GEOMETRY IN POTSDAM

Christian Bär

Geometric Wave Equations

Winter Term 2015/16

$$\partial_t^2 u = \sum_{ij} g^{ij}\partial_i\partial_j u + \sum_j b^j \partial_j u + cu$$

$$\partial_t^2 u = \sum_{ij} g^{ij}\partial_i\partial_j u + \sum_j b^j \partial_j u + cu$$

$$\partial_t^2 u = \sum_{ij} g^{ij}\partial_i\partial_j u + \sum_j b^j \partial_j u + cu$$

Version of April 13, 2017

Contents

Preface

These are the lecture notes of a course on geometric wave equations which I taught at the University of Potsdam in the winter term 2015/2016. The course gave an introduction to linear hyperbolic PDEs on Lorentzian manifolds. The geometric setup allows to apply the theory in general relativity, for instance.
Some basic knowledge of differential geometry was required. This includes concepts such as manifolds, vector bundles, connections etc. Since Lorentzian geometry is less standard than Riemannian geometry I collected the most relevant material in the first chapter on preliminaries, some of it without proofs. This chapter also contains an introduction to linear differential operators and distributions on manifolds.
The second chapter is devoted to the local study of normally hyperbolic equations. These are second-order wave equations. Local fundamental solutions are constructed and their Hadamard expansion is derived.
In the third chapter we study global solutions meaning solutions defined on the whole manifold. Of course, this requires the manifold to be "reasonable" which is made precise by the concept of global hyperbolicity. We construct global fundamental solutions, Green's operators and solutions to the Cauchy problem for normally hyperbolic operators. Symmetric hyperbolic systems turn out to be an important class of first-order equations for which we also derive the analytic basics. Using them one can for instance treat Maxwell's equations from electrodynamics. The existence of Green-operators already says a lot about the solution theory of a differential operator. This leads to the class of Green-hyperbolic operators which includes but is much larger than normally hyperbolic operators and symmetric hyperbolic systems. Finally, we use an argument due to Chernoff to show essential selfadjointness of many operators on Riemannian manifolds using symmetric hyperbolic systems.
Originally, I had planned to also include a chapter on non-linear wave equations known as wave maps. Unfortunately, this turned out to be unrealistic due to a lack of time.
I would like to thank all participants of the course for their active participation and the many hints they provided. Special thanks go to Claudia Grabs for taking notes and writing the first draft of these lecture notes. Many of the illustrations have also been created by her.

Potsdam, February 2016

Christian Bär

1 Preliminaries

Wave equations form a class of partial differential equations which describe many physical processes. The following table contains a collection of examples of equations which are, in principle, tractable by methods which we will develop in this course, at least the linear ones.

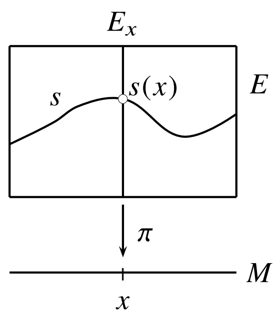
topic	**equation**	**describes**	**type**	**order**
electrodynamics	Maxwell's equations	field strength	linear	1st
	wave equation	four-potential	linear	2nd
quantum field theory	Klein-Gordon-equation	scalar field	linear	2nd
	Dirac equation	wave function of electron	linear	1st
general relativity	Einstein field equations	gravitational field	non-linear	2nd
	linearized Einstein equations	gravitational waves	linear	2nd
elasticity theory	equation of motion	deformation of elastic body	non-linear	2nd
differential geometry	wave maps	maps between manifolds	non-linear	2nd

Table 1.1: example equations

We will discuss these partial differential equations in a geometric formulation to be able to apply the results in geometric theories like electrodynamics, general relativity, and elasticity theory. The first chapter contains a somewhat diverse summary of the background material which we will need.

1.1 Linear differential operators on manifolds

1.1.1 Vector bundles and linear differential operators



Reminder. Let M be an n-dimensional differentiable manifold and let $\pi : E \to M$ be a vector bundle. A *section* of E is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.
We define

$$C^\infty(M, E) := \{\text{smooth sections of } E\}.$$

Definition 1.1.2. Let E and F be $\mathbb{K}$-vector bundles over M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
A **differential operator of order (at most) k** is a linear mapping $P : C^\infty(M, E) \to C^\infty(M, F)$, such that for any local coordinate system $x^1, \ldots, x^n$ on $U \subset M$ and any local trivialization $E|_U \xrightarrow{\approx} U \times \mathbb{K}^p$ and $F|_U \xrightarrow{\approx} U \times \mathbb{K}^q$ there exist smooth maps $A^\alpha : U \to \mathrm{Mat}(q \times p, \mathbb{K})$ such that

$$Pv|_U = \sum_{|\alpha| \leq k} A^\alpha(x) \frac{\partial^{|\alpha|} v}{(\partial x^1)^{\alpha_1} \ldots (\partial x^n)^{\alpha_n}}$$

for all $v \in C^\infty(M, E)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Notation 1.1.3. We define

$$\mathscr{D}\!\mathit{iff}_k(E, F) := \{P : C^\infty(M, E) \to C^\infty(M, F) \mid P \text{ differential operator of order} \leq k\}.$$

The vector spaces $\mathscr{D}\!\mathit{iff}_k(E, F)$ form a filtration,

$$\mathscr{D}\!\mathit{iff}_{k+1}(E, F) \supset \mathscr{D}\!\mathit{iff}_k(E, F) \supset \cdots \supset \mathscr{D}\!\mathit{iff}_0(E, F) = C^\infty(M, \mathrm{Hom}(E, F)).$$

Example 1.1.4. Let M be a semi-Riemannian manifold, let $E = M \times \mathbb{R}$ be the trivial real line bundle and $F = TM$ be the tangent bundle of M. The *gradient* is a differential operator of order 1 from E to F, grad $\in \mathscr{D}\!\mathit{iff}_1(M \times \mathbb{R}, TM)$. In local coordinates, we have:

$$\mathrm{grad}\, v = \sum_i g^{ij}(x) \frac{\partial v}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Comparing the coefficients in this formula with the coefficients A^α in Definition 1.1.2, we find:

$$A^{(0,\ldots,\overset{\overset{i}{\downarrow}}{1},\ldots,0)} = \left(g^{1i}, \ldots, g^{ni}\right)^\top, \qquad A^{(0\ldots,0)} = (0, \ldots, 0)^\top.$$

Example 1.1.5. Let M be a semi-Riemannian manifold, let $E = TM$ be the tangent bundle of M and let $F = M \times \mathbb{R}$ be the trivial real line bundle. The *divergence* is a differential operator of order 1 from E to F, div $\in \mathscr{D}\!\mathit{iff}_1(TM, \mathbb{R})$. In local coordinates, we have for $Y = \sum_i y^i \frac{\partial}{\partial x^i}$:

$$\mathrm{div}\,(Y) = \sum_i \frac{\partial y^i}{\partial x^i} + \sum_{ij} \Gamma^i_{ij} y^j.$$

The coefficients are

$$A^{(0,\ldots,\overset{\overset{i}{\downarrow}}{1},\ldots,0)} = (0, \ldots, \overset{\overset{i}{\downarrow}}{1}, \ldots, 0), \qquad A^{(0\ldots,0)} = \Big(\sum_i \Gamma^i_{i1}, \ldots, \sum_i \Gamma^i_{in}\Big).$$

Here Γ_{ij}^k denote the Christoffel symbols of the semi-Riemannian metric with respect to the coordinates $x^1, \ldots, x^n$.

Example 1.1.6. Let M be a Riemannian manifold and consider $E = \Lambda^m T^*M$ and $F = \Lambda^{m+1}T^*M$. The *exterior derivative* d is a differential operator of order 1 from E to F, $d \in \mathscr{Diff}_1(\Lambda^m T^*M, \Lambda^{m+1}T^*M)$.

Example 1.1.7. Let E be an arbitrary vector bundle over M with connection ∇ and let $F = T^*M \otimes E$. Then ∇ is a differential operator of first order from E to F.

Remark 1.1.8. Let $E, F, G \to M$ be vector bundles over a smooth manifold M. If $P \in \mathscr{Diff}_k(E, F)$ and $Q \in \mathscr{Diff}_l(F, G)$, then $Q \circ P \in \mathscr{Diff}_{k+l}(E, G)$.

Example 1.1.9. Let M be a semi-Riemannian manifold and consider $E = G = M \times \mathbb{R}$ and $F = TM$. Then $-\mathrm{div} \circ \mathrm{grad} \in \mathscr{Diff}_2(E, G)$. If M is Riemannian this operator is denoted by Δ and is called the *Laplace-Beltrami operator*. If M is Lorentzian then this operator is denoted by $\Box$ and is called the *d'Alembert operator*.

1.1.2 The principal symbol

For a given differential operator $P \in \mathscr{Diff}_k(E, F)$ and a covector $\xi \in T_x^*M$, we construct a linear mapping $\sigma_k(P, \xi) : E_x \to F_x$ as follows: We choose a smooth function $f : M \to \mathbb{R}$ such that $f(x) = 0$ and $df(x) = \xi$. We then set for $e \in E_x$:

$$\sigma_k(P, \xi) \cdot e := \frac{1}{k!} P(f^k \tilde{e})|_x, \tag{1.1}$$

where $\tilde{e} \in C^\infty(M, E)$ is any extension of e, i.e. $\tilde{e}(x) = e$. As we shall see, this definition is independent of the choice of $\tilde{e}$ and f. In local coordinates and local trivializations, we compute:

$$\begin{aligned}
\sigma_k(p, \xi) \cdot e &= \frac{1}{k!} \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial^{|\alpha|} \left(f^k \tilde{e}\right)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}(x) \\
&= \frac{1}{k!} \sum_{|\alpha| = k} A^\alpha(x) \frac{\partial^{|\alpha|} \left(f^k\right)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}(x) \cdot \tilde{e}(x) \\
&= \sum_{|\alpha| = k} A^\alpha(x) \cdot \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdot e\,.
\end{aligned} \tag{1.2}$$

The second equality holds because by assumption $f(x) = 0$, so that all terms vanish where f^k differentiated is less than k times. The last equality holds by a similar argument: If one of the factors in f^k is differentiated more than once, there is another factor which remains without differentiation and hence vanishes at x.
Since the right hand side of (1.2) is independent of the choice of $\tilde{e}$ and f, so is the left hand side. This shows that $\sigma_k(P, \xi)$ is well defined by (1.1).

For any $\xi \in T_x^*M$, we have constructed a homomorphism $\sigma_k(P,\xi) : E_x \to F_x$. Thus we have $\sigma_k(P,\cdot) \in \mathrm{Hom}(\pi^*E, \pi^*F)$ where $\pi : T^*M \to M$ is the projection to the foot point.

Definition 1.1.10. Let $E, F \to M$ be vector bundles over a smooth manifold M and let $P \in \mathscr{Diff}_k(E,F)$. Then $\sigma_k(P,\cdot) \in \mathrm{Hom}(\pi^*E,\pi^*F)$ is called the **principal symbol** of the operator P.

Remark 1.1.11. The principal symbol $\sigma_k(P,\cdot)$ contains the coefficients of the highest order derivatives of $P \in \mathscr{Diff}_k(E,F)$. In particular, we have

$$\sigma_k(P,\xi) = 0 \text{ for all } \xi \in T^*M \quad \Leftrightarrow \quad A^\alpha = 0 \text{ for all } |\alpha| = k \quad \Leftrightarrow \quad P \in \mathscr{Diff}_{k-1}(E,F).$$

In other words: The sequence

$$0 \to \mathscr{Diff}_{k-1}(E,F) \longrightarrow \mathscr{Diff}_k(E,F) \xrightarrow{\sigma_k(P,\cdot)} \mathrm{Hom}\,(\pi^*E,\pi^*F)$$

is exact.

Warning. In the literature the definition of $\sigma_k(P,\xi)$ often contains an additional factor i^k.

Convention. If k is clear from the context, we will write $\sigma(P,\xi)$ instead of $\sigma_k(P,\xi)$.

Example 1.1.13. We compute the principal symbol of the gradient, see Example 1.1.4. We fix a covector $\xi \in T_x^*M$. Since $E_x = \mathbb{R}$, we have to apply $\sigma(\mathrm{grad},\xi)$ to a real number, say 42. A convenient extension of 42 to a smooth section to E is the constant function $x \mapsto 42$.
Let $f : M \to \mathbb{R}$ be a smooth function, such that $f(x) = 0$ and $df(x) = \xi$. By the definition of $\sigma(\mathrm{grad},\xi)$, we have[1]

$$\begin{aligned}\sigma(\mathrm{grad},\xi)\cdot 42 &= \mathrm{grad}\,(f\cdot 42)(x)\\ &= 42\cdot \mathrm{grad}\, f(x)\\ &= 42\cdot df(x)^\sharp\\ &= 42\cdot \xi^\sharp.\end{aligned}$$

In short: $\sigma(\mathrm{grad},\xi) = \xi^\sharp$.

Example 1.1.14. We compute the principal symbol of the divergence, see Example 1.1.5. Here $E_x = T_xM$, so we have to apply $\sigma(\mathrm{div},\xi)$ to a tangent vector $Y \in T_xM$. Let $\tilde{Y}$ be a smooth

[1] Here $\xi^\sharp$ is the vector in T_xM dual to $\xi \in T_x^*M$ with respect to the Riemannian metric, i.e., for any $Y \in T_xM$ we have $\langle \xi^\sharp, Y\rangle = \xi(Y)$.

vector field such that $\tilde{Y}(x) = Y$. Again let $f : M \to \mathbb{R}$ be a smooth function such that $f(x) = 0$ and $df(x) = \xi$. Then we have

$$\begin{aligned}\sigma(\mathrm{div}, \xi)Y &= \mathrm{div}\,(f \cdot \tilde{Y})(x) \\ &= \underbrace{f(x)}_{=0} \cdot \mathrm{div}\,(\tilde{Y})(x) + \langle \mathrm{grad}\, f(x), \tilde{Y}(x)\rangle \\ &= \langle \xi^\sharp, Y\rangle \\ &= \xi(Y).\end{aligned}$$

Thus $\sigma(\mathrm{div}, \xi) = \xi$.

Example 1.1.15. We compute the principal symbol of the exterior derivative d, see Example 1.1.6. Let $\omega \in \Lambda^k T_x^* M$ and extend ω to a smooth k-form $\tilde{\omega} \in \Omega^k(M)$ such that $\tilde{\omega}(x) = \omega$. Then we have

$$\begin{aligned}\sigma(d, \xi)\omega &= d(f \cdot \tilde{\omega})(x) \\ &= (df \wedge \tilde{\omega} + f \cdot d\tilde{\omega})|_x \\ &= df(x) \wedge \omega + \underbrace{f(x)}_{=0} \cdot d\tilde{\omega}|_x \\ &= \xi \wedge \omega.\end{aligned}$$

Hence $\sigma(d, \xi) = \xi \wedge \cdot$.

Example 1.1.16. We compute the principal symbol of a connection ∇ on a vector bundle E, see Example 1.1.7. Let $e \in E_x$ and extend e to a smooth section $\tilde{e} \in C^\infty(M, E)$ such that $\tilde{e}(x) = e$. Then we have

$$\begin{aligned}\sigma(\nabla, \xi)e &= \nabla(f\tilde{e})|_x \\ &= (df \otimes \tilde{e} + f \cdot \nabla\tilde{e})|_x \\ &= df(x) \otimes e + \underbrace{f(x)}_{=0} \cdot (\nabla\tilde{e})|_x \\ &= \xi \otimes e.\end{aligned}$$

Thus $\sigma(\nabla, \xi) = \xi \otimes \cdot$.

Remark 1.1.17. Let E, F, G be vector bundles over a smooth manifold M, and let $P \in \mathscr{Diff}_k(E, F)$ and $Q \in \mathscr{Diff}_l(F, G)$. Then we have

$$\sigma_{k+l}(Q \circ P, \xi) = \sigma_l(Q, \xi) \circ \sigma_k(P, \xi).$$

Example 1.1.18. We compute the principal symbol of the Laplace-Beltrami operator Δ or d'Alembert operator $\Box$ from the principal symbols of div and grad :

$$\sigma_2(-\mathrm{div} \circ \mathrm{grad}, \xi) = -\sigma_1(\mathrm{div}) \cdot \sigma_1(\mathrm{grad}) = -\xi(\xi^\sharp) = -\langle \xi^\sharp, \xi^\sharp\rangle = -\langle \xi, \xi\rangle.$$

1.1.3 Formally adjoint and formally dual operator

In the following let M be a differentiable manifold equipped with a smooth positive volume density $d\mu$. The volume density is necessary for the integration of functions over M. In local coordinates, it takes the form $d\mu = \mu dx^1 \cdots dx^n$ where μ is a smooth positive function. Later we will use the volume density induced by a Riemannian or Lorentzian metric but for now this is irrelevant.

Moreover, let $E, F \to M$ be vector bundles whose fibers carry non-degenerate (but possibly indefinite) inner products $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_F$, respectively. They are supposed to depend smoothly on the base point.

Lemma 1.1.19. *For any $P \in \mathscr{Diff}_k(E,F)$ there exists a unique operator $P^t \in \mathscr{Diff}_k(F,E)$ such that*

$$\int_M \langle Pu, v\rangle_F \, d\mu = \int_M \langle u, P^t v\rangle_E \, d\mu \tag{1.3}$$

holds for all $u \in C^\infty(M,E)$ and $v \in C^\infty(M,F)$ with compact supports.

Definition 1.1.20. The operator $P^t \in \mathscr{Diff}_k(F,E)$ satisfying (1.3) is called the **operator formally adjoint** to P.

Proof of Lemma 1.1.19. Uniqueness:

Let $x^1, \ldots, x^n$ be local coordinates defined on $U \subset M$ and let local trivializations of E and F over U be fixed. Let $\mathcal{E}$ and $\mathcal{F}$ be the matrices representing the inner products of E and F with respect to the local trivializations. They are symmetric and invertible and depend smoothly on the footpoint in U.

Let $u \in C^\infty(M,E)$ and $v \in C^\infty(M,F)$ be sections with supports in U. We denote the canonical scalar product on $\mathbb{K}^m$ by $\langle\cdot,\cdot\rangle$ and we compute:

$$\begin{aligned}
\int_U \langle Pu, v\rangle_F \, d\mu &= \int_U \Big\langle \sum_{|\alpha|\le k} A^\alpha \frac{\partial^{|\alpha|} u}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}}, \mathcal{F} v\Big\rangle \mu \, dx \\
&= \sum_{|\alpha|\le k} \int_U \Big\langle \frac{\partial^{|\alpha|} u}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}}, \mu \cdot (A^\alpha)^\top \mathcal{F} v\Big\rangle \, dx \\
&\overset{\substack{\text{integr.}\\ \text{by parts}}}{=} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \int_U \Big\langle u, \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}} \Big(\mu \cdot (A^\alpha)^\top \mathcal{F} v\Big) \Big\rangle \, dx \\
&= \int_U \Big\langle u, \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{1}{\mu} \frac{\partial^{|\alpha|} (\mu \cdot (A^\alpha)^\top \mathcal{F} v)}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}} \Big\rangle \, d\mu
\end{aligned}$$

$$= \int_U \Big\langle u, \mathcal{E}^{-1} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{1}{\mu} \frac{\partial^{|\alpha|} (\mu \cdot (A^\alpha)^\top \mathcal{F} v)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} \Big\rangle_E \, d\mu.$$

Since this holds for all u a comparison with (1.3) yields

$$P^t v = \frac{1}{\mu} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \mathcal{E}^{-1} \frac{\partial^{|\alpha|} (\mu \cdot (A^\alpha)^\top \mathcal{F} v)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}. \tag{1.4}$$

This shows that $P^t v$ is uniquely determined provided its support is contained in a coordinate patch. Now let $v \in C^\infty(M, F)$ be an arbitrary section with compact support. We choose an open covering of M with local trivializations and a partition of unity subordinated to it. Then v is a finite sum of sections of the form considered above. Since P^t is required to be linear, it is uniquely determined by the local formula (1.4).

Existence: Let $v \in C^\infty(M, F)$ be a smooth section with compact support. We now use formula (1.4) to define $P^t v$ if v has support in U. For general v we use a partition of unity to write it as a sum of sections with supports contained in coordinate patches. It is tedious but straightforward to check that this definition is independent of the choice of coordinates, trivializations, and partition of unity. □

Remark 1.1.21. For any $P \in \mathscr{D}\!\mathit{iff}_k(E, F)$ we have $(P^t)^t = P$. This is obvious from equation (1.3) and the uniqueness of the formal adjoint.

Example 1.1.22. The gradient is a first order operator $\mathrm{grad} : C^\infty(M) \to C^\infty(M, TM)$, so grad^t maps vector fields to functions. By definition, for any function $u \in C^\infty(M)$ and any vector field $Y \in C^\infty(M, TM)$, both with compact support, we have

$$\begin{aligned} \int_M u \,\mathrm{grad}^t Y \, dvol &= \int_M \langle \mathrm{grad}\, u, Y \rangle \, dvol \\ &= \int_M (\mathrm{div}\,(uY) - u \,\mathrm{div}\, Y) \, dvol \\ &= -\int_M u \,\mathrm{div}\, Y \, dvol. \end{aligned}$$

In the last step we used the Gauss integration theorem. Thus $\mathrm{grad}^t = -\mathrm{div}$. By Remark 1.1.21 we also have $\mathrm{div}^t = -\mathrm{grad}$.

Remark 1.1.23. For differential operators $P \in \mathscr{D}\!\mathit{iff}_k(E, F)$ and $Q \in \mathscr{D}\!\mathit{iff}_l(F, G)$ we have

$$(Q \circ P)^t = P^t \circ Q^t.$$

Definition 1.1.24. Let M be a differentiable manifold with a smooth positive volume density. Let E be a vector bundle over M with a non-degenerate inner fiber metric. An operator $P \in \mathscr{D}\!\mathit{iff}_k(E,E)$ is called **formally selfadjoint** if $P = P^t$.

Example 1.1.25. Let $P = -\mathrm{div} \circ \mathrm{grad}$ on a semi-Riemannian manifold. We then have

$$P^t = -(\mathrm{div} \circ \mathrm{grad})^t = -\mathrm{grad}^t \circ \mathrm{div}^t = -(-\mathrm{div}) \circ (-\mathrm{grad}) = P.$$

Thus the Laplace-Beltrami operator and the d'Alembert operator are formally selfadjoint.

