$$\begin{tikzcd} S_n(X) \arrow[r] \arrow[d, "\partial"] & S_n(X,A) \arrow[d, "\bar{\partial}"] \\ S_{n-1}(X) \arrow[r] & S_{n-1}(X,A) \end{tikzcd} \tag{3.3}$$

commutes. Since $\partial \circ \partial = 0$ and since $S_n(X) \to S_n(X,A)$ is onto we also have that $\bar{\partial} \circ \bar{\partial} = 0$. Set

$$\begin{aligned} Z_n(X,A) = Z_n(X,A;R) &:= \ker\left(\bar{\partial} : S_n(X,A) \to S_{n-1}(X,A)\right), \\ B_n(X,A) = B_n(X,A;R) &:= \operatorname{im}\left(\bar{\partial} : S_{n+1}(X,A) \to S_n(X,A)\right). \end{aligned}$$

We define the *relative singular homology* of (X,A) by

$$H_n(X,A) = H_n(X,A;R) := \frac{Z_n(X,A;R)}{B_n(X,A;R)}.$$

Now consider the preimage of $Z_n(X,A)$ under $S_n(X) \to S_n(X,A)$ and set

$$\begin{aligned} Z'_n(X,A) &:= \{c \in S_n(X) \mid \partial c \in S_{n-1}(A)\}, \\ B'_n(X,A) &:= \{c \in S_n(X) \mid \exists b \in S_{n+1}(X) \text{ such that } c + \partial b \in S_n(A)\}. \end{aligned}$$

Since $Z_n(X,A) = Z'_n(X,A)/S_n(A)$ and $B_n(X,A) = B'_n(X,A)/S_n(A)$ we obtain

$$H_n(X,A) = \frac{Z'_n(X,A)/S_n(A)}{B'_n(X,A)/S_n(A)} = \frac{Z'_n(X,A)}{B'_n(X,A)}.$$

Remark 3.11. For $A = \emptyset$ we have the special cases

$$\begin{aligned} Z'_n(X,\emptyset) &= Z_n(X), \\ B'_n(X,\emptyset) &= B_n(X), \\ H_n(X,\emptyset) &= H_n(X). \end{aligned}$$

Example 3.12. Let $X = S^1 \times [0,1]$ be the cylinder over S^1 and $A = S^1 \times \{0\} \subset X$, see Figure 63. To construct an element in the relative homology $H_1(X,A)$ take the 1-simplex

$$\begin{aligned} \sigma : \Delta^1 &\to X, \\ (te_1 + (1-t)e_0) &\mapsto (\cos(2\pi t), \sin(2\pi t), 1), \end{aligned}$$

see Figure 64. Since σ is a closed curve in X we find for its boundary

$$\partial\sigma = \sigma(e_1) - \sigma(e_0) = 0.$$

Therefore $\sigma \in Z_1(X) \subset Z'_1(X,A)$ and, as we will see later, represents a nontrivial element in $H_1(X)$. For the 2-simplices σ_1 and σ_2 defined as indicated in Figure 65.
we find for the 2-chain $\sigma_1 + \sigma_2$ the boundary $\partial(\sigma_1 + \sigma_2) = \sigma + a$ where $a \in S_1(A)$. Hence, modulo $a \in S_1(A)$, we have $\sigma \in B_1(X,A)$ and therefore $0 = [\sigma] \in H_1(X,A)$.

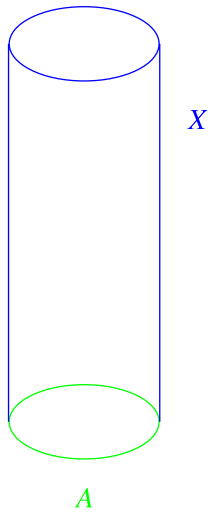


Figure 63. Cylinder relative to bottom

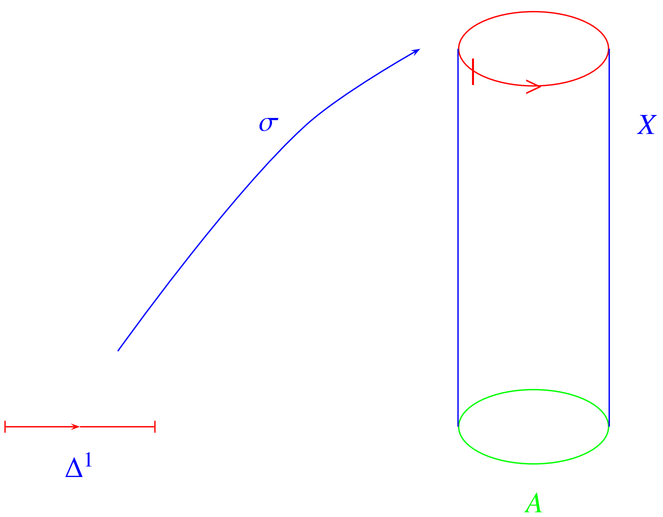


Figure 64. The representative σ

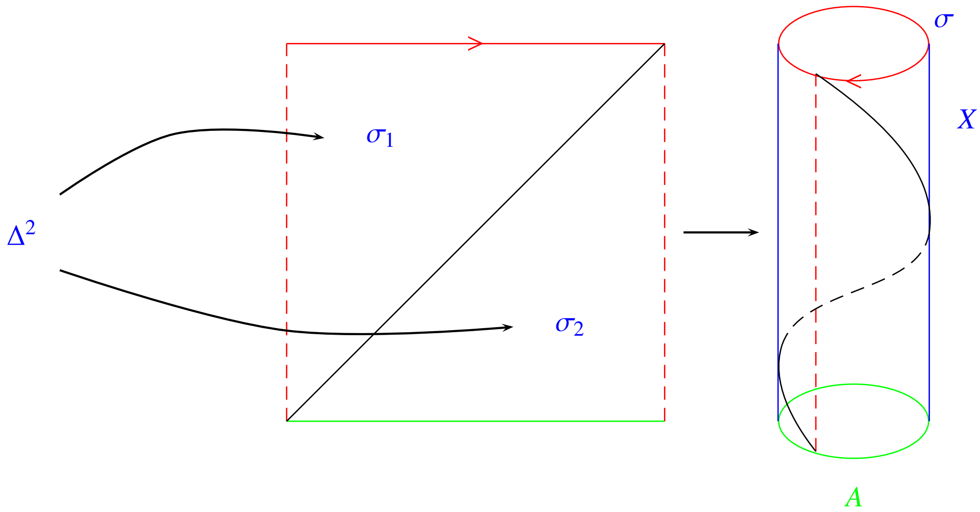


Figure 65. σ is null-homologous

Let (X, A) and (Y, B) be pairs of spaces and $f \in C((X, A), (Y, B))$. Then, for each $n \in \mathbb{N}_0$, we have the commutative diagram

$$\begin{tikzcd} S_n(X) \arrow[r, "S_n(f)"] & S_n(Y) \\ S_n(A) \arrow[u, hook] \arrow[r, "S_n(f|_A)"] & S_n(B) \arrow[u, hook] \end{tikzcd}$$

Thus we obtain a well-defined homomorphism $S_n(f)\colon S_n(X, A) \to S_n(Y, B)$ such that

$$\begin{tikzcd} S_n(X) \arrow[r, "S_n(f)"] \arrow[d] & S_n(Y) \arrow[d] \\ S_n(X, A) \arrow[r, "S_n(f)"] & S_n(Y, B) \end{tikzcd}$$

commutes. Combining with Lemma 3.10 and diagram (3.3) we get the commutative diagram

$$\begin{tikzcd} S_n(X, A) \arrow[d, "\bar{\partial}"] \arrow[rrr, bend left, "S_n(f)"] & S_n(X) \arrow[l] \arrow[r, "S_n(f)"] \arrow[d, "\partial"] & S_n(Y) \arrow[r] \arrow[d, "\partial"] & S_n(Y, B) \arrow[d, "\bar{\partial}"] \\ S_{n-1}(X, A) \arrow[rrr, bend right, "S_{n-1}(f)"'] & S_{n-1}(X) \arrow[l] \arrow[r, "S_{n-1}(f)"] & S_{n-1}(Y) \arrow[r] & S_{n-1}(Y, B) \end{tikzcd}$$

We have extended Lemma 3.10 to relative homology. In particular, we obtain a well-defined homomorphism $H_n(f)\colon H_n(X,A) \to H_n(Y,B)$ such that

$$\begin{tikzcd} Z_n(X,A) \arrow[r, "S_n(f)"] \arrow[d] & Z_n(Y,B) \arrow[d] \\ H_n(X,A) \arrow[r, "H_n(f)"] & H_n(Y,B) \end{tikzcd}$$

commutes.

Let (X,A) be a pair of spaces. The inclusion map $i : A \hookrightarrow X$ induces a homomorphism $H_n(i) : H_n(A) \to H_n(X)$. Furthermore we have the inclusion map $j : (X,\emptyset) \to (X,A)$ which induces the homomorphism $H_n(j) : H_n(X) = H_n(X,\emptyset) \to H_n(X,A)$.

We define the *connecting homomorphism* or *boundary operator*

$$\begin{aligned} \partial : H_n(X,A) &\to H_{n-1}(A), \\ [c] &\mapsto [\partial c], \end{aligned} \tag{3.4}$$

where $c \in Z'_n(X,A)$. Note that $\partial c \in Z_{n-1}(A)$ since $\partial^2 = 0$. The connecting homomorphism is well defined because replacing c by another representative $c + \partial b + a$ where $b \in S_{n+1}(X)$ and $a \in S_n(A)$ yields

$$c + \partial b + a \quad \mapsto \quad \partial(c + \partial b + a) = \partial c + \partial a.$$

Since $\partial a \in B_{n-1}(A)$ we get $[\partial c + \partial a] = [\partial c] \in H_{n-1}(A)$. Since $\partial : Z'_n(X,A) \to S_{n-1}(A)$ is a homomorphism, the connecting homomorphism is also a homomorphism.

Lemma 3.13. *The connecting homomorphism is* natural*, i.e. the diagram*

$$\begin{tikzcd} H_n(X,A) \arrow[r, "H_n(f)"] \arrow[d, "\partial"'] & H_n(Y,B) \arrow[d, "\partial"'] \\ H_{n-1}(A) \arrow[r, "H_{n-1}(f|_A)"] & H_{n-1}(B) \end{tikzcd}$$

commutes for every $f \in C((X,A),(Y,B))$ and $n \in \mathbb{N}$.

Proof. We compute

$$\begin{aligned} H_{n-1}(f|_A)\big(\partial\big(\big[\sum_i \alpha_i\sigma_i\big]\big)\big) &= H_{n-1}(f|_A)\big(\big[\partial \sum_i \alpha_i\sigma_i\big]\big) \\ &= H_{n-1}(f|_A)\big(\big[\sum_i \alpha_i \sum_j (-1)^j \sigma_j \circ F^j_{n-1}\big]\big) \\ &= \big[\sum_i \alpha_i \sum_j (-1)^j (f|_A) \circ (\sigma_i \circ F^j_{n-1})\big] \\ &= \big[\sum_i \alpha_i \sum_j (-1)^j (f \circ \sigma_i) \circ F^j_{n-1}\big] \end{aligned}$$

$$
\begin{aligned}
&= \big[\partial \sum_i \alpha_i (f \circ \sigma_i)\big] \\
&= \partial \big[\sum_i \alpha_i (f \circ \sigma_i)\big] \\
&= \partial \big(H_n(f)\big(\big[\sum_i \alpha_i \sigma_i\big]\big)\big).
\end{aligned}
$$
□

3.3. The Eilenberg-Steenrod axioms and applications

Now we list the most important properties of homology theory known as *Eilenberg-Steenrod axioms*. The first axiom is exactly Example 3.9.

Dimension Axiom.

$$
H_n(\{\text{point}\}; R) \cong \begin{cases} R, & n = 0 \\ 0, & \text{otherwise} \end{cases}
$$

The next axiom deals with homotopy invariance. Two continuous maps $f_0, f_1 : (X, A) \to (Y, B)$ of pairs of spaces are called homotopic (in symbols $f_0 \simeq f_1$) iff there exists an $H \in C(X \times [0, 1], Y)$ such that for all $x \in X$

$$
\begin{aligned}
H(x, 0) &= f_0(x), \\
H(x, 1) &= f_1(x), \\
H(A \times [0, 1]) &\subset B.
\end{aligned}
$$

Homotopy Axiom.

Let $f_0, f_1 \in C((X, A), (Y, B))$ be homotopic, $f_0 \simeq f_1$. Then the induced maps on homology coincide, i.e.,

$$
H_n(f_0) = H_n(f_1) : H_n(X, A) \to H_n(Y, B)
$$

holds for all n.

Remark 3.14. Let $(X, A) \simeq (Y, B)$, i.e., there exist $f \in C((X, A), (Y, B))$ and $g \in C((Y, B), (X, A))$ such that $f \circ g \simeq \mathrm{id}_{(Y,B)}$ and $g \circ f \simeq \mathrm{id}_{(X,A)}$. It then follows as before that $H_n(X, A; R) \cong H_n(Y, B; R)$.

Exactness Axiom.
For any pair of spaces (X, A) and inclusion maps $i : A \hookrightarrow X$, $j : (X, \emptyset) \hookrightarrow (X, A)$, the sequence

$$\begin{tikzcd}
H_{n+1}(X,A) \arrow[d, "\partial"'] & \cdots \arrow[l, "H_{n+1}(j)"'] & \\
H_n(A) \arrow[r, "H_n(i)"] & H_n(X) \arrow[r, "H_n(j)"] & H_n(X,A) \arrow[d, "\partial"'] \\
& \cdots & H_{n-1}(A) \arrow[l, "H_{n-1}(i)"']
\end{tikzcd}$$

is exact and natural.

Notation: For $A \subset X$ we call

$$\mathring{A} = \bigcup_{U \subset A} U \quad \text{with } U \text{ open in } X$$

the *interior* of A and

$$\bar{A} = \bigcap_{B \subset X} B \quad \text{with } B \text{ closed in } X$$

the *closure* of A.

Excision Axiom.
For every pair of spaces (X, A) and every $U \subset A$ with $\bar{U} \subset \mathring{A}$ the homomorphism

$$H_n(j) : H_n(X \setminus U, A \setminus U) \to H_n(X, A)$$

induced by the inclusion map

$$j : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$$

is an isomorphism.
The proofs of axioms A2, A3, and A4 will be given later. Before that we will show their usefulness by studying some basic examples.

Remark 3.15. (cf. Exercise 3.1)

1.) If $X \neq \emptyset$ is path-connected then $H_0(X; R) \cong R$ and generators are represented by every 0-simplex $\sigma : \Delta^0 \to X$. In other words, the isomorphism $H_0(X; R) \to R$ is given by $\sum_j \alpha_j \sigma_j \mapsto \sum_j \alpha_j$.

2.) If X_k, $k \in K$, are the path-components of X then $H_n(X; R) \cong \bigoplus_{k \in K} H_n(X_k; R)$.

Theorem 3.16. *For $n \geq 1$ we have*

$$H_m(S^n; R) \cong \begin{cases} R, & \text{if } m = 0 \text{ or } m = n \\ 0, & \text{otherwise} \end{cases}$$

$$H_m(D^n, S^{n-1}; R) \cong \begin{cases} R, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

Proof. a) For $n \geq 1$ the sphere S^n is path-connected and therefore $H_0(S^n; R) \cong R$.

For $n = 0$ we have that $S^0 = \{x, y\}$ with the discrete topology and therefore $H_0(S^0; R) \cong R \oplus R$.

b) We have the exact sequence

$$\begin{array}{ccccccc} H_0(S^0) & \longrightarrow & H_0(D^1) & \longrightarrow & H_0(D^1, S^0) & \longrightarrow & 0 \\ \wr\| & & \wr\| & & & & \\ R^2 & & R & & & & \\ \Uparrow & & \Uparrow & & & & \\ (a,b) & \longmapsto & a+b & & & & \end{array}$$

Since the map $(a, b) \mapsto a + b$ is onto, the map $H_0(D^1) \to H_0(D^1, S^0)$ must be zero. Hence $H_0(D^1, S^0) = 0$.

For $n \geq 2$ we have the following exact sequence

$$\begin{array}{ccccccc} H_0(S^{n-1}) & \longrightarrow & H_0(D^n) & \longrightarrow & H_0(D^n, S^{n-1}) & \longrightarrow & 0 \\ \wr\| & \text{id} & \wr\| & & & & \\ R & \longrightarrow & R & & & & \end{array}$$

Again the second arrow has to be trivial and therefore $H_0(D^n, S^{n-1}) = 0$. This settles the case $m = 0$.

c) We will now use the exact sequence

$$H_1(D^n) \longrightarrow H_1(D^n, S^{n-1}) \longrightarrow H_0(S^{n-1}) \longrightarrow H_0(D^n)$$

For $n = 1$ we have

$$\begin{array}{ccccccc} H_1(D^1) & \longrightarrow & H_1(D^1, S^0) & \longrightarrow & H_0(S^0) & \longrightarrow & H_0(D^1) \\ \wr\| & & & & \wr\| & & \wr\| \\ H_1(\{\text{point}\}) & & & & R^2 & & R \\ \| & & & & \Uparrow & & \Uparrow \\ 0 & & & & (a,b) & \longmapsto & a+b \end{array}$$

which implies $H_1(D^1, S^0) \cong \ker((a, b) \mapsto a + b) \cong R$.

For $n \geq 2$ we have

$$\begin{array}{ccccccc} H_1(D^n) & \longrightarrow & H_1(D^n, S^{n-1}) & \xrightarrow{0} & H_0(S^{n-1}) & \xrightarrow{\cong} & H_0(D^n) \\ \parallel & & & & & & \\ 0 & & & & & & \end{array}$$

from which we get $H_1(D^n, S^{n-1}) = 0$.

d) Consider the pair of spaces (S^n, D^n_-) where $D^n_- = \{x \in S^n \mid x_0 \leq 0\}$ is the lower hemisphere.[1]

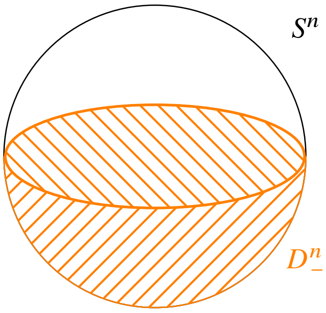


Figure 66. Sphere relative to southern hemisphere

For $n \geq 1$ both D^n_- and S^n are path-connected, hence the inclusion $D^n_- \hookrightarrow S^n$ induces an isomorphism $H_0(D^n_-) \to H_0(S^n)$. From

$$0 = H_1(D^n_-) \longrightarrow H_1(S^n) \longrightarrow H_1(S^n, D^n_-) \longrightarrow H_0(D^n_-) \xrightarrow{\cong} H_0(S^n)$$

we see that the connecting homomorphism $H_1(S^n, D^n_-) \to H_0(D^n_-)$ must be zero. Therefore $H_1(S^n) \to H_1(S^n, D^n_-)$ is onto. Since $H_1(D^n_-) = 0$ we get $H_1(S^n) \cong H_1(S^n, D^n_-)$.

e) Put $U^n_- := \{x \in S^n \mid x_0 < -\frac{1}{2}\}$.

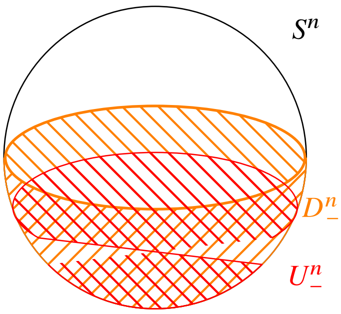


Figure 67. Excising a southern cap

By the excision axiom the inclusion

$$(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \hookrightarrow (S^n, D^n_-)$$

[1] For later use, we also define the upper hemisphere $D^n_+ = \{x \in S^n \mid x_0 \geq 0\}$.

induces an isomorphism

$$H_m(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \cong H_m(S^n, D^n_-)$$

since $\bar{U}^n_- \subset \mathring{D}^n_-$. We also have that

$$(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \simeq (D^n_+, S^{n-1}) \approx (D^n, S^{n-1})$$

where the homeomorphism is given by a vertical projection. This gives us the isomorphism

$$H_m(S^n \setminus U^n_-, D^n_- \setminus U^n_-) \cong H_m(D^n, S^{n-1}).$$

In particular,

$$H_1(S^n) \cong H_1(S^n, D^n_-) \cong H_1(D^n, S^{n-1}) = \begin{cases} R, & \text{for } n = 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

This concludes the case $m = 1$.

f) Finally we treat the case $m \geq 2$ by induction. Observe that

$$\begin{array}{ccccccc} H_m(D^n_-) & \longrightarrow & H_m(S^n) & \longrightarrow & H_m(S^n, D^n_-) & \longrightarrow & H_{m-1}(D^n) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array}$$

and

$$\begin{array}{ccccccc} H_m(D^n) & \longrightarrow & H_m(D^n, S^{n-1}) & \longrightarrow & H_{m-1}(S^{n-1}) & \longrightarrow & H_{m-1}(D^n) \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array}$$

This yields

$$H_m(S^n) \cong H_m(S^n, D^n_-) \cong H_m(D^n, S^{n-1}) \cong H_{m-1}(S^{n-1}) \tag{3.6}$$

Induction over m concludes the proof. □

Remark 3.17. Let us describe a generator of $H_1(S^1) \cong \mathbb{Z}$ geometrically. We use the isomorphisms in (3.5) and start with $H_1(D^1, S^0)$. The map $c : \Delta^1 \to D^1$, $t \mapsto \cos(\pi(1-t))$, is a singular 1-simplex with $\partial c = \mathrm{const}_1 - \mathrm{const}_{-1}$. Under the isomorphism $H_0(S^0) \cong R^2$ it maps to $(1, -1)$ which generates the kernel of the map $R^2 \to R$ given by $(a, b) \mapsto a + b$. Hence c represents a generator of $H_1(D^1, S^0)$.
The isomorphism $H_1(D^1_+, S^0) \to H_1(D^1, S^0)$ induced by vertical projection gives us a generator of $H_1(D^1_+, S^0)$, namely the homology class represented by $c' : \Delta^1 \to D^1_+$, $t \mapsto e^{i\pi(1-t)}$.
The isomorphism $H_1(D^1_+, S^0) \to H_1(S^1, D^1_-)$ is induced by the inclusion. Hence $[c'] \in H_1(S^1, D^1_-)$ is a generator. Now

$$F : \Delta^1 \times [0,1] \to S^1, \quad (t, s) \mapsto e^{i\pi(1-(s+1)t)},$$

is continuous and satisfies

$$\begin{aligned} F(t,0) &= c'(t), \\ F(t,1) &= e^{i\pi(1-2t)} =: c''(t), \\ F(0,s) &= -1 \in D^1_-, \\ F(1,s) &= e^{-i\pi s} \in D^1_-. \end{aligned}$$

Thus $c' \simeq c''$ as maps $(\Delta^1, \{0,1\}) \to (S^1, D^1_-)$. Using the homotopy axiom we compute

$$[c'] = [c' \circ \mathrm{id}_{\Delta^1}] = H_1(c')[\mathrm{id}_{\Delta^1}] = H_1(c'')[\mathrm{id}_{\Delta^1}] = [c'' \circ \mathrm{id}_{\Delta^1}] = [c''].$$

Therefore $[c''] \in H_1(S^1, D^1_-)$ is a generator. Since $\partial c'' = \mathrm{const}_{c''(1)} - \mathrm{const}_{c''(0)} = \mathrm{const}_{-1} - \mathrm{const}_{-1} = 0$, the 1-simplex c'' represents a homology class in $H_1(S^1)$. Moreover, since inclusion induces an isomorphism $H_1(S^1) \to H_1(S^1, D^1_-)$ we find that $[c''] \in H_1(S^1)$ is a generator.
Using the homotopy $(t,s) \mapsto e^{i\pi(s-2t)}$ we see that $t \mapsto e^{-2i\pi t}$ also represents a generator of $H_1(S^1)$. Finally, playing the same game with lower and upper hemispheres interchanged shows that $t \mapsto e^{2i\pi t}$ represents a generator of $H_1(S^1)$ as well.