Lemma 1.1.26. *Let M be a differentiable manifold with a smooth positive volume density $d\mu$. Let E and F be vector bundles over M equipped with non-degenerate inner fiber metrics. Let $P \in \mathscr{D}\!\mathit{iff}_k(E,F)$. Then for any $\xi \in T^*M$ we have*

$$\sigma_k(P^t,\xi) = (-1)^k \sigma_k(P,\xi)^t. \tag{1.5}$$

Proof. Since only the terms of order k contribute to the principal symbol $\sigma_k(P,\cdot)$, we write

$$Pu = \sum_{|\alpha|=k} A^\alpha(x) \frac{\partial^{|\alpha|} u}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} + \text{l.o.t.}$$

where "l.o.t." stands for "lower order terms". By this we mean expressions involving derivatives of u of order lower than k. By (1.4) the adjoint of P is given by

$$\begin{aligned} P^t v &= \frac{1}{\mu} \sum_{|\alpha|=k} (-1)^{|\alpha|} \mathcal{E}^{-1} \frac{\partial^{|\alpha|} (\mu \cdot (A^\alpha)^\top \mathcal{F} v)}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} + \text{l.o.t.} \\ &= \frac{1}{\mu} \sum_{|\alpha|=k} (-1)^k \mu \cdot \mathcal{E}^{-1} (A^\alpha)^\top \mathcal{F} \frac{\partial^{|\alpha|} v}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} + \text{l.o.t.} \\ &= \sum_{|\alpha|=k} (-1)^k (A^\alpha)^t \frac{\partial^{|\alpha|} v}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}} + \text{l.o.t.} \end{aligned}$$

Here we have denoted the transpose of A^α with respect to the fiber metrics of E and F by $(A^\alpha)^t$ and the transpose with respect to the standard fiber metrics induced by the local trivializations by $(A^\alpha)^\top$. They are related by $(A^\alpha)^t = \mathcal{E}^{-1} (A^\alpha)^\top \mathcal{F}$. Now (1.2) yields

$$\sigma_k(P^t,\xi) = (-1)^k \sum_{|\alpha|=k} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} A^\alpha(x)^t = (-1)^k \sigma_k(P,\xi)^t. \qquad \square$$

To conclude this section we present a variation of the concept of formally adjoint operators which does not require fiber metrics. We work with dual bundles instead. Let M be a differentiable manifold with volume density $d\mu$. Let E and F be $\mathbb{K}$-vector bundles over M.

Lemma 1.1.27. *For any $P \in \mathscr{Diff}_k(E, F)$ there exists a unique $P^* \in \mathscr{Diff}_k(F^*, E^*)$ with*

$$\int_M \eta\,(Pu)\,d\mu = \int_M (P^*\eta)\,(u)d\mu. \tag{1.6}$$

Proof. Uniqueness:
We will reduce to what we know already about formally adjoint operators. To this extent, we choose auxiliary metrics $\langle\cdot,\cdot\rangle_E$ on E and $\langle\cdot,\cdot\rangle_F$ on F. Hermitian and Riemannian metrics can always be constructed using a partition of unity.
Let $\mathcal{E}$ and $\mathcal{F}$ be the anti-linear isomorphisms defined by

$$\mathcal{E} : E \to E^*, \quad e \mapsto \langle\cdot, e\rangle_E$$
$$\mathcal{F} : F \to F^*, \quad f \mapsto \langle\cdot, f\rangle_F.$$

For the left hand side in

$$\int_M \langle Pu, v\rangle_F d\mu = \int_M \langle u, P^t v\rangle_E d\mu \tag{1.7}$$

we get $\int_M \langle Pu, v\rangle_F d\mu = \int_M (\mathcal{F}(v))\,(Pu)d\mu$ while right hand side of (1.7) is given by $\int_M \langle u, P^t v\rangle_E d\mu = \int_M (\mathcal{E}(P^t v))\,(u)d\mu$. Substituting $\eta = \mathcal{F}v$ we find that (1.7) is equivalent to

$$\int_M \eta(Pu)d\mu = \int \left(\mathcal{E}(P^t\mathcal{F}^{-1}\eta)\right)(u)d\mu\,.$$

Comparison with (1.6) leads to

$$P^* = \mathcal{E} \circ P^t \circ \mathcal{F}^{-1} \tag{1.8}$$

which proves uniqueness.
Existence: We define P^* by (1.8). The same calculation backwards shows that the this is the desired operator. □

Definition 1.1.28. P^* is called the **dual operator** to P.

There ist no notion of selfadjointness in the context of dual operators since P and P^* act on different bundels. The dual operator will become important when working with distributions.

Remark 1.1.29. We can see directly from the definining equation (1.6) that we again have

$$(Q \circ P)^* = P^* \circ Q^*.$$

1.1.30. Since E is of finite dimension, we can identify E with $(E^*)^*$ using the map

$$e \mapsto (l \mapsto l(e))\,.$$

Show

$$(P^*)^* = P.$$

1.2 Lorentzian geometry

Wave equations are closely related to Lorentzian geometry. In fact, the d'Alembert operator on a Lorentzian manifold yields the prototype of a wave equation. In this section, we collect some basic concepts of Lorentzian geometry.

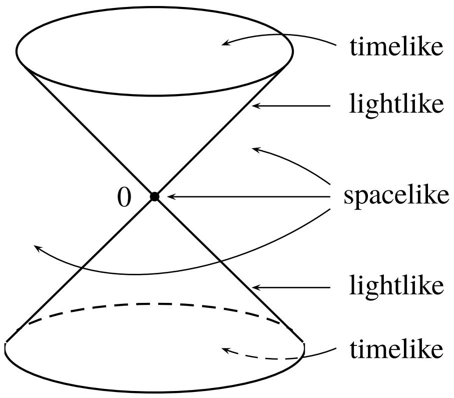
1.2.1 Future, past, and causality

For us, a Lorentzian metric is a semi-Riemannian metric with signature $(-, +, \ldots, +)$. In the literature, one also finds different conventions. Especially in the physics literature signature $(+, -, \ldots, -)$ is often used.

Definition 1.2.1. For a finite-dimensional real vector space $(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ with inner product of signature $(-, +, \ldots, +)$, we call $v \in V$

$$\begin{cases} \textbf{timelike}, & \text{if } \langle\!\langle v, v\rangle\!\rangle < 0 \\ \textbf{spacelike}, & \text{if } \langle\!\langle v, v\rangle\!\rangle > 0 \text{ or } v = 0 \\ \textbf{lightlike}, & \text{if } \langle\!\langle v, v\rangle\!\rangle = 0 \text{ and } v \neq 0 \\ \textbf{causal}, & \text{if } v \text{ is timelike or lightlike} \end{cases}$$

This definition will mostly be used for tangent vectors, i.e. if V is the tangent space of a Lorentzian manifold at some point.



Causality types of vectors

The set of timelike vectors in V consists of two connected components, similarly for lightlike and causal vectors. We want to call vectors in one component future-directed, the ones in the other component past-directed. Since there is no canonical choice of component we have to make one. On a Lorentzian manifold this choice needs to be done at each point, i.e. on each tangent space. This choice of connected component should depend continuously on the base point. This leads to the concept of time-orientation which we now define formally.

Definition 1.2.2. A vector field X is called timelike, spacelike, lightlike or causal, if $X(p)$ is timelike, spacelike, lightlike or causal, respectively, at every point $p \in M$.

Definition 1.2.3. Two timelike vector fields X, Y are equivalent, if for every $p \in M$ the vectors $X(p)$ and $Y(p)$ lie in the same connected component of timelike vectors, i.e. if $g(X(p), Y(p)) \leq 0$.
A **time orientation** on a Lorentzian manifold M is an equivalence class of continuous timelike vector fields.

Remark 1.2.4. There are Lorentzian manifolds that do not admit a time orientation.

Definition 1.2.5. A Lorentzian manifold is called **time orientable**, if it admits the choice of a time-orientation. A pair (M, Z) is called **time-oriented Lorentzian manifold**, if M is a Lorentzian manifold and Z is a time orientation on M.

Remark 1.2.6. It is also possible to define a time orientation as follows: The set of connected components of timelike vectors in the tangent spaces of a Lorentzian manifold carries a natural structure as a differentiable manifold. The footpoint map is then a twofold covering map. This is the time-orientation covering. A time orientation is then simply a continuous section of this covering.

In the following, by the usual abuse of notation, we also denote a time-oriented Lorentzian manifold (M, Z) simply by M.

Definition 1.2.7. Let (M, Z) be a time-oriented Lorentzian manifold. A causal vector $X \in T_p M$ is called **future-directed**, if $g(X(p), \xi(p)) \leq 0$ for $\xi \in Z$. The vector X is called **past-directed**, if $-X$ is future-directed.

Definition 1.2.8. A continuous piecewise C^1-curve $c : I \to M$ is called timelike, lightlike, spacelike, causal, future-directed or past-directed if $\dot{c}(t)$ is timelike, lightlike, spacelike, causal, future-directed or past-directed, respectively, for all $t \in I$ where $\dot{c}(t)$ is defined.

Definition 1.2.9. Let $I \subset \mathbb{R}$ be an open interval. A continuous curve $c : I \to M$ is called **inextendible**, if no reparametrization of the curve can be extended continuously.

Example 1.2.10. The line segment in $\mathbb{R}^n$ parametrized by $c : \mathbb{R} \to \mathbb{R}^n$, $c(t) = (\arctan(t), 0, \ldots, 0)$, is not inextendible because it can be reparametrized as $\tilde{c} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^n$, $\tilde{c}(t) = (t, 0, \ldots, 0)$, which has a continuous extension.
In contrast, the line $\mathbb{R} \to \mathbb{R}^n$, parametrized by $t \mapsto (t, 0, \ldots, 0)$, is inextendible.

Convention. From now on, M will always denote a time-oriented Lorentzian manifold, unless stated otherwise.

Definition 1.2.11. For $x \in M$ the set

$$I_+^M(x) := \left\{ y \in M \mid \text{there are future-directed timelike } C^1\text{-curves from } x \text{ to } y \right\}$$

is called the **chronological future** of x in M.

Remark 1.2.12. From the physical point of view, the chronological future of an event x consists of all events that can be influenced by x by means of signals slower than light.

Definition 1.2.13. For $x \in M$ the set

$$J_+^M(x) := \{x\} \cup \left\{y \in M \mid \text{there are future-directed causal } C^1\text{-curves from } x \text{ to } y\right\}$$

is called the **causal future of a point** of x in M .

Here the signals may travel with the speed of light.

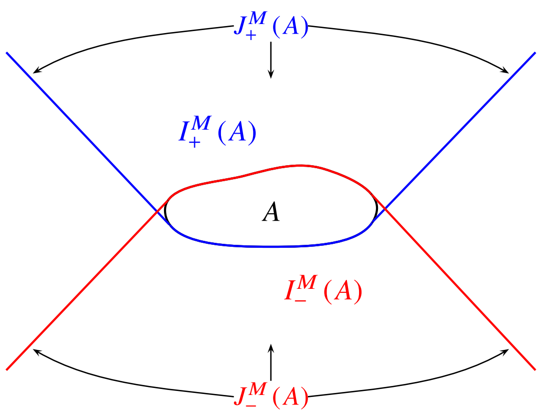
Definition 1.2.14. The chronological future of a subset $A \subset M$ is defined to be

$$I_+^M(A) := \bigcup_{x \in A} I_+^M(x)\,.$$

Similarly, the causal future of $A \subset M$ is

$$J_+^M(A) := \bigcup_{x \in A} J_+^M(x)\,.$$

Remark 1.2.15. In a similar way, one defines the chronological and causal pasts of a point x or a subset $A \subset M$. They are denoted by $I_-^M(x)$, $I_-^M(A)$, $J_-^M(x)$, and $J_-^M(A)$, respectively. We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$.



Causal and chronological future and past of subset A

Remark 1.2.16. The chronological and the causal futures have the following properties:

(i) $I_+^M(A)$ is always open;

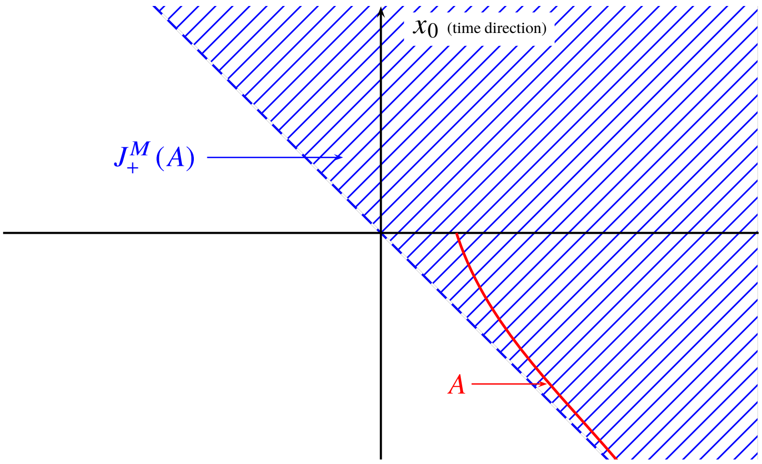
(ii) $I_+^M(A)$ is the interior of $J_+^M(A)$;

(iii) $J_+^M(A) \subset \overline{I_+^M(A)}$.

Similar statements hold for the pasts.

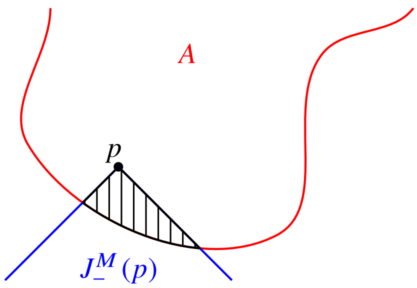
Warning. In (iii) equality does not always hold, $J_+^M(A)$ does not need to be closed, not even if A is closed.

Example 1.2.18. Let A be the curve as shown in the picture below; it is closed as a subset and asymptotic to a lightlike line in 2-dimensional Minkowski space. Its causal future $J_+^M(A)$ is the open half plane bounded by this lightlike line.



Causal future $J_+^M(A)$ is open

Definition 1.2.19. A subset $A \subset M$ is called **past compact** if $A \cap J_-^M(p)$ is compact for all $p \in M$. Similarly, one defines **future compact** subsets.



The subset A is past compact

Roughly speaking, past compact sets are possibly unbounded in the future, but bounded in the past.

Lemma 1.2.20. *Let M be a timeoriented Lorentzian manifold. Let $A \subset M$ be a subset. If for every $x \in M$ the intersection $A \cap J_-^M(x)$ is relatively compact in M, then $A \cap J_-^M(K)$ is relatively compact for every compact subset $K \subset M$.*
Similarly, if $A \cap J_+^M(x)$ is relatively compact for every $x \in M$, then $A \cap J_+^M(K)$ is relatively compact for every compact subset $K \subset M$.

Proof. It suffices to consider the first case. Let $K \subset M$ be compact. The family of open sets $\{I_-^M(y) \mid y \in M\}$ is an open cover of M. Since K is compact it is covered by a finite number of such sets, i.e. there exists a finite number of points $y_1, \dots, y_N \in M$ such that

$$K \subset I_-^M(y_1) \cup \dots \cup I_-^M(y_N).$$

We conclude

$$J_-^M(K) \subset J_-^M\left(I_-^M(y_1) \cup \dots \cup I_-^M(y_N)\right) \subset J_-^M(y_1) \cup \dots \cup J_-^M(y_N).$$

Since each $A \cap J_-^M(y_j)$ is relatively compact and

$$A \cap J_-^M(K) \subset \left(A \cap J_-^M(y_1)\right) \cap \dots \cap \left(A \cap J_-^M(y_N)\right)$$

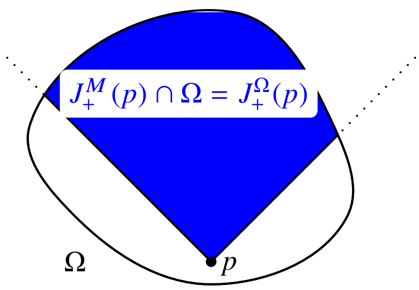
we have that $A \cap J_-^M(K)$ is relatively compact. $\square$

Definition 1.2.21. An open subset $\Omega \subset M$ in a time-oriented Lorentzian manifold is called **causally compatible** if

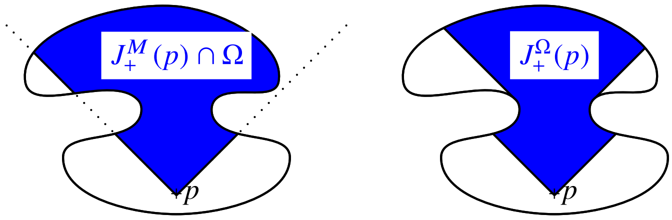
$$J_\pm^\Omega(x) = J_\pm^M(x) \cap \Omega$$

holds for all points $x \in \Omega$.

An open subset in a Lorentzian manifold is a Lorentzian manifold in its own right. If $\Omega \subset M$ is causally compatible, then whenever two points in Ω can be joined by a causal curve in M this can also be done inside Ω.



Causally compatible subset



Domain which is not causally compatible

Remark 1.2.22. If $\Omega \subset M$ is a causally compatible domain in a time-oriented Lorentzian manifold then

$$J_\pm^\Omega(A) = J_\pm^M(A) \cap \Omega$$

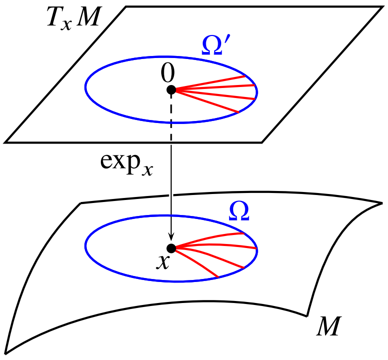
holds for arbitrary subsets $A \subset \Omega$.

Remark 1.2.23. Causal compatibility defines a transitive relation: If $\Omega \subset \Omega' \subset \Omega''$ are open subsets, if Ω is causally compatible in Ω', and if Ω' is causally compatible in Ω'', then so is Ω in Ω''.

1.2.2 Convexity

Using the Riemannian exponential map we can define starshaped and convex subsets of a semi-Riemannian manifold.

Definition 1.2.24. A domain $\Omega \subset M$ in a Lorentzian manifold is called **(geodesically) starshaped** with respect to a fixed point $x \in \Omega$ if there exists an open subset $\Omega' \subset T_x M$, starshaped with respect to 0 in the usual sense, such that the Riemannian exponential map $\exp_x$ maps Ω' diffeomorphically onto Ω.



Ω is geodesically starshaped w. r. t. x

The line segments in Ω' emanating from 0 are mapped by the Riemannian exponential map to segments of geodesics in Ω.

Definition 1.2.25. A domain $\Omega \subset M$ in a Lorentzian manifold is called **(geodesically) convex** if it is geodesically starshaped with respect to all of its points.

In particular, for any two points in Ω there is a unique geodesic segment in Ω connecting the points.

Remark 1.2.26. If Ω is starshaped with respect to x, then $\exp_x(I_\pm^{T_xM}(0) \cap \Omega') = I_\pm^\Omega(x)$.

Definition 1.2.27. Let Ω be a starshaped with respect to x. We define the smooth positive function $\mu_x : \Omega \to \mathbb{R}$ by

$$\mathrm{dvol} = \mu_x \cdot (\exp_x^{-1})^* (\mathrm{dz}),$$

where dvol is the Lorentzian volume form and dz is the standard Euclidean volume form on $T_x\Omega$. We call μ_x the **local density function**.

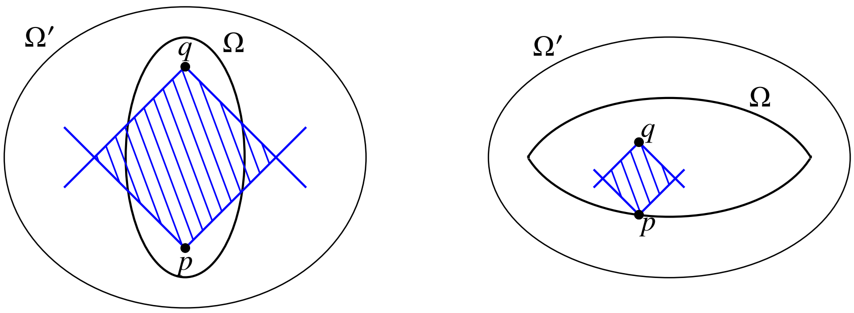
In normal coordinates about x we have $\mu_x = \sqrt{|\det(g_{ij})|}$.

Lemma 1.2.28. *For each open covering of a Lorentzian manifold there exists a refinement consisting of convex open subsets.*

For a proof see e.g. [13, Chap. 5, Lemma 10].

1.2.3 Cauchy hypersurfaces and global hyperbolicity

Definition 1.2.29. A domain Ω is called **causal** if $\overline{\Omega}$ is contained in a convex domain Ω' and if for any $x, y \in \overline{\Omega}$ the intersection $J_+^{\Omega'}(x) \cap J_-^{\Omega'}(y)$ is compact and contained in $\overline{\Omega}$.



convex, but not causal

causal

Convexity versus causality

Definition 1.2.30. A subset S of a connected time-oriented Lorentzian manifold is called **achronal** if each timelike curve meets S at most once. It is called **acausal** if each causal curve meets S at most once.

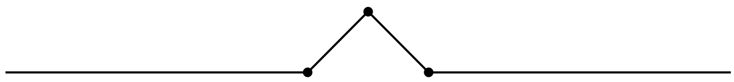
Every acausal subset is achronal, but the reverse implication does not hold.

Example 1.2.31. Let A be a segment of a lightlike line. Then A is achronal but not acausal.

Example 1.2.32. Let A be a segment of a spacelike line. Then A is achronal and acausal.

Definition 1.2.33. A subset $S \subset M$ of a connected time-oriented Lorentzian manifold M is a **Cauchy hypersurface** if each inextendible timelike curve in M meets S at exactly one point.

Example 1.2.34. Let M be the Minkowski space. Then every spacelike hyperplane is a Cauchy hypersurface. The hypersurface shown in the picture below is also a Cauchy hypersurface.



Cauchy hypersurface with piece of the light cone

By definition, the following implications hold:

$$\begin{array}{ccc} \text{Cauchy hypersurface} & \Rightarrow & \text{achronal} \\ & & \Uparrow \\ & & \text{acausal} \end{array}$$

No further implication holds true in general for these three notions.

Remark 1.2.35. We list some important facts about Cauchy hypersurfaces S:

(i) Every Cauchy hypersurface is a Lipschitz hypersurface, i.e. it can locally be written the graph of a Lipschitz function.[2] In particular, it is a closed topological submanifold of codimension 1.

(ii) The subset S is closed in M.

(iii) Every inextendible causal curve meets S.

(iv) Any two Cauchy hypersurfaces in M are homeomorphic.

(v) The causal future $J_+^M(S)$ is past compact and the causal past $J_-^M(S)$ is future compact.

Definition 1.2.36. A Lorentzian manifold is said to satisfy the **causality condition** if it does not contain any closed causal curve.

Remark 1.2.37. Physicists working in General Relativity prefer to work with spacetimes that satisfy the causality condition. The reason is that the existence of closed causal curves means that an event can affect itself. For example, you could change your own past, kill your grandfather before he gets to know your grandmother and thus prevent that you would ever have come to existence...

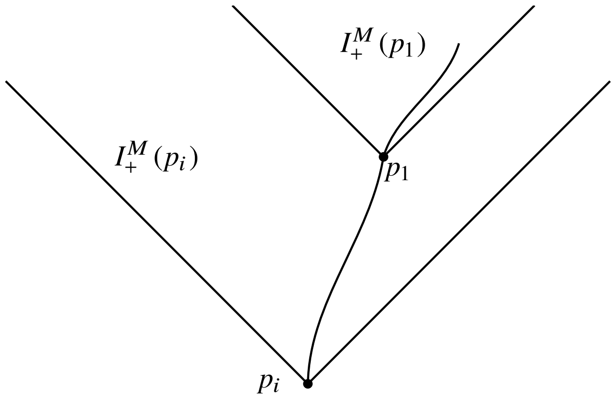
[2] This shows that Cauchy hypersurfaces are surprisingly regular. By Rademacher's theorem a Cauchy hypersurface is very close to a differentiable hypersurface.

Example 1.2.38. The Minkowski space satisfies the causality condition. More generally, any convex Lorentzian manifold satisfies the causality condition.