As a first application of Theorem 3.16 we now prove Brouwer's fixed point theorem in all dimensions.

Theorem 3.18 (Brouwer's fixed point theorem). *Let $f : D^n \to D^n$, $n \geq 1$, be a continuous map. Then f has a fixed point, i.e., there exists an $x \in D^n$ such that $f(x) = x$.*

Proof. We assume that the map f does not have a fixed point and then derive a contradiction. Let $n \geq 2$ (the case $n = 1$ having been treated in Remark 1.6).

Now consider the continuous map $g : D^n \to S^{n-1}$ as in the picture. Denote the inclusion map by $\iota : S^{n-1} \to D^n$ and note that $g \circ \iota = \mathrm{id}_{S^{n-1}}$.

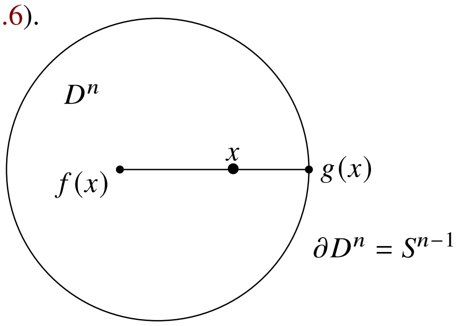


Figure 68. Constructing a retraction

By the functorial properties of homology we obtain the following commutative diagram:

$$\begin{tikzcd} H_{n-1}(S^{n-1};\mathbb{Z}) \cong \mathbb{Z} \arrow[rr, "H_{n-1}(\mathrm{id})=\mathrm{id}"] \arrow[dr, "H_{n-1}(\iota)"'] & & H_{n-1}(S^{n-1};\mathbb{Z}) \cong \mathbb{Z} \\ & H_{n-1}(D^n;\mathbb{Z}) = 0 \arrow[ur, "H_{n-1}(g)"'] & \end{tikzcd}$$

Since the identity $\mathbb{Z} \to \mathbb{Z}$ does not factor through 0 we run into a contradiction. □

Now we are in the position to answer the third question on page 3.

Theorem 3.19. *For $n \neq m$ the space $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*

Proof. Let us assume there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$$

is also a homeomorphism. Since $\mathbb{R}^n \setminus \{\text{point}\} \simeq S^{n-1}$ we obtain an isomorphism on the level of homology groups:

$$H_j(S^{n-1}) \cong H_j(\mathbb{R}^n \setminus \{0\}) \cong H_j(\mathbb{R}^m \setminus \{f(0)\}) \cong H_j(S^{m-1}).$$

By Theorem 3.16 this is a contradiction for $j = n - 1$ or $j = m - 1$ unless $n = m$. □

Proposition 3.20. *For $n \geq 1$ let $s : S^n \to S^n$ be the reflection given by*

$$s(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n).$$

Then the map

$$H_n(s) : H_n(S^n) \to H_n(S^n)$$

is given by $H_n(s) = -\operatorname{id}$.

Proof. The proof is given by induction. First consider the case $n = 1$. Let $c : \Delta^1 \to S^1$ be a 1-simplex generating $H_1(S^1)$ and let $S_1(s)(c)$ be its image under the induced homomorphism on 1-chains.

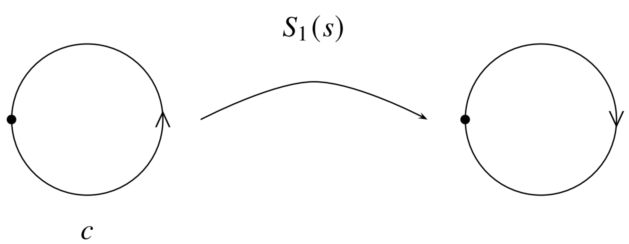


Figure 69. The reflection in one dimension

We need to show that $H_1(s)[c] = [S_1(s)c] = -[c]$. We construct two 2-simplices. The first one is as indicated in Figure 70.

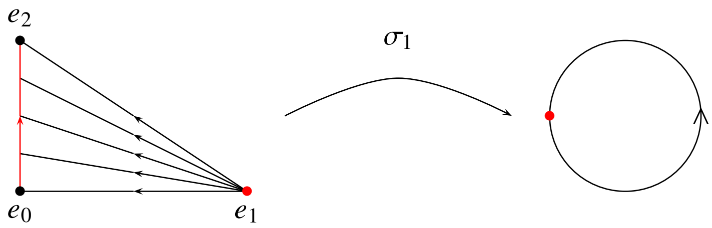


Figure 70. First 2-simplex σ_1

Applying the boundary operator yields

$$\partial\sigma_1 = S_1(s)c - \text{const} + c.$$

The second 2-simplex is the constant map.

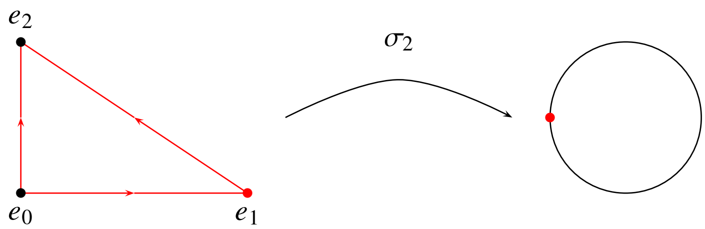


Figure 71. Second 2-simplex σ_2

We apply the boundary operator again and we get

$$\partial\sigma_2 = \text{const} - \text{const} + \text{const} = \text{const}$$

It follows that for the chain $\sigma_1 + \sigma_2$ that

$$\partial(\sigma_1 + \sigma_2) = S_1(s)c + c$$

and hence

$$0 = [\partial(\sigma_1 + \sigma_2)] = [S_1(s)c + c] = [S_1(s)c] + [c]\,.$$

This yields the desired result in the case of $n = 1$.
The induction step $n - 1 \Rightarrow n$ follows from the following commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"] \arrow[d, "H_n(s)"'] & H_n(S^n, D^n_-) \arrow[d, "H_n(s)"'] & H_n(D^n_+, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "H_n(s)"'] & H_{n-1}(S^{n-1}) \arrow[d, "H_{n-1}(s)"] \\
H_n(S^n) \arrow[r, "\cong"] & H_n(S^n, D^n_-) & H_n(D^n_+, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

where the horizontal isomorphisms are the ones in (3.6). Commutativity of the last square follows from Lemma 3.13 and that of the first and the second one from the fact that the horizontal isomorphisms are induced by inclusion maps (which commute with s). Note here that we have to choose the lower hemisphere with respect to a different coordinate than the one which is reflected by s so that $s(D^n_-) = D^n_-$. $\square$

Remark 3.21. Let $a : S^n \to S^n$ with $a(x) = -x$ for all $x \in S^n$ be the *antipodal map* then $H_n(a) = (-1)^{n+1}$. This follows from the fact that a is the composition of $n+1$ reflections.

Definition 3.22. A *vector field* on S^n is a map $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x) \perp x$ for all $x \in S^n$.

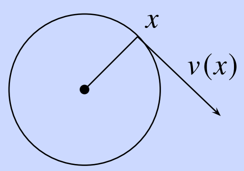


Figure 72. Vector field on S^n

Theorem 3.23 (Hairy ball theorem). *The n-dimensional sphere S^n admits a continuous vector field without zeros iff n is odd.*
In particular, every continuous vector field on S^2 has a zero.

Loosely speaking, this means that every continuously combed hedgehog has a "bald" spot.

Proof. If n is odd we simply set $v(x) := (-x_1, x_0, -x_3, x_2, \ldots, -x_n, x_{n-1})$. This defines a nowhere vanishing continuous vector field.

Now let n be even and let $v : S^n \to \mathbb{R}^{n+1}$ be a continuous vector field without a zero. Then we can put $w(x) := \frac{v(x)}{|v(x)|}$. We define the continuous map $F : S^n \times [0,1] \to S^n$ by

$$F(x,t) := x\cos(\pi t) + w(x)\sin(\pi t)\,.$$

Since $F(x,0) = x$ and $F(x,1) = -x = a(x)$ with $a(x)$ the antipodal map we have found that $a \simeq \mathrm{id}$. Hence $H_n(a) = H_n(\mathrm{id}) = 1$.

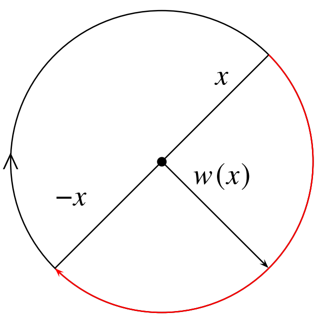


Figure 73. The retraction

This contradicts $H_n(a) = (-1)^{n+1} = -1$. □

3.4. The degree of a continuous map

For $n \geq 1$ we consider a continuous map $f : S^n \to S^n$. Then the homomorphism

$$H_n(f) : H_n(S^n;\mathbb{Z}) \cong \mathbb{Z} \to H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$$

is given by multiplication with a number which we denote $\deg(f) \in \mathbb{Z}$.

Definition 3.24. The number $\deg(f)$ is called the *degree* of the map f. The degree can be defined in the same way for continuous maps $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$.

Examples 3.25. 1.) For a reflection we have $\deg(s) = -1$.

2.) For the antipodal map we have seen that $\deg(a) = (-1)^{n+1}$.

Lemma 3.26. *The degree of a function has the following properties*

(i) $\deg(\mathrm{id}) = 1$*;*

(ii) $\deg(\mathrm{const}) = 0$*;*

(iii) $\deg(f \circ g) = \deg(f)\deg(g)$*;*

(iv) If $f \simeq g$ then $\deg(f) = \deg(g)$*;*

(v) If the map f is a homotopy equivalence then $\deg(f) = \pm 1$*;*

(vi) For $f : (D^{n+1}, S^n) \to (D^{n+1}, S^n)$ we have $\deg(f) = \deg(f|_{S^n})$.

Proof. The first and third assertion follow directly from the functorial property of $H_n(f)$. The fourth and fifth statement follow from the homotopy axiom. The second assertion follows from the fact that the homomorphism induced by a constant map factors through $H_n(pt) = 0$. The last statement of the lemma follows fromm the commutativity of the following diagram:

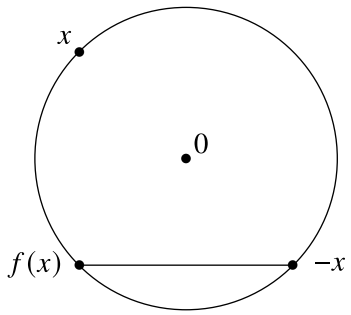
$$\begin{tikzcd}
H_{n+1}(D^{n+1}, S^n) \arrow[r, "H_{n+1}(f)"] \arrow[d, "\partial", "\cong"'] & H_{n+1}(D^{n+1}, S^n) \arrow[d, "\partial"', "\cong"] \\
H_n(S^n) \arrow[r, "H_{n+1}(f|_{S^n})"] & H_n(S^n)
\end{tikzcd}$$

Theorem 3.27. *(i) Every $f \in C(S^n, S^n)$ without fixed points satisfies* $\deg(f) = (-1)^{n+1}$.

(ii) Every $f \in C(S^n, S^n)$ without an antipodal point, i.e., $f(x) \neq -x$ for all $x \in S^n$, satisfies $\deg(f) = 1$.

(iii) For n even every $f \in C(S^n, S^n)$ has a fixed point or an antipodal point.

Proof. (i) Let $f \in C(S^n, S^n)$ be without a fixed point. Then the line segment joining $f(x)$ and $-x$ does not contain the origin.



Hence we can define the continuous map

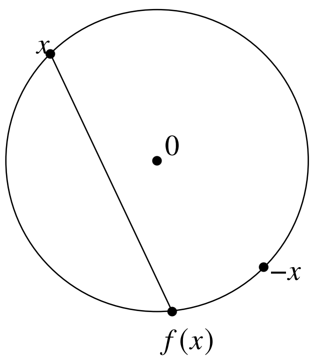
$$F : S^n \times [0,1] \to S^n, \quad F(x,t) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}.$$

Since $F(x,0) = f(x)$ and $F(x,1) = -x = a(x)$ with a being the antipodal map the map F is a homotopy for $f \simeq a$. Thus

$$\deg(f) = \deg(a) = (-1)^{n+1}.$$

(ii) Now let $f \in C(S^n, S^n)$ be without antipodal points. Then we can define

$$G(x,t) := \frac{(1-t)f(x) + tx}{|(1-t)f(x) + tx|}.$$



Since $G(x,0) = f(x)$ and $G(x,1) = x$ we have that $f \simeq \mathrm{id}$ and $\deg(f) = \deg(\mathrm{id}) = 1$ follows.

(iii) Finally, assume that $f \in C(S^n, S^n)$ has neither fixed points nor antipodal points. Then $\deg(f) = (-1)^{n+1}$ by ((i)) and by $\deg(f) = 1$ by ((ii)). Thus n must be odd. □

Let $\mu : S^n \times S^n \longrightarrow S^n$ for $n \geq 1$ be a continuous map. We choose $p \in S^n$ arbitrarily and define

$$\begin{aligned} j_1 : S^n &\longrightarrow S^n \times S^n, \quad x \mapsto (x, p), \\ j_2 : S^n &\longrightarrow S^n \times S^n, \quad x \mapsto (p, x)\,. \end{aligned}$$

We then get the following diagram:

$$\begin{tikzcd} \mathbb{Z}^2 \cong H_n(S^n) \oplus H_n(S^n) \arrow[r, "(H_n(j_1),H_n(j_2))"] \arrow[rr, bend right, "(d_1{,}d_2)"'] & H_n(S^n \times S^n) \arrow[r, "H_n(\mu)"] & H_n(S^n) \cong \mathbb{Z} \end{tikzcd}$$

with $d_\mu \in \mathbb{Z}$.

Definition 3.28. The pair of numbers $(d_1, d_2) \in \mathbb{Z}^2$ is called the *bidegree* of the map μ.

Remark 3.29. The bidegree does not depend on the choice of p. If one chooses another p', then a path from p to p' will yield a homotopy between the corresponding embedding maps j_ν and j'_ν. Hence they induce the same homomorphisms on homology and therefore the same bidegrees.

Examples 3.30. 1.) Consider the case $n = 1$, $S^1 \subset \mathbb{C}$ and let the map μ be given by $\mu(z_1, z_2) = z_1 z_2$. Choose $p = 1 \in S^1$. Then $\mu \circ j_1 = \mathrm{id}$ and thus

$$d_1 = \deg(\mu \circ j_1) = \deg(\mathrm{id}) = 1.$$

Similarly, we get $d_2 = 1$. Hence the bidegree by $(d_1, d_2) = (1, 1)$.

2.) In the case of $n = 3$, $S^3 \subset \mathbb{H}$ we consider the map μ given by quaternionic multiplication, $\mu(h_1, h_2) = h_1 h_2$. Similar reasoning shows that the bidegree is again given by $(d_1, d_2) = (1, 1)$.

Remark 3.31. The *quaternions* $\mathbb{H}$ form a division algebra isomorphic to $\mathbb{R}^4$ as a vector space. The algebra $\mathbb{H}$ is associate and noncommutative. The standard vector space basis we be denoted by $1, i, j, k$. Therefore any $h \in \mathbb{H}$ can be uniquely written as

$$h = h_0 + h_1 i + h_2 j + h_3 k\,.$$

Quaternionic multiplication is now determined by the relations $i^2 = j^2 = k^2 = ijk = -1$. With the help of the conjugation

$$h^* = h_0 - h_1 i - h_2 j - h_3 k$$

we can define $|h| := \sqrt{h^* h}$. We regard $S^3 \subset \mathbb{H}$ as the set of unit-length quaternions.

Proposition 3.32. *Let $n = 1$ or $n = 3$ and let $k \in \mathbb{Z}$. The map $f_k : S^n \to S^n$ with $f_k : z \mapsto z^k$ given by complex multiplication in the case $n = 1$ and by quaternionic multiplication in the case $n = 3$ has degree k.*

Proof. The proof is by induction on k. For $k = 0$ and $k = 1$ the statement is trivial because constant maps have degree 0 while the identity has degree 1. The case of $k = -1$ has already been shown for $n = 1$, since here $z \mapsto z^{-1} = \bar{z}$ is a reflection and hence $\deg(f_{-1}) = -1$. On the other hand for $k = -1$, $n = 3$ we note that $S^1 \subset S^2 \subset S^3$ regarding $\mathbb{C} \subset \mathbb{H}$. Now the following commutative diagram

$$\begin{tikzcd}
H_1(S^1) \arrow[r, phantom, "\cong"] \arrow[d, "H_1(f_{-1}|_{S^1})=\deg(f_{-1}|_{S^1})=-1"'] & H_2(S^2) \arrow[r, phantom, "\cong"] & H_3(S^3) \arrow[d, "H_3(f_{-1})=\deg(f_{-1})"] \\
H_1(S^1) \arrow[r, phantom, "\cong"] & H_2(S^2) \arrow[r, phantom, "\cong"] & H_3(S^3)
\end{tikzcd}$$

implies that $\deg(f_{-1}) = -1$ in the quaternionic case too. The horizontal isomorphisms are obtained as the composition of the isomorphisms

$$H_k(S^k) \cong H_{k+1}(D^{k+1}, S^k) \cong H_{k+1}(S^{k+1}, D_-^{k+1}) \cong H_{k+1}(S^{k+1}).$$

We used that the quaternionic multiplication restricted to the complex numbers $\mathbb{C}$ is just complex multiplication.
Now we are ready to carry out the induction over k. We consider $k > 0$ and we show that the statement for $k-1$ implies that for k. Let μ be complex (resp. quaternionic) multiplication. Then

$$\begin{aligned}
\deg(f_k) &= \deg(\mu \circ (f_{k-1}, f_1)) \\
&= (1, 1) \cdot \begin{pmatrix} \deg(f_{k-1}) \\ \deg(f_1) \end{pmatrix} \\
&= \deg(f_{k-1}) + \deg(f_1) \\
&= k - 1 + 1 = k.
\end{aligned}$$

The case $k < 0$ is treated similarly. □

Now we can show that the fundamental theorem of algebra 2.34 also holds for quaternionic polynomials.

Theorem 3.33 (Fundamental theorem of algebra for quaternions). *Every quaternionic polynomial*

$$p(z) = z^k + \alpha_1 z^{k-1} + \ldots + \alpha_k$$

of positive degree k has a quaternionic root.

Proof. Suppose the polynomial p has no root. Then we can define the continuous map $\hat{p} : S^3 \to S^3$ with $\hat{p}(z) := \frac{p(z)}{|p(z)|}$. Now consider $F(z,t) : S^3 \times [0,1] \to S^3$ with $F(z,t) := \frac{p(tz)}{|p(tz)|}$. We observe that $F(z,0) = \text{const}$ and $F(z,1) = \hat{p}(z)$, hence we have $\hat{p}(z) \simeq \text{const}$ and consequently $\deg(\hat{p}) = 0$.
On the other hand, we can put for $z \in S^3$ and $t > 0$

$$G(z,t) := \frac{t^k p\left(\frac{z}{t}\right)}{|t^k p\left(\frac{z}{t}\right)|}$$

and observe that the expression

$$t^k p\left(\frac{z}{t}\right) = z^k + t\alpha_1 z^{k-1} + \ldots + t^k \alpha$$

extends continuously to $t = 0$. The map $G : S^3 \times [0,1] \to S^3$ satisfies $G(z,0) = f_k(z)$ and again $G(z,1) = \hat{p}(z)$. Thus $\hat{p} \simeq f_k$ and hence $\deg(\hat{p}) = k$. This contradicts $k > 0$. $\square$

Now let $U \subset S^n$ be open, $n \geq 1$. Consider a continuous map $f : U \to S^n$ and a point $p \in S^n$ such that $f^{-1}(p)$ is compact. Then we have the following diagram (with $\deg_p(f) \in \mathbb{Z}$):

$$\begin{tikzcd}
\mathbb{Z} \cong H_n(S^n;\mathbb{Z}) \arrow[r, "(i)"] \arrow[d, "\deg_p(f)"'] & H_n(S^n, S^n \setminus f^{-1}(p);\mathbb{Z}) & H_n(U, U \setminus f^{-1}(p);\mathbb{Z}) \arrow[l, "(ii)"', "\cong"] \arrow[d, "H_n(f)"] \\
\mathbb{Z} \cong H_n(S^n;\mathbb{Z}) \arrow[rr, "(iii)", "\cong"'] & & H_n(S^n, S^n \setminus \{p\};\mathbb{Z})
\end{tikzcd} \tag{3.7}$$

We observe:

1. Concerning (i): This homomorphism is induced by the inclusion
$$S^n = (S^n, \emptyset) \hookrightarrow (S^n, S^n \setminus f^{-1}(p)).$$
2. Concerning (ii): The inclusion $(U, U \setminus f^{-1}(p)) \hookrightarrow (S^n, S^n \setminus f^{-1}(p))$ induces an isomorphism by the excision axiom.
3. Concerning (iii): Consider the exact homology sequence of $(S^n, S^n \setminus \{p\})$
$$0 = H_n(S^n \setminus \{p\}) \to H_n(S^n) \to H_n(S^n, S^n \setminus \{p\}) \to \underbrace{H_{n-1}(S^n \setminus \{p\})}_{=\begin{cases} 0 & \text{for } n \geq 2 \\ \mathbb{Z} & \text{for } n = 1 \end{cases}} \xrightarrow{\cong} H_{n-1}(S^n)\,.$$
Hence in both cases $n = 1$ and $n \geq 2$ the homomorphism (iii) is an isomorphism.

Definition 3.34. The number $\deg_p(f)$ is called the *local degree* of f over p.

Examples 3.35. 1.) If $p \notin \mathrm{im}(f)$ then $\deg_p(f) = 0$ because $H_n(S^n, S^n) = 0$.

2.) If $f : U \to S^n$ is the inclusion map then $\deg_p(f) = 1$ for all $p \in U$. Namely, in this case the homomorphisms (i) and (iii) in (3.7) coincide and so do $H_n(f)$ and (ii).

3.) For a homeomorphism $f : U \to f(U) \subset S^n$ we have that $\deg_p(f) = \pm 1$ for all $p \in f(U)$. Namely, the homomorphisms (i) and $H_n(f)$ in (3.7) are isomorphisms in this case, hence multiplication by $\deg_p(f)$ is an isomorphism, thus $\deg_p(f) = \pm 1$.

Proposition 3.36. *Assume that $f^{-1}(p) \subset K \subset V \subset U$ where K is compact and V open. Then the degree $\deg_p(f)$ is given by:*

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, "\deg_p(f)"'] & H_n(S^n, S^n \setminus K) & H_n(V, V \setminus K) \arrow[l, "\cong"'] \arrow[d, "H_n(f|_V)"] \\
H_n(S^n) \arrow[rr, "\cong"'] & & H_n(S^n, S^n \setminus \{p\})
\end{tikzcd}$$

Hence we can replace $f^{-1}(p)$ by a larger compact set in U and also U by a smaller open neighborhood of $f^{-1}(p)$. For this reason we call $\deg_p(f)$ local.