Every compact M does not satisfy the causality condition, because of the following Lemma:

Lemma 1.2.39. *Let M be a compact time-oriented Lorentzian manifold. Then it contains a closed causal curve.*

Proof. The family $\left\{I_+^M(p) \mid p \in M\right\}$ forms an open cover of M. Since M is compact, there is a finite subcover $\left\{I_+^M(p_1), \ldots, I_+^M(p_N)\right\}$. W.l.o.g. we can assume that $I_+^M(p_1) \not\subset I_+(p_i)$ for all $i = 2, \ldots n$; otherwise simply discard $I(p_1)$. Then $p_1 \notin I_+(p_i)$ for all $i \geq 2$ since otherwise $I_+(p_1)$ would have to be contained in $I_+(p_i)$, see the picture.



$I_+^M(p_1) \subset I_+^M(p_i)$

Therefore $p_1 \in I_+(p_1)$, i.e. p_1 is contained in its own future. Thus there is a closed future-directed timelike curve connecting p_1 to itself. □

Now we introduce the class of Lorentzian manifolds which will be suitable for the analysis of wave equations.

Definition 1.2.40. A time-oriented Lorentzian manifold is called **globally hyperbolic** if it satisfies the causality condition and if for all $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$[3]is compact.

Example 1.2.41. Minkowski space is globally hyperbolic.

Example 1.2.42. Compact Lorentzian manifolds are never globally hyperbolic because the causality condition fails by Lemma 1.2.39.

Example 1.2.43. If we remove one point from Minkowski space, the causality condition of course still holds but compactness of the causal diamonds fails for some points p and q (namely if p lies in the past of the "hole" and q in its future). Thus the punctured Minkowski space is not globally hyperbolic.

Remark 1.2.44. If M is a globally hyperbolic Lorentzian manifold, then a nonempty open subset $\Omega \subset M$ is itself globally hyperbolic if and only if for any $p, q \in \Omega$ the intersection $J_+^\Omega(p) \cap J_-^\Omega(q) \subset \Omega$ is compact. Indeed non-existence of closed causal curves in M directly implies non-existence of such curves in Ω.

The following important structure theorem tells us that globally hyperbolic manifolds can be characterized in several different ways.

Theorem 1.2.45 (Bernal-Sánchez). *Let M be a time-oriented Lorentzian manifold. Then the following are equivalent:*

(1) M is globally hyperbolic;

(2) There exists a Cauchy hypersurface in M;

(3) The manifold M is isometric to $\mathbb{R} \times S$ with metric $-N^2 dt^2 + g_t$ where $N : M \to \mathbb{R}$ is a smooth positive function[4], g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and all sets $\{t_0\} \times S$ are Cauchy hypersurfaces in M.

Proof. We first modify condition (1) to
(1'): M satisfies the *strong* causality condition[5] and the intersection $J_+^M(p) \cap J_-^M(q)$ is compact for all $p, q \in M$.
"(1') ⇒ (3)" has been shown by Bernal and Sánchez in [5, Thm. 1.1].
"(3) ⇒ (2)" is trivial.
"(2) ⇒ (1')" is well known, see e.g. [13, Cor. 39, p. 422].
"(1') ⇔ (1)" has been shown by Bernal and Sánchez in [8, Thm. 3.2]. □

[3] This intersection is sometimes called **causal diamond** of p and q

[4] N is known as the *lapse function*.

[5] A Lorentzian manifold is said to satisfy the **strong causality condition** if there are no *almost* closed causal curves in M. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.

Example 1.2.46. Let M be the Minkowski space. Then M is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. We have $M = \mathbb{R} \times S$ with $S = \mathbb{R}^{n-1}$, endowed with the time-independet Euclidean metric. The lapse function is $N \equiv 1$.

Example 1.2.47. Let $(S, \hat{g})$ be a Riemannian manifold and $I \subset \mathbb{R}$ an interval. Let $f : I \to \mathbb{R}$ be a smooth positive function. The manifold $M = I \times S$ with the metric $g = -dt^2 + f(t)^2 \cdot \hat{g}$ is globally hyperbolic if and only if $(S, \hat{g})$ is complete, see [3, Lem A.5.14]. This applies in particular if S is compact.
For example the well-known Robertson-Walker spacetimes and, in particular, the Friedman cosmological models are of this type.
Furthermore, the deSitter spacetime is of this type, where $I = \mathbb{R}$, $S = S^{n-1}$, $\hat{g}$ is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$.

Example 1.2.48. One can show that the Anti-deSitter spacetime is not globally hyperbolic.

Example 1.2.49. The interior and exterior Schwarzschild solutions are globally hyperbolic:
Let $m > 0$ be a real number. The physical interpretation of this constant is the mass of a central celestial body or a black hole. We set

$$M_{\text{ext}} = \mathbb{R} \times (2m, \infty) \times S^2,$$
$$M_{\text{int}} = \mathbb{R} \times (0, 2m) \times S^2.$$

The metric is given by

$$g = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 g_{S^2}$$

with the function $h(r) = 1 - \frac{2m}{r}$.
For the exterior Schwarzschild spacetime we have $M_{\text{ext}} = \mathbb{R} \times S$ with $S = (2m, \infty) \times S^2$. Here $N^2 = h$ and $g_t = \frac{1}{h(r)}dr^2 + r^2 g_{S^2}$ does not depend on t. The level sets $\{t = t_0\} \times (2m, \infty) \times S^2$ are Cauchy hypersurfaces.
On the interior Schwarzschild spacetime the function h is negative. So in order to write M_{int} as a product as in (3) of Theorem 1.2.45 we have to rearrage the metric. Now t is a spacelike function and r is timelike. The sets $\mathbb{R} \times \{r = r_0\} \times S^2$ are Cauchy hypersurfaces.

Corollary 1.2.50. *On every globally hyperbolic Lorentzian manifold M there exists a smooth function $h : M \to \mathbb{R}$ all of whose level sets are smooth spacelike Cauchy hypersurfaces.*

Proof. Define h to be the composition $t \circ \Phi$ where $\Phi : M \to \mathbb{R} \times S$ is the isometry given in Theorem 1.2.45 and $t : \mathbb{R} \times S \to \mathbb{R}$ is the projection onto the first factor. □

Lemma 1.2.51. *The gradient vector field of a function as in Corollary 1.2.50 is timelike.*

Proof. We choose local coordinates $x^2, \dots, x^n$ on S and complement them to coordinates $x^1 = h, x^2, \dots, x^n$ on M. In these coordinates the metric takes the form

$$(g_{ij}) = \begin{pmatrix} -N^2 & 0 & \cdots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix} \quad \text{and therefore} \quad (g^{ij}) = \begin{pmatrix} -N^{-2} & 0 & \cdots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

We compute

$$\operatorname{grad} h = \sum_{ij} g^{ij} \frac{\partial h}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_j g^{1j} \frac{\partial}{\partial x^j} = g^{11} \frac{\partial}{\partial x^1} = -\frac{1}{N^2} \frac{\partial}{\partial h}.$$

Since

$$g(\operatorname{grad} h, \operatorname{grad} h) = \frac{1}{N^4} \underbrace{g\left(\frac{\partial}{\partial h}, \frac{\partial}{\partial h}\right)}_{=-N^2} = -\frac{1}{N^2} < 0$$

the gradient $\operatorname{grad} h$ is timelike. □

Definition 1.2.52. A function h satisfying the properties given in Corollary 1.2.50 with past-directed gradient is called a **Cauchy time-function**.

The condition that $\operatorname{grad} h$ is past-directed can always be achieved by either reversing the time-orientation or by replacing h by $-h$. Since the gradient is past-directed timelike we have for any future-directed causal curve $s \mapsto c(s)$

$$\frac{\partial}{\partial h}(h \circ c) = g\left(\operatorname{grad} h, c'(s)\right) > 0.$$

Hence h increases along each future-directed curve.

We quote an enhanced form of Theorem 1.2.45, which will be needed in the global theory of wave equations.

Theorem 1.2.53 (Bernal-Sánchez). *Let M be a globally hyperbolic manifold and S be a spacelike smooth Cauchy hypersurface in M. Then there exists a Cauchy time-function $h : M \to \mathbb{R}$ such that $S = h^{-1}(\{0\})$.*

This result is also due to A. Bernal and M. Sánchez, see [7, Theorem 1.2].

Remark 1.2.54. Any given smooth spacelike Cauchy hypersurface in a (necessarily globally hyperbolic) Lorentzian manifold is therefore the leaf of a foliation by smooth spacelike Cauchy hypersurfaces.

Lemma 1.2.55. *Let M be a globally hyperbolic manifold. Then for every compact subset K of M the subsets $J_+^M(K)$ and $J_-^M(K)$ are closed.*

For a proof see [3, Lem. A.5.1].

Proposition 1.2.56. *Let M be a globally hyperbolic manifold. Let $S \subset M$ be a Cauchy hypersurface in M, let $K, K' \subset M$ be compact subsets of M. Then we have:*

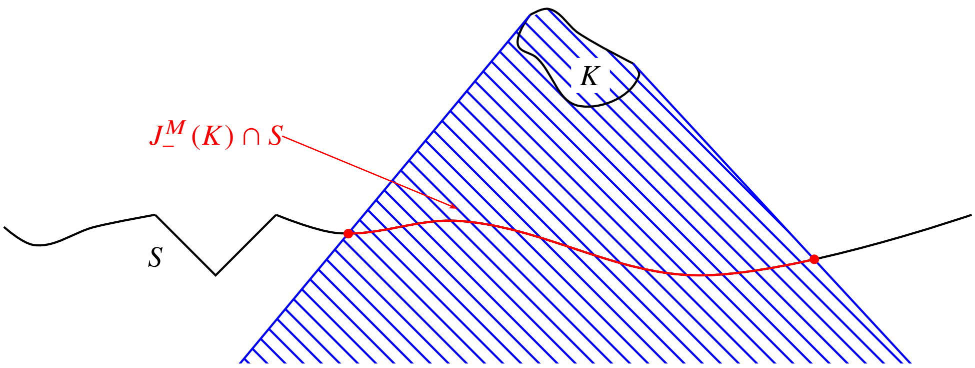
(i) $J_\pm^M(K) \cap S$ *is compact.*

(ii) $J_\pm^M(K) \cap J_\mp^M(S)$ *is compact*

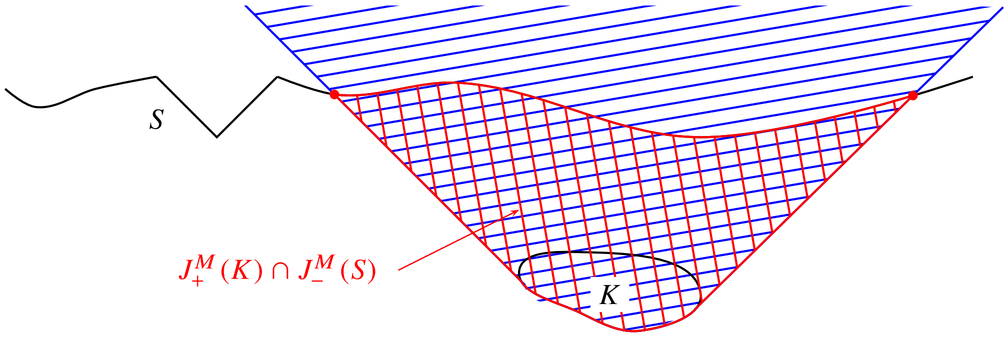
(iii) $J_+^M(K) \cap J_-^M(K')$ *is compact*

For the proofs see [3, Cor. A.5.4 and Lem. A.5.7].
We illustrate these facts:

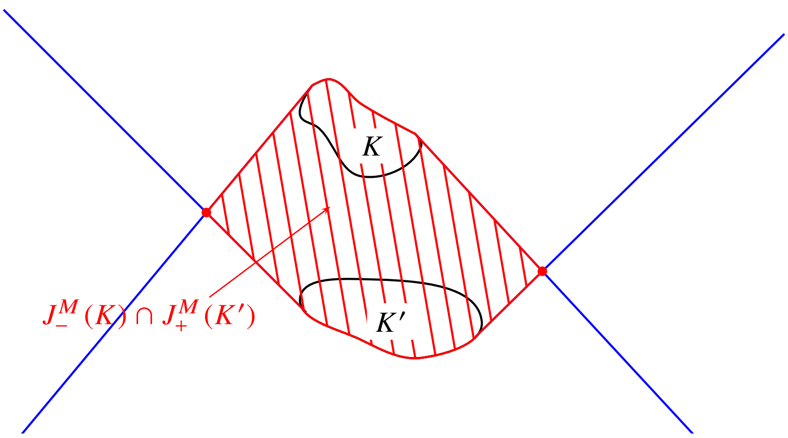
(i) $J_\pm^M(K) \cap S$ is compact



(ii) $J_\pm^M(K) \cap J_\mp^M(S)$ is compact



(iii) $J_+^M(K) \cap J_-^M(K')$ is compact



Notation 1.2.57. For two points $p, q \in M$, we define the relations

$$p < q \Leftrightarrow q \in I_+^M(p)$$
$$p \leq q \Leftrightarrow q \in J_+^M(p).$$

Since timelike and causal curves can be concatenated these relations are transitive.
In Riemannian geometry the length of curves gives rise to a distance function on the manifold. This metric is compatible with the given topology on the manifold. In Lorentzian geometry there is no such metric but the following provides a weak replacement for it.

Definition 1.2.58. Let $c : I \to M$ be a piecewise C^1-curve on a Lorentzian manifold M. The **length** $L[c]$ is defined by

$$L[c] := \int_a^b \sqrt{|g(\dot{c}(t), \dot{c}(t))|}\, dt.$$

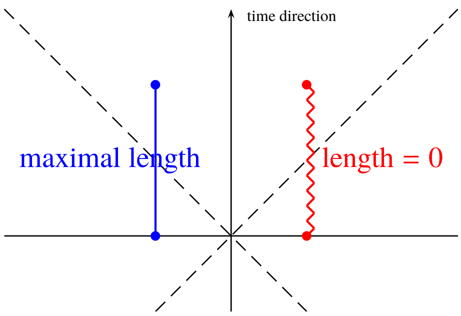
Here we take the absolute value in order to assure positiveness under the square root. We do this to avoid case distinctions by the causal type of c. We will really need the length only for causal curves so we could have replaced the absolute value by a minus sign.

Definition 1.2.59. The **time-separation** on a Lorentzian manifold M is the function $\tau : M \times M \to \mathbb{R} \cup \{\infty\}$ defined by

$$\tau(p, q) := \begin{cases} \sup\{L[c] \mid c \text{ future directed causal curve from } p \text{ to } q, & \text{if } p \leq q, \\ 0, & \text{otherwise,} \end{cases}$$

for all p, q in M.

Observe that we take the supremum of all connecting causal curves whereas in Riemannian geometry one take the infimum to define the distance function. To illustrate that this is reasonable one can check that in Minkowski space the line segment connection two points with $p \leq q$ is the *longest* causal curve connecting them. In fact, on convex subsets of a Lorentzian manifold or on globally hyperbolic manifolds the supremum is always attained by a causal geodesic. This is an analog to the Hopf-Rinow theorem for complete Riemannian manifolds.



Proposition 1.2.60. *Let M be a timeoriented Lorentzian manifold. Let p, q, and $r \in M$. Then*

1. $\tau(p, q) > 0$ *if and only if* $p < q$.
2. *The function τ is lower semi-continuous on $M \times M$. If M is convex or globally hyperbolic,*

then τ is finite and continuous.

3. *The function τ satisfies the* ***inverse triangle inequality****: If $p \le q \le r$, then*

$$\tau(p,r) \ge \tau(p,q) + \tau(q,r). \tag{1.9}$$

Proof. See e. g. Lemmas 16, 17, and 21 from Chapter 14 in [13] for a proof. □

1.2.4 Compactness properties

We introduce more concepts of compactness for closed subsets of globally hyperbolic manifolds and show their interrelation. We start by characterizing past-compact sets.

Lemma 1.2.61. *Let M be globally hyperbolic. For any closed subset $A \subset M$ the following are equivalent:*

(i) A is past compact;

(ii) there exists a smooth spacelike Cauchy hypersurface $S \subset M$ such that $A \subset J_+(S)$;

(iii) there exists a surjective Cauchy time function $t : M \to \mathbb{R}$ which is bounded from below on A.

Proof. "(ii) ⇒ (i)" Let $A \subset J_+(S)$ be a closed subset, let $x \in M$. Then $A \cap J_-(x) \subset J_+(S) \cap J_-(x)$. We know by Proposition 1.2.56 that $J_+(S) \cap J_-(x)$ is compact. Furthermore, A and $J_-(x)$ are closed sets, hence $A \cap J_-(x)$ is closed. This shows that $A \cap J_-(x)$ is compact as closed subset of a compact set for all $x \in M$. Hence A is past compact.
"(i) ⇒ (ii)" Let A be past compact. Then $J_+(A)$ is also past compact (and, in particular, closed). Moreover, $M' := M \setminus J_+(A)$ is an open subset of M with the property $J_-(M') = M'$. Hence M' is globally hyperbolic itself (exercise). Let S be a smooth spacelike Cauchy hypersurface in M'. Since $A \subset J_+(A) \subset J_+(S)$ it remains to show that S is also a Cauchy hypersurface in M.
Let c be an inextendible w.l.o.g. future-directed timelike curve in M. Once c has entered $J_+(A)$ it remains in $J_+(A)$. Since $J_+(A)$ is past compact and c is inextendible, c must also meet M'. Thus c is the concatenation of an inextendible future-directed timelike curve c_1 in M' and a (possibly empty) curve c_2 in $J_+(A)$. Since c_1 meets S exactly once, so does c. This shows that S is a Cauchy hypersurface in M as well.
"(iii) ⇒ (ii)" Choose $T < \inf(t(A))$. Then $S := t^{-1}(T)$ is a smooth spacelike Cauchy hypersurface such that $A \subset J_+(S)$.
"(ii) ⇒ (iii)" Let S be a smooth spacelike Cauchy hypersurface in M such that $A \subset J_+(S)$. By Theorem 1.2.53 there exists a Cauchy time function $t : M \to \mathbb{R}$ such that $S = t^{-1}(\{0\})$ W.l.o.g. we

can assume that t is surjective (otherwise compose with orientation-preserving diffeomorphism). Since $A \subset J_+(S)$ we have that $t \geq 0$ on A. $\square$

Reversing future and past, we see that a closed subset $A \subset M$ is future compact if and only if $A \subset J_-(S)$ for some Cauchy hypersurface $S \subset M$. This in turn is equivalent to the existence of a surjective Cauchy time function $t : M \to \mathbb{R}$ which is bounded from above on A.

Remark 1.2.62. The proof of "(ii) ⇒ (i)" did not use that the Cauchy hypersurface is smooth and spacelike. Therefore dropping the conditions "smooth and spacelike" in (ii) would yield another equivalent characterization of past-compact sets.

Lemma 1.2.63. *Let M be globally hyperbolic. For any past-compact subset $A \subset M$ there exists a past-compact subset $A' \subset M$ such that A is contained in the interior of A'.*

Proof. Let $A \subset M$ be past compact. Choose a Cauchy hypersurface $S \subset M$ such that $A \subset J_+(S)$. Choose a second Cauchy hypersurface $S' \subset I_-(S)$. Then $A' := J_+(S')$ does the job. $\square$

Analogous statements hold for future-compact sets and for temporally-compact sets as defined here:

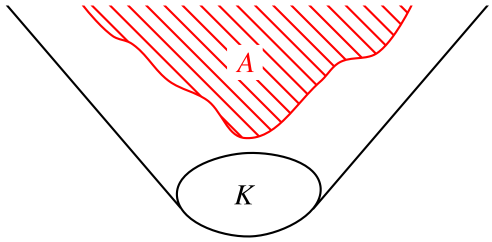
Definition 1.2.64. Let M be globally hyperbolic. We call a subset $A \subset M$

(a) **strictly past compact** if it is closed and there is a compact subset $K \subset M$ such that $A \subset J_+(K)$.

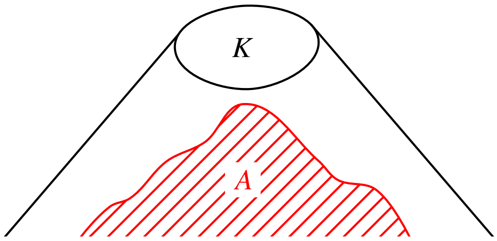
(b) **strictly future compact** if it is closed and there is a compact subset $K \subset M$ such that $A \subset J_-(K)$.

(c) **spatially compact** if A is closed and there exists a compact subset $K \subset M$ with $A \subset J(K)$.

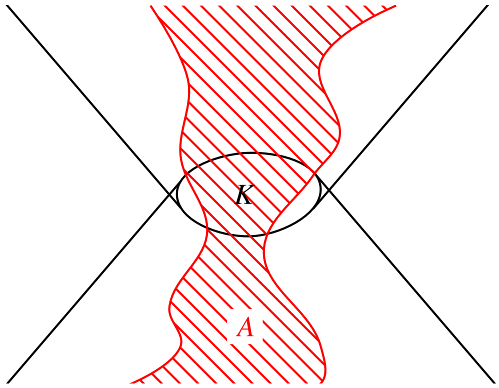
(d) **temporally compact** if A is past compact and future compact.



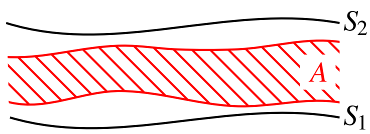
(a) strictly past compact



(b) strictly future compact



(c) spatially compact



(d) temporally compact

Remark 1.2.65. By Lemma 1.2.61 for past and future-compact sets, A is temporally compact if and only if $A \subset J_+(S_1) \cap J_-(S_2)$ for some Cauchy hypersurfaces $S_1, S_2 \subset M$.

We have the following analog to Lemma 1.2.61:

Lemma 1.2.66. *Let M be globally hyperbolic. For any closed subset $A \subset M$ the following holds:*

(i) A is strictly past compact if and only if there exists a smooth spacelike Cauchy hypersurface $S \subset M$ and a compact subset $K_S \subset S$ such that $A \subset J_+(K_S)$;

(ii) A is strictly future compact if and only if there exists a smooth spacelike Cauchy hypersurface $S \subset M$ and a compact subset $K_S \subset S$ such that $A \subset J_-(K_S)$;

(iii) A is spatially compact if and only if for all smooth spacelike Cauchy hypersurfaces $S \subset M$ there exists a compact subset $K_S \subset S$ such that $A \subset J(K_S)$.

Proof. (i) "$\Leftarrow$" is trivial: if $A \subset J_+(K_S)$, then A is strictly past compact by definition.
(i) "$\Rightarrow$" Let A be strictly past compact and let $K \subset M$ be a compact subset such that $A \subset J_+(K)$. Then choose a smooth spacelike Cauchy hypersurface $S \subset M$ such that $K \subset J_+(S)$ and put

$K_S := S \cap J_-(K)$. Then K_S is compact by Proposotion 1.2.56 and

$$A \subset J_+(K) \subset J_+(J_+(S) \cap J_-(K)) = J_+(S \cap J_-(K)) = J_+(K_S).$$

(ii) The proof is analogous done by reversing future and past.
(iii) "$\Leftarrow$" is trivial. If A is spatially compact and $S \subset M$ a Cauchy hypersurface, then $K_S := S \cap J(K)$ does the job.
(iii) "$\Rightarrow$" Let A be spatially compact and K a compact set such that $A \subset K$. Let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Then $S \cap J(K)$ is compact by Proposition 1.2.56. Since $K \subset J(K_S)$ we have $J(K) \subset J(K_S)$. Hence since $A \subset J(K)$ we found $A \subset J(K_S)$.