Proof. The assertion follows from the commutativity of the following diagram:

$$\begin{tikzcd}
& H_n(S^n, S^n \setminus f^{-1}(p)) & H_n(U, U \setminus f^{-1}(p)) \arrow[l, "\cong"'] \arrow[dr, "H_n(f)"] & & \\
H_n(S^n) \arrow[ur] \arrow[dr] & & & H_n(S^n, S^n \setminus \{p\}) & H_n(S^n) \arrow[l, "\cong"'] \\
& H_n(S^n, S^n \setminus K) \arrow[uu] & H_n(V, V \setminus K) \arrow[l, "\cong"'] \arrow[uu] \arrow[ur, "H_n(f|_V)"'] & &
\end{tikzcd}$$

where all but two arrows are induced by inclusions. □

Corollary 3.37. *For $f : S^n \to S^n$ we have that $\deg(f) = \deg_p(f)$ for all $p \in S^n$.*

Proof. Choose $K = V = S^n$ in Proposition 3.36. □

Lemma 3.38. *Let $f : (D^n_+, S^{n-1}) \to (D^n_+, S^{n-1})$ be continuous and $p \in \mathring{D}^n_+$ such that $f^{-1}(p) \subset \mathring{D}^n_+$ is compact. Then*

$$\deg(f) = \deg_p\left(f|_{\mathring{D}^n_+}\right).$$

Proof. We extend f to a continuous map $F : (S^n, D^n_-) \to (S^n, D^n_-)$ by mapping the circular arc from a point x on the equator S^{n-1} to the south pole to the corresponding arc from $f(x)$ to the south pole. In formulas,

$$F(x_0, x') = \begin{cases} f(x_0, x') & \text{if } x_0 \geq 0, \\ \left(x_0, \|x'\| \cdot \left(f(0, x'/\|x'\|)\right)'\right) & \text{if } -1 < x_0 < 0, \\ (-1, 0) & \text{if } x_0 = -1. \end{cases}$$

Here we wrote $x = (x_0, x') = (x_0, x_1, \ldots, x_n) \in S^n \subset \mathbb{R}^{n+1}$. Then by Corollary 3.37 and Proposition 3.36 with $K = f^{-1}(p)$, $V = \mathring{D}^n_+$ and $U = S^n$ we find

$$\deg(F) = \deg_p(F) = \deg_p(F|_{\mathring{D}^n_+}) = \deg_p(f|_{\mathring{D}^n_+}) . \tag{3.8}$$

On the other hand, the commutative diagram

$$\begin{tikzcd} (D^n_+, S^{n-1}) \arrow[r] \arrow[d, "f"] & (S^n, D^n_-) \arrow[d, "F"] & S^n \arrow[l] \arrow[d, "F"] \\ (D^n_+, S^{n-1}) \arrow[r] & (S^n, D^n_-) & S^n \arrow[l] \end{tikzcd}$$

yields on the level of homology

$$\begin{tikzcd} H_n(D^n_+, S^{n-1}) \arrow[r, "\cong"] \arrow[d, "\cdot \deg(f)"] & H_n(S^n, D^n_-) \arrow[d, "H_n(F)"] & H_n(S^n) \arrow[l, "\cong"'] \arrow[d, "\cdot \deg(F)"] \\ H_n(D^n_+, S^{n-1}) \arrow[r, "\cong"] & H_n(S^n, D^n_-) & H_n(S^n) \arrow[l, "\cong"'] \end{tikzcd}$$

Hence $\deg(F) = \deg(f)$. This together with (3.8) concludes the proof. □

Proposition 3.39. *Let $f : U \subset S^n \to S^n$ be continuous, let $p \in S^n$ with $f^{-1}(p)$ compact and let $g \in C(S^n, S^n)$. Then*

$$\deg_p(f \circ g) = \deg_p(f) \cdot \deg(g).$$

Proof. This follows from the commutative diagram:

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[dr, "\deg_p(f)"] & H_n(S^n, S^n \setminus f^{-1}(p)) \cong H_n(U, U \setminus f^{-1}(p)) \arrow[d, "H_n(f)"] \\
& H_n(S^n) \cong H_n(S^n, S^n \setminus \{p\}) \\
H_n(S^n) \arrow[r] \arrow[uu, "\deg(g)=H_n(g)"] \arrow[ur, "\deg_p(f\circ g)"'] & H_n(S^n, S^n \setminus (f \circ g)^{-1}(p)) \cong H_n(g^{-1}(U), g^{-1}(U) \setminus (f \circ g)^{-1}(p)) \arrow[u, "H_n(f\circ g)"']
\end{tikzcd}$$

□

Remark 3.40. If X_i are the path-components of X then

$$H_m(X) \cong \bigoplus_i H_m(X_i),$$

see Exercise 3.1. The isomorphism is induced by the inclusion maps of the connected components into X. Similarly one sees that for $A \subset X$ and $A_i = A \cap X_i$

$$H_m(X, A) \cong \bigoplus_i H_m(X_i, A_i)\,.$$

Proposition 3.41 (Additivity of the local degree). *Let $f : U \to S^n$ be a continuous map. Let $p \in S^n$ be such that $f^{-1}(p)$ is compact. Let $U_\lambda \subset U$ be open and put $f_\lambda := f|_{U_\lambda}$, $\lambda = 1, \ldots, r$. Assume that $f^{-1}(p)$ is the disjoint union of the $f_\lambda^{-1}(p)$, i.e. $f^{-1}(p) = \sqcup_{\lambda=1}^r f_\lambda^{-1}(p)$. Then*

$$\deg_p(f) = \sum_{\lambda=1}^{r} \deg_p(f_\lambda)\,.$$

Before proving the proposition we give some examples.

Example 3.42
Assume that $f^{-1}(p)$ is a finite set, i.e. $f^{-1}(p) = \{p_1, \dots, p_r\}$. Now choose U_λ such that $p_\lambda \in U_\lambda$ and $p_\nu \notin U_\lambda$ for $\nu \neq \lambda$. It follows that

$$\deg_p(f) = \sum_{\lambda=1}^{r} \deg_p(f_\lambda)\,.$$

If the map f is a local homeomorphism, then $\deg_p(f_\lambda) = \pm 1$ for every λ.

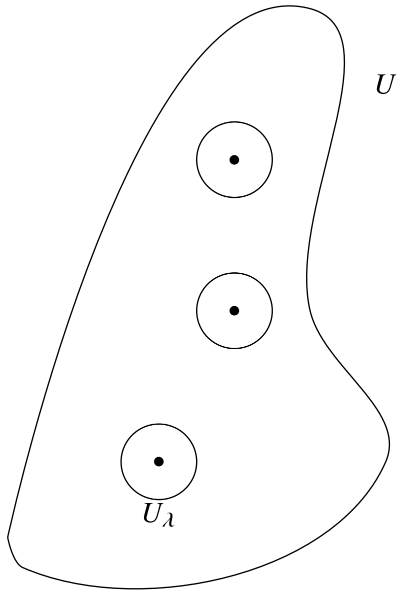


Figure 74. The case of finite preimage

Example 3.43. Consider the map $f_k : S^1 \to S^1$ with $f(z) = z^k$, $k > 0$, and set $p = 1$. We write

$$f_k^{-1}(1) = \{k\text{-th unit roots}\} = \{\xi_1, \dots, \xi_k\}$$

and find that $f_k|_{\text{small neighborhood of } \xi_\lambda}$ is a homeomorphism. Hence $\deg(f_{k,\lambda}) = \pm 1$. Since $k > 0$, the restriction of f_k to a small neighborhood of ξ_λ is homotopic to an embedding of a (k times larger) neighborhood of ξ_λ into S^1. Hence $\deg_1(f_{k,\lambda}) = 1$. We conclude $\deg_1(f_k) = k$.

Proof of Proposition 3.41. Choose open neighborhoods V_λ such that $f_\lambda^{-1}(p) \subset V_\lambda \subset U_\lambda$ with $V_\lambda \cap V_\mu = \emptyset$ for $\lambda \neq \mu$. Now put $V = \cup_{\lambda=1}^{r} V_\lambda$. Proposition 3.36 tells us $\deg_p(f) = \deg_p(f|_V)$. The commutative diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, "\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}"'] \arrow[rrr, bend left, "\deg_p(f|_V)"] & H_n(V, V \setminus f^{-1}(p)) \arrow[r, "H_n(f|_V)"] & H_n(S^n, S^n \setminus \{p\}) & H_n(S^n) \arrow[l, "\cong"'] \\
\bigoplus_{\lambda=1}^{r} H_n(S^n) \arrow[r] \arrow[rrr, bend right, "\begin{pmatrix}\deg_p(f_1) & & \\ & \ddots & \\ & & \deg_p(f_r)\end{pmatrix}"'] & \bigoplus_{\lambda=1}^{r} H_n(V_\lambda, V_\lambda \setminus f_\lambda^{-1}(p)) \arrow[u, "\cong"] \arrow[r, "\oplus_\lambda H_n(f_\lambda)"] & \bigoplus_{\lambda=1}^{r} H_n(S^n, S^n \setminus \{p\}) \arrow[u, "(1,\dots,1)"] & \bigoplus_{\lambda=1}^{r} H_n(S^n) \arrow[l, "\cong"'] \arrow[u, "(1,\dots,1)"]
\end{tikzcd}$$

yields

$$\deg_p(f|_V) = (1,\dots,1)\cdot\begin{pmatrix}\deg_p(f_1) & & \\ & \ddots & \\ & & \deg_p(f_r)\end{pmatrix}\cdot\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \deg_p(f_1)+\dots+\deg_p(f_r). \qquad \square$$

3.5. Homological algebra

Before continuing with topological considerations we clarify some of the underlying algebra. Throughout this section let R be a commutative ring with unit element.

Definition 3.44. A *complex* of R-modules K_* is a sequence

$$\dots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \longrightarrow \dots$$

of R-modules K_n together with homomorphisms ∂_n such that

$$\partial_n \circ \partial_{n+1} = 0 \tag{3.9}$$

for all $n \in \mathbb{Z}$. We define the space of *n-cycles*

$$Z_nK_* := \{x \in K_n \mid \partial_n x = 0\} = \ker(\partial_n)\,,$$

and the space of *n-boundaries*

$$B_nK_* := \{\partial_{n+1}x \in K_n \mid x \in K_{n+1}\} = \operatorname{im}(\partial_{n+1}).$$

By (3.9) $B_nK \subset Z_nK$ so that we can define the *n-th homology* by

$$H_nK_* := \frac{Z_nK_*}{B_nK_*}\,.$$

Example 3.45. For

$$K_n = \begin{cases} S_n(X,A;R) = S_n(X;R)/S_n(A;R), & n \geq 0 \\ 0, & n < 0 \end{cases}$$

we obtain relative singular homology $H_nK_* = H_n(X,A;R)$.

Definition 3.46. Given two complexes K_* and K'_* a *chain map* $\varphi_* : K_* \to K'_*$ is a sequence of homomorphisms $\varphi_n : K_n \to K'_n$ such that

$$\begin{array}{ccc} K_{n+1} & \xrightarrow{\varphi_{n+1}} & K'_{n+1} \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial'_{n+1}} \\ K_n & \xrightarrow{\varphi_n} & K'_n \end{array}$$

commutes for all $n \in \mathbb{Z}$.

Example 3.47. For a continuous map $f : (X, A) \to (Y, B)$ the homomorphisms $\varphi_n = S_n(f)$ constitute a chain map.

Given a general chain map φ_* we conclude from

$$\varphi_n \circ \partial_{n+1} = \partial'_{n+1} \circ \varphi_{n+1}$$

that $\varphi_n(Z_n K_*) \subset Z_n K'_*$ and also that $\varphi_n(B_n K_*) \subset B_n K'_*$. Hence φ_* induces homomorphisms $H_n(\varphi_*) : H_n K_* \to H_n K'_*$ by $H_n(\varphi_*)([z]) = [\varphi_n z]$. This construction is functorial in the sense that

$$H_n(\varphi_* \circ \psi_*) = H_n(\varphi_*) \circ H_n(\psi_*) \quad \text{and} \quad H_n(\mathrm{id}_{K_*}) = \mathrm{id}_{H_n K_*}\,.$$

Definition 3.48. A sequence $\cdots \longrightarrow K'_* \xrightarrow{\varphi_*} K_* \xrightarrow{\psi_*} K''_* \longrightarrow \cdots$ of chain maps is called *exact* iff the sequence $\cdots \longrightarrow K'_n \xrightarrow{\varphi_n} K_n \xrightarrow{\psi_n} K''_n \longrightarrow \cdots$ is exact for every $n \in \mathbb{Z}$.

Proposition 3.49. *If* $0 \to K'_* \xrightarrow{i_*} K_* \xrightarrow{p_*} K''_* \to 0$ *is an exact sequence of complexes then the sequence* $H_n K'_* \xrightarrow{H_n(i_*)} H_n K_* \xrightarrow{H_n(p_*)} H_n K''_*$ *is exact for every* $n \in \mathbb{Z}$.

Proof. a) By assumption we have that $p_* \circ i_* = 0$ and hence

$$0 = H_n(p_* \circ i_*) = H_n(p_*) \circ H_n(i_*).$$

Therefore $\operatorname{im} H_n(i_*) \subset \ker H_n(p_*)$.

b) It remains to show that $\ker H_n(p_*) \subset \operatorname{im} H_n(i_*)$.

Let $z \in Z_n K_*$ represent $[z] \in \ker H_n(p_*)$. Hence $p_n z = \partial'' x''$ for some $x'' \in K''_{n+1}$. Since p_{n+1} is surjective we can choose $x \in K_{n+1}$ with $p_{n+1} x = x''$. We compute

$$p_n(z - \partial x) = \partial'' x'' - \partial'' p_{n+1} x = \partial'' x'' - \partial'' x'' = 0\,.$$

By exactness of the complex there exists $y' \in K'_n$ such that $z - \partial x = i_n y'$. Now we get

$$i_{n-1}\partial' y' = \partial i_n y' = \partial[z - \partial x] = \partial z - \partial\partial x = 0\,.$$

Since i_{n-1} is injective it follows that $\partial' y' = 0$, i.e., $y' \in Z_n K'_*$ represents an element in homology. Finally we see

$$H_n(i_*)([y']) = [i_n y'] = [z - \partial x] = [z].$$

This shows $\ker H_n(p_*) \subset \operatorname{im} H_n(i_*)$. □

Definition 3.50. Let $0 \to K'_* \xrightarrow{i_*} K_* \xrightarrow{p_*} K''_* \to 0$ be an exact sequence of complexes. We construct the *connecting homomorphism*

$$\partial_* : H_n K''_* \to H_{n-1} K'_*$$

as follows: Let $z'' \in Z_n K''_*$ represent an element $[z''] \in H_n K''_*$. Since p_n is surjective we can choose $x \in K_n$ with $p_n x = z''$. For $\partial x \in K_{n-1}$ we observe

$$p_{n-1}\partial x = \partial'' p_n x = \partial'' z'' = 0.$$

By exactness there is a unique $y' \in K'_{n-1}$ with $i_{n-1} y' = \partial x$. Moreover,

$$i_{n-2}\partial' y' = \partial i_{n-1} y' = \partial\partial x = 0.$$

Since i_{n-2} is injective we have $\partial' y' = 0$, i.e., $y' \in Z_{n-1} K'_*$. Now put

$$\partial_*[z''] := [y'].$$

Lemma 3.51. *The conncecting homomorphism $\partial_* : H_n K''_* \to H_{n-1} K'_*$ is well defined, i.e., independent of the choices made in its construction.*

Proof. There are two choices in the construction of the connecting homomorphism: that of the preimage x with $p_n x = z''$ and that of the representing cycle z'' itself.
As to the choice of x, let $p_n x = p_n \bar{x} = z''$. For the corresponding elements $y', \bar{y}' \in K'_{n-1}$ we have and $i_{n-1} y' = \partial x$ and $i_{n-1}\bar{y}' = \partial \bar{x}$. Since $p_n(x - \bar{x}) = 0$ there exists an $\omega' \in K'_n$ with $x - \bar{x} = i_n(\omega')$. We compute

$$i_{n-1}(y' - \bar{y}') = \partial(x - \bar{x}) = \partial(i_n \omega') = i_{n-1}\partial\omega'\,,$$

from which we conclude that $y - \bar{y}' = \partial\omega'$ and therefore $[y'] = [\bar{y}']$.

As to the choice of the representing cycle z'', it suffices to show that if z'' is a boundary then $[y'] = 0$. Let $z'' = \partial''\zeta''$ be a boundary. Since p_{n+1} is onto we can choose $\xi \in K_{n+1}$ with $p_{n+1}\xi = \zeta''$. Then $x = \partial\xi$ is an admissible choice because

$$p_n x = p_n \partial\xi = \partial'' p_{n+1}\xi = \partial''\zeta'' = z''.$$

Thus $\partial x = \partial\partial\xi = 0$ and hence $y' = 0$. □

It is easy to see that the connecting homomorphism is indeed a homomorphism. Now we are ready to prove the Exactness Axiom.

Proposition 3.52. *If* $0 \to K'_* \xrightarrow{i_*} K_* \xrightarrow{p_*} K''_* \to 0$ *is an exact sequence of complexes then the long homology sequence*

$$\cdots \longrightarrow H_{n+1}K''_* \xrightarrow{\partial_*} H_nK'_* \xrightarrow{H_n(i_*)} H_nK_* \xrightarrow{H_n(p_*)} H_nK''_* \xrightarrow{\partial_*} H_{n-1}K'_* \longrightarrow \cdots$$

is also exact.

Proof. In view of Proposition 3.49 it remains to show $\ker H(i) = \operatorname{im}\partial_*$ and $\ker\partial_* = \operatorname{im} H(p)$.

a) $\operatorname{im}\partial_* \subset \ker H(i)$:
Using the notation of Definition 3.50 we compute

$$H(i)\partial_*[z''] = H(i)\partial_*[px] = H(i)[i^{-1}\partial x] = [\partial x] = 0\,.$$

b) $\ker H(i) \subset \operatorname{im}\partial_*$:
Let $H(i)[z'] = 0$. Then we have $[iz'] = 0$ and therefore $iz' = \partial x$. Put $z'' := px$. Then $\partial'' z'' = \partial'' px = p\partial x = piz' = 0$. Hence $z'' \in Z_nK_*$ represents an element in homology. We compute

$$\partial_*[z''] = \partial_*[px] = [i^{-1}\partial x] = [z']\,.$$

Hence $[z'] \in \operatorname{im}\partial_*$.

c) $\operatorname{im} H(p) \subset \ker\partial_*$:
This follows from

$$\partial_* H(p)[z] = \partial_*[pz] = [i^{-1}\underbrace{\partial z}_{=0}] = 0\,.$$

d) $\ker\partial_* \subset \operatorname{im} H(p)$:
Let $\partial_*[z''] = 0$. We write $z'' = px$ and compute

$$0 = \partial_*[z''] = \partial_*[px] = [i^{-1}\partial x]\,.$$

This implies that $i^{-1}\partial x$ is a boundary in K'. Hence we have that $i^{-1}\partial x = \partial' x'$ and therefore

$$\partial(x - ix') = \partial x - i\partial' x' = 0\,.$$

This finally leads to

$$H(p)[x - ix'] = [px - pix'] = [px] = [z'']\,,$$

hence $[z''] \in \operatorname{im} H(p)$. □

Example 3.53. For $K'_n = S_n(A)$, $K_n = S_n(X)$ and

$$K''_n = S_n(X, A) = \frac{S_n(X)}{S_n(A)}$$

Proposition 3.52 now yields the Exactness Axiom for singular homology.

Example 3.54. Consider a triple of spaces (X, A, B) with $B \subset A \subset X$ and set

$$\begin{aligned} K'_n &= S_n(A)/S_n(B) = S_n(A, B) \\ K_n &= S_n(X)/S_n(B) = S_n(X, B) \\ K''_n &= S_n(X)/S_n(A) = S_n(X, A)\,. \end{aligned}$$

Then the canonical sequence

$$0 \longrightarrow K'_n \longrightarrow K_n \longrightarrow K''_n \longrightarrow 0$$

is again exact. By Proposition 3.52 we obtain the long exact homology sequence for a triple:

$$\begin{tikzcd} & \dots \arrow[r] & H_{n+1}(X, A) \arrow[dll, "\partial_*"'] \\ H_n(A, B) \arrow[r] & H_n(X, B) \arrow[r] & H_n(X, A) \arrow[dll, "\partial_*"'] \\ H_{n-1}(A, B) \arrow[r] & \dots & \end{tikzcd}$$

Proposition 3.55. *The long homology sequence is natural, i.e., if the diagram of chain maps*

$$\begin{tikzcd} 0 \arrow[r] & K'_* \arrow[r, "i_*"] \arrow[d, "\varphi'_*"] & K_* \arrow[r, "p_*"] \arrow[d, "\varphi_*"] & K''_* \arrow[r] \arrow[d, "\varphi''_*"] & 0 \\ 0 \arrow[r] & L'_* \arrow[r, "j_*"] & L_* \arrow[r, "q_*"] & L''_* \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows then the diagram

$$\begin{tikzcd}
\dots \arrow[r, "H_{n+1}(p_*)"] & H_{n+1}K''_* \arrow[r, "\partial_*"] \arrow[d, "H_{n+1}(\varphi''_*)"] & H_nK'_* \arrow[r, "H_n(i_*)"] \arrow[d, "H_n(\varphi'_*)"] & H_nK_* \arrow[r, "H_n(p_*)"] \arrow[d, "H_n(\varphi_*)"] & H_nK''_* \arrow[r, "\partial_*"] \arrow[d, "H_n(\varphi''_*)"] & \dots \\
\dots \arrow[r, "H_{n+1}(q_*)"] & H_{n+1}L''_* \arrow[r, "\partial_*"] & H_nL'_* \arrow[r, "H_n(j_*)"] & H_nL_* \arrow[r, "H_n(q_*)"] & H_nL''_* \arrow[r, "\partial_*"] & \dots
\end{tikzcd}$$

is commutative as well.

Proof. By assumption we have $\varphi \circ i = j \circ \varphi'$ and $\varphi'' \circ p = q \circ \varphi$, hence

$$\begin{aligned} H_n(\varphi) \circ H_n(i) &= H_n(j) \circ H_n(\varphi'), \\ H_n(\varphi'') \circ H_n(p) &= H_n(q) \circ H_n(\varphi). \end{aligned}$$

We are left to show that the diagram

$$\begin{tikzcd}
H_nK'' \arrow[r, "\partial_*"] \arrow[d, "H_n(\varphi'')"'] & H_{n-1}K' \arrow[d, "H_{n-1}(\varphi')"] \\
H_nL'' \arrow[r, "\partial_*"] & H_nL'
\end{tikzcd}$$

commutes. We calculate

$$\begin{aligned} H_{n-1}(\varphi')\partial_*[px] &= H_{n-1}(\varphi')[i^{-1}\partial x] = [\varphi' i^{-1}\partial x] = [j^{-1}\varphi\partial x] \\ &= [j^{-1}\partial\varphi x] = \partial_*[p\varphi x] = \partial_*[\varphi'' px] = \partial_* H_n(\varphi'')[px]\,, \end{aligned}$$

which proves the assertion. □

Example 3.56. Let $f \in C((X,A),(Y,B))$. For $K'_n = S_n(A)$, $K_n = S_n(X)$, $K''_n = S_n(X,A)$ and $\varphi'_n = S_n(f|_A)$, $\varphi_n = S_n(f)$, and φ''_n the induced homomorphism on $S_n(X,A)$ we recover Lemma 3.13.