We have the following diagram of implications of possible properties of a closed subset of a globally hyperbolic manifold M:

$$\begin{tikzcd}
 & \text{spatially compact} & \\
\text{strictly past compact} \arrow[ur, Rightarrow] \arrow[d, Rightarrow] & & \text{strictly future compact} \arrow[ul, Rightarrow] \arrow[d, Rightarrow] \\
 & \text{compact} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[dd, Rightarrow] & \\
\text{past compact} & & \text{future compact} \\
 & \text{temporally compact} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

None of the reverse implications in the diagram holds in general.

Remark 1.2.67. The terminology "spatially compact" is justified by the following observation: Let $A \subset M$ be spatially compact and let $S \subset M$ be a Cauchy hypersurface. Then $A \cap S \subset J(K) \cap S$ which is compact by Proposition 1.2.56. Hence $A \cap S$ is compact for any Cauchy hypersurface.

In a special case the diagram simplifies considerably, namely if M itself is spatially compact. Recall that since all Cauchy hypersurfaces are homeomorphic they are all compact or all non-compact.

Lemma 1.2.68. *The globally hyperbolic manifold M is spatially compact if and only if it has compact Cauchy hypersurfaces.*

Proof. If the Cauchy hypersurfaces are compact, let S be one of them. Then $M = J(S)$, hence M is spatially compact.
Conversely, if M is spatially compact, then Remark 1.2.67 with $A = M$ shows that the Cauchy hypersurfaces are compact. □

Lemma 1.2.69. *Let M be globally hyperbolic and spatially compact. Let $A \subset M$ be closed. Then the following are equivalent:*

(i) A is strictly past compact;

(ii) A is past compact;

(iii) some Cauchy time function $t : M \to \mathbb{R}$ attains its minimum on A;

(iv) all Cauchy time functions $t : M \to \mathbb{R}$ attain their minima on A.

Proof. "(iv)⇒(iii)" is clear.
"(iii)⇒(ii)" Let $t : M \to \mathbb{R}$ be a Cauchy time function which attains its minimum on A. By composing with an orientation-preserving diffeomorphism $t(M) \to \mathbb{R}$, we may w.l.o.g. assume that t is surjective. Now Lemma 1.2.61 shows that A is past compact.
"(ii)⇒(i)" Let A be past compact. Then Lemma 1.2.61 shows that $A \subset J_+(S)$ for a Cauchy hypersurface $S \subset M$. From Lemma 1.2.68 we know that S is compact and therefore A is strictly past compact by Lemma 1.2.66 with $K_S = S$.
"(i)⇒(iv)" Let $A \subset J_+(K)$ for some compact subset $K \subset M$ and let t be a Cauchy time function. Choose T larger than the infimum of t on A. Since $A \cap J_-(t^{-1}(T))$ is contained in the compact set $J_+(K) \cap t^{-1}((-\infty, T]) = J_+(K) \cap J_-(t^{-1}(T))$, the function t attains its minimum t_0 on this set. On the rest of A, the values of t are even larger than T, hence t_0 is the minimum of t on all of A.

Remark 1.2.70. If M is spatially compact, then every closed subset of $A \subset M$ is spatially compact. Moreover, if A is temporally compact then any Cauchy time function $t : M \to \mathbb{R}$ attains its maximum s_+ and its minimum s_- by Lemma 1.2.69. Thus $A \subset t^{-1}([s_-, s_+]) \approx S \times [s_-, s_+]$ where $S = t^{-1}(s_-)$ is a Cauchy hypersurface. Since S is compact, so is A.
Summarizing, the diagram of implications for closed subsets simplifies as follows for spatially compact M:

$$\begin{array}{ccccc}
\text{strictly past compact} & \Longleftarrow & \text{compact} & \Longrightarrow & \text{strictly future compact} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{past compact} & \Longleftarrow & \text{temporally compact} & \Longrightarrow & \text{future compact}
\end{array}$$

We will need the following duality result:

Lemma 1.2.71. *Let M be globally hyperbolic and let $A \subset M$ be closed. Then the following holds:*

(i) A is past compact if and only if $A \cap B$ is compact for all strictly future compact sets B;

(ii) A is future compact if and only if $A \cap B$ is compact for all strictly past compact sets B;

(iii) A is temporally compact if and only if $A \cap B$ is compact for all spatially compact sets B;

(iv) A is strictly past compact if and only if $A \cap B$ is compact for all future compact sets B;

(v) A is strictly future compact if and only if $A \cap B$ is compact for all past compact sets B;

(vi) A is spatially compact if and only if $A \cap B$ is compact for all temporally compact sets B.

Proof. (i) "$\Leftarrow$
If $A \cap B$ is compact for every strictly future compact B, then, in particular, $A \cap J_-(x)$ is compact for every $x \in M$. Hence A is past compact.
(i) "$\Rightarrow$"
Let A be past compact and B be strictly future compact. Let $S \subset M$ be some Cauchy hypersurface with $A \subset J_+(S)$ and $K \subset M$ be some compact subset with $B \subset J_-(K)$. Then $A \cap B \subset J_+(S) \cap J_-(K)$, hence $A \cap B$ is contained in a compact set, hence compact itself.
(ii)
The proof is analogous.
(iii) "$\Leftarrow$"
If $A \cap B$ is compact for every spatially compact B, then, in particular, $A \cap J_+(x)$ and $A \cap J_-(x)$ are compact for every $x \in M$. Hence A is temporally compact.
(iii) "$\Rightarrow$"
Let A be temporally compact and B be spatially compact. We choose a compact $K \subset M$ with $B \subset J(K)$. By (i), $A \cap J_-(K)$ is compact and by (ii), $A \cap J_+(K)$ is compact. Thus $A \cap B \subset A \cap J(K) = (A \cap J_+(K)) \cup (A \cap J_-(K))$ is compact.
(iv) "$\Rightarrow$"
By (ii) the intersection of a strictly past compact set and a future compact set is compact.
(iv) "$\Leftarrow$"
Now assume A is not strictly past compact. We have to find a future compact set B such that $A \cap B$ is noncompact. Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset M$ be an exhaustion by compact subsets. We choose the exhaustion such that every compact subset of M is contained in K_j for sufficiently large j. Since A is not strictly past compact there exists $x_j \in A \setminus J_+(K_j)$ for every j. The set $B := \{x_1, x_2, x_3, \ldots\}$ is not compact because otherwise, for sufficiently large j, we would have $B \subset K_j \subset J_+(K_j)$ contradicting the choice of the x_i. But B is future compact. Namely, let $x \in M$. Then $x \in K_j$ for j large and therefore $B \cap J_+(x) \subset B \cap J_+(K_j) \subset \{x_1, \ldots, x_{j-1}\}$ is finite, hence compact. Now $A \cap B = B$ is not compact which is what we wanted to show.
(v)
The proof is analogous.
(vi) "$\Rightarrow$"

We know already by (iii) that the intersection of a temporally compact and a spatially compact set is always compact.
(vi) "$\Leftarrow$"
If A is not spatially compact, then the same construction as in the proof of (iv) with $J_+(K_j)$ replaced by $J(K_j)$ yields a noncompact set $B \subset A$ which is temporally compact. This concludes the proof. $\square$

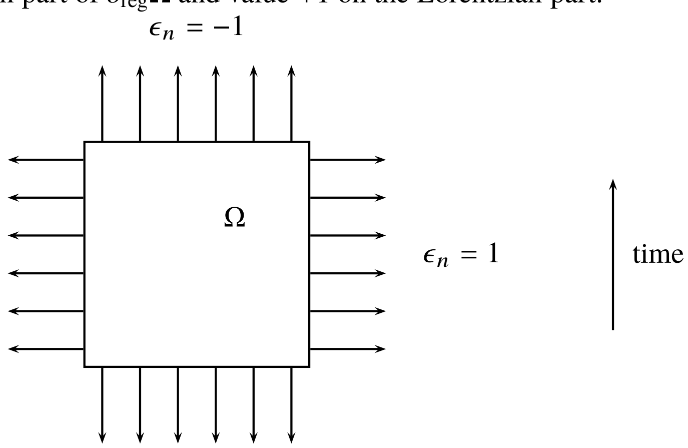
1.2.5 Gauss' divergence theorem

We continue our collection of facts in Lorentzian Geometry by stating Gauss' divergence theorem. We turn our attention to the signs arising in the Lorentzian case.

Theorem 1.2.72 (Gauss' divergence theorem). *Let M be a Lorentzian manifold and let $\Omega \subset M$ be a domain with piecewise C^1-boundary. We assume that the induced metric on the regular part of the boundary of Ω is non-degenerate, i. e., it is either Riemannian or Lorentzian on each connected component. Let n denote the exterior normal field along the regular part of the boundary $\partial_{\mathrm{reg}}\Omega$, normalized to $\epsilon_n := g(n, n) = \pm 1$.*
Then for every C^1-vector field X on M with $\mathrm{supp}(X) \cap \overline{\Omega}$ compact we have

$$\int_{\Omega} \mathrm{div}(X)\mathrm{dvol} = \int_{\partial_{\mathrm{reg}}\Omega} \epsilon_n g(X, n)\, \mathrm{dA}.$$

Remark 1.2.73. The singular part of $\partial\Omega$ forms a null set. Thus we may as well integrate over all of $\partial\Omega$ in the right hand side of this formula. The function ϵ_n is locally constant with value -1 on the Riemannian part of $\partial_{\mathrm{reg}}\Omega$ and value $+1$ on the Lorentzian part.



The divergence theorem can easily be derived from the Stokes' theorem which does not involve any metric. One replaces the vector field by a dual $(n-1)$-form and expresses the divergence by means of the exterior derivative.

Now we come back to the local density function μ_x. For the local calculation of solutions of wave equations we need to know some derivatives of μ_x.

Lemma 1.2.74. *Let $\Omega \subset M$ be geodesically starshaped with respect to $x \in \Omega$. Then the function μ_x defined in (1.2.27) satisfies*

$$\mu_x(x) = 1, \quad d\mu_x|_x = 0, \quad \mathrm{Hess}(\mu_x)|_x = -\frac{1}{3}\mathrm{ric}_x, \quad (\Box\mu_x)(x) = \frac{1}{3}\mathrm{scal}(x),$$

where Hess *denotes the Hessian,*[6] ric_x *the Ricci curvature considered as a bilinear form on* $T_x\Omega$ *and* scal *is the scalar curvature.*

For a proof see [3, Lem. 1.3.17] .

Corollary 1.2.75. *Let $\Omega \subset M$ be geodesically starshaped with respect to $x \in \Omega$. For the function μ_x we have*

$$(\Box\mu_x^{-1/2})(x) = -\frac{1}{6}\mathrm{scal}(x).$$

Proof. For C^2-functions $f : M \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ a straightforward computation yields the useful equation

$$\Box(F \circ f) = -(F'' \circ f)\langle df, df\rangle + (F' \circ f)\Box f.$$

Using this with $f = \mu_x$ and $F(t) = t^{-1/2}$ and Lemma 1.2.74 we compute

$$(\Box\mu_x^{-1/2})(x) = 0 - \frac{1}{2} \cdot \mu_x^{-\frac{3}{2}}|_x \cdot \Box\mu_x|_x = -\frac{1}{6}\mathrm{scal}(x). \qquad \Box$$

We continue our preparations with another function that we are going to need in connection to Riesz-Distributions.

Definition 1.2.76. Let $\Omega \subset M$ be open and geodesically starshaped with respect to $x \in \Omega$. We put

$$\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R}$$

where $\gamma : T_xM \to \mathbb{R}$ is defined by $\gamma(X) = -g(X, X)$.

[6] For a C^2-function f the **Hessian** at x is the symmetric bilinear form $\mathrm{Hess}(f)|_x : T_xM \times T_xM \to \mathbb{R}$, $\mathrm{Hess}(f)|_x(X,Y) := g\left(\nabla_X \mathrm{grad}\, f, Y\right)$. The d'Alembert operator can be written by $\Box f := -\mathrm{tr}(\mathrm{Hess}(f))$

Lemma 1.2.77. *Let $\Omega \subset M$ be open and geodesically starshaped with respect to $x \in \Omega$. Let $n = \dim(M)$. Then the following holds on Ω:*

1. $g\left(\operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\right) = -4\Gamma_x$*;*
2. *On $I_+^\Omega(x)$ (or on $I_-^\Omega(x)$) the gradient $\operatorname{grad}\Gamma_x$ is a past-directed (or future-directed, respectively) timelike vector field;*
3. $\Box\Gamma_x = 2n - g\left(\operatorname{grad}\Gamma_x, \operatorname{grad}\left(\log(\mu_x)\right)\right)$.

The general proof can be found in [3, Lem. 1.3.19]. We do it here in the case that M is the Minkowski space. Since M and T_xM are canonically isometric via the exponential map there is no essential difference between γ and Γ and we regard γ as a function on M. In standard coordinates $(x^1, \ldots, x^n)$ we have

$$g = -(dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2,$$

and

$$\gamma = (x^1)^2 - (x^2)^2 - \ldots - (x^n)^2.$$

Moreover, note that

$$(g^{ij}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

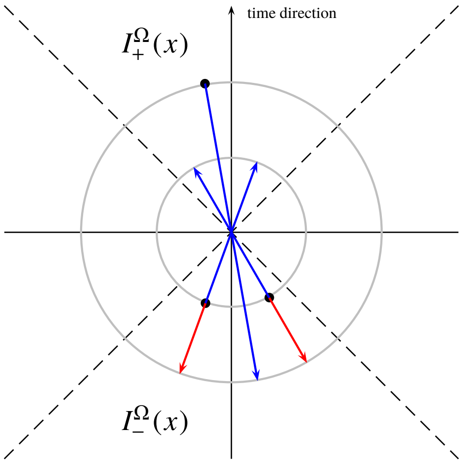
The gradient of γ turns out to be

$$\begin{aligned} \operatorname{grad}\gamma &= \sum_i g^{ij}(x)\,\frac{\partial\gamma}{\partial x^i}\frac{\partial}{\partial x^j} \\ &= g^{11}\cdot 2x^1\frac{\partial}{\partial x^1} - g^{22}\cdot 2x^2\frac{\partial}{\partial x^2} - \ldots - 2g^{nn}\cdot\frac{\partial}{\partial x^n} \\ &= -2\sum_i x^i\frac{\partial}{\partial x^i}\,. \end{aligned}$$

This implies the first assertion:

$$\begin{aligned} g(\operatorname{grad}\gamma, \operatorname{grad}\gamma) &= g\Big(-2\sum_i x^i\frac{\partial}{\partial x^i}, -2\sum_i x^i\frac{\partial}{\partial x^i}\Big) \\ &= 4\left(-(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2\right) \\ &= -4\gamma\,. \end{aligned}$$

Since $\gamma > 0$ on $I_\pm(0)$ equation (i) shows that $\operatorname{grad}\gamma$ is timelike there. Since $\operatorname{grad}\gamma$ is a negative multiple of the position vector field it is past-directed on $I_+(0)$ and future-directed on $I_-(0)$.



red: $x^i \frac{\partial}{\partial x^i}$
blue: $\operatorname{grad}\gamma$

This proves the second statement. Finally, we verify the third assertion:

$$\begin{aligned}\Box\gamma &= -\left(-\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \ldots + \frac{\partial^2}{(\partial x^n)^2}\right)\left((x^1)^2 - (x^2)^2 - \ldots - (x^n)^2\right) \\ &= 2 + 2 + \ldots + 2 \\ &= 2n\,.\end{aligned}$$

Since μ_x is constant the term $-g\left(\operatorname{grad}\Gamma_x, \operatorname{grad}\left(\log(\mu_x)\right)\right)$ vanishes which concludes the proof.

Remark 1.2.78. One can check that there is a connection between τ and Γ, namely if Ω is convex and τ is the time-separation function of Ω, then

$$\tau(p,q) = \begin{cases} \sqrt{\Gamma_p(q)}, & \text{if } p < q, \\ 0, & \text{otherwise.} \end{cases}$$

So $\tau(p,q)$ is actually the length of the geodesic segment connecting p and q. The supremum is attained and the geodesic segement is the longest causal curve.

1.3 Distributions

Wave equations will allow for very irregular solutions. There are no such nice features as elliptic regularity theory for wave equations. Due to this fact we need distributions. They have the advantage that, while possibly being very irregular, they can be differentiated arbitrarily often. Roughly speaking, while functions can be evaluated at a point, distributions only have "smeared" values. For this reason they are functionals on test functions. We imagine a test function as having its support very close to the point. From the point of view of physics, this is a very reasonable concept because measuring instruments are never able to measure a quantity exactly at a given point.

1.3.1 Distributions on manifolds

Let M be a differentiable manifold with volume density $d\mu$. Let $E \to M$ be a $\mathbb{K}$-vector bundle with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Notation 1.3.1. We denote the space of compactly supported smooth sections of E by

$$\mathcal{D}(M, E) := C_c^\infty(M, E).$$

The elements of $\mathcal{D}(M, E)$ are referred to as **test sections** in E.

Definition 1.3.2. Let $k \in \mathbb{N}$ be a positive number and $K \subset M$ be a compact subset. Let ∇ denote connections on E and T^*M and $\langle \cdot, \cdot \rangle$ positive definite fiberwise scalar products on E and T^*M. For $u \in \mathcal{D}(M, E)$ we define the C^k**-seminorm** by

$$\|u\|_{k,\nabla,\langle\cdot,\cdot\rangle,K} := \max_{j=0,\ldots,k} \max_{x\in K} \|\nabla^j u(x)\|.$$

Remark 1.3.3. Note that $\nabla^j u \in C^\infty(M, \underbrace{T^*M \otimes \cdots \otimes T^*M}_{j \text{ factors}} \otimes E)$. For $\|\nabla^j u(x)\|$ to be defined we need metrics on E and T^*M. They induce metrics on all bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$.
The seminorm $\|\cdot\|_{k,\nabla,\langle\cdot,\cdot\rangle,K}$ is not a norm in general because in case $\operatorname{supp} u \cap K = \emptyset$ the seminorm $\|u\|_{k,\nabla,\langle\cdot,\cdot\rangle,K}$ vanishes although u need not be the zero section on all of M.

Different choices of the metrics and the connections yield equivalent seminorms for fixed k and K. For this reason we will usually drop the metric and the connection in the notation and simply write $\|\cdot\|_{C^k(K)}$ instead of $\|\cdot\|_{k,\nabla,\langle\cdot,\cdot\rangle,K}$.
We define a notion of convergence of test sections.

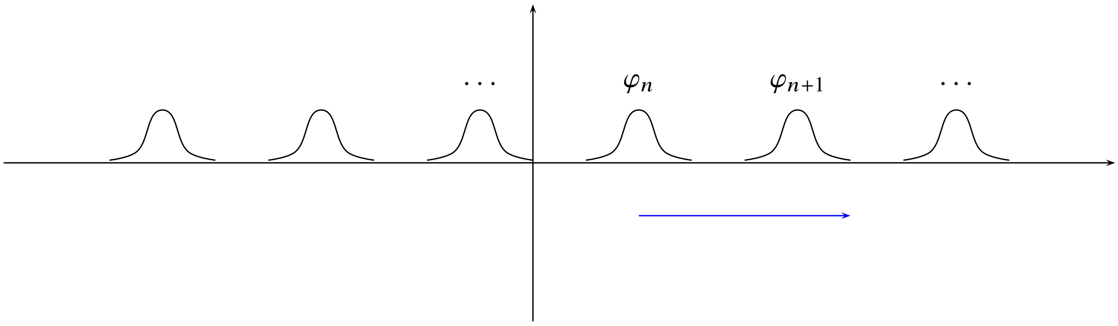
Definition 1.3.4. Let $\varphi, \varphi_n \in \mathcal{D}(M, E)$. We say that the sequence $(\varphi_n)_n$ **converges to** φ **in** $\mathcal{D}(M, E)$ if the following two conditions hold:

1. There is a compact set $K \subset M$ such that the supports of φ and all φ_n are contained in K, i. e., $\operatorname{supp}(\varphi), \operatorname{supp}(\varphi_n) \subset K$ for all n.

2. The sequence $(\varphi_n)_n$ converges to φ in all C^k-norms over K, i. e., for each $k \in \mathbb{N}_0$

$$\|\varphi - \varphi_n\|_{C^k(K)} \xrightarrow{n\to\infty} 0.$$

Remark 1.3.5. The first condition means that the sequence $(\varphi_n)_n$ cannot converge to zero by its supports escapting to infinity.



We fix a finite-dimensional $\mathbb{K}$-vector space W. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex.

Definition 1.3.6. A $\mathbb{K}$-linear map $T : \mathcal{D}(M, E^*) \to W$ is called a **distribution in** E **with values in** W if it is sequentially continuous, i.e., for all convergent sequences $\varphi_n \to \varphi$ in $\mathcal{D}(M, E^*)$ one has $T[\varphi_n] \to T[\varphi]$. We write $\mathcal{D}'(M, E, W)$ for the space of all W-valued distributions in E. In the case $W = \mathbb{K}$ write $\mathcal{D}'(M, E)$.

Note that since W is finite-dimensional all norms on W yield the same topology on W. Hence there is no need to specify a norm on W for Definition 1.3.6 to make sense.

Lemma 1.3.7. *For $T \in \mathcal{D}'(M, E, W)$ and $K \subset M$ compact there is a nonnegative integer $k \in \mathbb{N}_0$ and a constant $C > 0$ such that for all $\varphi \in \mathcal{D}(M, E^*)$ with* $\mathrm{supp}(\varphi) \subset K$ *we have*

$$|T[\varphi]| \leq C \cdot \|\varphi\|_{C^k(K)}. \tag{1.10}$$

The smallest k for which inequality (1.10) holds is called the **order** of F over K.

Proof. Assume (1.10) does not hold for any pair of C and k. Then for every positive integer k we can find a $\varphi_k \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi_k) \subset K$ and $|T[\varphi_k]| > k \cdot \|\varphi_k\|_{C^k}$. We define sections $\psi_k := \frac{1}{|F[\varphi_k]|}\varphi_k$. Obviously, these ψ_k satisfy $\mathrm{supp}(\psi_k) \subset K$ and

$$\|\psi_k\|_{C^k(K)} = \frac{1}{|F[\varphi_k]|}\|\varphi_k\|_{C^k(K)} \leq \frac{1}{k}.$$

Hence for $j \geq k$

$$\|\psi_j\|_{C^k(K)} \leq \|\psi_j\|_{C^j(K)} \leq \frac{1}{j}.$$

Therefore the sequence $(\psi_j)_k$ converges to 0 in $\mathcal{D}(M, E^*)$. Since T is a distribution we get $T[\psi_j] \to F[0] = 0$ for $j \to \infty$. On the other hand, $|F[\psi_j]| = \left|\frac{1}{|F[\varphi_j]|}F[\varphi_j]\right| = 1$ for all j, which yields a contradiction. ◻

Lemma 1.3.7 states that the restriction of any distribution to a (relatively) compact set is of finite order. We say that a distribution F is of **order** k if k is the smallest integer such that for each compact subset $K \subset M$ there exists a constant C so that

$$|F[\varphi]| \leq C \cdot \|\varphi\|_{C^k(K)}$$

for all $\varphi \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi) \subset K$. Such a distribution extends uniquely to a continuous linear map on $C^k_c(M, E^*)$, the space of C^k-sections in E^* with compact support.
Here, convergence in $C^k_c(M, E^*)$ is defined similarly to that of test sections. We say that φ_n converge to φ in $C^k_c(M, E^*)$ if the supports of the φ_n and φ are contained in a common compact subset $K \subset M$ and $\|\varphi - \varphi_n\|_{C^k(K)} \to 0$ as $n \to \infty$.
Next we give two important examples of distributions.

Example 1.3.8. Pick a bundle $E \to M$ and a point $x \in M$. The **delta-distribution** δ_x is an E^*_x-valued distribution in E.
For $\varphi \in \mathcal{D}(M, E^*)$ it is defined by

$$\delta_x[\varphi] = \varphi(x).$$

The distribution δ_x is of order 0, since we have $|\delta_x(\varphi)| = |\varphi(x)| \leq \|\varphi\|_{C^0(K)}$ for all test functions with support in K.

Example 1.3.9. Let $f \in L^1_{\mathrm{loc}}(M, E)$ be a locally integrable section in E. We set for any $\varphi \in \mathcal{D}(M, E^*)$

$$T_f[\varphi] := \int_M \varphi(x)(f(x))d\mu(x).$$

Again, as a distribution, T_f is of order 0, since $|T_f[\varphi]| \leq \int_K \|\varphi\|_{C^0(K)}|f(x)|\mathrm{d}\mu \leq C_f(K) \cdot \|\varphi\|_{C^0(K)}$.
One usually writes $f[\varphi]$ instead of $T_f[\varphi]$ and interpretes f itself as a $\mathbb{K}$-valued distribution in E.

Notation 1.3.10. Let $E \to M$ and $F \to N$ be $\mathbb{K}$-vector bundles. Then $E \boxtimes F$ denotes the vector bundle over $M \times N$ whose fiber over $(x, y) \in M \times N$ is given by $E_x \otimes F_y$.

Lemma 1.3.11. *Let M and N be differentiable manifolds equipped with smooth volume densities. Let $E \to M$ and $F \to N$ be vector bundles. Let $K \subset N$ be compact and let $\varphi \in C^k(M \times N, E \boxtimes F^*)$ be such that $supp(\varphi) \subset M \times K$. Let $m \leq k$ and let $T \in \mathcal{D}'(N, F)$ be a distribution of order m. Then the following statements hold:*

1. *The map*

$$f : M \to E,$$
$$x \mapsto (\mathrm{id}_{E_x} \otimes T)\left[\varphi(x, \cdot)\right] =: T[\varphi(x, \cdot)],$$

   defines a C^{k-m}-section in E.

2. *The support of f is contained in the projection of* $\mathrm{supp}(\varphi)$ *to the first factor, i. e.,* $\mathrm{supp}(f) \subset \{x \in M \mid \exists\, y \in K$ *such that* $(x, y) \in \mathrm{supp}(\varphi)\}$.

3. *If P is a linear differential operator of order $\leq k - m$ acting on sections in E, then*

$$Pf = T[P_x \varphi(x, \cdot)].$$

   In other words, derivatives up to order $k-m$ in directions tangent to M may be interchanged with T.

For a proof see [3, Lem. 1.1.6].
Next we will see how differential operators act on distributional sections. Let $P \in \mathscr{Diff}_k(E, F)$ be a linear differential operator. Then P extends uniquely to a continuous linear operator $P : \mathcal{D}'(M, E, W) \to \mathcal{D}'(M, F, W)$ by

$$(PT)[\varphi] := T[P^*\varphi]$$

where $\varphi \in \mathcal{D}(M, F^*)$.
For this definition to be correct, we have to check:

- that $P^*(\varphi)$ is again a test section with compact support. This is clear because a differential operator never enlarges the support of a section.
- that PT is again a distribution, i.e., it is sequentially continuous and linear. Since T and P^* are sequentially continuous and linear so is their composition. Therefore PT is again a distribution.

1.3.12. Let $f \in C^\infty(M, E)$ be a section and denote the associated distribution by $T_f \in \mathcal{D}'(M, E)$. Show that for any linear differential operator $P \in \mathscr{Diff}(E, F)$ the following holds:

$$PT_f = T_{Pf}.$$

This means that applying P in the classical sense is the same thing as applying it in the distributional sense.

Definition 1.3.13. We equip the space $\mathcal{D}'(M, E, W)$ of distributions in E with the **weak topology**. This means that $T_n \to T$ in $\mathcal{D}'(M, E, W)$ if and only if $T_n[\varphi] \to T[\varphi]$ for all $\varphi \in \mathcal{D}(M, E^*)$.

Remark 1.3.14. Linear differential operators P are always continuous with respect to the weak topology. Namely, if $T_n \to T$, then we have for every $\varphi \in \mathcal{D}(M, E^*)$

$$PT_n[\varphi] = T_n[P^*\varphi] \to T[P^*\varphi] = PT[\varphi].$$

Hence

$$PT_n \to PT.$$

Definition 1.3.15. The **support** of a distribution $T \in \mathcal{D}'(M, E, W)$ is defined as the set

$$\begin{aligned} \mathrm{supp}(T) := \{x \in M \mid \forall \text{ neighborhood } U \text{ of } x\ \exists\, \varphi \in \mathcal{D}(M, E^*) \text{ with} \\ \mathrm{supp}(\varphi) \subset U \text{ and } T[\varphi] \neq 0\}. \end{aligned}$$

It follows from the definition that the support of T is a closed subset of M. In case $T = T_f$ with $f \in L^1_{\mathrm{loc}}$ we have that $\mathrm{supp}(T_f) = \mathrm{ess} - \mathrm{supp}(f)$. Thus the concept of support for distributions generalizes the usual support of sections.

Remark 1.3.16. If for $\varphi \in \mathcal{D}(M, E^*)$ and $T \in \mathcal{D}'(M, E, W)$ the supports of φ and T are disjoint, then $T[\varphi] = 0$. Namely, for each $x \in \mathrm{supp}(\varphi)$ there is a neighborhood U of x such that $T[\psi] = 0$ whenever $\mathrm{supp}(\psi) \subset U$. Cover the compact set $\mathrm{supp}(\varphi)$ by finitely many such open sets $U_1, \ldots, U_k$. Using a partition of unity one can write $\varphi = \psi_1 + \cdots + \psi_k$ with $\psi_j \in \mathcal{D}(M, E^*)$ and $\mathrm{supp}(\psi_j) \subset U_j$. Then

$$T[\varphi] = T[\psi_1 + \cdots + \psi_k] = T[\psi_1] + \cdots + T[\psi_k] = 0.$$

Warning. Iit is not sufficient to assume that φ vanishes on $\mathrm{supp}(T)$ in order to ensure $T[\varphi] = 0$.

Example 1.3.18. Let $M = \mathbb{R}$ and E be the trivial $\mathbb{K}$-line bundle. Let $T \in \mathcal{D}'(\mathbb{R}, \mathbb{K})$ be given by $T[\varphi] = \varphi'(0)$. Then $\mathrm{supp}(T) = \{0\}$ but $T[\varphi] = \varphi'(0)$ may well be nonzero while $\varphi(0) = 0$. For instance, if $\varphi(t) = t$ near 0, then φ does the job.

If a distribution has compact support then we can evaluate it even on "test sections" with non-compact support. More generally, let $T \in \mathcal{D}'(M, E, W)$ and $\varphi \in C^\infty(M, E^*)$ with $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ compact. Pick a function $\sigma \in \mathcal{D}(M, \mathbb{R})$ that is constant 1 on a neighborhood of $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$. Then we define the evalution $T[\varphi]$ by

$$T[\varphi] := T[\sigma\varphi].$$

This definition is independent of the choice of σ since for another choice σ' we have

$$T[\sigma\varphi] - T[\sigma'\varphi] = T[(\sigma - \sigma')\varphi] = 0$$

because $\mathrm{supp}((\sigma - \sigma')\varphi)$ and $\mathrm{supp}(T)$ are disjoint.

Definition 1.3.19. Let $T \in \mathcal{D}'(M, E, W)$ and let $\Omega \subset M$ be an open subset. For $\varphi \in \mathcal{D}(\Omega, E^*)$. The extension of φ by 0 yields a test section $\mathrm{ext}_M \varphi \in \mathcal{D}(M, E^*)$. This defines an embedding $\mathcal{D}(\Omega, E^*) \subset \mathcal{D}(M, E^*)$. We define the restriction of T to Ω by

$$(T|_\Omega)[\varphi] := T[\mathrm{ext}_M \varphi].$$

Definition 1.3.20. The **singular support** $\mathrm{sing\,supp}(T)$ of a distribution $T \in \mathcal{D}'(M, E, W)$ is the set of points which do not have a neighborhood restricted to which T coincides with a smooth section, i.e.

$$\mathrm{sing\,supp}(T) := \{x \in M \mid \forall \text{ neighborhood } \Omega \text{ of } x \text{ the restriction } T|_\Omega \text{ does not coincide with a smooth section}\}.$$

By definition, the singular support is also a closed subset, just as for the support. Moreover, we clearly have $\mathrm{sing\,supp}\,(T) \subset \mathrm{supp}\,(T)$.

Example 1.3.21. For the delta-distribution δ_x we have $\mathrm{supp}\,(\delta_x) = \mathrm{sing\,supp}(\delta_x) = \{x\}$.

Lemma 1.3.22. *Let $T_n, T \in C^0(M, E)$ and suppose $T_n \to T$ locally uniformly. Consider T_n and T as distributions.*
Then $T_n \to T$ in $\mathcal{D}'(M, E)$. In particular, for every linear differential operator P we have $PT_n \to PT$ in the sense of distributions.

Proof. Let $\varphi \in \mathcal{D}(M, E^*)$. Put $K := \operatorname{supp}(\varphi)$. We compute

$$\begin{aligned}
\lim_{n\to\infty} T_n[\varphi] &= \lim_{n\to\infty} \int_M \varphi(x)\,(T_n(x))\,\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty} \int_K \varphi(x)\,(T_n(x))\,\mathrm{d}\mu(x) \\
&= \int_K \lim_{n\to\infty} \varphi(x)\,(T_n(x))\,\mathrm{d}\mu(x) \\
&= \int_K \varphi(x) \left(\lim_{n\to\infty} T_n(x)\right) \mathrm{d}\mu(x) \\
&= \int_K \varphi(x)\,(T(x))\,\mathrm{d}\mu(x) \\
&= T[\varphi].
\end{aligned}$$

which implies $T_n \to T$ in $\mathcal{D}'(M, E)$.
The limit may be interchanged with the integral by Lebesgue's dominated convergence theorem.□

The following situation will arise frequently. Let E, F, and G be $\mathbb{K}$-vector bundles over M. Let $\varphi \in C^k(M, E \otimes F)$ and $\psi \in C^k(M, F^* \otimes G)$. We define $\varphi \cdot \psi \in C^k(M, E \otimes G)$ by means of the natural pairing $F \otimes F^* \to \mathbb{K}$. The pairing is given by evaluation of the second factor on the first and yields a vector bundle homomorphism $E \otimes F \otimes F^* \otimes G \to E \otimes G$. Then $\varphi \cdot \psi$ is the the contraction $F \otimes F^* \to \mathbb{K}$ applied to $\varphi \otimes \psi$.[7]

Lemma 1.3.23. *For all C^k-sections φ in $E \otimes F$ and ψ in $F^* \otimes G$ and all $K \subset M$ compact we have*

$$\|\varphi \cdot \psi\|_{C^k(K)} \le 2^k \cdot \|\varphi\|_{C^k(K)} \cdot \|\psi\|_{C^k(K)}.$$

Proof. We use induction on k.
The case $k = 0$ follows from the Cauchy-Schwarz inequality as follows: For fixed $x \in M$ we choose an orthonormal frame $f_1, \dots, f_r$ for F_x. Let $f_1^*, \dots, f_r^*$ be the dual frame for F_x^*. We write $\varphi(x) = \sum_{i=1}^r e_i(x) \otimes f_i(x)$ and similarly $\psi(x) = \sum_{i=1}^r f_i^*(x) \otimes g_i(x)$. Then $\varphi(x) \cdot \psi(x) = \sum_{i=1}^r e_i(x) \otimes g_i(x)$ and we see, using the Cauchy-Schwarz inequality twice:

$$\begin{aligned}
|\varphi \cdot \psi|^2 &= |\sum_{i=1}^r e_i \otimes g_i|^2 \\
&= \sum_{i,j=1}^r \langle e_i \otimes g_i, e_j \otimes g_j \rangle
\end{aligned}$$

[7] If one identifies $E \otimes F$ with $\mathrm{Hom}(E^*, F)$ and $F^* \otimes G$ with $\mathrm{Hom}(F, G)$, then $\varphi \cdot \psi$ corresponds to the composition of ψ and φ.

$$
\begin{aligned}
&= \sum_{i,j=1}^{r} \langle e_i, e_j\rangle\langle g_i, g_j\rangle \\
&\le \sqrt{\sum_{i,j=1}^{r} \langle e_i, e_j\rangle^2} \cdot \sqrt{\sum_{i,j=1}^{r} \langle g_i, g_j\rangle^2} \\
&\le \sqrt{\sum_{i,j=1}^{r} |e_i|^2|e_j|^2} \cdot \sqrt{\sum_{i,j=1}^{r} |g_i|^2|g_j|^2} \\
&= \sqrt{\sum_{i=1}^{r} |e_i|^2 \sum_{j=1}^{r} |e_j|^2} \cdot \sqrt{\sum_{i=1}^{r} |g_i|^2 \sum_{j=1}^{r} |g_j|^2} \\
&= \sum_{i=1}^{r} |e_i|^2 \cdot \sum_{i=1}^{r} |g_i|^2 \\
&= |\varphi|^2 \cdot |\psi|^2.
\end{aligned}
$$

Now we perform the induction step.

$$
\begin{aligned}
\|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(K)} &\le \|\nabla(\varphi \cdot \psi)\|_{C^k(K)} \\
&= \|(\nabla\varphi) \cdot \psi + \varphi \cdot \nabla\psi\|_{C^k(K)} \\
&\le \|(\nabla\varphi) \cdot \psi\|_{C^k(K)} + \|\varphi \cdot \nabla\psi\|_{C^k(K)} \\
&\le 2^k \cdot \|\nabla\varphi\|_{C^k(K)} \cdot \|\psi\|_{C^k(K)} + 2^k \cdot \|\varphi\|_{C^k(K)} \cdot \|\nabla\psi\|_{C^k(K)} \\
&\le 2^k \cdot \|\varphi\|_{C^{k+1}(K)} \cdot \|\psi\|_{C^{k+1}(K)} + 2^k \cdot \|\varphi\|_{C^{k+1}(K)} \cdot \|\psi\|_{C^{k+1}(K)} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(K)} \cdot \|\psi\|_{C^{k+1}(K)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\varphi \cdot \psi\|_{C^{k+1}(K)} &= \max\{\|\varphi \cdot \psi\|_{C^k(K)}, \|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(K)}\} \\
&\le \max\{2^k \cdot \|\varphi\|_{C^k(K)} \cdot \|\psi\|_{C^k(K)}, 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(K)} \cdot \|\psi\|_{C^{k+1}(K)}\} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(K)} \cdot \|\psi\|_{C^{k+1}(K)}.
\end{aligned}
$$
□

This lemma allows us to estimate the C^k-norm of products of sections in terms of the C^k-norms of the factors. The next lemma allows us to deal with compositions of functions. We recursively define the following constants:

$$\alpha(k, 0) := 1,$$

$$\alpha(k, j) := 0$$

for $j > k$ and for $j < 0$ and

$$\alpha(k + 1, j) := \max\{\alpha(k, j),\ 2^k \cdot \alpha(k, j - 1)\} \tag{1.11}$$

if $1 \le j \le k$. The precise values of the $\alpha(k, j)$ are not important. The definition is made in such a way that the following lemma holds.

Lemma 1.3.24. *Let Γ be a real valued C^k-function on a Lorentzian manifold M and let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^k-function. Then for all $K \subset M$ compact, $k \in \mathbb{N}_0$ and $I \subset \mathbb{R}$ such that $\Gamma(K) \subset I$ we have*

$$\|\sigma \circ \Gamma\|_{C^k(K)} \leq \|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k,j) \|\Gamma\|^j_{C^k(K)}.$$

Proof. Again, we perform an induction on k. The case $k = 0$ is obvious. By Lemma 1.3.23

$$\begin{aligned}
\|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(K)} &= \|\nabla^k[(\sigma' \circ \Gamma) \cdot \nabla\Gamma]\|_{C^0(K)} \\
&\leq \|(\sigma' \circ \Gamma) \cdot \nabla\Gamma\|_{C^k(K)} \\
&\leq 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(K)} \cdot \|\nabla\Gamma\|_{C^k(K)} \\
&\leq 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(K)} \cdot \|\Gamma\|_{C^{k+1}(K)} \\
&\leq 2^k \cdot \|\sigma'\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k,j)\|\Gamma\|^j_{C^{k+1}(K)} \cdot \|\Gamma\|_{C^{k+1}(K)} \\
&\leq 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k} \alpha(k,j)\|\Gamma\|^{j+1}_{C^{k+1}(K)} \\
&= 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k,j-1)\|\Gamma\|^j_{C^{k+1}(K)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\sigma \circ \Gamma\|_{C^{k+1}(K)} &= \max\{\|\sigma \circ \Gamma\|_{C^k(K)}, \|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(K)}\} \\
&\leq \max\{\|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k,j)\|\Gamma\|^j_{C^k(K)}, \\
&\qquad 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k,j-1)\|\Gamma\|^j_{C^{k+1}(K)}\} \\
&\leq \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \max\{\alpha(k,j), 2^k\alpha(k,j-1)\}\|\Gamma\|^j_{C^{k+1}(K)} \\
&= \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \alpha(k+1,j)\|\Gamma\|^j_{C^{k+1}(K)}.
\end{aligned}$$

□

1.3.2 Riesz distributions on Minkowski space

In this subsection we construct the basic building blocks for solutions of the d'Alembert equation on Minkowski space. Let V be an n-dimensional $\mathbb{R}$-vector space, let $\langle\cdot,\cdot\rangle$ be an inner product of index 1 on V. Hence $(V, \langle\cdot,\cdot\rangle)$ is isometric to Minkowski space. Endow V with a time orientation such that $I_+(0)$ and $I_-(0)$ are defined. Let $\gamma : V \to \mathbb{R}$ be the function $\gamma(X) := -\langle X, X\rangle$ qw in 1.2.76.

Definition 1.3.25. For any complex number α with $\Re(\alpha) > n$ let $R_+(\alpha)$ and $R_-(\alpha)$ be the complex-valued continuous functions on V defined by

$$R_\pm(\alpha)(X) := \begin{cases} C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}, & \text{if } X \in J_\pm(0), \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$ and $z \mapsto (z-1)!$ is the Gamma function.

Remark 1.3.26. Note that the functions $R_\pm(\alpha)$ are continuous because γ vanishes on the boundary of $J_\pm(0)$ and the exponent $\frac{\alpha-n}{2}$ is assumed to have positive real part. Indeed, if we increase the real part of the exponent then the function vanishes to higher order along the boundary and hence becomes more regular. Concretely, for $\Re(\alpha) > n + k$ we have $R_\pm(\alpha) \in C^k(V, \mathbb{C})$.

Lemma 1.3.27. *For all $\alpha \in \mathbb{C}$ with $\Re(\alpha) > n$ we have*

(1) $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2)R_\pm(\alpha + 2)$*;*

(2) $(\operatorname{grad}\gamma) \cdot R_\pm(\alpha) = 2\alpha \operatorname{grad} R_\pm(\alpha + 2)$*;*

(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$*.*

Moreover, the map $\alpha \mapsto R_\pm(\alpha)$ extends uniquely to all of $\mathbb{C}$ as a holomorphic family of distributions. This means that for each test function φ the function $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is holomorphic.

Proof. Identity (1) follows from

$$\frac{C(\alpha,n)}{C(\alpha+2,n)} = \frac{2^{(1-\alpha)}\ (\frac{\alpha+2}{2}-1)!\ (\frac{\alpha+2-n}{2})!}{2^{(1-\alpha-2)}\ (\frac{\alpha}{2}-1)!\ (\frac{\alpha-n}{2})!} = \alpha\,(\alpha - n + 2).$$

To show (2) we choose a Lorentzian orthonormal basis $e_1, \dots, e_n$ of V and we denote differentiation in direction e_i by ∂_i. We fix a test function φ and integrate by parts:

$$\begin{aligned}
\partial_i\gamma \cdot R_\pm(\alpha)[\varphi] &= C(\alpha,n)\int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n}{2}}\ \partial_i\gamma(X)\varphi(X)\ dX \\
&= \frac{2C(\alpha,n)}{\alpha+2-n}\int_{J_\pm(0)} \partial_i(\gamma(X)^{\frac{\alpha-n+2}{2}})\varphi(X)\ dX \\
&= -2\alpha C(\alpha+2,n)\int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n+2}{2}}\partial_i\varphi(X)\ dX \\
&= -2\alpha R_\pm(\alpha+2)[\partial_i\varphi] \\
&= 2\alpha\partial_i R_\pm(\alpha+2)[\varphi],
\end{aligned}$$

which proves (2). Furthermore, it follows from (2) that

$$\begin{aligned}
\partial_i^2 R_\pm(\alpha+2) &= \partial_i\left(\frac{1}{2\alpha}\partial_i\gamma\cdot R_\pm(\alpha)\right)\\
&= \frac{1}{2\alpha}\left(\partial_i^2\gamma\cdot R_\pm(\alpha)+\partial_i\gamma\cdot\left(\frac{1}{2(\alpha-2)}\partial_i\gamma\cdot R_\pm(\alpha-2)\right)\right)\\
&= \frac{1}{2\alpha}\partial_i^2\gamma\cdot R_\pm(\alpha)+\frac{1}{4\alpha(\alpha-2)}(\partial_i\gamma)^2\frac{(\alpha-2)(\alpha-n)}{\gamma}\cdot R_\pm(\alpha)\\
&= \left(\frac{1}{2\alpha}\partial_i^2\gamma+\frac{\alpha-n}{4\alpha}\cdot\frac{(\partial_i\gamma)^2}{\gamma}\right)\cdot R_\pm(\alpha),
\end{aligned}$$

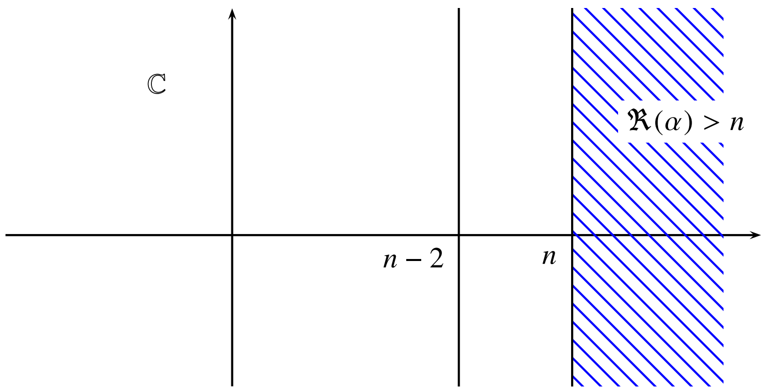
so that

$$\begin{aligned}
\Box R_\pm(\alpha+2) &= \left(\frac{n}{\alpha}+\frac{\alpha-n}{4\alpha}\cdot\frac{4\gamma}{\gamma}\right)R_\pm(\alpha)\\
&= R_\pm(\alpha).
\end{aligned}$$

To show the final assertion we first note that for fixed $\varphi\in\mathcal{D}(V,\mathbb{C})$ the map $\{\mathfrak{R}(\alpha)>n\}\to\mathbb{C}$, $\alpha\mapsto R_\pm(\alpha)[\varphi]$, is holomorphic. For $\mathfrak{R}(\alpha)>n-2$ we set

$$\widetilde{R}_\pm(\alpha) := \Box R_\pm(\alpha+2). \tag{1.12}$$

This defines a distribution on V. The map $\alpha\mapsto\widetilde{R}_\pm(\alpha)$ is then holomorphic on $\{\alpha\in\mathbb{C}\mid\mathfrak{R}(\alpha)>n-2\}$. By (3) we have $\widetilde{R}_\pm(\alpha)=R_\pm(\alpha)$ for $\mathfrak{R}(\alpha)>n$, so that $\alpha\mapsto\widetilde{R}_\pm(\alpha)$ extends $\alpha\mapsto R_\pm(\alpha)$ holomorphically to $\{\alpha\in\mathbb{C}\mid\mathfrak{R}(\alpha)>n-2\}$.