Definition 3.57. Let $\varphi, \psi : K \longrightarrow K'$ be chain maps. A *chain homotopy* between φ and ψ is a sequence of homomorphisms

$$h_n : K_n \longrightarrow K'_{n+1}$$

such that

$$\varphi_n - \psi_n = \partial'_{n+1} h_n + h_{n-1}\partial_n\,,$$

for all $n \in \mathbb{Z}$. The maps φ and ψ are called *homotopic* if such a homotopy exists. In this case we write $\varphi \simeq \psi$.

Proposition 3.58. *(i) The relation "$\simeq$" is an equivalence relation on the set of all chain maps $K \longrightarrow K'$.*

(ii) If $\varphi \simeq \psi : K \longrightarrow K'$ and $\varphi' \simeq \psi' : K' \longrightarrow K''$ then $\varphi'\varphi \simeq \psi'\psi : K \longrightarrow K''$.

Proof. (i) First we show that "$\simeq$" is an equivalence relation.

a) To show that $\varphi \simeq \varphi$ choose $h = 0$.

b) The statement $\varphi \simeq \psi \Rightarrow \psi \simeq \varphi$ follows from replacing h by $-h$.

c) If $\varphi \underset{h}{\simeq} \psi$ and $\psi \underset{k}{\simeq} \chi$ then $\varphi \underset{h+k}{\simeq} \chi$.

(ii) Assume that $\varphi \underset{h}{\simeq} \psi$ and $\varphi' \underset{h'}{\simeq} \psi'$. Now we calculate

$$\varphi'\varphi - \varphi'\psi = \varphi'(\partial' h + h\partial) = \partial\varphi' h + \varphi' h\partial\,,$$

hence $\varphi'\varphi \underset{\varphi' h}{\simeq} \varphi'\psi$. Similarly we see

$$\varphi'\psi - \psi'\psi = (\partial' h' + h'\partial)\psi = \partial' h'\psi + h'\psi\partial\,,$$

hence $\varphi'\psi \underset{h'\psi}{\simeq} \psi'\psi$. Combining both equivalences we get the desired result, namely

$$\varphi'\varphi \simeq \varphi'\psi \simeq \psi'\psi\,.$$

Homotopic chain maps induce the same homomorphisms on homology:

Proposition 3.59. *If $\varphi \simeq \psi : K \longrightarrow K'$ then $H_n(\varphi) = H_n(\psi) : H_nK \longrightarrow H_nK'$ for all n.*

Proof. Let $z \in Z_nK$. We compute

$$H_n(\varphi)[z] = [\varphi z] = [\psi z + \partial' hz + h\partial z] = [\psi z] = H_n(\psi)[z]$$

because $\partial z = 0$ and $[\partial' hz] = 0$. □

Definition 3.60. A chain map $\varphi : K \longrightarrow K'$ is called a *homotopy equivalence* if there exists a chain map $\psi : K' \longrightarrow K$ such that $\psi\varphi \simeq \mathrm{id}_K$ and $\varphi\psi \simeq \mathrm{id}_{K'}$. If such a homotopy equivalence exists we write $K \simeq K'$.

Corollary 3.61. *If $K \underset{\varphi}{\simeq} K'$ then $H_nK \underset{H_n(\varphi)}{\cong} H_nK'$ for all n.*

3.6. Proof of the homotopy axiom

We put $I := [0,1]$ and define the affine linear map

$$T_n^j : \Delta^{n+1} \to \Delta^n \times I$$

for $j = 0, \dots, n$ by

$$T_n^j(e_k) = \begin{cases} (e_k, 0), & k \leq j, \\ (e_{k-1}, 1), & k > j. \end{cases}$$

In the case $n = 0$ we have

$$T_0^0 : \Delta^1 \to \Delta^0 \times I = \{e_0\} \times I\,,$$

which is visualized in Figure 75.

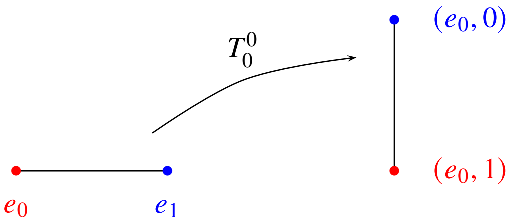


Figure 75. T_0^0

In the case $n = 1$ we find

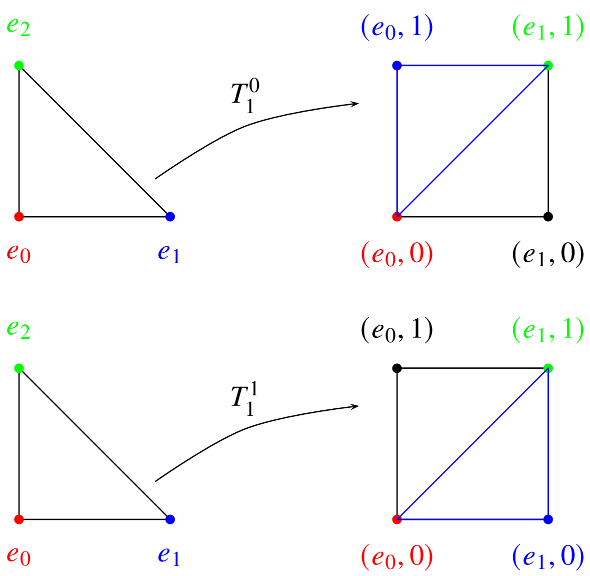


Figure 76. T_1^0 and T_0^1

Lemma 3.62. *The composition of the operators T and the affine linear map F (defined in* (3.1) *on page* 80*) yields:*

$$\begin{aligned} T_n^{j+1} \circ F_{n+1}^i &= (F_n^i \times \mathrm{id}) \circ T_{n-1}^j && (j \geq i), \\ T_n^j \circ F_{n+1}^{i+1} &= (F_n^i \times \mathrm{id}) \circ T_{n-1}^j && (j < i), \\ T_n^i \circ F_{n+1}^i &= T_n^{i-1} \circ F_{n+1}^i && (1 \leq i \leq n), \\ T_n^0 \circ F_{n+1}^0 &= i_1, \\ T_n^n \circ F_{n+1}^{n+1} &= i_0, \end{aligned}$$

with $i_t : \Delta^n \to \Delta^n \times I$ being the inclusion map $x \mapsto (x, t)$.

Proof. Let us prove the first formula where $j \geq i$. If $k < i$ then

$$T_n^{j+1} \circ F_{n+1}^i(e_k) = T_n^{j+1}(e_k) = (e_k, 0) = (F_n^i \times \mathrm{id})(e_k, 0) = (F_n^i \times \mathrm{id}) \circ T_{n-1}^j(e_k).$$

If $i \leq k \leq j$ then

$$T_n^{j+1} \circ F_{n+1}^i(e_k) = T_n^{j+1}(e_{k+1}) = (e_{k+1}, 0) = (F_n^i \times \mathrm{id})(e_k, 0) = (F_n^i \times \mathrm{id}) \circ T_{n-1}^j(e_k).$$

If $k > j$ then

$$T_n^{j+1} \circ F_{n+1}^i(e_k) = T_n^{j+1}(e_{k+1}) = (e_k, 1) = (F_n^i \times \mathrm{id})(e_{k-1}, 1) = (F_n^i \times \mathrm{id}) \circ T_{n-1}^j(e_k).$$

The proofs of the other formulas are similar exercises in index shifting. □

Now let $\sigma : \Delta^n \to X$ be a singular n-simplex. Note that $(\sigma \times \mathrm{id}) \circ T_n^j \in C(\Delta^{n+1}, X \times I)$. We define $P\sigma \in S_{n+1}(X \times I)$ by

$$P\sigma := \sum_{j=0}^{n} (-1)^j (\sigma \times \mathrm{id}) \circ T_n^j \,.$$

We extend P linearly to chains and get a linear map

$$P : S_n(X) \to S_{n+1}(X \times I).$$

This homomorphism descends to relative chains

$$P : S_n(X, A) \to S_{n+1}(X \times I, A \times I).$$

The operator P is called *prism operator*.

Lemma 3.63. *Let (X, A) be a pair of spaces. The operator P is a chain homotopy for*

$$S(i_0^X), S(i_1^X) : S(X, A) \to S(X \times I, A \times I),$$

where $i_t^X : X \to X \times I$ denotes the inclusion map $x \mapsto (x, t)$.

Proof. We have to show that

$$S(i_0^X) - S(i_1^X) = P\partial + \partial P. \tag{3.10}$$

Let $\sigma : \Delta^n \to X$ be a singular n-simplex. We compute, using Lemma 3.62:

$$\begin{aligned}
P\partial\sigma &= P\sum_{i=0}^{n}(-1)^i(\sigma \circ F_n^i) \\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j(\sigma \circ F_n^i \times \mathrm{id}) \circ T_{n-1}^j \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}(\sigma \times \mathrm{id}) \circ (F_n^i \times \mathrm{id}) \circ T_{n-1}^j \\
&\quad + \sum_{0\le i\le j\le n-1}(-1)^{i+j}(\sigma \times \mathrm{id}) \circ (F_n^i \times \mathrm{id}) \circ T_{n-1}^j \\
&= -\sum_{0\le j<i\le n}(-1)^{i+j+1}(\sigma \times \mathrm{id}) \circ T_n^j \circ F_{n+1}^{i+1} \\
&\quad - \sum_{0\le i\le j\le n-1}(-1)^{i+j+1}(\sigma \times \mathrm{id}) \circ T_n^{j+1} \circ F_{n+1}^i.
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
\partial P\sigma &= \partial\sum_{j=0}^{n}(-1)^j(\sigma \times \mathrm{id}) \circ T_n^j \\
&= \sum_{j=0}^{n}(-1)^j\sum_{i=0}^{n+1}(-1)^i(\sigma \times \mathrm{id}) \circ T_n^j \circ F_{n+1}^i \\
&= \sum_{0\le i<j\le n}(-1)^{i+j}(\sigma \times \mathrm{id}) \circ T_n^j \circ F_{n+1}^i \qquad (i<j) \\
&\quad + \sum_{i=0}^{n}(\sigma \times \mathrm{id}) \circ T_n^i \circ F_{n+1}^i \qquad (i=j) \\
&\quad - \sum_{i=1}^{n+1}(\sigma \times \mathrm{id}) \circ T_n^{i-1} \circ F_{n+1}^i \qquad (i=j+1) \\
&\quad + \sum_{1\le j+1<i\le n+1}(-1)^{i+j}(\sigma \times \mathrm{id}) \circ T_n^j \circ F_{n+1}^i \qquad (i>j+1)
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{0\le i\le j'\le n-1} (-1)^{i+j'+1}(\sigma\times\mathrm{id})\circ T_n^{j'+1}\circ F_{n+1}^{i} \\
& +(\sigma\times\mathrm{id})\circ i_1-(\sigma\times\mathrm{id})\circ i_0 \\
& +\sum_{0\le j<i'\le n} (-1)^{i'+j+1}(\sigma\times\mathrm{id})\circ T_n^{j}\circ F_{n+1}^{i'+1}
\end{aligned}
$$

In the last step we changed the summation indices to $j' = j-1$ and $i' = i-1$. We see that

$$
\begin{aligned}
P\partial\sigma+\partial P\sigma &= (\sigma\times\mathrm{id})\circ i_1-(\sigma\times\mathrm{id})\circ i_0 \\
&= i_1^X\circ\sigma-i_0^X\circ\sigma \\
&= S_n(i_1^X)\sigma-S_n(i_0^X)\sigma
\end{aligned}
$$

which proves the lemma. □

Proposition 3.64. *If $f\simeq g:(X,A)\to(Y,B)$ then we have*

$$S(f)\simeq S(g):S(X,A)\to S(Y,B)$$

Proof. Let F be a homotopy for f and g, i.e., $F\in C((X\times I,A\times I),(Y,B))$ with

$$f=F\circ i_1^X,\qquad g=F\circ i_0^X.$$

Then by (3.10) we get

$$S(f)-S(g)=S(F)S(i_1^X)-S(F)S(i_0^X)=S(F)P\partial+S(F)\partial P=S(F)P\partial+\partial S(F)P.$$

Hence $S(F)P$ is a chain homotopy for $S(f)$ and $S(g)$. □

Corollary 3.65. *The homotopy axiom holds for singular homology.*

3.7. Proof of the excision axiom

We want to show that the inclusion $(X\setminus U,A\setminus U)\hookrightarrow(X,A)$ induces an isomorphism

$$H_n(X\setminus U,A\setminus U)\cong H_n(X,A)$$

provided $\bar U\subset\mathring{A}$.

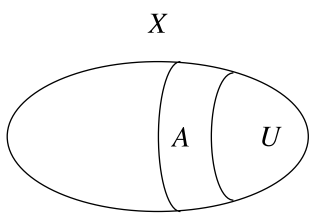


Figure 77. Setup for the excision axiom

Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. A chain $\sigma = \sum_j \alpha_j \sigma_j \in S_n(X)$ is called *$\mathcal{U}$-small* iff for each j there exists an i such that $\sigma_j(\Delta^n) \subset U_i$. We denote

$$\begin{aligned} S_n^{\mathcal{U}}(X) &:= \{\sigma \in S_n(X) \mid \sigma \text{ is } \mathcal{U}\text{-small}\} \\ &= \operatorname{im}\left(\bigoplus_{i\in I} S_n(U_i) \xrightarrow{\oplus_i S_n(j_i)} S_n(X)\right) \end{aligned}$$

with $j_i : U_i \to X$ the inclusion map. For $A \subset X$ we put

$$S_n^{\mathcal{U}}(X,A) := \frac{S_n^{\mathcal{U}}(X)}{S_n^{\mathcal{U}}(A)}.$$

Theorem 3.66 (Small chain theorem). *The inclusion $S_n^{\mathcal{U}}(X,A) \to S_n(X,A)$ induces an isomorphism in homology.*

Before coming to the proof we show that the small chain theorem implies the excision axiom. To this extent let (X,A) be a pair of spaces and let $U \subset A$ be such that $\bar{U} \subset \mathring{A}$. Now set $U_1 := \mathring{A}$ and $U_2 := X \setminus \bar{U}$. Then $\mathcal{U} = \{U_1, U_2\}$ forms an open cover of the space X. We compute

$$\begin{aligned} S_n^{\mathcal{U}}(X,A) &= \frac{S_n^{\mathcal{U}}(X)}{S_n^{\mathcal{U}}(A)} \\ &= \frac{S_n(\mathring{A}) + S_n(X\setminus\bar{U})}{S_n(\mathring{A}) + S_n(A\setminus\bar{U})} \\ &= \frac{S_n(X\setminus\bar{U})}{(S_n(\mathring{A}) + S_n(A\setminus\bar{U})) \cap S_n(X\setminus\bar{U})} \\ &= \frac{S_n(X\setminus\bar{U})}{S_n(A\setminus\bar{U})}. \end{aligned}$$

Similarly we get

$$S_n^{\mathcal{U}}(X\setminus U, A\setminus U) = \frac{S_n(\mathring{A}\setminus U) + S_n(X\setminus\bar{U})}{S_n(\mathring{A}\setminus U) + S_n(A\setminus\bar{U})} = \frac{S_n(X\setminus\bar{U})}{S_n(A\setminus\bar{U})}.$$

Thus

$$S_n^{\mathcal{U}}(X,A) = S_n^{\mathcal{U}}(X\setminus U, A\setminus U).$$

In the following commutative diagram all arrows are induced by inclusions.

$$\begin{tikzcd} S_m^{\mathcal{U}}(X\setminus U, A\setminus U) \arrow[r, "="] \arrow[d] & S_n^{\mathcal{U}}(X,A) \arrow[d] \\ S_m(X\setminus U, A\setminus U) \arrow[r] & S_n(X,A) \end{tikzcd}$$

By the small chain theorem the vertical arrows induce isomorphisms on homology and the excision theorem is proved.

It remains to prove Theorem 3.66. We define the *barycenter* of Δ^n by

$$B_n := \frac{1}{n+1}\sum_{j=0}^{n} e_j .$$

Example 3.67. For example,

$$\begin{aligned} B_0 &= e_0, \\ B_1 &= \frac{1}{2}(e_0+e_1), \\ B_2 &= \frac{1}{3}(e_0+e_1+e_2). \end{aligned}$$

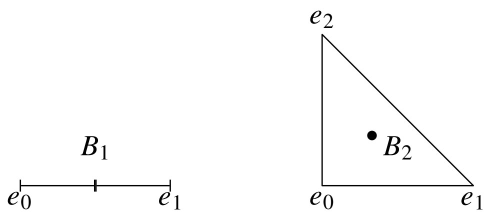


Figure 78. Barycenters in 1 and 2 dimensions

For an affine map $\sigma : \Delta^n \to \Delta^{n+1}$ we define the affine map $C_n\sigma : \Delta^{n+1} \to \Delta^{n+1}$ by

$$(C_n\sigma)(e_k) = \begin{cases} B_{n+1}, & k = 0, \\ \sigma(e_{k-1}), & k \geq 1. \end{cases}$$

Example 3.68. For the example $\sigma = F_2^0 : \Delta^1 \to \Delta^2$ see Figure 79.

Now we set

$$S_k^{\text{aff}}(\Delta^n) := \left\{\sigma \in S_k(\Delta^n) \,\middle|\, \sigma = \sum_j \alpha_j\sigma_j \text{ and each } \sigma_j \text{ is affine}\right\}.$$

By linear extension we get a homomorphism

$$C_n : S_n^{\text{aff}}(\Delta^{n+1}) \to S_{n+1}^{\text{aff}}(\Delta^{n+1}).$$

Lemma 3.69. *The homomorphism C_n has the following properties:*

(i) $\partial C_0(c) = c - (\sum_j \alpha_j)B_0$ *where* $c = \sum_j \alpha_j\sigma_j$;

(ii) $\partial C_n(c) = c - C_{n-1}\partial c$ *for* $n \geq 1$.

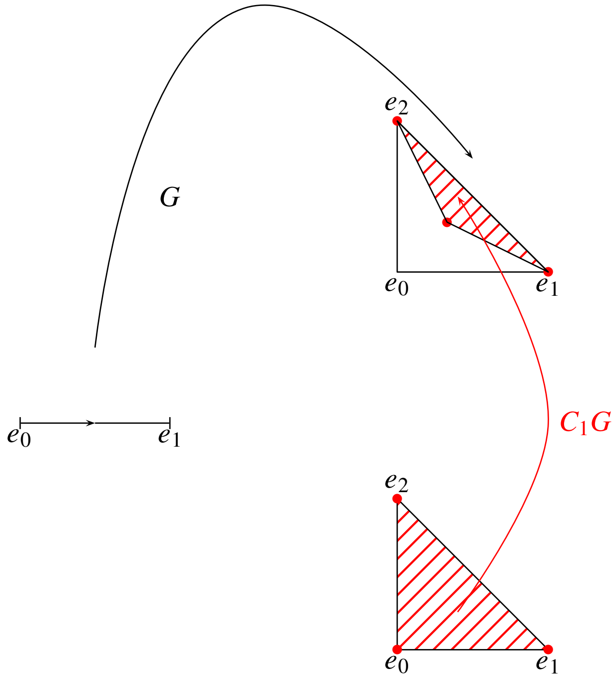


Figure 79. C_1

Proof. (i) It suffices to show the assertion for an affine simplex c. We then find

$$(C_0c)(e_0) = B_0, \qquad (C_0c)(e_1) = c(e_0)$$

and hence

$$\partial C_0(c)(e_0) = (C_0(c))(e_1) - (C_0(c))(e_0) = c(e_0) - B_0 = (c - 1B_0)(e_0)$$

as desired.

(ii) is left as an exercise. □

Lemma 3.70. *To each topological space X and each $n \in \mathbb{N}_0$ we can associate homomorphisms*

$$\begin{aligned} \mathrm{Sd}_n &: S_n(X) \longrightarrow S_n(X), \\ Q_n &: S_n(X) \longrightarrow S_{n+1}(X), \end{aligned}$$

such that

(i) Sd_* *is a chain map, i.e.,* $\partial \circ \mathrm{Sd}_n = \mathrm{Sd}_{n-1} \circ \partial$*;*

(ii) Q_* *is a chain homotopy between* id *and* Sd_**, i.e.,* $\mathrm{id} - \mathrm{Sd}_n = \partial \circ Q_n + Q_{n-1} \circ \partial$*;*

(iii) Sd_* *and* Q_* *are natural, i.e., for every* $f \in C(X, Y)$ *the following diagrams commute:*

$$\begin{tikzcd} S_n(X) \arrow[r, "\mathrm{Sd}_n"] \arrow[d, "S_n(f)"'] & S_n(X) \arrow[d, "S_n(f)"] \\ S_n(Y) \arrow[r, "\mathrm{Sd}_n"] & S_n(Y) \end{tikzcd} \qquad \begin{tikzcd} S_n(X) \arrow[r, "Q_n"] \arrow[d, "S_n(f)"'] & S_{n+1}(X) \arrow[d, "S_{n+1}(f)"] \\ S_n(Y) \arrow[r, "Q_n"] & S_{n+1}(Y) \end{tikzcd}$$

(iv) If the map $\sigma : \Delta^n \to \Delta^n$ *is affine then each simplex* σ_j *occuring in* $\mathrm{Sd}_n(\sigma)$ *or in* $Q_n(\sigma)$ *is affine and*

$$\mathrm{diam}(\sigma_j) \le \frac{n}{n+1}\,\mathrm{diam}(\sigma).$$

Proof. The construction of Sd_n and Q_n will be done recursively, the proof is by induction over n. We start by considering the case $n = 0$. We put $\mathrm{Sd}_0 := \mathrm{id} : S_0(X) \to S_0(X)$ and $Q_0 := 0$. It is obvious that the four assertions hold.
In the case $n \ge 1$ we assume that Sd_{n-1} and Q_{n-1} are already defined and we set for a singular simplex $\sigma : \Delta^n \to X$:

$$\mathrm{Sd}_n(\sigma) := S_n(\sigma)\Big(\underbrace{C_{n-1}\big(\underbrace{\mathrm{Sd}_{n-1}(\underbrace{\partial\, \underbrace{(\mathrm{id}_{\Delta^n})}_{\in S_n^{\mathrm{aff}}(\Delta^n)}}_{\in S_{n-1}^{\mathrm{aff}}(\Delta^n)})}_{\in S_{n-1}^{\mathrm{aff}}(\Delta^n)}\big)}_{\in S_n^{\mathrm{aff}}(\Delta^n)}\Big)$$

and

$$Q_n(\sigma) = S_{n+1}(\sigma)\big(\underbrace{C_n(\underbrace{\mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n}}_{\in S_n^{\mathrm{aff}}(\Delta^n)})}_{\in S_{n+1}^{\mathrm{aff}}(\Delta^n)}\big).$$

We have to verify assertions (i)-(iv). We check (iii):

$$\begin{aligned} \mathrm{Sd}_n(S_n(f)(\sigma)) &= S_n(S_n(f)(\sigma))(C_{n-1}(\mathrm{Sd}_{n-1}(\partial(\mathrm{id}_{\Delta^n})))) \\ &= S_n(f \circ \sigma)(C_{n-1}(\mathrm{Sd}_{n-1}(\partial(\mathrm{id}_{\Delta^n})))) \\ &= S_n(f) \circ S_n(\sigma)(C_{n-1}(\mathrm{Sd}_{n-1}(\partial(\mathrm{id}_{\Delta^n})))) \\ &= S_n(f)(\mathrm{Sd}_n(\sigma)). \end{aligned}$$

The computation for Q_n is similar.