Proceeding inductively, we obtain a holomorphic extension of $\alpha\mapsto R_\pm(\alpha)$ to all of $\mathbb{C}$, which is necessarily unique. □

Lemma 1.3.27 defines $R_\pm(\alpha)$ for all $\alpha\in\mathbb{C}$, not as functions but as distributions.

Definition 1.3.28. We call $R_+(\alpha)$ the **advanced Riesz distribution** and $R_-(\alpha)$ the **retarded Riesz distribution** on V for $\alpha \in \mathbb{C}$.

We next want to collect more important facts on Riesz distributions. As preparation, we first identify the set of all $\alpha \in \mathbb{C}$ that lead to vanishing $C(\alpha, n)$.

Remark 1.3.29. We defined $R_\pm(\alpha)(X) = C(\alpha, n)\gamma(X)^{\frac{\alpha-n}{2}}$ where $C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$ and $z \mapsto (z-1)!$ is the Gamma function. The Gamma function has no zeros but simple poles at the non-positive integers. For the two factors of the denominator of $C(\alpha, n)$ we have:

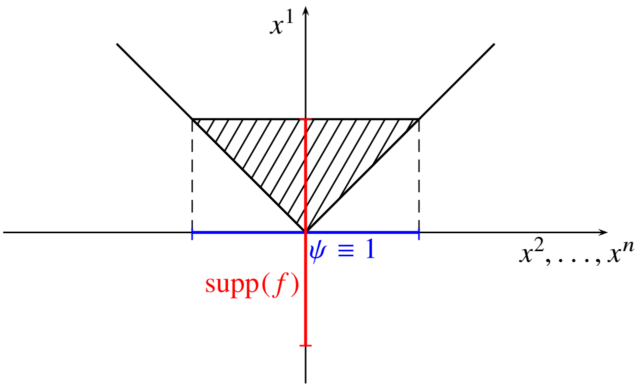
- $(\frac{\alpha}{2} - 1)!$ has a pole iff $\alpha \in \{0, -2, -4, \ldots\}$
- $(\frac{\alpha-n}{2})!$ has a iff $\alpha \in \{n-2, n-4, \ldots\}$

Observe that there is a difference between even and odd dimensional V. In odd dimension, the sets are disjoint and all negative numbers lead to vanishing $C(\alpha, n)$. In even dimension, the two sets overlap and $C(\alpha, n)$ has zeros with double multiplicity.

To complete our preparations, we derive a more explicit formula for the Riesz distributions evaluated on test functions of a particular form.
Introduce linear coordinates $x^1, \ldots, x^n$ on V such that $\gamma(x) = -(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$ and such that the x^1-axis is future directed. Let $f \in \mathcal{D}(\mathbb{R}, \mathbb{C})$ and choose $\psi \in \mathcal{D}(\mathbb{R}^{n-1}, \mathbb{C})$ such that $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ defined by

$$\varphi(x) := f(x^1)\psi(x^2, \ldots, x^n) \tag{1.13}$$

satisfies $\varphi(x) = f(x^1)$ on $J_+(0)$.



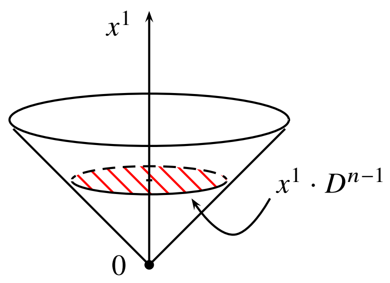
Lemma 1.3.30. *For such test functions and if $\Re(\alpha) \geq 1$ then*

$$R_+(\alpha)[\varphi] = \frac{1}{(\alpha-1)!}\int_0^\infty r^{\alpha-1} f(r)dr. \tag{1.14}$$

Proof. First we note that it suffices to show the formula for $\Re(\alpha) > n$. Namely, since both sides of the equation are holomorphic in α for $\Re(\alpha) > 1$ the identity theorem for holomorphic functions will then imply that equation (1.14) holds for all α with $\Re(\alpha) > 1$. By continuity, we then also get it for all α with $\Re(\alpha) = 1$.
Let therefore $\Re(\alpha) > n$. We abbreviate $\hat{x} = (x^2, \ldots, x^n)$. We compute

$$\begin{aligned}
R_+(\alpha)[\varphi] &= C(\alpha,n)\int_{J_+(0)} \varphi(X)\gamma(X)^{\frac{\alpha-n}{2}}dX \\
&= C(\alpha,n)\int_0^\infty \int_{\{|\hat{x}|\le x^1\}} \varphi(x^1,\hat{x})((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}d\hat{x}\, dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_{\{|\hat{x}|\le x^1\}} ((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}d\hat{x}\, dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_0^{x^1}\int_{S^{n-2}} ((x^1)^2-\varrho^2)^{\frac{\alpha-n}{2}}\varrho^{n-2}d\omega\, d\varrho\, dx^1,
\end{aligned}$$

where S^{n-2} is the $(n-2)$-dimensional round sphere and $d\omega$ its standard volume element.



Renaming x^1 we get

$$R_+(\alpha)[\varphi] = \mathrm{vol}(S^{n-2})\, C(\alpha,n)\int_0^\infty f(r)\int_0^r (r^2-\varrho^2)^{\frac{\alpha-n}{2}}\varrho^{n-2}d\varrho\, dr.$$

Using $\int_0^r (r^2-\varrho^2)^{\frac{\alpha-n}{2}}\varrho^{n-2}d\varrho = \frac{1}{2}r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}$ we obtain

$$\begin{aligned}
R_+(\alpha)[\varphi] &= \frac{\mathrm{vol}(S^{n-2})}{2}C(\alpha,n)\int_0^\infty f(r)r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}dr \\
&= \frac{1}{2}\frac{2\pi^{(n-1)/2}}{(\frac{n-1}{2}-1)!}\cdot\frac{2^{1-\alpha}\pi^{1-n/2}}{(\alpha/2-1)!(\frac{\alpha-n}{2})!}\cdot\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr \\
&= \frac{\sqrt{\pi}\cdot 2^{1-\alpha}}{(\alpha/2-1)!(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr.
\end{aligned}$$

Legendre's duplication formula (see [12, p. 218])

$$\left(\frac{\alpha}{2}-1\right)!\left(\frac{\alpha+1}{2}-1\right)! = 2^{1-\alpha}\sqrt{\pi}\,(\alpha-1)! \tag{1.15}$$

yields the claim. □

Now we are ready to state and prove the following collection of important facts on Riesz distributions:

Proposition 1.3.31. *The following holds for all $\alpha \in \mathbb{C}$:*

1. $\gamma \cdot R_{\pm}(\alpha) = \alpha(\alpha - n + 2)\, R_{\pm}(\alpha + 2)$,
2. $(\operatorname{grad}\gamma) R_{\pm}(\alpha) = 2\alpha \operatorname{grad}\,(R_{\pm}(\alpha + 2))$,
3. $\Box R_{\pm}(\alpha + 2) = R_{\pm}(\alpha)$,
4. *For every $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$, we have* $\operatorname{supp}\,(R_{\pm}(\alpha)) = J_{\pm}(0)$ *and* $\operatorname{sing\,supp}\,(R_{\pm}(\alpha)) \subset \partial J_{\pm}(0)$.
5. *For every $\alpha \in \{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\}$, we have* $\operatorname{supp}\,(R_{\pm}(\alpha)) = \operatorname{sing\,supp}\,(R_{\pm}(\alpha)) \subset \partial J_{\pm}(0)$.
6. *For $n \geq 3$ and $\alpha = n-2, n-4, \dots, 1$ or 2 respectively, we have* $\operatorname{supp}\,(R_{\pm}(\alpha)) = \operatorname{sing\,supp}\,(R_{\pm}(\alpha)) = \partial J_{\pm}(0)$.
7. $R_{\pm}(0) = \delta_0$.
8. *For $\Re(\alpha) > 0$ the order of $R_{\pm}(\alpha)$ is bounded from above by $n + 1$.*
9. *If $\alpha \in \mathbb{R}$, then $R_{\pm}(\alpha)$ is real, i. e., $R_{\pm}(\alpha)[\varphi] \in \mathbb{R}$ for all $\varphi \in \mathcal{D}(V, \mathbb{R})$.*

Proof. Assertions (1), (2), and (3) hold for $\Re(\alpha) > n$ by Lemma 1.3.27. Since, after insertion of a fixed $\varphi \in \mathcal{D}(V, \mathbb{C})$, all expressions in these equations are holomorphic in α they hold for all α.

Proof of (4). Let $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\operatorname{supp}(\varphi) \cap J_{\pm}(0) = \emptyset$. Since $\operatorname{supp}(R_{\pm}(\alpha)) \subset J_{\pm}(0)$ for $\Re(\alpha) > n$, it follows for those α that

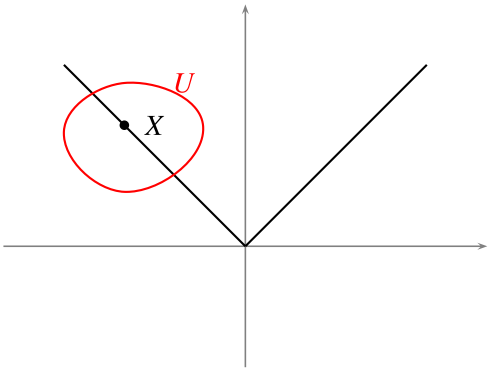
$$R_{\pm}(\alpha)[\varphi] = 0,$$

and then for all α by holomorphicity. Therefore $\operatorname{supp}(R_{\pm}(\alpha)) \subset J_{\pm}(0)$ for all α.

On the other hand, if $X \in I_{\pm}(0)$, then $\gamma(X) > 0$ and the map $\alpha \mapsto C(\alpha, n)\gamma(X)^{\frac{\alpha-n}{2}}$ is well defined and holomorphic on all of $\mathbb{C}$. Again by holomorphicity, we have for $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\operatorname{supp}(\varphi) \subset I_{\pm}(0)$

$$R_{\pm}(\alpha)[\varphi] = \int_{\operatorname{supp}(\varphi)} C(\alpha, n)\gamma(X)^{\frac{\alpha-n}{2}}\varphi(X)dX$$

for *all* $\alpha \in \mathbb{C}$. Thus $R_\pm(\alpha)$ coincides on $I_\pm(0)$ with the smooth function $C(\alpha,n)\gamma(\cdot)^{\frac{\alpha-n}{2}}$ and therefore $\operatorname{sing\,supp}(R_\pm(\alpha)) \subset C_\pm(0)$. By Remark 1.3.29 the map $\alpha \mapsto C(\alpha,n)$ vanishes only on $\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$, so we have $I_\pm(0) \subset \operatorname{supp}(R_\pm(\alpha))$ for every $\alpha \in \mathbb{C} \setminus (\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\})$. Thus $\operatorname{supp}(R_\pm(\alpha)) = J_\pm(0)$. This proves (4).
Proof of (5). For $\alpha \in \{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$ we have $C(\alpha,n) = 0$ and therefore $R_\pm(\alpha)|_{I_\pm(0)} \equiv 0$, so $\operatorname{supp}(R_\pm(\alpha)) \subset \partial J_\pm(0)$. Since $\operatorname{sing\,supp}(R_\pm(\alpha)) \subset \operatorname{supp}(R_\pm(\alpha))$ is clear, it remains to show $\operatorname{supp}(R_\pm(\alpha)) \subset \operatorname{sing\,supp}(R_\pm(\alpha))$. Let $X \notin \operatorname{sing\,supp}(R_\pm(\alpha))$, $X \in \partial J_\pm(0)$. Then there exits a neighboorhood U of X with $R_\pm(\alpha)|_U$ is smooth.



Since $U \setminus \partial J_\pm(0)$ is dense in U continuity implies $R_\pm(\alpha)|_U \equiv 0$. Thus $X \notin \operatorname{supp}(R_\pm(\alpha))$. This proves (5).
To show (6) recall first from (5) that we know already

$$\operatorname{sing\,supp}(R_\pm(\alpha)) = \operatorname{supp}(R_\pm(\alpha)) \subset \partial J_\pm(0)$$

for $\alpha = n-2, n-4, \dots, 2$ or 1 respectively. Note also that the distribution $R_\pm(\alpha)$ is invariant under timeorientation-preserving Lorentz transformations, that is, for any such transformation A of V we have

$$R_\pm(\alpha)[\varphi \circ A] = R_\pm(\alpha)[\varphi]$$

for every test function φ. Hence $\operatorname{supp}(R_\pm(\alpha))$ as well as $\operatorname{sing\,supp}(R_\pm(\alpha))$ are also invariant under the group of those transformations. Under the action of this group the orbit decomposition of $\partial J_\pm(0)$ is given by

$$\partial J_\pm(0) = \{0\} \cup (\partial J_\pm(0) \setminus \{0\}).^{8}$$

So $\operatorname{supp}(R_\pm(\alpha))$ is made up of orbits. Thus $\operatorname{supp}(R_\pm(\alpha)) = \operatorname{sing\,supp}(R_\pm(\alpha))$ coincides either with $\emptyset$, $\{0\}$, $\partial J_\pm(0)$ or a union of these components. Since the support is closed we have the possibilities that $\operatorname{supp}(R_\pm(\alpha)) = \operatorname{sing\,supp}(R_\pm(\alpha))$ is $\emptyset$, $\{0\}$ or $\partial J_+(0)$.

[8] To see that $(\partial J_\pm(0) \setminus \{0\})$ is one orbit, look at Lorentz boosts and space rotations.

We now consider test functions φ as in (1.13). We showed that for such φ

$$R_+(\alpha)[\varphi] = \int_0^\infty r^{\alpha-1} f(r) dr \neq 0$$

for suitable $f \in \mathcal{D}(\mathbb{R}, \mathbb{C})$. Simply choose f nonnegative everywhere and positive somewhere in $\mathbb{R}^+$. This concludes the proof of (6).
Proof of (7). Fix a compact subset $K \subset V$. Let $\sigma_K \in \mathcal{D}(V, \mathbb{R})$ be a compactly supported function such that $\sigma_{K\,|_K} \equiv 1$. For any $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\mathrm{supp}(\varphi) \subset K$ write

$$\varphi(x) = \varphi(0) + \sum_{j=1}^n x^j \varphi_j(x)$$

with suitable smooth functions φ_j, see Exercise 1.3.32 below. Then

$$\begin{aligned} R_\pm(0)[\varphi] &= R_\pm(0)[\sigma_K \varphi] \\ &= R_\pm(0)[\varphi(0)\sigma_K + \sum_{j=1}^n x^j \sigma_K \varphi_j] \\ &= \varphi(0) \underbrace{R_\pm(0)[\sigma_K]}_{=:c_K} + \sum_{j=1}^n \underbrace{(x^j R_\pm(0))}_{=0 \text{ by } (2)}[\sigma_K \varphi_j] \\ &= c_K \varphi(0). \end{aligned}$$

The constant c_K actually does not depend on K because for $K' \supset K$ and $\mathrm{supp}(\varphi) \subset K (\subset K')$,

$$c_{K'} \varphi(0) = R_+(0)[\varphi] = c_K \varphi(0),$$

so that $c_K = c_{K'} =: c$. It remains to show $c = 1$.
We again look at test functions φ as in (1.13) and compute, using (3),

$$\begin{aligned} c \cdot \varphi(0) &= R_+(0)[\varphi] \\ &= (\Box R_+(2))[\varphi] \\ &= R_+(2)[\Box\varphi] \\ &= \int_0^\infty r f''(r) dr \\ &= -\int_0^\infty f'(r) dr \\ &= f(0) \\ &= \varphi(0). \end{aligned}$$

This concludes the proof of (7).
Proof of (8). By its definition, the distribution $R_\pm(\alpha)$ is a continuous function if $\Re(\alpha) > n$, therefore it is of order 0. Since $\Box$ is a differential operator of order 2, the order of $\Box R_\pm(\alpha)$ is at most that of $R_\pm(\alpha)$ plus 2. It then follows from (3) that:

• If n is even: for every α with $\Re(\alpha) > 0$ we have $\Re(\alpha) + n = \Re(\alpha) + 2 \cdot \frac{n}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than n (and so $n + 1$).
• If n is odd: for every α with $\Re(\alpha) > 0$ we have $\Re(\alpha) + n + 1 = \Re(\alpha) + 2 \cdot \frac{n+1}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than $n + 1$.
This concludes the proof of (8).
Assertion (9) is clear by definition whenever $\alpha > n$. For general $\alpha \in \mathbb{R}$ choose $k \in \mathbb{N}$ so large that $\alpha + 2k > n$. Using (3) we get for any $\varphi \in \mathcal{D}(V, \mathbb{R})$

$$R_\pm(\alpha)[\varphi] \quad = \quad \Box^k R_\pm(\alpha + 2k)[\varphi] \quad = \quad R_\pm(\alpha + 2k)[\Box^k \varphi] \in \mathbb{R}$$

because $\Box^k \varphi \in \mathcal{D}(V, \mathbb{R})$ as well. □

1.3.32. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^k-function, $k \geq 1$.

a) Show that there exist C^{k-1}-functions $\varphi_1, \ldots, \varphi_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x^j \cdot \varphi_j(x)$$

for all $x \in \mathbb{R}^n$.

b) Show that the φ_j need not have compact support even if φ does.

Shortly after the definition of distributions, we saw that a distribution extends uniquely to a continuous linear map on the space of C^k-sections in E with compact support, if the order is not too high. For the Riesz distributions, in Proposition 1.3.31 (8) we found an upper bound for the order. This leads us to a slight generalization of Lemma 1.3.27 that we will need later on.

Corollary 1.3.33. *For $\varphi \in \mathcal{D}^k(V, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ defines a holomorphic function on $\{\alpha \in \mathbb{C} \mid \Re(\alpha) > n - 2[\frac{k}{2}]\}$.*

Proof. Let $\phi \in \mathcal{D}^k(V, \mathbb{C})$. On $\{\Re(\alpha) > n\}$, by the definition of $R_\pm(\alpha)$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi] = C(\alpha, n) \int_{J_\pm} \gamma^{\frac{\alpha-n}{2}} \varphi dX$ is clearly holomorphic . With (3) of Proposition 1.3.31 we have $R_\pm(\alpha)[\varphi] = \Box R_\pm(\alpha + 2)[\varphi] = R_\pm(\alpha + 2)[\Box\varphi]$, so $R_\pm(\alpha)[\varphi]$ is holomorphic for $\Re(\alpha) > n - 2$ in case $k \geq 2$. We iterate this argument $[\frac{k}{2}]$-times (to stay in the case of (8) of Proposition 1.3.31) and we get the holomorphic extension to the set $\{\Re(\alpha) > n - 2[\frac{k}{2}]\}$. □

Remark 1.3.34. Combining (3) and (7) of Proposition 1.3.31 we find $\Box R_\pm(2) = R_\pm(0) = \delta_0$. We say that $R_\pm(2)$ are *fundamental solutions* for the d'Alembert operator on Minkowski space, a concept we will study in detail. Using fundamental solutions, it is easy to solve the inhomogeneous equation $\Box u = f$ arbitrary right side f by convolution.

There are qualitative differences in the properties of the solutions depending on the parity of the dimension:
If $n \geq 4$ is even, we have $\mathrm{supp}\,(R_\pm(2)) = \partial J_\pm(0)$. Interprete the right hand side of $\Box R_\pm(2) = \delta_0$ as point source at 0 of a signal that propagates with constant speed. Inside the future light cone the solution is zero, the wave propagates strictly on the cone. An observer with timelike world line would note the signal for just one moment as he crosses $\partial J_\pm(0)$. This is known as the *Huygens property*. It is familiar to us from light and sound waves propagating in 3 space dimensions (hence $n = 4$ spacetime dimensions).
If $n \geq 3$ is odd then $\mathrm{supp}\,(R_\pm(2)) = J_\pm(0)$ and the Huygens property does not hold. In this case, the signal of a point source propagates also inside the light cone. For an observer, the wave is noticable not only at a single moment but still after the signal has arrived. An example of such waves are waves on 2-dimensional surfaces like water waves.

1.3.3 Riesz distributions on a domain

Riesz distributions have been defined on all spaces isometric to Minkowski space. They are therefore defined on the tangent spaces at all points of a Lorentzian manifold. We now show how to construct Riesz distributions defined in small open subsets of the Lorentzian manifold itself. The passage from the tangent space to the manifold will be provided by the Riemannian exponential map.
Let Ω be a domain in a timeoriented n-dimensional Lorentzian manifold, $n \geq 2$. Suppose Ω is geodesically starshaped with respect to some point $x \in \Omega$. In particular, the Riemannian exponential function $\exp_x$ restricts to a diffeomorphism $\Omega' \to \Omega$ where Ω' is an open subset of T_xM, starshaped with respect to 0. Let $\mu_x : \Omega \to \mathbb{R}$ be defined as in (1.2.27). We define for every test function $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$,

$$R_\pm^\Omega(\alpha, x)[\varphi] := R_\pm(\alpha)[(\mu_x\varphi) \circ \exp_x].$$

Note that $\mathrm{supp}((\mu_x\varphi) \circ \exp_x)$ is contained in Ω'. Extending the function $(\mu_x\varphi) \circ \exp_x$ by zero we can regard it as a test function on $T_x\Omega$ and thus apply $R_\pm(\alpha)$ to it.

Definition 1.3.35. We call $R_+^\Omega(\alpha, x)$ the **advanced Riesz distribution** and $R_-^\Omega(\alpha, x)$ the **retarded Riesz distribution** on Ω at x for $\alpha \in \mathbb{C}$.

The relevant properties of the Riesz distributions are collected in the following proposition.

Proposition 1.3.36. *The following holds for all $\alpha \in \mathbb{C}$ and all $x \in \Omega$:*

1. *If $\Re(\alpha) > n$, then $R_\pm^\Omega(\alpha, x)$ is the continuous function*

$$R_\pm^\Omega(\alpha, x) = \begin{cases} C(\alpha, n)\,\Gamma_x^{\frac{\alpha-n}{2}} & \text{on } J_\pm^\Omega(x), \\ 0 & \text{elsewhere.} \end{cases}$$

2. *For every fixed test function* φ *the map* $\alpha \mapsto R_\pm^\Omega(\alpha, x)[\varphi]$ *is holomorphic on* $\mathbb{C}$.
3. $\Gamma_x \cdot R_\pm^\Omega(\alpha, x) = \alpha(\alpha - n + 2)\, R_\pm^\Omega(\alpha + 2, x)$
4. $\mathrm{grad}\,(\Gamma_x) \cdot R_\pm^\Omega(\alpha, x) = 2\alpha \mathrm{grad}\, R_\pm^\Omega(\alpha + 2, x)$
5. *If* $\alpha \neq 0$, *then* $\Box R_\pm^\Omega(\alpha + 2, x) = \left(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_\pm^\Omega(\alpha, x)$
6. $R_\pm^\Omega(0, x) = \delta_x$
7. *For every* $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \ldots\} \cup \{n-2, n-4, \ldots\})$ *we have*
$$\mathrm{supp}\left(R_\pm^\Omega(\alpha, x)\right) = J_\pm^\Omega(x) \quad \textit{and} \quad \mathrm{sing\,supp}\left(R_\pm^\Omega(\alpha, x)\right) \subset C_\pm^\Omega(x).$$
8. *For every* $\alpha \in \{0, -2, -4, \ldots\} \cup \{n-2, n-4, \ldots\}$ *we have*
$$\mathrm{supp}\left(R_\pm^\Omega(\alpha, x)\right) = \mathrm{sing\,supp}\left(R_\pm^\Omega(\alpha, x)\right) \subset C_\pm^\Omega(x).$$
9. *For* $n \geq 3$ *and* $\alpha = n-2, n-4, \ldots, 1$ *or* 2, *respectively, we have*
$$\mathrm{supp}\left(R_\pm^\Omega(\alpha, x)\right) = \mathrm{sing\,supp}\left(R_\pm^\Omega(\alpha, x)\right) = C_\pm^\Omega(x).$$
10. *For* $\Re(\alpha) > 0$ *we have* $\mathrm{ord}(R_\pm^\Omega(\alpha, x)) \leq n + 1$. *Moreover, there exists a neighborhood* U *of* x *and a constant* $C > 0$ *such that*
$$|R_\pm^\Omega(\alpha, x')[\varphi]| \leq C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$$
for all $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$ *and all* $x' \in U$.
11. *If* $U \subset \Omega$ *is an open neighborhood of* x *such that* Ω *is geodesically starshaped with respect to all* $x' \in U$ *and if* $V \in \mathcal{D}(U \times \Omega, \mathbb{C})$, *then the function* $U \to \mathbb{C}$, $x' \mapsto R_\pm^\Omega(\alpha, x')[y \mapsto V(x', y)]$, *is smooth.*
12. *If* $U \subset \Omega$ *is an open neighborhood of* x *such that* Ω *is geodesically starshaped with respect to all* $x' \in U$, *if* $\Re(\alpha) > 0$, *and if* $V \in \mathcal{D}^{n+1+k}(U \times \Omega, \mathbb{C})$, *then the function* $U \to \mathbb{C}$, $x' \mapsto R_\pm^\Omega(\alpha, x')[y \mapsto V(x', y)]$, *is* C^k.
13. *For every* $\varphi \in \mathcal{D}^k(\Omega, \mathbb{C})$ *the map* $\alpha \mapsto R_\pm^\Omega(\alpha, x)[\varphi]$ *is a holomorphic function on* $\{\alpha \in \mathbb{C} \mid \Re(\alpha) > n - 2[\frac{k}{2}]\}$.
14. *If* $\alpha \in \mathbb{R}$, *then* $R_\pm^\Omega(\alpha, x)$ *is real, i. e.,* $R_\pm^\Omega(\alpha, x)[\phi] \in \mathbb{R}$ *for all* $\phi \in \mathcal{D}(\Omega, \mathbb{R})$.

Proof. Proof of (1). Let $\Re(\alpha) > n$ and $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$. Then
$$R_\pm^\Omega(\alpha, x)[\varphi] \quad = \quad R_\pm^\Omega(\alpha, x)[(\mu_x \cdot \varphi) \circ \exp_x]$$

$$
\begin{aligned}
&= C(\alpha,n)\int_{J_\pm(0)} \gamma^{\frac{\alpha-n}{2}}(\mu_x\cdot\varphi)\circ\exp_x\, dX\\
&= C(\alpha,n)\int_{J_\pm^\Omega(x)} \Gamma_x^{\frac{\alpha-n}{2}}\cdot\varphi\ \mathrm{dV}.
\end{aligned}
$$

Proof of (2). This follows directly from the definition of $R_+^\Omega(\alpha,x)$ and from Lemma 1.3.27.
Proof of (3). By (1) this obviously holds for $\Re(\alpha)>n$ since $C(\alpha,n)=\alpha(\alpha-n+2)C(\alpha+2,n)$. By analyticity of $\alpha\mapsto R_+^\Omega(\alpha,x)$ it must hold for all α.
Proof of (4). Consider α with $\Re(\alpha)>n$. By (1) the function $R_\pm^\Omega(\alpha+2,x)$ is then C^1. On $J_\pm^\Omega(x)$ we compute

$$
\begin{aligned}
2\alpha\,\mathrm{grad}\,R_\pm^\Omega(\alpha+2,x) &= 2\alpha C(\alpha+2,n)\,\mathrm{grad}\left(\Gamma_x^{\frac{\alpha+2-n}{2}}\right)\\
&= \underbrace{2\alpha C(\alpha+2,n)\frac{\alpha+2-n}{2}}_{C(\alpha,n)}\Gamma_x^{\frac{\alpha-n}{2}}\,\mathrm{grad}\,\Gamma_x\\
&= R_\pm^\Omega(\alpha,x)\,\mathrm{grad}\,\Gamma_x.
\end{aligned}
$$

For arbitrary $\alpha\in\mathbb{C}$ assertion (4) follows from analyticity of $\alpha\mapsto R_\pm^\Omega(\alpha,x)$.
Proof of (5). Let $\alpha\in\mathbb{C}$ with $\Re(\alpha)>n+2$. Since $R_\pm^\Omega(\alpha+2,x)$ is then C^2, we can compute $\Box R_\pm^\Omega(\alpha+2,x)$ classically. This will show that (5) holds for all α with $\Re(\alpha)>n+2$. Analyticity then implies (5) for all α.

$$
\begin{aligned}
\Box R_\pm^\Omega(\alpha+2,x) &= -\mathrm{div}\left(\mathrm{grad}\,R_\pm^\Omega(\alpha+2,x)\right)\\
&\overset{(4)}{=} -\frac{1}{2\alpha}\,\mathrm{div}\left(R_\pm^\Omega(\alpha,x)\cdot\mathrm{grad}\,(\Gamma_x)\right)\\
&= \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x)-\frac{1}{2\alpha}\langle\mathrm{grad}\,\Gamma_x,\mathrm{grad}\,R_\pm^\Omega(\alpha,x)\rangle\\
&\overset{(4)}{=} \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x)-\frac{1}{2\alpha\cdot 2(\alpha-2)}\langle\mathrm{grad}\,\Gamma_x,\mathrm{grad}\,\Gamma_x\cdot R_\pm^\Omega(\alpha-2,x)\rangle\\
&\overset{\text{Lemma 1.2.77.1}}{=} \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x)+\frac{1}{\alpha(\alpha-2)}\Gamma_x\cdot R_\pm^\Omega(\alpha-2,x)\\
&\overset{(3)}{=} \frac{1}{2\alpha}\Box\Gamma_x\cdot R_\pm^\Omega(\alpha,x)+\frac{(\alpha-2)(\alpha-n)}{\alpha(\alpha-2)}R_\pm^\Omega(\alpha,x)\\
&= \left(\frac{\Box\Gamma_x-2n}{2\alpha}+1\right)R_\pm^\Omega(\alpha,x).
\end{aligned}
$$

Proof of (6). Let φ be a test function on Ω. Then by Proposition 1.3.31 (7)

$$
\begin{aligned}
R_\pm^\Omega(0,x)[\varphi] &= R_\pm(0)[(\mu_x\varphi)\circ\exp_x]\\
&= \delta_0[(\mu_x\varphi)\circ\exp_x]\\
&= ((\mu_x\varphi)\circ\exp_x)(0)\\
&= \mu_x(x)\varphi(x)\\
&= \varphi(x)\\
&= \delta_x[\varphi].
\end{aligned}
$$

The assertions (7), (8) and (9) follow directly from the corresponding properties of the Riesz distributions on Minkowski space. Namely, exp_x is a diffeomorphism and therefore maps (singular) supports to (singular) supports. Moreover, μ_x is a smooth positive function and hence does not affect the (singular) supports.
Proof of (11). Let $A(x, x') : T_x\Omega \to T_{x'}\Omega$ be a timeorientation preserving linear isometry. Then

$$\begin{aligned} R_\pm^\Omega(\alpha, x')[V(x', \cdot)] &= R_\pm^\Omega(\alpha, x)[V(x', \cdot) \circ A(x, x')] \\ &= R_\pm(\alpha)[(\mu_{x'} \cdot V(x', \cdot)) \circ \exp_{x'} \circ A(x, x')] \end{aligned}$$

where $R_\pm(\alpha)$ is, as before, the Riesz distribution on $T_x\Omega$. Hence if we choose $A(x, x')$ to depend smoothly on x', then $(\mu_{x'} \cdot V(x', y)) \circ \exp_{x'} \circ A(x, x')$ is smooth in x' and y and the assertion follows from Lemma 1.3.11.
Proof of (10). Since $\mathrm{ord}(R_\pm(\alpha)) \le n+1$ by Proposition 1.3.31 (8) we have $\mathrm{ord}(R_\pm^\Omega(\alpha, x)) \le n+1$ as well. We now have to show that the constant C may be chosen locally uniformly in x. We choose $A(x, x')$ as in the proof of (11) and consider the case $V = \varphi$ independent of x'. We find

$$\begin{aligned} \left|R_\pm^\Omega(\alpha, x')[\varphi]\right| &= \left|R_\pm^\Omega(\alpha)[(\mu_{x'} \cdot \varphi) \circ \exp_{x'} \circ A(x, x')]\right| \\ &\le C\,\|(\mu_{x'} \cdot \varphi) \circ \exp_{x'} \circ A(x, x')]\|_{C^{n+1}} \\ &\le C'\,\|\varphi\|_{C^{n+1}} \end{aligned}$$

where C' contains derivatives of μ, exp and A up to order $n+1$.
Proof of (12). By (10) we can apply $R_\pm^\Omega(\alpha, x')$ to $V(x', \cdot)$. Now the same argument as for (11) shows that the assertion follows from Lemma 1.3.11.
Assertion (13) is a consequence of Corollary 1.3.33. Furthermore, (14) follows from Proposition 1.3.31.9 because μ_x is real as well. □

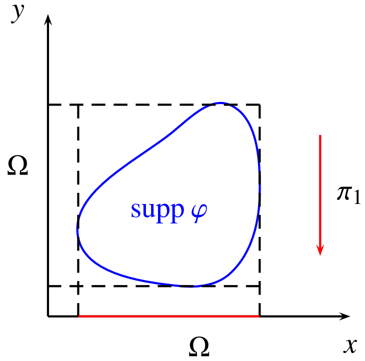
Advanced and retarded Riesz distributions are related as follows.

Lemma 1.3.37. *Let Ω be a convex timeoriented Lorentzian manifold. Let $\alpha \in \mathbb{C}$. Then for all $u \in \mathcal{D}(\Omega \times \Omega, \mathbb{C})$ we have*

$$\int_\Omega R_+^\Omega(\alpha, x)\ [y \mapsto u(x, y)]\ \mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha, y)\ [x \mapsto u(x, y)]\ \mathrm{dV}(y).$$

Proof. The convexity condition for Ω ensures that the Riesz distributions $R_\pm^\Omega(\alpha, x)$ are defined for all $x \in \Omega$. By Proposition 1.3.36.11 the integrands are smooth.

Since u has compact support contained in $\Omega \times \Omega$ the integrand $R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]$ (as a function of x) has compact support contained in Ω. Namely, its support is contained in the projection $\pi_1(\mathrm{supp} u)$, which is compact as continuous image of a compact set. Here $\pi_1 : \Omega \times \Omega \to \Omega$ is the projection to the first factor.

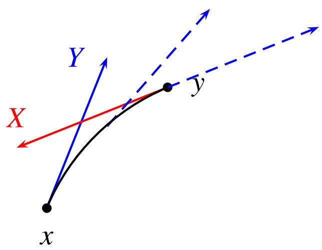


A similar statement holds for the integrand of the right hand side. Hence the integrals exist. By Proposition 1.3.36.13 they are holomorphic in α. Thus it suffices to show the equation for α with $\Re(\alpha) > n$.
For such an $\alpha \in \mathbb{C}$ the Riesz distributions $R_+(\alpha, x)$ and $R_-(\alpha, y)$ are continuous functions. From the explicit formula (1) in Proposition 1.3.36 we see

$$R_+(\alpha, x)(y) = R_-(\alpha, y)(x)$$

for all $x, y \in \Omega$. We just have to check that $\Gamma_x(y) = \Gamma_y(x)$.
This can be seen as follows:

We set $X := \exp_y^{-1}(x)$ and $Y := \exp_x^{-1}(y)$. By Definition 1.2.76 we then have $\Gamma_x(y) = -g|_x(Y,Y)$ and $\Gamma_y(x) = -g|_y(X,X)$. The definition of the exponential map then tells us that X is minus the vector Y obtained by parallel transport from $T_x\Omega$ to $T_y\Omega$ along the unique geodesic connecting x and y. It follows that $g|_x(Y,Y) = g|_y(X,X)$ and hence $\Gamma_x(y) = \Gamma_y(x)$.



By Fubini's theorem we then get

$$\begin{aligned}
\int_\Omega R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]\;\mathrm{dV}(x) &= \int_\Omega \left(\int_\Omega R_+^\Omega(\alpha, x)(y)\; u(x,y)\;\mathrm{dV}(y) \right)\;\mathrm{dV}(x) \\
&= \int_\Omega \left(\int_\Omega R_-^\Omega(\alpha, y)(x)\; u(x,y)\;\mathrm{dV}(x) \right)\;\mathrm{dV}(y) \\
&= \int_\Omega R_-^\Omega(\alpha, y)\,[x \mapsto u(x,y)]\;\mathrm{dV}(y)
\end{aligned}$$

which concludes the proof. □

As a technical tool we will also need a version of Lemma 1.3.37 for certain nonsmooth sections.

Lemma 1.3.38. *Let Ω be a causal domain in a timeoriented Lorentzian manifold of dimension n. Let $\Re(\alpha) > 0$ and let $k \geq n+1$. Let K_1, K_2 be compact subsets of $\overline{\Omega}$ and let $u \in C^k(\overline{\Omega} \times \overline{\Omega}, \mathbb{C})$ so that* $\mathrm{supp}(u) \subset J_+^\Omega(K_1) \times J_-^\Omega(K_2)$. *Then*

$$\int_\Omega R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]\,\mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha, y)\,[x \mapsto u(x,y)]\,\mathrm{dV}(y).$$

Proof. For fixed x, the support of the function $y \mapsto u(x,y)$ is contained in $J_-^\Omega(K_2)$. Since Ω is causal, it follows from Lemma 1.2.20 (with $A = J_+(x)$) that the subset $J_-^\Omega(K_2) \cap J_+^\Omega(x)$ is relatively compact in $\overline{\Omega}$. So we have that $\mathrm{supp}(u(x,\cdot)) \cap \mathrm{supp}(R_+^\Omega(\alpha, x))$ is closed and contained in a relatively compact set and therefore compact. By Proposition 1.3.36.10 one can then apply $R_+^\Omega(\alpha, x)$ to the C^k-function $y \mapsto u(x,y)$.
Furthermore, the support of the continuous function $x \mapsto R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]$ is contained in $J_+^\Omega(K_1) \cap J_-^\Omega(\mathrm{supp}(y \mapsto u(x,y))) \subset J_+^\Omega(K_1) \cap J_-^\Omega(J_-^\Omega(K_2)) = J_+^\Omega(K_1) \cap J_+^\Omega(K_2)$, which is relatively compact in $\overline{\Omega}$, again by Lemma 1.2.20 (with $A = J_-^\Omega(K_2)$).
Hence the function $x \mapsto R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]$ has compact support in $\overline{\Omega}$, so that the left-hand-side makes sense. Analogously the right-hand-side is well defined.
Our considerations also show that the integrals depend only on the values of u on $\left(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\right) \times \left(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\right)$ which is a relatively compact set. Applying a cut-off function argument we may assume without loss of generality that u has compact support. Proposition 1.3.36.13 says that the integrals depend holomorphically on α on the domain $\{\Re(\alpha) > 0\}$. Therefore it suffices to show the equality for α with sufficiently large real part, which can be done exactly as in the proof of Lemma 1.3.37. □

1.4 Sobolev spaces

Let N be a compact manifold without boundary of $\dim N = n$. Let μ be a positive volume density. Moreover, let $E \to N$ be a Riemannian or Hermitian vector bundle. The L^2-norm for $u \in C^\infty(N, E)$ is

$$\|u\|_0^2 = \int_N |u(x)|^2\, d\mu(x)$$

where the norm $|u(x)|$ is induced by the metrics on the fibers of E.
For a metric connection ∇ on E, define $\Delta := \nabla^*\nabla + \mathrm{id} \in \mathscr{Diff}_2(E,E)$. This elliptic operator is formally selfadjoint by construction, moreover it is essentially selfadjoint. Hence we can use spectral calculus to define any function of this operator.

Definition 1.4.1. The **Sobolev norm** for $u \in C^\infty(N,E)$ and $k \in \mathbb{R}$ is given by

$$\|u\|_k^2 = \left\|\Delta^{\frac{k}{2}} u\right\|_0 .$$

Remark 1.4.2. Here $\Delta^{\frac{k}{2}}$ is defined by spectral calculus. In case $k \in 2\mathbb{N}$ this definition coincides with the usual definition as composition and therefore yields a differential operator of order k in that case.

Definition 1.4.3. The **Sobolev space** $H^k(N,E)$ for $k \in \mathbb{R}$ is the completion of $C^\infty(N,E)$ with respect to $\|\|_k$.

We collect some properties of Sobolev spaces. First we see that for growing k the spaces get smaller.

Proposition 1.4.4. *For $k < l$ there is a continuous embedding*

$$H^l(N,E) \hookrightarrow H^k(N,E).$$

Proof. For $k < l$ we calculate $\|u\|_k = \left\|\Delta^{\frac{k}{2}} u\right\|_0 = \left\|\Delta^{\frac{k-l}{2}} \Delta^{\frac{l}{2}} u\right\|_0 \le \left\|\Delta^{\frac{l}{2}} u\right\|_0 = \|u\|_l$. Note here that the operator norm of $\Delta^{\frac{k-l}{2}}$ is bounded by 1 because the function $\lambda \mapsto \lambda^{\frac{k-l}{2}}$ is bounded by 1 on $[1,\infty)$ and hence on the spectrum of Δ. $\square$

The refined version of this is the Rellich-Kondrachov theorem.

Theorem 1.4.5 (Rellich-Kondrachov theorem). *For $k < l$ the embedding $H^l(N,E) \hookrightarrow H^k(N,E)$ is compact.*

Remark 1.4.6. If the embedding is compact then any bounded sequence in $H^l(N,E)$ has a subsequence that converges in $H^k(N,E)$. Note that the Rellich-Kondrachov embedding theorem is not in general true for non-compact manifolds.

Sobolev sections can be considered as distributional sections, $H^k(N,E) \subset \mathcal{D}'(N,E)$, via

$$u[\varphi] := (\Delta^{-\frac{k}{2}} u)[\Delta^{\frac{k}{2}} \varphi] = \int_N \left(\Delta^{\frac{k}{2}} \varphi(x)\right) \left(\Delta^{-\frac{k}{2}} u(x)\right) d\mu(x)$$

for $\varphi \in C_c^\infty(N,E^*)$. Note here that $\Delta^{-\frac{k}{2}} u \in L^2(M,E)$ so that we already know how to consider it as a distribution.
Every distributional section turns out to be of a certain Sobolev regularity. Namely, we have:

Proposition 1.4.7. *The union of all Sobolev spaces equals the space of distributional sections,i.e.*

$$\bigcup_{k\in\mathbb{R}} H^k(N,E) = \mathcal{D}'(N,E).$$

Concerning the relation between the C^k-norms and the Sobolev norms, we first note that $C^k(N,E) \subset H^k(N,E)$ for $k \in \mathbb{N}$ because the Sobolev norm $\|\cdot\|_k$ can obviously be estimated by $\|\cdot\|_{C^k}$. In the converse direction we need more Sobolev regularity to control classical C^k-regularity.

Theorem 1.4.8 (Sobolev embedding theorem). *There is a continuous embedding*

$$H^k(N,E) \hookrightarrow C^l(N,E)$$

for $k > l + \frac{n}{2}$.

In particular, this implies

Proposition 1.4.9. *The intersection of all Sobolev spaces equals the space of all smooth sections, i.e.*

$$\bigcap_{k\in\mathbb{R}} H^k(N,E) = C^\infty(N,E).$$

Finally we note:

Proposition 1.4.10. *Any* $P \in \mathscr{D}\!\mathit{iff}_l(E,F)$ *extends to a bounded linear map* $H^k(N,E) \to H^{k-l}(N,F)$.

Remark 1.4.11. For fixed $k \in \mathbb{R}$ different choices of the volume density μ, the metric on E and the connection ∇ on E give rise to equivalent $\|\cdot\|_k$-norms. Hence the Sobolev spaces $H^k(N,E)$ are defined as topological vector spaces independently of those choices.

1.5 Miscellanea

Grönwall's lemma is often very useful as it turns an implicit estimate into an explicit one.

Lemma 1.5.1 (Grönwall's inequality). *Let $\alpha, \beta, h : [t_0, t_1] \to [0, \infty)$ be continuous and let α be monotonically increasing. If*

$$h(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s)h(s)ds$$

holds for all $t \in [t_0, t_1]$ then so does

$$h(t) \leq \alpha(t) \cdot \exp\left(\int_{t_0}^{t} \beta(s)ds\right).$$

Proof. We only need to prove the implication for $t = t_1$. Let $\varepsilon > 0$. Then

$$h(t) \leq \alpha(t) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds\,. \tag{1.16}$$

By assumption on $h(t)$ and since α is monotonically increasing, we find for the time derivative of the right hand side of (1.16):

$$\begin{aligned}
\frac{d}{dt}\left(\alpha(t_1) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds\right) &= \beta(t)h(t) \\
&\leq \beta(t)\left(\alpha(t) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds\right) \\
&\leq \beta(t)\left(\alpha(t_1) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds\right).
\end{aligned}$$

Division by the strictly positive term $\alpha(t_1) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds$ yields for the logarithmic derivative

$$\frac{d}{dt}\log\left(\alpha(t_1) + \varepsilon + \int_{t_0}^{t} \beta(s)h(s)ds\right) \leq \beta(t).$$

Integrating over $[t_0, t_1]$ we get

$$\log\left(\alpha(t_1) + \varepsilon + \int_{t_0}^{t_1} \beta(s)h(s)ds\right) - \log\left(\alpha(t_1) + \varepsilon\right) \leq \int_{t_0}^{t_1} \beta(s)ds.$$

Putting $\log\left(\alpha(t_1) + \varepsilon\right)$ on the other side and exponentiating we find

$$\alpha(t_1) + \varepsilon + \int_{t_0}^{t_1} \beta(s)h(s) \leq \left(\alpha(t_1) + \varepsilon\right) \cdot \exp\left(\int_{t_0}^{t_1} \beta(s)ds\right).$$

Hence by (1.16) for $t = t_1$ we find that $h(t_1)$ can be estimated

$$\begin{aligned} h(t_1) &\leq \alpha(t_1) + \varepsilon + \int_{t_0}^{t_1} \beta(s)h(s) \\ &\leq (\alpha(t_1) + \varepsilon) \cdot \exp\left(\int_{t_0}^{t_1} \beta(s)ds\right). \end{aligned}$$

Letting $\varepsilon \to 0$ then yields

$$h(t_1) \leq \alpha(t_1) \cdot \exp\left(\int_{t_0}^{t_1} \beta(s)ds\right).$$ □

Theorem 1.5.2 (Arzelà-Ascoli theorem). *Let X, Y be metric spaces and let X be compact. We equip $C(X,Y)$ with the metric $d(u,v) = \max_{x\in X} d^Y(u(x), v(x))$, i.e. with the topology of uniform convergence. Let $F \subset C(X,Y)$. Then the following two statements are equivalent:*

(i) $F \subset C(X,Y)$ is relatively compact.