Next we check (i): First we consider the case that $X = \Delta^n$ and $\sigma = \mathrm{id}_{\Delta^n}$.

$$\begin{aligned}
\partial(\mathrm{Sd}_n(\mathrm{id}_{\Delta^n})) &= \partial(C_{n-1}(\mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n}))) \\
&\overset{\text{L. 3.69}}{=} \mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) - C_{n-2}\partial(\mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \\
&\overset{\text{ind. hyp.}}{=} \mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) - C_{n-2}\mathrm{Sd}_{n-2}\underbrace{\partial\partial}_{=0}\mathrm{id}_{\Delta^n} \\
&= \mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n}).
\end{aligned}$$

For a general simplex $\sigma : \Delta^n \to X$ we then find

$$\begin{aligned}
\partial(\mathrm{Sd}_n(\sigma)) &= \partial(S_n(\sigma)(\mathrm{Sd}_n(\mathrm{id}_{\Delta^n}))) \\
&= S_n(\sigma)(\partial(\mathrm{Sd}_n(\mathrm{id}_{\Delta^n}))) \\
&= S_n(\sigma)(\mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \\
&\overset{(iii)}{=} \mathrm{Sd}_{n-1}(S_n(\sigma)(\partial(\mathrm{id}_{\Delta^n}))) \\
&= \mathrm{Sd}_{n-1}(\partial S_n(\sigma)(\mathrm{id}_{\Delta^n})) \\
&= \mathrm{Sd}_{n-1}(\partial\sigma).
\end{aligned}$$

Now we check (ii): Again we first consider the case $X = \Delta^n$ and $\sigma = \mathrm{id}_{\Delta^n}$.

$$\begin{aligned}
\partial Q(\mathrm{id}_{\Delta^n}) &= \partial C_n(\mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n}) \\
&\overset{\text{L. 3.69}}{=} \mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n} \\
&\qquad - C_{n-1}\partial\Big(\mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n}\Big) \\
&\overset{\text{ind. hyp.}}{=} \mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n} \\
&\qquad - C_{n-1}\Big(\partial\,\mathrm{id}_{\Delta^n} - \partial\mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - (\partial\,\mathrm{id}_{\Delta^n} - \mathrm{Sd}_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \\
&\qquad\qquad + Q_{n-2}(\underbrace{\partial\partial}_{=0}\mathrm{id}_{\Delta^n}))\Big) \\
&\overset{(i)}{=} \mathrm{id}_{\Delta^n} - \mathrm{Sd}_n(\mathrm{id}_{\Delta^n}) - Q_{n-1}\partial\,\mathrm{id}_{\Delta^n}\,.
\end{aligned}$$

The passage to general σ can now be done as before.
Finally we check (iv): It is clear from the recursive definition of Sd_n and of Q_n that each simplex σ_j occuring in $\mathrm{Sd}_n(\sigma)$ or in $Q_n(\sigma)$ is again affine. The diameter of an affine simplex is the maximal distance of any two of its vertices. We distinguish two cases:

1. The vertices p, q of σ_j of maximal distance lie on $\partial\sigma$.

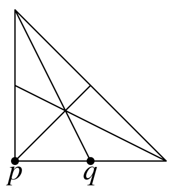


Figure 80. Vertices of maximal distance

Then we find by the induction hypothesis

$$d(p,q) \quad \leq \quad \frac{n-1}{n}\,\mathrm{diam}(\text{face of }\sigma) \quad < \frac{n}{n+1}\,\mathrm{diam}(\sigma)\,.$$

2. One vertex is B_n:

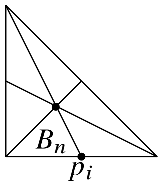


Figure 81. Barycenter is a vertex

Then we find

$$\begin{aligned} d(p_i,B_n) &= \left| p_i - \frac{1}{n+1}\sum_{j=0}^{n} p_j \right| \\ &= \left| \frac{1}{n+1}\sum_{j=0}^{n}(p_i - p_j) \right| \\ &\leq \frac{1}{n+1}\sum_{j=0}^{n} \qquad \underbrace{|p_i - p_j|}_{\leq \mathrm{diam}(\sigma) \text{ and } =0 \text{ for } j=i} \\ &\leq \frac{n}{n+1}\,\mathrm{diam}(\sigma)\,. \end{aligned}$$

□

Conclusion 3.71. *We have that* $\mathrm{Sd} \simeq \mathrm{id}$ *and therefore*

$$\begin{aligned} \mathrm{Sd} - \mathrm{Sd}^2 &= \mathrm{Sd}\circ\partial\circ Q + \mathrm{Sd}\circ Q\circ\partial \\ &= \partial\circ\mathrm{Sd}\circ Q + \mathrm{Sd}\circ Q\circ\partial. \end{aligned}$$

We conclude that $\mathrm{Sd}^2 \simeq \mathrm{Sd} \simeq \mathrm{id}$. *Iterating this procedure we find that* $\mathrm{Sd}^r \simeq \mathrm{id}$ *for all* $r \in \mathbb{N}$. *Hence there exist homomorphisms* $Q_n^{(r)} : S_n(X) \longrightarrow S_{n+1}(X)$ *with*

$$\mathrm{Sd}_n^r - \mathrm{id} = \partial\circ Q_n^{(r)} + Q_{n-1}^{(r)}\circ\partial\,.$$

For σ *affine, we have for every* σ_j *occuring in* $\mathrm{Sd}^r(\sigma)$ *that*

$$\mathrm{diam}(\sigma_j) \leq \left(\frac{n}{n+1}\right)^r \mathrm{diam}(\sigma)$$

Lemma 3.72. *For $\sigma : \Delta^n \to X$ continuous there exists a $\varepsilon > 0$ such that all ε-balls $\cap \Delta^n$ are completely contained in $\sigma^{-1}(U_i)$ for some $U_i \in \mathcal{U}$.*

Proof. Assume the assertion were false. Then for $\varepsilon_k = 1/k$ there exists a point $p_k \in \Delta^n$ such that

$$B_{\frac{1}{k}}(p_k) = \{x \in \Delta^n \mid |x - p_k| < \varepsilon_k\}$$

is not contained in any of the $\sigma^{-1}(U_i)$. After passing to a subsequence we have that $p_k \to p \in \Delta^n$ by compactness of Δ^n. Choose i_0 with $p \in \sigma^{-1}(U_{i_0})$. Since $\sigma^{-1}(U_{i_0})$ is open there exists a $\delta > 0$ such that $B_\delta(p) \subset \sigma^{-1}(U_{i_0})$. Now choose k so large, that $|p_k - p| < \delta/2$ and $\varepsilon_k = \frac{1}{k} < \delta/2$. We then find

$$B_{\frac{1}{k}}(p_k) \subset B_\delta(p) \subset \sigma^{-1}(U_{i_0}),$$

a contradiction. □

Corollary 3.73. *Assume that $\sigma : \Delta^n \to X$ is continuous. Then there exists an $r(\sigma) \in \mathbb{N}$ such that every simplex σ_j occuring in $\mathrm{Sd}^r(\sigma)$ or in $Q^{(r)}(\sigma)$ for $r \geq r(\sigma)$ is completely contained in one of the U_i, i.e., $\mathrm{Sd}^r(\sigma), Q^{(r)}(\sigma) \in S_n^{\mathcal{U}}(X)$.*

Finally we can prove the small chain theorem.

Proof of Theorem 3.66. a) First we consider the case $A = \emptyset$.

i) We show injectivity: Assume that $z \in Z_n^{\mathcal{U}}(X)$ with $H_n(j)([z]_{H_n^{\mathcal{U}}}) = 0$. Then there exists an $x \in S_{n+1}(X)$ with $\partial x = z$. Now we calculate

$$\begin{aligned} \partial \underbrace{\mathrm{Sd}^r x}_{\in S_{n+1}^{\mathcal{U}}(X) \text{ for large } r} &= \mathrm{Sd}^r \partial x \\ &= \mathrm{Sd}^r z \\ &= z - \partial Q^{(r)} z - Q^{(r)} \underbrace{\partial z}_{=0}. \end{aligned}$$

Hence

$$z = \partial(\underbrace{\mathrm{Sd}^r x + Q^{(r)} z}_{\in S_n^{\mathcal{U}}(X) \text{ for large } r})$$

and therefore $[z]_{H_n^{\mathcal{U}}} = 0$.

ii) We show surjectivity: Let $[z] \in H_n(X)$ be given. We know that $\mathrm{Sd}^r z \in S_n^{\mathcal{U}}(X)$ for r large enough. We compute

$$\underbrace{[\mathrm{Sd}_n^r z]}_{\in H_n(j)(H_n^{\mathcal{U}}(X))} = [z - \partial Q_n^{(r)} z - Q_{n-1}^{(r)} \underbrace{\partial z}_{=0}] = [z].$$

b) Now we pass to general (X, A). Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
H_n^{\mathcal{U}}(A) \arrow[r] \arrow[d, "\cong"'] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\cong"'] & H_n^{\mathcal{U}}(X,A) \arrow[r] \arrow[d] & H_{n-1}^{\mathcal{U}}(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}^{\mathcal{U}}(X) \arrow[d, "\cong"] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

By part a) of the proof we know that the outer four arrows are isomorphisms. The Five Lemma (Exercise 3.11) implies that the map $H_n^{\mathcal{U}}(X, A) \to H_n(X, A)$ is also an isomorphism. □

3.8. The Mayer-Vietoris sequence

Let X be a topological space and let $A \subset X$ be a subset. Assume that $U_1, U_2 \subset X$ are open with $U_1 \cup U_2 = X$, hence $\mathcal{U} = \{U_1, U_2\}$ forms an open cover of X. Consider the exact sequence of chain complexes

$$0 \longrightarrow S_n(U_1 \cap U_2) \xrightarrow{\begin{pmatrix} S_n(i_1) \\ -S_n(i_2) \end{pmatrix}} S_n(U_1) \oplus S_n(U_2) \xrightarrow{(S_n(j_1), S_n(j_2))} S_n^{\mathcal{U}}(X) \longrightarrow 0$$

with the inclusion maps $i_\nu : U_1 \cap U_2 \to U_\nu$ and $j_\nu : U_\nu \to X$. We then get the following long exact homology sequence:

$$\cdots \to H_n(U_1 \cap U_2) \xrightarrow{\begin{pmatrix} H_n(i_1) \\ -H_n(i_2) \end{pmatrix}} H_n(U_1) \oplus H_n(U_2) \xrightarrow{(H_n(j_1), H_n(j_2))} H_n^{\mathcal{U}}(X) \xrightarrow{\partial} H_{n-1}(U_1 \cap U_2) \to \cdots$$

By the small chain theorem $H_n^{\mathcal{U}}(X)$ is canonically isomorphic to $H_n(X)$. Using this isomorphism we can replace $H_n^{\mathcal{U}}(X)$ in the above exact homology sequence by $H_n(X)$.

$$\begin{tikzcd}
\cdots \to H_n(U_1) \oplus H_n(U_2) \arrow[r, "{(H_n(j_1), H_n(j_2))}"] \arrow[dr, "{(H_n(j_1), H_n(j_2))}"'] & H_n^{\mathcal{U}}(X) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(U_1 \cap U_2) \to \cdots \\
& H_n(X) \arrow[ur, "\partial^{MV}"'] &
\end{tikzcd}$$

The same reasoning applies to relative homology and we obtain

Theorem 3.74 (Mayer-Vietoris sequence). *Let X be a topological space, let $A \subset X$ and let $U_1, U_2 \subset X$ be open such that $U_1 \cup U_2 = X$. Set $A_\nu := A \cap U_\nu$ and let*

$$\begin{aligned} i_\nu :& (U_1 \cap U_2, A_1 \cap A_2) \to (U_\nu, A_\nu), \\ j_\nu :& (U_\nu, A_\nu) \to (X, A), \end{aligned}$$

be the inclusion maps, $\nu = 1, 2$. Then the following sequence is exact and natural

$$\begin{tikzcd} \cdots \arrow[r, "\partial^{MV}"] & H_n(U_1 \cap U_2, A_1 \cap A_2) \arrow[r, "\begin{pmatrix} H_n(i_1) \\ -H_n(i_2) \end{pmatrix}"] & H_n(U_1, A_1) \oplus H_n(U_2, A_2) \arrow[dll, "(H_n(j_1), H_n(j_2))"] \\ H_n(X, A) \arrow[r, "\partial^{MV}"'] & H_{n-1}(U_1 \cap U_2, A_1 \cap A_2) \arrow[r] & \cdots \end{tikzcd}$$

Example 3.75. We give a new computation of the homology of S^1. To this extent we cover the circle as indicated in the picture.

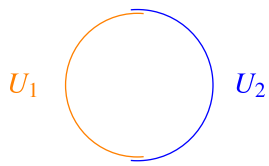


Figure 82. Open cover of S^1

We directly see that

$$U_1 \approx U_2 \approx (0, 1) \simeq \{p\}$$

and

$$U_1 \cap U_2 \approx (0, 1) \sqcup (0, 1) \simeq \{p_1, p_2\}$$

Consider the following part of the Mayer-Vietoris sequence.

$$\begin{tikzcd} H_n(U_1 \cap U_2) = H_n(\{p_1, p_2\}) \arrow[r] & H_n(U_1) \oplus H_n(U_2) = H_n(\{p\}) \oplus H_n(\{p\}) \arrow[r] & H_n(S^1) \arrow[dll, "\partial^{MV}"] \\ H_{n-1}(U_1 \cap U_2) = H_{n-1}(\{p_1, p_2\}) \arrow[r] & H_{n-1}(U_1) \oplus H_{n-1}(U_2) = H_{n-1}(\{p\}) \oplus H_{n-1}(\{p\}) & \end{tikzcd}$$

If $n \geq 2$ then all homologies of the point occuring in this diagram vanish. Hence $H_n(S^1) = 0$ for all $n \geq 2$. Since S^1 is path-connected we have that $H_0(S^1) \cong R$. In the case $n = 1$ we find

$$\begin{tikzcd} 0 \arrow[r] & H_1(S^1) \arrow[r] & R^2 \arrow[r, "\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)"] & R^2 \end{tikzcd}$$

To compute $H_1(S^1)$ we calculate

$$H_1(S^1) \cong \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} -x \\ x \end{pmatrix} \middle| x \in R \right\} \cong R.$$

Example 3.76. Now we consider the space $X = G_2$ as given in the picture.

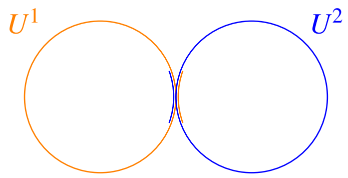


Figure 83. Open cover of figure 8

It is easy to see that $U_1 \approx U_2 \simeq S^1$ and $U_1 \cap U_2 \simeq \{p\}$. Since G_2 is path-connected we find $H_0(G_2) \cong R$.

$$H_n(U_1 \cap U_2) \to \underbrace{H_n(U_1) \oplus H_n(U_2)}_{\cong H_n(S^1) \oplus H_n(S^1)} \to H_n(G_2) \to \underbrace{H_{n-1}(U_1 \cap U_2)}_{\cong H_{n-1}(\{p\})}$$

In the case of $n \geq 2$ we find the exact sequence

$$0 \longrightarrow H_n(G_2) \longrightarrow 0$$

and hence $H_n(G_2) = 0$. For $n = 1$ we find

$$\underbrace{H_1(\{p\})}_{=0} \to \underbrace{H_1(S^1) \oplus H_1(S^1)}_{\cong R^2} \to H_1(G_2) \to \underbrace{H_0(\{p\})}_{\cong R} \to \underbrace{H_0(S^1) \oplus H_0(S^1)}_{\cong R^2}.$$

The last map in this diagram is given by

$$\begin{tikzcd} H_0(\{p\}) \cong R \arrow[r] \arrow[r, bend right, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"'] & H_0(S^1) \oplus H_0(S^1) \cong R^2 \end{tikzcd}$$

and hence injective. Consequently we find the isomorphism

$$H_1(G_2) \cong H_1(U_1) \oplus H_1(U_2) \cong R^2.$$

3.9. Generalized Jordan curve theorem

In this section we will use the Mayer-Vietoris sequence to prove the Jordan curve theorem in arbitrary dimensions.

Lemma 3.77. *Let X be a topological space and let $U_i \subset X$ be open with $U_i \subset U_{i+1}$ and $\cup_{i\in\mathbb{N}} U_i = X$. Furthermore let $\iota_n : U_n \hookrightarrow X$ and $\iota_{n,m} : U_n \hookrightarrow U_m$ for $m \geq n$ be the inclusion maps. Then we have:*

(i) For each $\alpha \in H_k(X;R)$ there exists an n_0 such that $\alpha \in \operatorname{im}(H_k(\iota_n))$ for all $n \geq n_0$.

(ii) For each $\alpha_n \in H_k(U_n;R)$ with $H_k(\iota_n)(\alpha_n) = 0$ there exists an m_0 such that $H_k(\iota_{n,m})(\alpha_n) = 0$ for all $m \geq m_0$.

Proof. (i) Let $\alpha \in H_k(X)$. We represent α by

$$\sum_{j=1}^{l} \alpha_j \sigma_j \in Z_k(X;R), \qquad \alpha_j \in R, \quad \sigma_j \in C(\Delta^k, X).$$

Note that $\sigma_j(\Delta^k) \subset X$ is a compact subset and therefore $C := \bigcup_{j=1}^{l} \sigma_j(\Delta^k) \subset X$ is compact. Hence there exists an n_0 such that for all $n \geq n_0$ we have $C \subset U_n$. We conclude that

$$\sum_{j=1}^{l} \alpha_j \sigma_j \in Z_k(U_n;R)$$

for all $n \geq n_0$ and therefore

$$\alpha = \Big[\sum \alpha_j \sigma_j\Big]_{H_k(X)} = H_n(\iota_n)\left(\Big[\sum \alpha_j \sigma_j\Big]_{H_k(U_n)}\right).$$

(ii) Again represent $\alpha_n \in H_k(U_n)$ by $\sum_{j=1}^{l} \alpha_j \sigma_j$. From $H_n(\iota_n)(\alpha_n) = 0$ we know that there exists a $\beta \in C_{k+1}(X;R)$ such that $\sum_{j=1}^{l} \alpha_j \sigma_j = \partial\beta$. As before there exists a compact subset $C' \subset X$ such that $\beta \in C_{k+1}(C';R)$. Since there exists an m_0 with $C' \subset U_m$ for all $m \geq m_0$ and thus $\beta \in C_{k+1}(U_m;R)$ we have that $H_k(\iota_{n,m})(\alpha_n) = 0$. □

By an *embedding* we mean a continuous map $f : X \to Y$ which is a homeomorphism onto its image. In other words, f is continuous, open and injective. In the considerations which follow the domain will be compact and the target will be Hausdorff so that f is automatically open.

Proposition 3.78. *Let $n \in \mathbb{N}$ and let Y be a compact topological space such that for any embedding $f : Y \to S^n$*

$$H_*(S^n \setminus f(Y);R) \cong H_*(\text{point};R).$$

Then the space $[0,1] \times Y$ also has this property.

Proof. Let $f : [0,1] \times Y \to S^n$ be an embedding. Suppose $0 \neq \alpha \in H_i(S^n \setminus f([0,1] \times Y))$. Put

$$U_0 := S^n \setminus \underbrace{f([0, \tfrac{1}{2}] \times Y)}_{\text{compact}} \quad \text{and} \quad U_1 := S^n \setminus \underbrace{f([\tfrac{1}{2}, 1] \times Y)}_{\text{compact}}.$$

Both U_0 and U_1 are open. We also find

$$U_0 \cap U_1 = S^n \setminus f([0,1] \times Y) \quad \text{and} \quad U_0 \cup U_1 = S^n \setminus f(\underbrace{\{\tfrac{1}{2}\} \times Y}_{\approx Y}).$$

The Mayer-Vietoris sequence yields:

$$\begin{tikzcd} 0 = H_{i+1}(S^n \setminus f(\{\tfrac{1}{2}\} \times Y)) \arrow[r] & H_i(S^n \setminus f([0,1] \times Y)) \arrow[d] \\ & H_i(S^n \setminus f([0,\tfrac{1}{2}] \times Y)) \oplus H_i(S^n \setminus f([\tfrac{1}{2},1] \times Y)) \end{tikzcd}$$

Thus the inclusions $S^n \setminus f([0,1] \times Y) \hookrightarrow S^n \setminus f([0,\tfrac{1}{2}] \times Y)$ and $S^n \setminus f([0,1] \times Y) \hookrightarrow S^n \setminus f([\tfrac{1}{2},1] \times Y)$ induce an injective homomorphism

$$H_i(S^n \setminus f([0,1] \times Y)) \to H_i(S^n \setminus f([0,\tfrac{1}{2}] \times Y) \oplus H_i(S^n \setminus f([\tfrac{1}{2},1] \times Y)).$$

Hence

$$\begin{aligned} 0 &\neq H_i(\text{inclusion})(\alpha) \in H_i(S^n \setminus f([0,\tfrac{1}{2}] \times Y)) \text{ or} \\ 0 &\neq H_i(\text{inclusion})(\alpha) \in H_i(S^n \setminus f([\tfrac{1}{2},1] \times Y)). \end{aligned}$$

By iterating this procedure we obtain a sequence of intervals I_k such that

$$I_0 = [0,1], \qquad I_{k+1} \subset I_k, \qquad |I_k| = 2^{-k}$$

and

$$0 \neq H_i(\iota_{0,k})(\alpha) \in H_i(S^n \setminus f(I_k \times Y))$$

for all k. Here $V_k := S^n \setminus f(I_k \times Y)$ is open in S^n and $\iota_{k,l} : V_k \hookrightarrow V_l$ for $l \geq k$ denotes the inclusion map. We find

$$\begin{aligned} \cap_{k\in\mathbb{N}} I_k &= \{t\} \\ \cup_{k\in\mathbb{N}} V_k &= S^n \setminus f(\{t\} \times Y) =: X \end{aligned}$$

Now we apply the previous Lemma 3.77 for the inclusion $\iota : V_0 \hookrightarrow X$. Hence, for $i \geq 1$,

$$0 \neq H_i(\iota)(\alpha) \in H_i(X) = H_i(S^n \setminus f(\underbrace{\{t\} \times Y}_{\approx Y})) = 0$$

giving a contradiction. The proof for $i = 0$ is similar (or in fact the same if one uses augmented homology). $\square$

Corollary 3.79. *If $f : D^r \to S^n$ is an embedding then*

$$H_*(S^n \setminus f(D^r)) \cong H_*(\{\text{point}\})$$

Proof. The proof is by induction on r. For $r = 0$ we find

$$S^n \setminus f(D^0) \approx \mathbb{R}^n \simeq \{\text{point}\}$$

and hence $H_*(S^n \setminus f(D^r)) \cong H_*(\{\text{point}\})$.
For the induction step "$r-1 \Rightarrow r$" we observe $D^r \approx W^r = [0,1] \times W^{r-1} \approx [0,1] \times D^{r-1}$ and hence Proposition 3.78 applies. □

Theorem 3.80. *Let $r < n$ and let $f : S^r \to S^n$ be an embedding. Then*

$$H_*(S^n \setminus f(S^r)) \cong H_*(S^{n-r-1}).$$

Proof. Again the proof is done by induction on r. For $r = 0$ we find

$$S^n \setminus f(S^0) = S^n \setminus \{p, q\} \approx \mathbb{R}^n \setminus \{0\} \simeq S^{n-1}.$$

For the induction step "$r-1 \Rightarrow r$" we write $S^r = D^r_+ \cup D^r_-$. Then $D^r_+ \cap D^r_- = S^{r-1}$. We put $U_+ := S^n \setminus f(D^r_+)$ and $U_- := S^n \setminus f(D^r_-)$. Both sets U_+, U_- are open because $f(D^r_\pm)$ is compact. In addition

$$U_+ \cap U_- = S^n \setminus f(S^r) \quad \text{and} \quad U_+ \cup U_- = S^n \setminus f(S^{r-1}).$$

Now we look at the Mayer-Vietoris sequence and use Corollary 3.79:

$$\begin{tikzcd}
\underbrace{H_{i+1}(U_+) \oplus H_{i+1}(U_-)}_{\cong H_{i+1}(\text{point}) \oplus H_{i+1}(\text{point}) = 0} \arrow[r] & H_{i+1}(S^n \setminus f(S^{r-1})) \arrow[r] & H_i(S^n \setminus f(S^r)) \arrow[d] \\
& & \underbrace{H_i(U_+) \oplus H_i(U_-)}_{\substack{\cong H_i(\text{point}) \oplus H_i(\text{point}) = 0 \\ \text{if } i \geq 1}}
\end{tikzcd}$$

Thus for $i \geq 1$ (and by similar reasoning also for $i = 0$) we get an isomorphism

$$H_i(S^n \setminus f(S^r)) \cong H_{i+1}(S^n \setminus f(S^{r-1})) \cong H_{i+1}(S^{n-r}) \cong H_i(S^{n-r-1}).$$

This proves the theorem. □

Theorem 3.81 (Jordan-Brouwer separation theorem). *For any embedding $f : S^{n-1} \to S^n$ the complement $S^n \setminus f(S^{n-1})$ consists of exactly two path-components U and V. Both U and V are open in S^n and $\partial U = \partial V = f(S^{n-1})$.*

Proof. a) We know that $H_0(S^n \setminus f(S^{n-1})) \cong H_0(S^0) \cong R^2$, hence $S^n \setminus f(S^{n-1})$ has exactly two path-components U and V.

b) Since $f(S^{n-1})$ is compact the union $U \cup V = S^n \setminus f(S^{n-1})$ is open and therefore both connected components U and V are open.

c) Since U and V are open they contain no boundary point of U, thus $\partial U \subset f(S^{n-1})$. Assume that there exists $p \in S^{n-1}$ with $f(p) \notin \partial U$. Then there exists $W \subset S^n$ open with $f(p) \in W$ and $W \cap U = \emptyset$. Since f is continuous we can choose an open ball $B \subset S^{n-1}$ with $f(B) \subset W$. Since $S^{n-1} \setminus B \approx D^{n-1}$ we find that the space $Y := S^n \setminus f(S^{n-1} \setminus B)$ is path-connected because of Corollary 3.79:

$$H_0(Y) = H_0(S^n \setminus f(S^{n-1} \setminus B)) \cong H_0(\text{point}) \cong R\,.$$

Moreover, we have

$$Y = U \cup V \cup f(B) \subset U \cup V \cup W \quad \text{and} \quad U \cap (V \cup W) = \emptyset$$

and hence

$$Y = \underbrace{(Y \cap U)}_{U} \sqcup \underbrace{(Y \cap (V \cup W))}_{\supset V}$$

can be written as a disjoint union of open non-empty subsets. This contradicts Y being path-connected. □

Corollary 3.82 (Generalized Jordan curve theorem). *For every embedding $f : S^{n-1} \to \mathbb{R}^n$ the complement $\mathbb{R}^n \setminus f(S^{n-1})$ consists of exactly two path components U and V. Both U and V are open, U is bounded, V is unbounded and $\partial U = \partial V = f(S^{n-1})$.*

Proof. We use the stereographic projection to identify $\mathbb{R}^n$ with a subset of S^n,

$$S^n = \mathbb{R}^n \cup \{\infty\}, \qquad f : S^{n-1} \to \mathbb{R}^n \subset S^n.$$

By the Jordan-Brouwer separation theorem the two path-components $\tilde{U}$ and $\tilde{V}$ of $S^n \setminus f(S^{n-1})$ are both open and $\partial\tilde{U} = \partial\tilde{V} = f(S^{n-1})$. Let $\tilde{V}$ be the component containing ∞. Then $U = \tilde{U}$ and $V = \tilde{V} \setminus \{\infty\}$ are the two path-components of $\mathbb{R}^n \setminus f(S^{n-1})$. Clearly, V is unbounded and $\partial U = \partial\tilde{U} = \partial V = \partial\tilde{V} = f(S^{n-1})$. If U were unbounded then $\infty \in \partial U$ which is not the case. Hence U is bounded. □

Remark 3.83. Consider an embedding $f : S^{n-1} \to S^n$. If $i \geq 1$ we find for the homology of the components of the complement

$$H(U \sqcup V) = H_i(U) \oplus H_i(V) \cong H_i(S^0) = 0$$

and hence U and V have the same homology as a point. This makes us suspect that $U, V \approx \mathring{D}^n$. For $n = 2$ this is indeed true, but it is false for $n \geq 3$. The *Alexander horned sphere* is an example of an embedding of S^2 into S^3 where one component of the complement is not even simply connected:

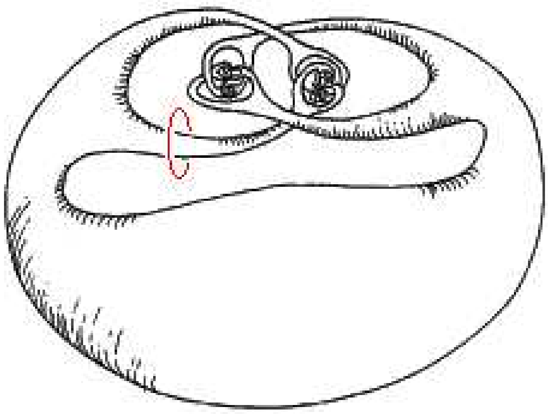


Figure 84. Alexander horned sphere[2]

The red circle in the picture is a non-contractible loop giving rise to a non-trivial element in the fundamental group. A very nice video illustrating this embedding of S^2 can be found at http://www.youtube.com/watch?v=d1Vjsm9pQlc.

3.10. CW-complexes

We now describe a type of topological spaces for which there is a particularly efficient way to compute their homology. These space are obtained by gluing together balls of various dimensions.

Definition 3.84. A *finite CW-complex* is a pair $(X, \mathcal{X})$ where X is a Hausdorff space, $\mathcal{X} = \coprod_{n \in \mathbb{N}_0} \mathcal{X}_n$, $\mathcal{X}_n \subset \mathcal{P}(X)$ and $|\mathcal{X}| < \infty$ with the following properties:

(i) $X = \coprod_{\sigma \in \mathcal{X}} \sigma$.

(ii) Set $X^n := \cup_{\substack{\sigma \in \mathcal{X}_m \\ m \leq n}} \sigma \subset X$. For every $\sigma \in \mathcal{X}_n$ we have $\bar{\sigma} \setminus \sigma \subset X^{n-1}$.

(iii) For every $\sigma \in \mathcal{X}_n$ there exists a surjective continuous map

$$\varphi_\sigma : D^n \to \bar{\sigma} \subset X$$

[2]Taken from https://matteocapucci.wordpress.com/2019/02/05/you-wont-believe-what-this-space-is-homeomorphic-to

such that $\varphi_\sigma|_{\mathring{D}^n} : \mathring{D}^n \to \sigma$ is a homeomorphism.

Definition 3.85. An element $\sigma \in \mathcal{X}_n$ is called an *n-cell*. The map φ_σ is called the *characteristic map* of σ and X^n is called the *n-skeleton* of $(X, \mathcal{X})$. The map $\varphi_\sigma|_{S^{n-1}=\partial D^n} : S^{n-1} \to X^{n-1}$ is called the *attaching map* of σ.

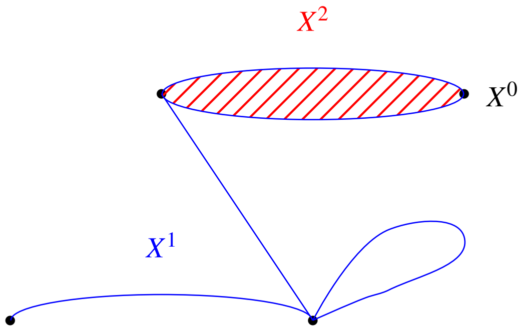


Figure 85. A 2-dimensional CW-complex

Example 3.86. Consider $X = S^n$ for $n \geq 1$. Then the choice

$$\begin{aligned} \mathcal{X}_0 &= \{\{e_1\}\}, \\ \mathcal{X}_n &= \{\sigma_n = S^n \setminus \{e_1\}\}, \\ \mathcal{X}_m &= \emptyset, \text{ otherwise,} \end{aligned}$$

turns the n-sphere into a CW-complex.

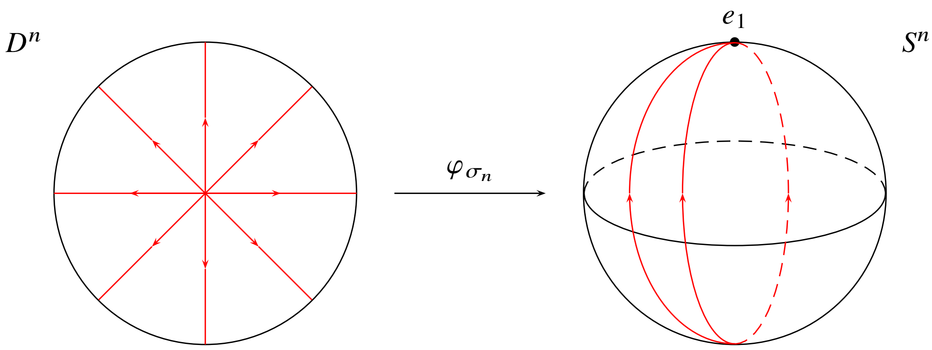


Figure 86. Attaching an n-cell to a point to obtain an n-sphere

The attaching map to the n-cell is the constant map $S^{n-1} \to \{e_1\}$. We have

$$\begin{aligned} &X^0 = X^1 = X^2 = \ldots = X^{n-1} = \{e_1\}, \\ &S^n = X^n = X^{n+1} = \ldots \end{aligned}$$

Example 3.87. If a space X has the structure of a CW-complex there are in general many different ways to write X as a CW-complex, i.e., there are many different $\mathcal{X}$ for the same X. Let us look again at $X = S^n$. We start with the case $n = 0$. Here the CW-structure is unique:

$$\mathcal{X}_m = \begin{cases} \{\{e_1\}, \{-e_1\}\}, & m = 0 \\ \emptyset, & m > 0\,. \end{cases}$$

For $n > 0$ we use that $S^{n-1} \subset S^n$ and define recursively

$$\mathcal{X}_m^{S^n} := \begin{cases} \mathcal{X}_m^{S^{n-1}}, & m \le n-1 \\ \{\mathring{D}_+^n, \mathring{D}_-^n\}, & m = n \\ \emptyset, & m > n \end{cases}$$

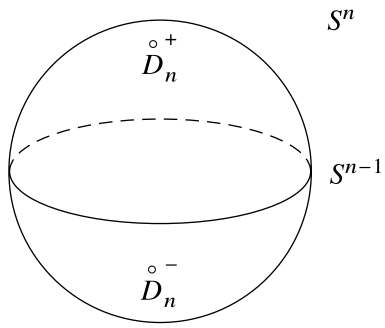


Figure 87. Cell decomposition of S^n with two n-cells

Therefore we have in this case

$$X^0 = S^0,\ X^1 = S^1, \dots,\ X^n = S^n.$$

Example 3.88. *Real projective space* $\mathbb{RP}^n$. We define the *real projective space* as

$$\mathbb{RP}^n = \{\text{1-dimensional real vector subspace of } \mathbb{R}^{n+1}\} = \mathbb{R}^{n+1} \setminus \{0\}/\!\sim$$

where

$$x = (x_0, \dots, x_n) \sim y = (y_0, \dots, y_n)$$

iff there exists a $t \neq 0$ such that $x = ty$. We consider the canonical projection map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n, \quad x \mapsto [x_0, \dots, x_n].$$

The restriction $\psi := \pi\big|_{S^n} : S^n \to \mathbb{RP}^n$ is continuous and surjective. Thus $\mathbb{RP}^n$ is compact. Clearly $\psi(x) = \psi(y)$ iff $x = \pm y$. We set

$$\sigma_k := \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_{k+1} = \dots = x_n = 0,\ x_k \neq 0\}\,.$$

Then

$$\bar{\sigma}_k = \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_{k+1} = \dots = x_n = 0\} \approx \mathbb{RP}^k.$$

For the characteristic map $\varphi_k : D^k \to \bar{\sigma}_k$ we can take

$$\xi \mapsto [\xi_1, \dots, \xi_k, \sqrt{1 - |\xi|^2}, 0, \dots, 0]\,.$$

It is clear that the map φ_k is continuous. Now we check that it is also surjective. Let $[x', x_k, 0] \in \bar{\sigma}_k$ where $x' = (x_0, \dots, x_{k-1})$. Without loss of generality we assume that $x_k \geq 0$. Set $\xi := \frac{x'}{\sqrt{|x'|^2 + |x_k|^2}} \in D^k$. We compute

$$\begin{aligned}
\varphi_k(\xi) &= \left[\frac{x'}{\sqrt{|x'|^2 + |x_k|^2}}, \sqrt{1 - \frac{|x'|^2}{|x'|^2 + |x_k|^2}}, 0\right] \\
&= \left[\frac{x'}{\sqrt{|x'|^2 + |x_k|^2}}, \sqrt{\frac{|x_k|^2}{|x'|^2 + |x_k|^2}}, 0\right] \\
&= [x', |x_k|, 0] \\
&= [x', x_k, 0]\,.
\end{aligned}$$

Next we show that $\varphi_k|_{\mathring{D}^k}$ is injective. Let $\varphi_k(\xi) = \varphi_k(\eta)$ for $\xi, \eta \in \mathring{D}^k$. This leads to the two equations:

$$\xi = t\eta \quad \text{and} \quad \sqrt{1 - |\xi|^2} = t\sqrt{1 - |\eta|^2}$$

for some $t \neq 0$. Squaring and adding both equations we find

$$|\xi|^2 + 1 - |\xi|^2 = t^2|\eta|^2 + t^2(1 - |\eta|^2)$$

leading to $t^2 = 1$ and consequently $t = \pm 1$. Since $|\xi|, |\eta| < 1$ it follows that $\sqrt{1 - |\xi|^2} > 0$ and $\sqrt{1 - |\eta|^2} > 0$ and therefore $t > 0$. Hence $t = 1$ and thus $\xi = \eta$.
The map $\varphi_k : D^k \to \bar{\sigma}_k$ is closed, therefore the restriction

$$\varphi_k\big|_{\mathring{D}_k} : \mathring{D} \to \sigma_k$$

is bijective, continuous and closed, hence a homeomorphism. We find:

$$X^0 = \{\text{point}\} \subset \underbrace{X^1}_{\approx \mathbb{RP}^1} \subset \underbrace{X^2}_{\approx \mathbb{RP}^2} \subset \dots \subset \underbrace{X^n}_{= \mathbb{RP}^n}.$$

Finally let us discuss the gluing map. For $\xi \in \partial D^k = S^{k-1}$, i.e. $|\xi| = 1$, the gluing map is given by $\varphi_k(\xi) = [\xi, 0, 0]$ and hence $\varphi_k = \psi : S^{k-1} \to X^{k-1} \approx \mathbb{RP}^{k-1}$.

Example 3.89. *Complex projective space* $\mathbb{CP}^n$. For the *complex projective space* the discussion is analogous to the real case with complex parameters instead of real parameters,

$$\mathbb{CP}^n = \{\text{1-dimensional complex vector subspace of } \mathbb{C}^{n+1}\} = \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

with $x \sim y$ iff $x = ty$ for some $t \in \mathbb{C}$, $t \neq 0$. We find

$$|\mathcal{X}_m| = \begin{cases} 1, & m \text{ even and } 0 \leq m \leq 2n \\ 0, & \text{otherwise} \end{cases}$$

and

$$X^0 = \{\text{point}\} = X^1 \subset \underbrace{X^2}_{\approx\mathbb{CP}^1} = X^3 \subset \underbrace{X^4}_{\approx\mathbb{CP}^2} = X^5 \subset \cdots \subset X^{2n-1} = \underbrace{X^{2n}}_{=\mathbb{CP}^n}.$$

Example 3.90. *Quaternionic projective space* $\mathbb{HP}^n$. Similarly, for the *quaternionic projective space*

$$\mathbb{HP}^n = \{\text{1-dimensional quaternionic vector subspace of } \mathbb{H}^{n+1}\} = \mathbb{H}^{n+1} \setminus \{0\}/\sim$$

with $x \sim y$ iff $x = ty$ for some $t \in \mathbb{H}$, $t \neq 0$, we find that

$$|\mathcal{X}_m| = \begin{cases} 1, & m \text{ divisible by 4 and } 0 \leq m \leq 4n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$X^0 = \{\text{point}\} = X^1 = X^2 = X^3 \subset \underbrace{X^4}_{\approx\mathbb{HP}^1} = \ldots \subset X^{4n-3} = \cdots = \underbrace{X^{4n}}_{=\mathbb{HP}^n}.$$

Remark 3.91. Every compact differentiable manifold can be triangulated and is consequently a finite CW-complex.

3.11. Homology of CW-complexes

Throughout this section we assume that $(X, \mathcal{X})$ is a finite CW-complex. Our goal is to prove Theorem 3.100 which will provide us with an efficient way to compute the homology of X.

Lemma 3.92. *The map*

$$\bigoplus_{\sigma \in \mathcal{X}_n} H_i(D^n, S^{n-1}) \xrightarrow{\oplus_{\sigma\in\mathcal{X}_n} H_i(\varphi_\sigma)} H_i(X^n, X^{n-1})$$

is an isomorphism.

Once we have this lemma, Theorem 3.16 implies

Corollary 3.93. *We have the isomorphisms*

$$H_i(X^n, X^{n-1}) \cong \begin{cases} R^{|\mathcal{X}_n|}, & \text{for } i = n \\ 0, & \text{otherwise} \end{cases}$$

Proof of Lemma 3.92. We set $\dot{D}^n := D^n \setminus \{0\}$ and

$$\dot{X}^n := X^{n-1} \cup \bigcup_{\sigma \in \mathcal{X}_n} \varphi_\sigma(\dot{D}^n) = X^n \setminus \{\varphi_\sigma(0) \mid \sigma \in \mathcal{X}_n\}.$$

The inclusion $S^{n-1} \hookrightarrow \dot{D}^n$ is a homotopy equivalence with homotopy inverse $x \mapsto \frac{x}{|x|}$.

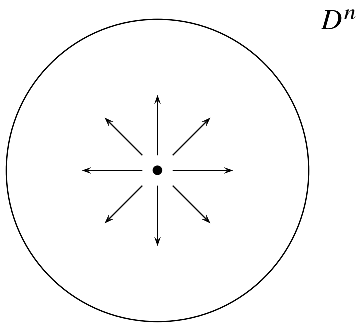


Figure 88. Punctured n-disk is homotopy equivalent to S^{n-1}

We define

$$Y^n := \coprod_{\sigma \in \mathcal{X}_n} D^n, \qquad Y^{n-1} := \coprod_{\sigma \in \mathcal{X}_n} S^{n-1}, \qquad \dot{Y}^n := \coprod_{\sigma \in \mathcal{X}_n} \dot{D}^n$$

The inclusions yield homotopy equivalences $Y^{n-1} \to \dot{Y}^n$ and $X^{n-1} \to \dot{X}^n$.

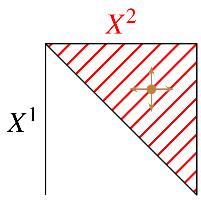


Figure 89. Punctured n-skeleton is homotopy equivalent to $(n-1)$-skeleton

Now consider:

$$(Y^n, Y^{n-1}) \hookrightarrow (Y^n, \dot{Y}^n) \xrightarrow{\Phi := \cup_{\sigma \in \mathcal{X}_n} \varphi_\sigma} (X^n, \dot{X}^n) \hookleftarrow (X^n, X^{n-1})$$

Both inclusions induce isomorphisms on H_i. Due to the diagram

$$\begin{tikzcd}
H_i(Y^{n-1}) \arrow[r] \arrow[d, "\cong"] & H_i(Y^n) \arrow[r] \arrow[d, "="] & H_i(Y^n, Y^{n-1}) \arrow[r] \arrow[d] & H_{i-1}(Y^{n-1}) \arrow[r] \arrow[d, "\cong"] & H_{i-1}(Y^n) \arrow[d, "="] \\
H_i(\dot{Y}^n) \arrow[r] & H_i(Y^n) \arrow[r] & H_i(Y^n, \dot{Y}^n) \arrow[r] & H_{i-1}(\dot{Y}^n) \arrow[r] & H_{i-1}(Y^n)
\end{tikzcd}$$

and the Five Lemma the map $H_i(Y^n, Y^{n-1}) \to H_i(Y^n, \dot{Y}^n)$ is also an isomorphism. Similarly, we get that the inclusion $(X^n, X^{n-1}) \hookrightarrow (X^n, \dot{X}^n)$ induces an isomorphism $H_i(X^n, X^{n-1}) \to H_i(X^n, \dot{X}^n)$. The inclusions

$$\begin{aligned}
(Y^n \setminus Y^{n-1}, \dot{Y}^n \setminus Y^{n-1}) &\hookrightarrow (Y^n, \dot{Y}^n), \\
(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1}) &\hookrightarrow (X^n, \dot{X}^n),
\end{aligned}$$

induce isomorphisms on homology by the excision axiom. In addition, we have

$$(Y^n \setminus Y^{n-1}, \dot{Y}^n \setminus Y^{n-1}) \approx (X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1})$$

Hence we find

$$\begin{aligned}
\bigoplus_{\sigma \in \mathcal{X}_n} H_i(D^n, S^{n-1}) &\cong H_i(Y^n, Y^{n-1}) \\
&\cong H_i(Y^n, \dot{Y}^n) \\
&\cong H_i(Y^n \setminus Y^{n-1}, \dot{Y}^n \setminus Y^{n-1}) \\
&\cong H_i(X^n \setminus X^{n-1}, \dot{X}^n \setminus X^{n-1}) \\
&\cong H_i(X^n, \dot{X}^n) \\
&\cong H_i(X^n, X^{n-1}) \, .
\end{aligned}$$

□

Set $K_n(X, \mathcal{X}) := H_n(X^n, X^{n-1}) \cong R^{|\mathcal{X}_n|}$. We define a homomorphism

$$\partial_n : K_n(X, \mathcal{X}) \to K_{n-1}(X, \mathcal{X})$$

by the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[drr, "\partial_n"] & H_{n-1}(X^{n-1}) \arrow[r] \arrow[d, "="] & H_{n-1}(X^n) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_{n-1}(X^{n-2}) \arrow[r] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 3.94. *The sequence of groups $K_n(X, \mathcal{X})$ together with ∂_n forms a complex.*

The pair $(K_*(X, \mathcal{X}), \partial_*)$ is called the *cellular complex* of $(X, \mathcal{X})$.

Proof. The diagram

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[drr, bend left=10, "\partial_n"] & H_{n-1}(X^{n-1}) \arrow[d, "="] & & & \\
& H_{n-1}(X^{n-1}) \arrow[r] \arrow[rr, bend right, "=0"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\partial"] \arrow[drr, bend left=10, "\partial_{n-1}"] & H_{n-1}(X^{n-2}) \arrow[d, "="] & \\
& & & H_{n-1}(X^{n-2}) \arrow[r] & H_{n-2}(X^{n-2}, X^{n-3})
\end{tikzcd}$$

shows $\partial_{n-1} \circ \partial_n = 0$ which is the claim. □

Lemma 3.95. *For $p \geq q \geq n$ or $n > p \geq q$ we have $H_n(X^p, X^q) = 0$.*

Proof. The proof is by induction on $p - q$. The assertion is certainly true for $p - q = 0$. To analyze the situation $p - q > 0$ we look at the exact homology sequence for the triple (X^p, X^{q+1}, X^q):

$$H_n(X^{q+1}, X^q) \longrightarrow H_n(X^p, X^q) \longrightarrow H_n(X^p, X^{q+1}) \stackrel{\text{ind. hyp.}}{=} 0.$$

Since either $q \geq n$ or $q < p < n$ we have $n \neq q + 1$. Thus $H_n(X^{q+1}, X^q) = 0$ by Lemma 3.92 and hence $H_n(X^p, X^q) = 0$. □

Corollary 3.96. *For $n > p$ we have that $H_n(X^p) = 0$.*

Proof. Lemma 3.95 with $q = 0$ says $H_n(X^p, X^0) = 0$. The claim now follows from the exact sequence

$$0 = H_n(X^0) \longrightarrow H_n(X^p) \longrightarrow H_n(X^p, X^0) = 0.$$

Corollary 3.97. *For $q \geq n$ we have $H_n(X, X^q) = 0$.*

Proof. Choose $p \geq q$ so large that $X^p = X$ and use Lemma 3.95. □

Corollary 3.98. *For $r > n$ the inclusion $X^r \hookrightarrow X$ induces an isomorphism*

$$H_n(X) \cong H_n(X^r).$$

Proof. The assertion follows from Corollary 3.97 and the exact sequence

$$0 = H_{n+1}(X, X^r) \longrightarrow H_n(X^r) \longrightarrow H_n(X) \longrightarrow H_n(X, X^r) = 0.$$

Lemma 3.99. *For $r > n$ and $r \geq q$ the inclusion induces an isomorphism*

$$H_n(X, X^q) \cong H_n(X^r, X^q)\,.$$

Proof. Since $r \geq n+1$, Corollary 3.97 gives us $H_{n+1}(X, X^r) = H_n(X, X^r) = 0$. The assertion now follows from the exact homology sequence of the triple (X, X^r, X^q):

$$H_{n+1}(X, X^r) \longrightarrow H_n(X^r, X^q) \longrightarrow H_n(X, X^q) \longrightarrow H_n(X, X^r).$$

Theorem 3.100. *We have the following isomorphism:*

$$H_n K_*(X, \mathcal{X}) \cong H_n(X)$$

Proof. Consider the commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"'] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-2}) = 0 \arrow[d] \\
0 = H_n(X^{n-1}) \arrow[r] & H_n(X^n) \arrow[r, "i_*"] \arrow[d] & H_n(X^n, X^{n-1}) \arrow[r] \arrow[dr, "\partial_{n-1}"] & H_{n-1}(X^{n-1}) \arrow[d] \\
 & H_n(X^{n+1}) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^{n+1}, X^n) = 0 & &
\end{tikzcd}$$

Now we compute

$$\begin{aligned} H_n(X) &\cong H_n(X^{n+1}) \\ &\cong \frac{H_n(X^n)}{\partial H_{n+1}(X^{n+1}, X^n)} \end{aligned}$$

$$
\begin{aligned}
&\cong \frac{i_* H_n(X^n)}{i_* \partial H_{n+1}(X^{n+1}, X^n)} \\
&\cong \frac{\ker(H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}))}{\partial_n H_{n+1}(X^{n+1}, X^n)} \\
&\cong \frac{\ker(\partial_{n-1})}{\partial_n H_{n+1}(X^{n+1}, X^n)} \\
&= H_n K_*(X, \mathcal{X})
\end{aligned}
$$

and the theorem is proved. □

In the following examples we set $\alpha_n := |\mathcal{X}_n|$.

Example 3.101. We consider $X = S^n$ for $n \geq 2$. The CW-decomposition of S^n from Example 3.86 has

$$\alpha_0 = \alpha_n = 1, \qquad \alpha_j = 0 \text{ otherwise.}$$

Hence we have to compute the homology of the complex

$$R \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow R \longleftarrow 0 \longleftarrow \cdots$$

Since all arrows are 0 the homology coincides with the complex, hence

$$H_j(S^n) \cong \begin{cases} R, & j = 0 \text{ or } n, \\ 0, & \text{otherwise,} \end{cases}$$

which confirms our earlier findings.

Example 3.102. Now look at $X = \mathbb{CP}^n$. Then we have for the CW-decomposition from Example 3.89

$$\alpha_0 = \alpha_2 = \ldots = \alpha_{2n} = 1, \qquad \alpha_j = 0 \text{ otherwise.}$$

Again all arrows in the complex

$$R \longleftarrow 0 \longleftarrow \cdots \longleftarrow 0 \longleftarrow R \longleftarrow 0 \longleftarrow \cdots$$

must be zero, hence the homology is given by

$$H_j(\mathbb{CP}^n) \cong \begin{cases} R, & j = 0, 2, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.103. Consider $X = \mathbb{HP}^n$. For the CW-decomposition from Example 3.90 we have

$$\alpha_0 = \alpha_4 = \alpha_8 = \ldots = \alpha_{4n} = 1, \qquad \alpha_j = 0 \text{ otherwise}$$

and all arrows in

$$0 \longleftarrow R \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow R \longleftarrow \cdots \longleftarrow 0 \longleftarrow R \longleftarrow 0 \longleftarrow \cdots$$

must be zero. We find for the homology

$$H_j(\mathbb{HP}^n) \cong \begin{cases} R, & j = 0, 4, 8, \dots, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

3.12. Betti numbers and the Euler number

Throughout this section let R be a field.

Definition 3.104. The dimension $b_j(X; R) := \dim_R H_j(X; R)$ is called j-th *Betti number* of the space X (over the field R).

Now let $(X, \mathcal{X})$ be a finite CW-complex. Denote the number of j-cells in $(X, \mathcal{X})$ by α_j. Then we get the following estimate for the Betti numbers:

$$\begin{aligned} b_j(X; R) &= \dim_R H_j(X; R) \\ &= \dim_R \frac{\ker(\partial_j : K_j(X, \mathcal{X}) \to K_{j-1}(X, \mathcal{X}))}{\operatorname{im}(\partial_{j+1} : K_{j+1}(X, \mathcal{X}) \to K_j(X, \mathcal{X}))} \\ &\leq \dim_R \ker(\partial_j : K_j(X, \mathcal{X}) \to K_{j-1}(X, \mathcal{X})) \\ &\leq \dim_R K_j(X, \mathcal{X}) \\ &= \alpha_j. \end{aligned}$$

We conclude that $b_j(X; R) \leq \alpha_j$. In particular, the Betti numbers are finite, $b_j(X; R) < \infty$.

Definition 3.105. For a finite CW-complex $(X, \mathcal{X})$ we call

$$\chi(X, \mathcal{X}) = \sum_{i=0}^{\infty} (-1)^i \alpha_i$$

the *Euler number* or *Euler-Poincaré characteristic*.

Note that the sum in this definition is finite. It ends at the highest dimension occuring in the cell decomposition.

Proposition 3.106. *We have the following relation between Euler and Betti numbers:*

$$\chi(X, \mathcal{X}) = \sum_{i=0}^{\infty} (-1)^i \, b_i(X; R).$$

In particular, the Euler number does not depend on the cell decomposition because the Betti numbers don't. On the other hand, the Euler number does not depend on the coefficient field because the α_i's don't. We will henceforth write $\chi(X)$ instead of $\chi(X, \mathcal{X})$.
In order to prove the proposition we use the following

Lemma 3.107. *Let*

$$0 \longleftarrow V_0 \xleftarrow{d_1} V_1 \xleftarrow{d_2} \cdots \xleftarrow{d_n} V_n \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

be a complex of finite-dimensional R-vector spaces. Then

$$\sum_{j=0}^{\infty}(-1)^j \dim H_j V_* = \sum_{j=0}^{\infty}(-1)^j \dim V_j\,.$$

Proof of Lemma 3.107. To show the lemma we compute, using the dimension formula from linear algebra,

$$\begin{aligned}\sum_j(-1)^j \dim V_j &= \sum_j(-1)^j(\dim(d_jV_j) + \dim\ker(d_j))\\ &= \sum_j(-1)^j(\dim\ker(d_j) - \dim(d_{j+1}V_{j+1}))\\ &= \sum_j(-1)^j \dim\left(\frac{\ker(d_j)}{d_{j+1}V_{j+1}}\right)\\ &= \sum_j(-1)^j \dim H_jV_*\,.\end{aligned}$$ □

Proof of Proposition 3.106. This follows from Lemma 3.107 with $V_j = K_j(X, \mathcal{X}) \cong R^{\alpha_j}$. □

Example 3.108. For $X = S^n$ we have

$$b_0 = b_n = 1, \qquad b_j = 0 \text{ otherwise}$$

and therefore

$$\chi(S^n) = \begin{cases} 2, & n \text{ even} \\ 0, & n \text{ odd}\,. \end{cases}$$

The case $n = 2$ contains Euler's classical formula for polyhedra as a special case. It says that for the alternating sum of the number of vertices, edges and faces of a polyhedron is always equal to 2. In particular, for platonic solids we have the following list:

	α_0	α_1	α_2
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

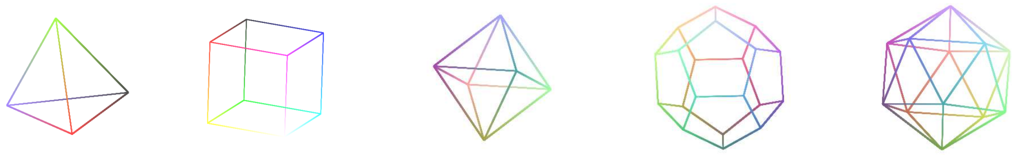


Figure 90. Platonic solids

Examples 3.109. We compute the Euler numbers of the projective spaces.

1.) For $X = \mathbb{CP}^n$ we have $b_0 = b_2 = \ldots = b_{2n} = 1$ and $b_j = 0$ otherwise. Hence $\chi(\mathbb{CP}^n) = n+1$.

2.) For $X = \mathbb{HP}^n$ we have $b_0 = b_4 = \ldots = b_{4n} = 1$ and $b_j = 0$ otherwise. Hence $\chi(\mathbb{HP}^n) = n+1$.

3.) In the case of $X = \mathbb{RP}^n$ we do not know the Betti numbers yet. So we use Proposition 3.106 to compute the Euler number. For the cell decomposition described in Example 3.88 we have

$$\alpha_0 = \ldots = \alpha_n = 1, \qquad \alpha_j = 0 \text{ otherwise.}$$

Hence

$$\chi(\mathbb{RP}^n) = \begin{cases} 1, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

3.13. Incidence numbers

We return to a general commutative ring R with unit 1. Throughout this section let $(X, \mathcal{X})$ be a finite CW-complex. In order to compute the cellular homology of $(X, \mathcal{X})$ we need a better understanding of the homomorphism $\partial_{n+1} : K_{n+1}(X, \mathcal{X}) \to K_n(X, \mathcal{X})$. In the case of $n = 0$ we find

$$\begin{tikzcd}
K_1(X,\mathcal{X}) \arrow[r, "\partial_1"] \arrow[d, equal] & K_0(X,\mathcal{X}) \arrow[d, equal] \\
H_1(X^1, X^0) \arrow[r, "\partial"] \arrow[d, equal] & H_0(X^0) \arrow[d, "\cong"] \\
\bigoplus_{\tau \in \mathcal{X}_1} H_1(\varphi_\tau)(H_1(D^1, S^0)) & \bigoplus_{\sigma \in \mathcal{X}_0} R
\end{tikzcd}$$

Hence ∂_1 is given by the $(\alpha_0 \times \alpha_1)$-matrix (∂^τ_σ) where $\tau \in \mathcal{X}_1$ and $\sigma \in \mathcal{X}_0$. The entries $\partial^\tau_\sigma \in R$ of this matrix are easily computed. Namely, recall that a generator of $H_1(D^1, S^0)$ is represented by $c : \Delta^1 = [0,1] \to D^1 = [-1,1]$ with $c(t) = 2t - 1$. Then

$$\partial_1 H_1(\varphi_\tau)([c]) = \partial[\varphi_\tau \circ c] = \varphi_\tau(c(1)) - \varphi_\tau(c(0)) = \varphi_\tau(1) - \varphi_\tau(-1).$$

Thus if $\varphi_\tau(-1) = \varphi_\tau(1)$ then $\partial^\tau_\sigma = 0$ for all σ. If $\varphi_\tau(-1) \neq \varphi_\tau(1)$ then

$$\partial^\tau_\sigma = \begin{cases} 1, & \text{for } \sigma = \varphi_\tau(1), \\ -1, & \text{for } \sigma = \varphi_\tau(-1), \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.110. We compute the homology of the following CW-complex consisting of two 0-cells and three 1-cells:

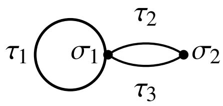


Figure 91. A cell decomposition of the figure 8

Clearly, $\partial^{\tau_1}_{\sigma_1} = \partial^{\tau_1}_{\sigma_2} = 0$. Depending on how the characteristic maps parametrize the 1-cells τ_2 and τ_3 we get

$$\partial^{\tau_2}_{\sigma_1} = \partial^{\tau_3}_{\sigma_1} = 1 \quad \text{and} \quad \partial^{\tau_2}_{\sigma_2} = \partial^{\tau_3}_{\sigma_2} = -1,$$

or possibly different signs which will not affect the homology however. Thus the cellular complex is

$$\cdots \longleftarrow 0 \longleftarrow R^2 \xleftarrow{\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}} R^3 \longleftarrow 0 \longleftarrow \cdots$$

The image of the matrix is $\{(x, -x) \mid x \in R\} = R \cdot (1, -1)$ and hence

$$H_0(X; R) \cong R^2 / R \cdot (1, -1) \cong R$$

where the latter isomorphism is induced by $R^2 \to R$, $(x, y) \mapsto x + y$. Moreover,

$$H_1(X; R) \cong \ker \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \{(x, y, -y) \mid x, y \in R\} \cong R^2.$$

We summarize

$$H_j(X; R) \cong \begin{cases} R & \text{if } j = 0, \\ R^2 & \text{if } j = 1, \\ 0 & \text{else.} \end{cases}$$

Returning to our general CW-complex $(X, \mathcal{X})$ we find for $n \geq 1$:

$$
\begin{tikzcd}
K_{n+1}(X,\mathcal{X}) \arrow[rr, "\partial_{n+1}"] \arrow[d, equal] & & K_n(X,\mathcal{X}) \arrow[d, equal] \\
H_{n+1}(X^{n+1}, X^n) \arrow[d, equal] & & H_n(X^n, X^{n-1}) \arrow[d, equal] \\
\bigoplus_{\tau \in \mathcal{X}_{n+1}} H_{n+1}(\varphi_\tau)(H_{n+1}(D^{n+1}, S^n)) \arrow[d, "\cong"] & & \bigoplus_{\sigma \in \mathcal{X}_n} H_n(\varphi_\sigma)(H_n(D^n, S^{n-1})) \arrow[d, "\cong"] \\
R^{\alpha_{n+1}} \arrow[rr, "(\partial^\tau_\sigma)"] & & R^{\alpha_n}
\end{tikzcd}
$$

Hence ∂_{n+1} is given by the $(\alpha_n \times \alpha_{n+1})$-matrix (∂^τ_σ) where $\tau \in \mathcal{X}_{n+1}$ and $\sigma \in \mathcal{X}_n$. The entries $(\partial^\tau_\sigma) \in R$ of this matrix are called the *incidence numbers*. We want to see how we can compute them.
Fix $\sigma \in \mathcal{X}_n$, $\tau \in \mathcal{X}_{n+1}$, and $p \in \mathring{D}^n$. From the commutative diagram

$$
\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[rr, bend left, "\partial_{n+1}"] & H_n(X^n) \arrow[r] & H_n(X^n, X^{n-1}) \arrow[r, "\mathrm{pr}_\sigma"] & H_n(\varphi_\sigma)H_n(D^n, S^{n-1}) \\
& & & H_n(D^n, S^{n-1}) \arrow[u, "H_n(\varphi_\sigma)", "\cong"'] \\
H_{n+1}(D^{n+1}, S^n) \arrow[uu, "H_{n+1}(\varphi_\tau)"] \arrow[r, "\partial", "\cong"'] & H_n(S^n) \arrow[uu, "H_n(\varphi_\tau|_{S^n})"] \arrow[rr, "\deg_p(\varphi_\sigma^{-1} \circ \varphi_\tau : \varphi_\tau^{-1}(\sigma) \to D^n \subset S^n)"] \arrow[d] & & H_n(S^n) \arrow[u, "\cong"] \\
& H_n(S^n, S^n \setminus \varphi_\tau^{-1}(\varphi_\sigma(p))) & & H_n(\varphi_\tau^{-1}(\sigma), \varphi_\tau^{-1}(\sigma) \setminus \varphi_\tau^{-1}(\varphi_\sigma(p))) \arrow[u, "H_n(\varphi_\sigma^{-1} \circ \varphi_\tau)"] \arrow[ll, "\cong"]
\end{tikzcd}
$$

we conclude that

$$
\partial^\tau_\sigma = \deg_p(\varphi_\sigma^{-1} \circ \varphi_\tau|_{\varphi_\tau^{-1}(\sigma)} : \varphi_\tau^{-1}(\sigma) \to D^n \subset S^n).
$$

We used the canonical isomorphism $H_n(S^n) \cong H_n(D^n, S^{n-1})$. We have intepreted the incidence numbers as certain local mapping degrees. In applications they can often be computed by counting preimages.

Example 3.111. Look at $X = \mathbb{RP}^n$. We want to compute the homology of

$$
0 \longleftarrow R \xleftarrow{\partial_1} R \xleftarrow{\partial_2} R \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_n} R \longleftarrow 0 \longleftarrow \cdots
$$

The operator ∂_{j+1} is given by the (1×1)-matrix (∂^τ_σ) with $\sigma \in \mathcal{X}_j$ and $\tau \in \mathcal{X}_{j+1}$. The gluing map $\varphi_\tau|_{S^j} : S^j \to X^j \approx \mathbb{RP}^j$ is given by the canonical projection. The image point $\varphi_\sigma(p) \in X^j$ has exactly two preimages under φ_τ which we call $x, -x \in S^j$. Put $\Phi := \varphi_\sigma^{-1} \circ \varphi_\tau : \varphi_\tau^{-1}(\sigma) \to D^j$. From the additivity of the local degree we obtain

$$\deg_p(\Phi : \varphi_\tau^{-1}(\sigma) \to D^j) = \deg_p(\Phi|_{U(x)} : U(x) \to D^j) + \deg_p(\Phi|_{U(-x)} : U(-x) \to D^j)$$

where $U(x)$ is a small neighborhood of x. W.l.o.g. we assume $U(-x) = -U(x)$. Since $\Phi|_{U(x)}$ is a homeomorphism onto its image we have

$$\deg_p(\Phi|_{U(x)} : U(x) \to D^j) = \pm 1 =: \varepsilon\,.$$

Let $a : S^j \to S^j$ be the antipodal map. Then $\Phi = \Phi \circ a$ and hence

$$\begin{aligned}
\deg_p(\Phi|_{U(-x)} : U(-x) \to D^j) &= \deg_p(\Phi \circ a : U(-x) \to D^j) \\
&= \deg_p(\Phi : U(x) \to D^n) \cdot \deg(a) \\
&= \varepsilon \cdot (-1)^{j+1}.
\end{aligned}$$

We conclude that $\partial_{j+1} = \varepsilon \cdot (1 + (-1)^{j+1})$. For n even we obtain the complex

$$0 \longleftarrow R \xleftarrow{0} R \xleftarrow{\pm 2} \cdots \xleftarrow{0} R \xleftarrow{\pm 2} R \longleftarrow 0$$

whereas for n odd we find

$$0 \longleftarrow R \xleftarrow{0} R \longleftarrow \cdots \xleftarrow{\pm 2} R \xleftarrow{0} R \longleftarrow 0.$$

For $R = \mathbb{Z}/2\mathbb{Z}$ all arrows are zero so that

$$H_j(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & j = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

For $R = \mathbb{Z}$ the homology is computed to

$$H_j(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, \\ \mathbb{Z}/2\mathbb{Z}, & j = 1, 3, n-1 \text{ or } n-2 \text{ resp.}, \\ \mathbb{Z}, & j = n \text{ odd}, \\ 0, & j = n \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

For $R = \mathbb{Q}$ we find

$$H_j(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & j = 0 \text{ or } j = n \text{ odd}, \\ 0, & \text{else.} \end{cases}$$

3.14. Homotopy versus homology

The final goal of this chapter is to compare homotopy groups and homology groups. We start by examining the fundamental group of CW-complexes.

Proposition 3.112. *Let $(X, \mathcal{X})$ be a finite CW-complex and let $x_0 \in X^0$. Then the inclusion map $j : X^2 \hookrightarrow X$ induces an isomorphism*

$$j_\# : \pi_1(X^2; x_0) \longrightarrow \pi_1(X; x_0)\,.$$

Proof. We have to show that attaching a k−cell for $k \geq 3$ does not alter π_1 in the sense that the inclusion induces an isomorphism on π_1. We assume that X^2 is path-connected, since otherwise we may replace X^2 by the path-component that contains x_0.
Let Y be a path-connected finite CW-complex and let $\tilde{Y}$ be obtained from Y by attaching a k-cell. More precisely, $\tilde{Y} = Y \cup_\varphi D^k = Y \sqcup D^k/\!\sim$ where $x \sim \varphi(x)$ for all $x \in S^{k-1}$. Here $\varphi : S^{k-1} \longrightarrow Y$ is a continuous map. We have to show that the inclusion map induces an isomorphism

$$j_\# : \pi_1(Y; x_0) \longrightarrow \pi(\tilde{Y}; x_0)$$

if $k \geq 3$. Let $D^k(\frac{1}{2}) \subset \mathring{D}^k$ be the closed k-dimensional subball of radius $\frac{1}{2}$. Cover $\tilde{Y}$ by the two open subsets $U_1 = \mathring{D}^k$ and $U_2 = \tilde{Y} \setminus D^k(\frac{1}{2}) \simeq Y$.

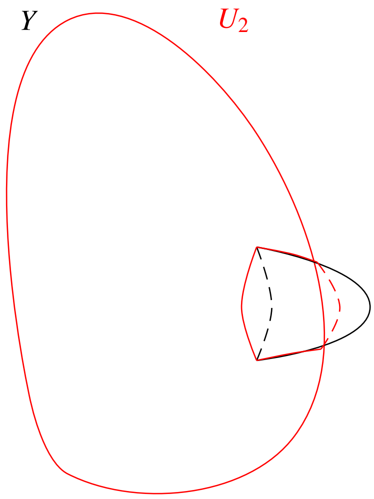


Figure 92. Attaching a cell of dimension ≥ 3

We then find $U_1 \cap U_2 = \mathring{D}^k \setminus D^k(\frac{1}{2}) \simeq S^{k-1}$. To apply the Seifert-van-Kampen Theorem 2.68 we calculate

$$\begin{aligned} \pi_1(U_1) &= \{1\}, \\ \pi_1(U_2) &\cong \pi_1(Y), \\ \pi_1(U_1 \cap U_2) &\cong \pi_1(S^{k-1}) = \{1\}, \qquad \text{(here we use } k-1 \geq 2) \end{aligned}$$

and we find

$$\pi_1(\tilde{Y}) \cong \frac{\pi_1(U_1) \star \pi_1(U_2)}{\operatorname{im} \pi_1(U_1 \cap U_2)} \cong \pi_1(Y)$$

the isomorphisms being induced by inclusions. □

Example 3.113. Consider complex-projective space $X = \mathbb{CP}^n$. We use the cell decomposition from Example 3.89:

$$X^0 = \{\text{point}\} = X^1 \subset \underbrace{X^2}_{\approx \mathbb{CP}^1} = X^3 \subset \underbrace{X^4}_{\approx \mathbb{CP}^2} = X^5 \subset \ldots \subset X^{2n-1} \subset \underbrace{X^{2n}}_{= \mathbb{CP}^n}$$

We are now able to calculate the fundamental group

$$\pi_1(\mathbb{CP}^n) \cong \pi_1(X^2) \cong \pi_1(\mathbb{CP}^1) = \pi_1(S^2) = \{1\}.$$

Hence complex-projective space $\mathbb{CP}^n$ is simply connected.

Example 3.114. Similarly, for $X = \mathbb{HP}^n$ we use the cell decomposition from Example 3.90:

$$X^0 = \{\text{point}\} = X^1 = X^2 = X^3 \subset \underbrace{X^4}_{\approx \mathbb{HP}^1} = \ldots \subset X^{4n-3} = \ldots \subset \underbrace{X^{4n}}_{= \mathbb{HP}^n}$$

For the fundamental group we find

$$\pi_1(\mathbb{HP}^n) = \pi_1(X^2) = \pi_1(\{\text{point}\}) = \{1\}.$$

Thus $\mathbb{HP}^n$ is also simply connected.

Remark 3.115. Proposition 3.112 can be generalized as follows: For a finite CW-complex the inclusion map $j : X^{n+1} \hookrightarrow X$ always induces an isomorphism $j_\# : \pi_n(X^{n+1}; x_0) \to \pi_n(X; x_0)$ where $x_0 \in X^0$.

Now we relate homotopy and homology groups. Recall that for the n-dimensional cube $W^n = [0,1]^n$ we have $(W^n, \partial W^n) \approx (D^n, S^{n-1})$. Fix a generator $\alpha_n \in H_n(W^n, \partial W^n; \mathbb{Z}) \cong H_n(D^n, S^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$. The elements of $\pi_n(X; x_0)$ are homotopy classes relative to ∂W^n of maps $f : W^n \to X$ with $f(\partial W^n) = \{x_0\}$. Then $H_n(f)(\alpha_n) \in H_n(X, \{x_0\}; \mathbb{Z})$. The long exact homology sequence of the pair $(X, \{x_0\})$ yields for $n \geq 1$

$$0 = H_n(\{x_0\}; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow H_n(X, \{x_0\}; \mathbb{Z}) \xrightarrow{0} H_{n-1}(\{x_0\}; \mathbb{Z}).$$

Namely, if $n \geq 2$ then $H_{n-1}(\{x_0\}; \mathbb{Z}) = 0$. For $n = 1$ the arrow emanating from $H_0(\{x_0\}; \mathbb{Z})$ is injective so that the incoming arrow must again be zero. In either case the inclusion $j : (X, \emptyset) \hookrightarrow (X, \{x_0\})$ induces an isomorphism $H_n(j) : H_n(X; \mathbb{Z}) \xrightarrow{\cong} H_n(X, \{x_0\}; \mathbb{Z})$. Now set

$$h([f]) := H_n(j)^{-1} H_n(f)(\alpha_n) \in H_n(X; \mathbb{Z}).$$

Due to homotopy invariance the expression $h([f])$ only depends on the homotopy class of the map f. Hence we have constructed a well-defined map

$$h : \pi_n(X; x_0) \to H_n(X; \mathbb{Z}), \qquad n \geq 1.$$

Proposition 3.116. *The map $h : \pi_n(X; x_0) \to H_n(X; \mathbb{Z})$ is a homomorphism.*

Proof. Consider the map $s_1 : W_1^n = [0, \frac{1}{2}] \times [0,1]^{n-1} \to W^n$ given by $(x_1, \dots, x_n) \mapsto (2x_1, x_2, \dots, x_n)$ and the map $s_2 : W_2^n = [\frac{1}{2}, 1] \times [0,1]^{n-1} \to W^n$ defined by $(x_1, \dots, x_n) \mapsto (2x_1 - 1, x_2, \dots, x_n)$.

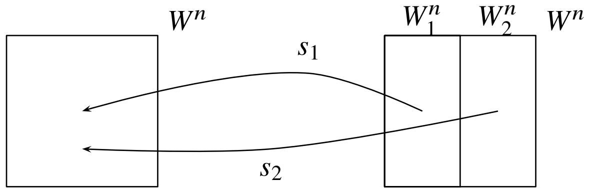


Figure 93. The maps s_1 and s_2

For $[f_1], [f_2] \in \pi_n(X; x_0)$ we have $[f_1] + [f_2] = [g]$ with

$$g(x) = \begin{cases} f_1(s_1(x)), & x_1 \le \frac{1}{2}, \\ f_2(s_2(x)), & x_1 \ge \frac{1}{2}. \end{cases}$$

We represent α_n by $c_1 + c_2 \in S_n(W^n; \mathbb{Z})$, where $c_\nu \in S^n(W_\nu^n; \mathbb{Z})$ and c_ν represents the generator $H_n(s_\nu)^{-1}(\alpha_n)$ of $H_n(W_\nu^n, \partial W_\nu^n; \mathbb{Z})$.

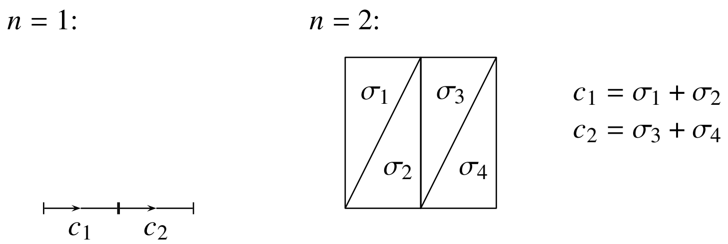


Figure 94. Representative of α_1 and α_2

Now the proposition follows from

$$\begin{aligned} h([f_1] \cdot [f_2]) &= h([g]) \\ &= H_n(j)^{-1} H_n(g)(\alpha_n) \\ &= H_n(j)^{-1} H_n(g)([c_1 + c_2]) \\ &= H_n(j)^{-1} H_n(g)([c_1]) + H_n(j)^{-1} H_n(g)([c_2]) \\ &= H_n(j)^{-1} H_n(f_1 \circ s_1)([c_1]) + H_n(j)^{-1} H_n(f_2 \circ s_2)([c_2]) \end{aligned}$$

$$\begin{aligned}
&= H_n(j)^{-1}H_n(f_1)H_n(s_1)([c_1]) + H_n(j)^{-1}H_n(f_2)H_n(s_2)([c_1]) \\
&= H_n(j)^{-1}H_n(f_1)(\alpha_n) + H_n(j)^{-1}H_n(f_2)(\alpha_n) \\
&= h(f_1) + h(f_2).
\end{aligned}$$ □

Definition 3.117. The map $h : \pi_n(X;x_0) \to H_n(X;\mathbb{Z})$ is called *Hurewicz homomorphism*.

Proposition 3.118. *The Hurewicz homomorphism h is natural, i.e., for every $f \in C(X,Y)$ with $f(x_0) = y_0$ the following diagram commutes:*

$$\begin{tikzcd}
\pi_n(X;x_0) \arrow[r, "h"] \arrow[d, "f_\#"'] & H_n(X;\mathbb{Z}) \arrow[d, "H_n(f)"'] \\
\pi_n(Y;y_0) \arrow[r, "h"] & H_n(Y;\mathbb{Z})
\end{tikzcd}$$

Proof. The inclusion maps

$$\begin{aligned}
j_X : (X,\emptyset) &\hookrightarrow (X,\{x_0\}) \\
j_Y : (Y,\emptyset) &\hookrightarrow (Y,\{y_0\})
\end{aligned}$$

satisfy

$$\begin{aligned}
(j_Y \circ f)(x) &= j_Y(f(x)) = (f(x), y_0) \\
(f \circ j_X)(x) &= f((x,x_0)) = (f(x), f(x_0)) = (f(x), y_0)
\end{aligned}$$

and therefore $j_Y \circ f = f \circ j_X$. On the level of homology groups we find

$$H_n(j_Y) \circ H_n(f) = H_n(f) \circ H_n(j_X)$$

and consequently

$$H_n(f) \circ H_n(j_X)^{-1} = H_n(j_Y)^{-1} \circ H_n(f)\,.$$

Thus

$$\begin{aligned}
(H_n(f) \circ h)([\sigma]) &= H_n(f) \circ H_n(j_X)^{-1} \circ H_n(\sigma)(\alpha_n) \\
&= H_n(j_Y)^{-1} \circ H_n(f) \circ H_n(\sigma)(\alpha_n) \\
&= H_n(j_Y)^{-1} \circ H_n(f \circ \sigma)(\alpha_n) \\
&= h([f \circ \sigma]) \\
&= (h \circ f_\#)([\sigma])
\end{aligned}$$

as claimed. □

Theorem 3.119 (Hurewicz). *Let X be a topological space, $x_0 \in X$ and $n \geq 2$. Assume that*

$$\pi_0(X;x_0) = \pi_1(X;x_0) = \ldots = \pi_{n-1}(X;x_0) = \{1\}\,.$$

Then

$$h : \pi_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z})$$

is an isomorphism.

In particular, we conclude that $H_1(X;\mathbb{Z}) = \ldots = H_{n-1}(X;\mathbb{Z}) = 0$. For a proof of this theorem see e.g. [2, Sec. 4.2].

Remark 3.120. For $n = 1$ this theorem cannot be true as it stands because $H_1(X;\mathbb{Z})$ is always abelian while $\pi_1(X;x_0)$ is not in general. However, h induces a homomorphism

$$\bar{h} : \pi_1(X;x_0)/_{[\pi_1(X;x_0),\pi_1(X;x_0)]} \to H_1(X;\mathbb{Z}).$$

Already Poincaré showed that if $\pi_0(X,x_0) = \{1\}$ (i.e., X is path-connected) then the map $\bar{h}$ is an isomorphism.

Example 3.121. Consider $X = S^n$ for $n \geq 2$. We already know that S^n is 1-connected. Now apply the Hurewicz isomorphism with $n = 2$:

$$\pi_2(S^n) \cong H_2(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 2, \\ 0, & n \geq 3. \end{cases}$$

For $n \geq 3$ we find due to Hurewicz:

$$\pi_3(S^n) \cong H_3(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 3, \\ 0, & n \geq 4. \end{cases}$$

By induction we then deduce that $\pi_1(S_n) = \ldots = \pi_{n-1}(S^n) = 0$ and $\pi_n(S^n) \cong \mathbb{Z}$.

Example 3.122. We know that $X = \mathbb{CP}^n$ is 1-connected. With the help of the Hurewicz isomorphism we calculate

$$\pi_2(\mathbb{CP}^n) \cong H_2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}\,.$$

Example 3.123. We also know that $X = \mathbb{HP}^n$ is 1-connected. We apply the Hurewicz isomorphism three times and we get

$$\begin{aligned} \pi_2(\mathbb{HP}^n) &\cong H_2(\mathbb{HP}^n;\mathbb{Z}) = 0, \\ \pi_3(\mathbb{HP}^n) &\cong H_3(\mathbb{HP}^n;\mathbb{Z}) = 0, \\ \pi_4(\mathbb{HP}^n) &\cong H_4(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}. \end{aligned}$$

Example 3.124. Now let us analyze the space $X = \mathbb{RP}^n$ for $n \geq 2$. The map $\psi : S^n \to \mathbb{RP}^n$ is a two-fold covering and by Theorem 2.102

$$\pi_1(S^n) \xrightarrow{\psi_\sharp} \pi_1(\mathbb{RP}^n) \to \pi_0(\{p, q\}) \to \pi_0(S^n)$$

is exact. Since S^n is simply connected $\pi_1(\mathbb{RP}^n) \to \pi_0(\{p, q\})$ is an isomorphism of pointed sets, thus $\pi_1(\mathbb{RP}^n)$ has exactly two elements. There is only one group with two elements, hence $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$. By Corollary 2.105 $\psi_\# : \pi_k(S^n) \to \pi_k(\mathbb{RP}^n)$ is an isomorphism for all $k \geq 2$. We conclude that $\pi_2(\mathbb{RP}^n) = \ldots = \pi_{n-1}(\mathbb{RP}^n) = 0$ and $\pi_n(\mathbb{RP}^n) \cong \mathbb{Z}$.

Remark 3.125. Under the assumptions of the theorem of Hurewicz 3.119 not much can be said about $h : \pi_k(X, x_0) \to H_k(X; \mathbb{Z})$ for $k > n$. For example, consider the Hopf fibration $S^3 \to S^2$ with fiber S^1. By Theorem 2.102 it induces an isomorphism $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. But $H_3(S^2; \mathbb{Z}) = 0$ and hence $h : \pi_3(S^2) \to H_3(S^2; \mathbb{Z})$ is not injective.
On the other hand, for the 2-torus T^2 we have again by Corollary 2.105 $\pi_2(T^2) \cong \pi_2(\mathbb{R}^2) = 0$ while one can compute $H_2(T^2; \mathbb{Z}) \cong \mathbb{Z}$. Thus $h : \pi_2(T^2) \to H_2(T^2; \mathbb{Z})$ is not surjective.

Remark 3.126. We have seen that $H_k(S^n; \mathbb{Z}) = 0$ whenever $k > n$. But in general this is not true for the higher homotopy groups of the sphere, e.g. $\pi_3(S^2) \cong \mathbb{Z}$. The computation of $\pi_k(S^n)$ for $k > n$ is a difficult problem and many of these groups are not known to date.

3.15. Exercises

3.1. Let X be a topological space. Show:

a) If X is path-connected then

$$H_0(X; R) \cong R$$

b) If X_k, $k \in K$, are the path-components of X then

$$H_n(X; R) \cong \bigoplus_{k \in K} H_n(X_k; R)$$

3.2. Let $Y_k = \{1, ..., k\}$ be equipped with the discrete topology. Compute $H_n(Y_k; R)$ without using Exercise 3.1. Instead use the Eilenberg-Steenrod axioms.
Hint: Consider the pair (Y_k, Y_{k-1}).

3.3. Let X be a topological space. The *augmented boundary operator* is defined by

$$\partial^\# : S_0(X; R) \to R,$$

$$\partial^\# \left(\sum \alpha_i \sigma_i \right) = \sum \alpha_i,$$

where $\sigma_i \in C(\Delta^0, X)$.

a) Verify $\partial^\# \circ \partial = 0$.

b) Compute the *augmented homology*

$$H_0^\#(X;R) := \frac{\ker(\partial^\# : S_0(X;R) \to R)}{\operatorname{im}(\partial : S_1(X;R) \to S_0(X;R))}$$

for $X = \{\text{point}\}$.

3.4. a) Show that homeomorphisms $\sigma : \Delta^n \to D^n$ represent generators of $H_n(D^n, S^{n-1}; R)$.

b) Describe generators of $H_n(S^n;\mathbb{Z})$. Make a drawing for $n = 2$.

3.5 (Topological invariance of the dimension). Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and nonempty. Show: If U and V are homeomorphic then $n = m$.
Hint: For $p \in U$ and $q \in V$ consider the pairs $(U, U \setminus \{p\})$ and $(V, V \setminus \{q\})$.

3.6. Let p be a complex polynomial without zeros on the unit circle $S^1 \subset \mathbb{C}$. Show: The degree of the map

$$\hat{p} : S^1 \to S^1, \quad \hat{p}(z) = \frac{p(z)}{|p(z)|},$$

coincides with the number of zeros of p in the interior of the unit disk (counted with multiplicities).

3.7 (Homotopy invariance of the local mapping degree). Let $V \subset S^n$ be open and $F : V \times [0,1] \to S^n$ continuous. We put $f_t(x) := F(x,t)$. Let $p \in S^n$ such that $\bigcup_{t\in[0,1]} f_t^{-1}(p)$ is compact. Show:

$$\deg_p(f_1) = \deg_p(f_0).$$

3.8. Let (X, A) be a pair such that A is closed and a strong deformation retract of an open neighborhood U. Show that $H_n(X, A) = H_n(X/A)$ for $n \neq 0$.

3.9. Let $Z = S^1 \times [0,1]$ be the cylinder. Compute $H_n(Z, S^1 \times \{0\} \cup S^1 \times \{1\})$ for all n. Sketch generators of the nontrivial homology groups.
Hint: Use the homology sequence of the triple

$$(Z, S^1 \times \{0\} \cup S^1 \times \{1\}, S^1 \times \{0\}).$$

3.10. Let (X, A) be a pair. Describe the 0^{th} singular relative homology group $H_0(X, A)$.

3.11 (Five lemma). Let the rows in the following commutative diagram of abelian groups be exact:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

Show that if φ_1, φ_2, φ_4, and φ_5 are isomorphisms then so is φ_3.
Hint: φ_3 is injective if φ_1 is surjective and φ_2, φ_4 are injective; φ_3 is surjective if φ_5 is injective and φ_2, φ_4 are surjective.

3.12. Suppose
$$\cdots \to G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \to \cdots$$
is a long exact sequence of abelian groups and
$$\cdots \to G'_{n+1} \xrightarrow{f'_{n+1}} G'_n \xrightarrow{f'_n} G'_{n-1} \to \cdots$$
is a subsequence, i.e., $G'_n \subset G_n$ and $f'_n = f_n|_{G'_n}$. Prove that the subsequence is exact if and only if the quotient sequence
$$\cdots \to G_{n+1}/G'_{n+1} \to G_n/G'_n \to G_{n-1}/G'_{n-1} \to \cdots$$
is exact.

3.13. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Show that there exists $f \in C(S^n, S^n)$ with $\deg(f) = m$.

3.14. Let $f \in C(S^n, S^n)$ with $f(D^n_+) \subset D^n_+$ and $f(D^n_-) \subset D^n_-$. We identify S^{n-1} with $D^n_+ \cap D^n_-$ and therefore have $f(S^{n-1}) \subset S^{n-1}$. Show
$$\deg(f) = \deg(f|_{S^{n-1}}).$$

3.15. Show that $H_1(\mathbb{R}, \mathbb{Q}; \mathbb{Z})$ is a free abelian group and find a basis as a $\mathbb{Z}$-module.

3.16. Let M be an n-dimensional manifold, $n \geq 3$. Let $p \in M$. Show that the inclusion map $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism
$$H_j(M \setminus \{p\}; R) \cong H_j(M; R)$$
for all $j \in \{1, ..., n-2\}$.

3.17. a) Show that for disjoint closed subsets $A, B \subset \mathbb{R}^2$ we have
$$H_1(\mathbb{R}^2 \setminus (A \cup B)) \cong H_1(\mathbb{R}^2 \setminus A) \oplus H_1(\mathbb{R}^2 \setminus B).$$

b) Let $p_1, ..., p_n \in \mathbb{R}^2$ be pairwise distinct. Compute $H_1(\mathbb{R}^2 \setminus \{p_1, ..., p_n\})$ and sketch generators.

3.18. Use the Mayer-Vietoris sequence to compute

a) the homology of the 2-torus;

b) the homology of surfaces F_g of genus $g \geq 2$.

Sketch generators.

3.19. Let $(X, \mathcal{X})$ be a finite CW-complex. Show that X is compact.

3.20. Describe a CW-decomposition for the surfaces of genus $g \geq 1$.

3.21. Show:

a) Each nonempty CW-complex has at least one 0-cell.

b) Each CW-complex consisting of exactly two cells is homeomorphic to a sphere.

3.22. Let $n \geq 2$ and $k \geq 1$. Let X be the topological space obtained from k copies of S^n by identifying them all at one point. More formally,

$$X = \Big(\bigcup_{j=1}^{k} \{j\} \times S^n\Big) \big/ \sim$$

where $(j, x) \sim (j', x')$ iff $x = x' = e_1$. Compute the homology of X.

3.23. Find a CW-decomposition of the 2-torus with exactly one 0-cell, two 1-cells, and one 2-cell and use it to compute the homology.

A. Appendix

A.1. Free module generated by a set

Let R be a commutative ring with unit 1 and let S be a set. Then the set X of all maps from S to R forms an R-module.[1] Addition and multiplication with scalars are defined pointwise, for any $f, g\colon S \to R$ and $\alpha \in R$ we have, by definition,

$$(f+g)(s) = f(s) + g(s), \quad (\alpha \cdot f)(s) = \alpha \cdot f(s)$$

for all $s \in S$.
Now let $Y \subset X$ be the set of all $f \in X$ for which $f(s) = 0$ for all but *finitely many* $s \in S$. Then Y is an R-submodule of X. The module Y has a natural basis. Namely, for each $s \in S$ define

$$f_s(s') := \begin{cases} 1, & \text{if } s' = s, \\ 0, & \text{if } s' \neq s. \end{cases}$$

Then for each $f \in Y$ we have

$$f = \sum_{s \in S} f(s) f_s. \tag{A.1}$$

Note that we need to sum only over those s for which $f(s) \neq 0$ which leaves us with a finite sum. Thus the set $\{f_s \mid s \in S\}$ generates Y. The set is also linearly independent. Namely, if

$$\alpha_1 f_{s_1} + \ldots + \alpha_m f_{s_m} = 0$$

for pairwise different s_j then by inserting s_i we find

$$0 = (\alpha_1 f_{s_1} + \ldots + \alpha_m f_{s_m})(s_i) = \alpha_1 f_{s_1}(s_i) + \ldots + \alpha_m f_{s_m}(s_i) = \alpha_i.$$

Thus $\{f_s \mid s \in S\}$ is indeed a basis of Y. Hence the dimension of Y is given by the cardinality of S. In particular, Y is infinite-dimensional if S is an infinite set.

Definition A.1. The R-module Y is called the *free R-module generated by S*.

We usually use a somewhat sloppy notation and will not distinuish between an element $s \in S$ and the corresponding function $f_s\colon S \to R$. Thus, instead of (A.1) we will write

$$f = \sum_{s \in S} f(s) s.$$

[1] If you are not familiar with modules over rings think of the special case that R is a field. Then an R-module is just an R-vector space.

In Section 3.1 we consider the free R-module generated by $S = C(\Delta^n, X)$ and write $S_n(X; R)$ instead of Y.

List of Figures

Bibliography

[1] Greenberg, M. J.; Harper, J. R.: *Algebraic topology. A first course*, Benjamin/Cummings Publishing, 1981

[2] Hatcher, A.: *Algebraic topology*, Cambridge University Press, 2002

[3] Hilbert, D.: *Ueber die stetige Abbildung einer Line auf ein Flächenstück*, Math. Ann. **38** (1891), *459–460*

[4] Lück, W.: *Algebraische Topologie. Homologie und Mannigfaltigkeiten*, Vieweg, 2004

[5] Ossa, E.: *Topologie*, Vieweg, 1992

[6] Peano, G.: *Sur une courbe, qui remplit toute une aire plane*, Math. Ann. **36** (1890), *157–160*

[7] tom Dieck, T.: *Topologie*, Walter de Gruyter, 1991

[8] Warner, F. W.: *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1983

Index

Symbols

T

U

V

W