(ii) For all $x \in X$ the set $\{f(x) \mid f \in F\} \subset Y$ is relatively compact and the family of maps F is equicontinuous.

1.6 Exercises

1.6.1. Let M be a manifold and let E and F be $\mathbb{K}$-vector bundles over M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $P \in \mathscr{Diff}_1(E,F)$. Show that for any $u \in C^\infty(M,E)$ and any smooth function $f : M \to \mathbb{K}$ one has the "Leibnitz rule"

$$P(fu) = \sigma_1(P, df)u + fPu.$$

1.6.2. Let M be a manifold and let E, F and G be $\mathbb{K}$-vector bundles over M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $P \in \mathscr{Diff}_k(E,F)$ and $Q \in \mathscr{Diff}_\ell(F,G)$. Show that for any $\xi \in T^*M$

$$\sigma_{k+\ell}(Q \circ P, \xi) = \sigma_\ell(Q,\xi) \circ \sigma_k(P,\xi).$$

1.6.3. Let $a, b, c : \mathbb{R} \to \mathbb{C}$ be smooth functions and let $P = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$. Here the underlying manifold is $M = \mathbb{R}$ and $E = F = M \times \mathbb{C}$ is the trivial complex line bundle with the usual Hermitian metric. Compute P^t.

1.6.4. Let $M = \mathbb{R}/\mathbb{Z} = S^1$ and let $E = F = M \times \mathbb{C}$ be the trivial complex line bundle with the usual Hermitian metric. We consider functions on M as periodic functions on $\mathbb{R}$.
Let $a, b : M \to \mathbb{C}$ be smooth and let $a(x) \neq 0$ for all x. Let $P = a(x)\frac{d}{dx} + b(x)$.

a) Show that $\dim(\ker(P)) \in \{0, 1\}$.

b) Show by example that both cases in a) occur.

c) Show that $\dim(\ker(P)) = \dim(\ker(P^t))$.

1.6.5. Let M be a Lorentzian manifold and $A \subset M$ a subset. The *Cauchy development* $D(A)$ of A is the set of those points p in M for which all inextendible causal curves through p intersect A.

a) Show that the Cauchy development satisfies:

$$D(D(A)) = D(A).$$

b) Give an example where A is a closed subset of M but $D(A)$ is not.

1.6.6. Two Lorentz metrics g and $\bar{g}$ on an manifold M are called *conformally equivalent* if there is a smooth positive function $f : M \to \mathbb{R}$ such that $\bar{g} = f^2 g$.
Show that in this case (M, g) is globally hyperbolic if and only if $(M, \bar{g})$ is.

1.6.7. We fix $m > 0$ and define the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(r) = (r - 2m) \exp\left(\frac{r}{2m} - 1\right)$.

a) Show that φ is a diffeomorphism from $(0, \infty)$ onto $(-2me^{-1}, \infty)$.

b) We consider $M = \{(u, v) \in \mathbb{R} \mid uv > -2me^{-1}\}$ and the function $r : M \to \mathbb{R}^+$, $r = \varphi^{-1}(uv)$. The manifold M together with the metric $g = \frac{8m^2}{r} \exp\left(1 - \frac{r}{2m}\right)(du \otimes dv + dv \otimes du)$ is called the *Kruskal plane*. It is closely related to the Schwarzschild solution.

   Indicate the two possible time-orientations in a drawing.

c) Find a smooth spacelike Cauchy hypersurface in the Kruskal plane.

1.6.8. Let M be a time-oriented Lorentzian manifold. Show that future-compact subsets of M are closed.

1.6.9. Let M be a manifold equipped with a volume density and two vector bundles $E, F \to M$. Let $P \in \mathscr{Diff}_k(E, F)$ and $f \in C^k(M, E)$.
Show that the application of P to f in the classical sense coincides with the application in the distributional sense.

1.6.10. Give an example of a distribution on $\mathbb{R}$ which does not have finite order.

1.6.11. For $\varepsilon > 0$ we define the function $u_\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ by

$$u_\varepsilon(t) = \begin{cases} \frac{1}{t}, & \text{if } |t| \geq \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

a) Show that the limit

$$\lim_{\varepsilon \searrow 0} u_\varepsilon$$

exists in $\mathcal{D}'(\mathbb{R}, \mathbb{R})$ but not in $L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$. This limit is called the *principal value* of $\frac{1}{t}$ and often denoted by $\mathrm{PV}(\frac{1}{t})$.

b) Prove $(\log(|\cdot|))' = \mathrm{PV}(\frac{1}{t})$.

1.6.12. On $\mathbb{R}^n$, $n \geq 2$, we define the funtion $r(x) = \|x\|$ where $\|\cdot\|$ denotes the Euclidean norm. We put for $x \in \mathbb{R}^n \setminus \{0\}$ and constants c_n

$$E := \begin{cases} \frac{1}{2\pi} \log(r), & \text{if } n = 2, \\ c_n r^{2-n}, & \text{if } n \geq 3. \end{cases}$$

Moreover, we set $E(0) := 0$.

a) Show that E is locally integrable on $\mathbb{R}^n$ and hence defines a distribution.

b) Show that for suitable choice of c_n the equation $\Delta E = \delta_0$ holds in the distributional sense.

1.6.13. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^k-function, $k \geq 1$.

a) Show that there exist C^{k-1}-functions $\varphi_1, \ldots, \varphi_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x^j \cdot \varphi_j(x)$$

for all $x \in \mathbb{R}^n$.

b) Show that the φ_j need not have compact support even if φ does.

1.6.14. Determine the support and the singular support of $R_\pm(\alpha)$ for $\alpha \in \{0, -2, -4, \ldots\}$.

1.6.15. Show that the Riesz distributions $R_\pm(\alpha)$ on Minkowski space are *tempered* distributions (see e.g. [14, p. 134] for a definition).
Hint: Show it first for $\Re(\alpha)$ sufficiently large and observe that holomorphicity of $\alpha \mapsto R_\pm(\alpha)[\varphi]$ and $R_\pm(\alpha+2)[\Box\varphi] = R_\pm(\alpha)[\varphi]$ still hold true for test functions φ of Schwartz class.

1.6.16. Let M be the $1+1$-dimensional Minkowski space. Check if $A \subset M$ is compact, spatially compact, future compact, strictly future compact, (strictly) past compact, or temporally compact (no detailed proofs required) where

a) $A = \{(t, x) \mid -1 \leq t \leq 1\}$;

b) $A = \{(t, x) \mid -1 \leq x \leq 1\}$;

c) $A = J^+(0)$;

d) $A = M \setminus I^+(0)$;

e) $A = \{(t, x) \mid t \geq 0\}$;

f) $A = \{(t, x) \mid t \geq -\frac{|x|}{2}\}$.

1.6.17. Let M be a globally hyperbolic Lorentzian manifold. Let $M' \subset M$ be an open subset with $J^-(M') = M'$. Show that M' is itself globally hyperbolic.

2 Linear wave equations - local theory

We now start to investigate the theory of linear wave equations. In particular, we will construct fundamental solutions. We first do this in small domains of the manifold - this is what we mean by the local theory.

2.1 Normally hyperbolic operators

We start by defining the type of differential operators which give rise to wave equations.

Definition 2.1.1. Let M be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. A linear differential operator $P \in \mathscr{Diff}_2(E, E)$ is called **normally hyperbolic** if its principal symbol is given by the metric,

$$\sigma_2(P, \xi) = -\langle \xi, \xi \rangle$$

for all $\xi \in T^*M$.

Remark 2.1.2. In other words, if we choose local coordinates $x^1, \ldots, x^n$ on M and a local trivialization of E, then a normally hyperbolic operator P is given by

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B_1(x)$$

where A_j and B_1 are matrix-valued coefficients depending smoothly on x and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$ with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Remark 2.1.3. On a Riemannian manifold this type of operator would be called a *Laplace-type operator* which would be an elliptic operator.

Example 2.1.4. Let E be the trivial line bundle so that sections in E are just functions. The d'Alembert operator $P = \Box$ is normally hyperbolic, see Example 1.1.18. Adding terms of order zero yields two more examples which are of physical relevance:
For $m > 0$ the operator $P = \Box + m^2$ is normally hyperbolic; it is called the **Klein-Gordon operator** with mass m. The operator $P = \Box + m^2 + \xi\mathrm{scal}$ for some constant ξ is normally hyperbolic; it is called a **covariant Klein-Gordon operator**.

Example 2.1.5. Let E be a vector bundle and let ∇ be a connection on E. This connection together with the Levi-Civita connection on T^*M induces a connection on $T^*M \otimes E$, again denoted ∇. We define the **connection-d'Alembert operator** $\Box^\nabla$ by the following commutative diagram:

$$\begin{tikzcd} C^\infty(M,E) \arrow[r, "\nabla"] \arrow[d, "\Box^\nabla"'] & C^\infty(M,T^*M \otimes E) \arrow[d, "\nabla"] \\ C^\infty(M,E) & C^\infty(M,T^*M \otimes T^*M \otimes E) \arrow[l, "-\mathrm{tr}\otimes\mathrm{id}_E"'] \end{tikzcd}$$

where $\mathrm{tr} : T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$.
We compute the principal symbol

$$\begin{aligned} \sigma_{\Box^\nabla}(\xi)s &= \sigma_0(-(\mathrm{tr} \otimes \mathrm{id}_E, \xi) \circ \sigma_1(\nabla, \xi) \circ \sigma_1(\nabla, \xi)(s) \\ &= -(\mathrm{tr} \otimes \mathrm{id}_E)(\xi \otimes \xi \otimes s) \\ &= -\langle \xi, \xi \rangle\, s. \end{aligned}$$

Hence $\Box^\nabla$ is normally hyperbolic.

Example 2.1.6. Let $E = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one, see [6, p. 34] for details. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = d\delta + \delta d$ is normally hyperbolic.

The following lemma says that each normally hyperbolic operator is a connection-d'Alembert operator up to a term of order zero.

Lemma 2.1.7. *Let $P :\in \mathscr{Diff}_2(E,E)$ be a normally hyperbolic operator on a Lorentzian manifold M. Then there exists a unique connection ∇ on E and a unique endomorphism field $B \in C^\infty(M, \mathrm{End}(E,E))$ such that*

$$P = \Box^\nabla + B.$$

Proof. First we prove uniqueness of such a connection. Let ∇ be an arbitrary connection on E. For any section $s \in C^\infty(M,E)$ and any function $f \in C^\infty(M)$ we get

$$\begin{aligned} \Box^\nabla(f \cdot s) &= -(\mathrm{tr} \otimes \mathrm{id}_E)\,(\nabla\,(\nabla\,(f \cdot s))) \\ &= -(\mathrm{tr} \otimes \mathrm{id}_E)\,(\nabla\,(df \otimes s + f \cdot \nabla s)) \\ &= -(\mathrm{tr} \otimes \mathrm{id}_E)\,(\nabla df \otimes s + 2df \otimes \nabla s + f \cdot \nabla\nabla s) \\ &= (\Box f) \cdot s - 2\nabla_{\mathrm{grad} f} s + f \cdot (\Box^\nabla s). \end{aligned} \tag{2.1}$$

Now suppose that ∇ satisfies the condition in Lemma 2.1.7. Then $B = P - \Box^\nabla$ is an endomorphism field and we obtain

$$f \cdot \left(P(s) - \Box^\nabla s\right) = P(f \cdot s) - \Box^\nabla (f \cdot s).$$

By (2.1) this yields

$$\nabla_{\operatorname{grad} f} s = \tfrac{1}{2} \left\{ f \cdot P(s) - P(f \cdot s) + (\Box f) \cdot s \right\}. \tag{2.2}$$

At a given point $x \in M$ every tangent vector $X \in T_x M$ can be written in the form $X = \operatorname{grad}_x f$ for some suitably chosen function f. Thus (2.2) shows that ∇ is determined by P and $\Box$ (which is determined by the Lorentzian metric). Since ∇ and hence $\Box^\nabla$ is determined by P and the Lorentzian metric, so is B.

To show existence one could use (2.2) to define a connection ∇ as in the statement. We follow an alternative path. Let ∇' be some connection on E. Since P and $\Box^{\nabla'}$ are both normally hyperbolic operators acting on sections in E, the difference $P - \Box^{\nabla'}$ is a differential operator of first order and can therefore be written in the form

$$P - \Box^{\nabla'} = A' \circ \nabla' + B',$$

for some $A' \in C^\infty(M, \operatorname{Hom}(T^*M \otimes E, E))$ and $B' \in C^\infty(M, \operatorname{Hom}(E, E))$. Set for every vector field X on M and section s in E

$$\nabla_X s := \nabla'_X s - \frac{1}{2} A'(X^\flat \otimes s).$$

This defines a new connection ∇ on E.

Let $e_1, \ldots, e_n$ be a local Lorentz orthonormal basis of TM , i.e. $\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij}$. We assume it to be ∇-synchronous at a given point $p \in M$, i.e. we have $\nabla_X e_j|_p = 0$.

Then we compute at p

$$\begin{aligned}
\Box^{\nabla'} s + A' \circ \nabla' s &= \sum_{j=1}^n \varepsilon_j \Big\{ - \nabla'_{e_j} \nabla'_{e_j} s + A'(e_j^\flat \otimes \nabla'_{e_j} s) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ - (\nabla_{e_j} + \frac{1}{2} A'(e_j^\flat \otimes \cdot))(\nabla_{e_j} s + \frac{1}{2} A'(e_j^\flat \otimes s)) \\
&\qquad + A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{2} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ - \nabla_{e_j} \nabla_{e_j} s - \frac{1}{2} \nabla_{e_j} (A'(e_j^\flat \otimes s)) \\
&\qquad + \frac{1}{2} A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{4} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \Box^\nabla s + \underbrace{\frac{1}{4} \sum_{j=1}^n \varepsilon_j \Big\{ A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) - 2(\nabla_{e_j} A')(e_j^\flat \otimes s) \Big\}}_{=:Qs},
\end{aligned}$$

where ∇ in $\nabla_{e_j} A'$ stands for the induced connection on $\mathrm{Hom}(T^*M \otimes E, E)$.
We observe that $Q(s) = \Box^{\nabla'} s + A' \circ \nabla' s - \Box^{\nabla} s$ is of order zero. Hence

$$P = \Box^{\nabla'} + A' \circ \nabla' + B' = \Box^{\nabla} s + Q(s) + B'(s)$$

is the desired expression with $B = Q + B'$. □

Definition 2.1.8. The connection in Lemma 2.1.7 will be called the ***P*-compatible** connection.

We shall henceforth always work with the P-compatible connection.
We restate (2.2) as a lemma.

Lemma 2.1.9. *Let $P = \Box^{\nabla} + B$ be normally hyperbolic. For $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$ one gets*

$$P(f \cdot s) = f \cdot P(s) - 2\nabla_{\operatorname{grad} f} s + \Box f \cdot s.$$

□

2.2 Fundamental solutions

Our next aim is to construct fundamental solutions in small domains of a Lorentzian manifold.

Definition 2.2.1. Let M be a timeoriented Lorentzian manifold, let $E \to M$ be a vector bundle and let $P \in \mathscr{Diff}_2(E, E)$ be normally hyperbolic. Let $x \in M$. A **fundamental solution** of P at x is a distribution $F \in \mathcal{D}'(M, E, E_x^*)$ such that

$$PF = \delta_x.$$

A fundamental solution F at x is called

$$\begin{cases} \text{an } \textbf{advanced} \text{ fundamental solution} & \text{if } \operatorname{supp}(F(x)) \subset J_+^\Omega(x) \\ \text{a } \textbf{retarded} \text{ fundamental solution} & \text{if } \operatorname{supp}(F(x)) \subset J_-^\Omega(x) \end{cases}$$

Example 2.2.2. Let M be the Minkowski space. Then $R_+(2)$ is an advanced fundamental solution and $R_-(2)$ is a retarded fundamental solution of $P = \Box$ at $x = 0$.

In the following we will construct local fundamental solutions for an arbitrary normally hyperbolic operator. The construction consists of three steps.

2.2.1 Formal fundamental solutions

First we write down a formal series in Riesz distributions with unknown coefficients. We then find recursive relations for these Hadamard coefficients known as *transport equations*. The transport equations are singular ordinary differential equations of first order along geodesics. We will see that they can be solved uniquely without the need to specify initial values. There is no reason why the formal solution constructed in this way should be convergent.
Let Ω be geodesically starshaped with respect to some fixed point $x \in \Omega$ so that the Riesz distributions $R_\pm^\Omega(\alpha, x)$ are defined. Let $E \to \Omega$ be a real or complex vector bundle and let P be a normally hyperbolic operator acting on $C^\infty(\Omega, E)$.
We make the following formal ansatz:

$$\mathcal{R}_\pm(x) := \sum_{k=0}^{\infty} V_x^k \; R_\pm^\Omega(2 + 2k, x)$$

where $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$ are smooth sections yet to be found.
For $\varphi \in \mathcal{D}(\Omega, E^*)$ the function $V_x^k \cdot \varphi$ is an E_x^*-valued test function and we have $(V_x^k \cdot R_\pm^\Omega(2 + 2k, x))[\varphi] = R_\pm^\Omega(2+2k, x)[V_x^k \cdot \varphi] \in E_x^*$. Hence each summand $V_x^k \cdot R_\pm^\Omega(2+2k, x)$ is a distribution in $\mathcal{D}'(\Omega, E, E_x^*)$.
We want $\mathcal{R}_\pm(x)$ to be a fundamental solution, i.e. we want

$$P\mathcal{R}_\pm(x) = \delta_x.$$

Here the application of P to the formal series $\mathcal{R}_\pm(x)$ is to be understood termwise. By Proposition 1.3.31.6 $\delta_x = R_\pm^\Omega(0, x)$, so

$$P\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} P(V_x^k R_\pm^\Omega(2 + 2k, x)) = R_\pm^\Omega(0, x).$$

This leads us to conditions on the V_x^k. Using Lemma 2.1.9 and properties (4) and (5) in Proposition 1.3.31 we compute

$$\begin{aligned}
R_\pm^\Omega(0, x) &= \sum_{k=0}^{\infty} P(V_x^k R_\pm^\Omega(2 + 2k, x)) \\
&= \sum_{k=0}^{\infty} \{V_x^k \cdot \Box R_\pm^\Omega(2 + 2k, x) - 2\,\nabla_{\mathrm{grad}\; R_\pm^\Omega(2+2k,x)} V_x^k + PV_x^k \cdot R_\pm^\Omega(2 + 2k, x)\} \\
&= V_x^0 \cdot \Box R_\pm^\Omega(2, x) - 2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0 \\
&\quad + \sum_{k=1}^{\infty} \left\{ V_x^k \cdot \left(\frac{\frac{1}{2}\Box\Gamma_x - n}{2k} + 1 \right) R_\pm^\Omega(2k, x) - \tfrac{2}{4k}\, \nabla_{\mathrm{grad}\,\Gamma_x\, R_\pm^\Omega(2k,x)} V_x^k \right. \\
&\qquad \left. + PV_x^{k-1} \cdot R_\pm^\Omega(2k, x) \right\} \\
&= V_x^0 \cdot \Box R_\pm^\Omega(2, x) - 2\,\nabla_{\mathrm{grad}\, R_\pm^\Omega(2,x)} V_x^0 &&(2.3) \\
&\quad + \sum_{k=1}^{\infty} \tfrac{1}{2k} \left\{ \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k \right) V_x^k - \nabla_{\mathrm{grad}\,\Gamma_x} V_x^k + 2k\, PV_x^{k-1} \right\} R_\pm^\Omega(2k, x). &&(2.4)
\end{aligned}$$

Comparing the coefficients of $R_\pm^\Omega(2k,x)$ we get the conditions

$$2\,\nabla_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 - \Box R_\pm^\Omega(2,x)\cdot V_x^0 + R_\pm^\Omega(0,x) = 0 \qquad \text{for } k=0 \text{ and} \tag{2.5}$$

$$\nabla_{\operatorname{grad}\Gamma_x} V_x^k - \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right) V_x^k = 2k\, PV_x^{k-1} \qquad \text{for } k \geq 1. \tag{2.6}$$

We take a look at what condition (2.6) would mean for $k=0$. We multiply this equation by $R_\pm^\Omega(\alpha,x)$ and get

$$\nabla_{\operatorname{grad}\Gamma_x\, R_\pm^\Omega(\alpha,x)} V_x^0 - \left(\tfrac{1}{2}\Box\Gamma_x - n\right)\, V_x^0\cdot R_\pm^\Omega(\alpha,x) = 0.$$

By Proposition 1.3.36.4 and 5 we obtain

$$\nabla_{2\alpha \operatorname{grad} R_\pm^\Omega(\alpha+2,x)} V_x^0 - \left(\alpha\Box R_\pm^\Omega(\alpha+2,x) - \alpha R_\pm^\Omega(\alpha,x)\right)\, V_x^0 = 0.$$

Division by α and the limit $\alpha\to 0$ yield

$$2\nabla_{\operatorname{grad} R_\pm^\Omega(2,x)} V_x^0 - \left(\Box R_\pm^\Omega(2,x) - R_\pm^\Omega(0,x)\right)\, V_x^0 = 0.$$

Therefore we recover condition (2.5) if and only if $V_x^0(x) = \mathrm{id}_{E_x}$.

To get formal fundamental solutions $\mathcal{R}_\pm(x)$ for P we hence need $V_x^k \in C^\infty(\Omega, E\otimes E_x^*)$ satisfying

$$\nabla_{\operatorname{grad}\Gamma_x} V_x^k - \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right) V_x^k = 2k\, PV_x^{k-1} \tag{2.7}$$

for all $k\geq 0$ with "initial condition" $V_x^0(x) = \mathrm{id}_{E_x}$. In particular, we have the same conditions on V_x^k for $\mathcal{R}_+(x)$ and for $\mathcal{R}_-(x)$. Equations (2.7) are known as *transport equations*.

Definition 2.2.3. Let Ω be timeoriented and geodesically starshaped with respect to $x\in\Omega$. Sections $V_x^k \in C^\infty(\Omega, E\otimes E_x^*)$ are called **Hadamard coefficients** for P at x if they satisfy the transport equations (2.7) for all $k\geq 0$ and $V_x^0(x) = \mathrm{id}_{E_x}$.

The transport equations (2.7) will allow us to solve for the Hadamard coefficients recursively. First one solves for V_x^0 where the right hand side in (2.7) vanishes. Then we proceed inductively, given V_x^{k-1} we solve for V_x^k.

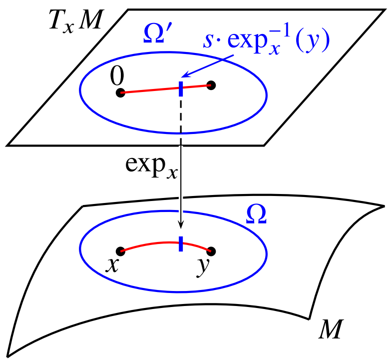
We observe that the transport equations are linear first order ordinary differential equations along the integral curves of $\operatorname{grad}\Gamma_x$. These integral curves are precisely the geodesics emanating from x. Naively, one might now think that there is a unique solution for V_x^k given some freely chosen initial value at x. The problem is that the transport equations are singular at x because $\operatorname{grad}\Gamma_x$ vanishes there. This is why the standard Picard-Lindelöf theorem does not apply. Therefore we have to analyze the transport equations in more detail.

For $y\in\Omega$ we denote the ∇-parallel translation along the (unique) geodesic from x to y by

$$\Pi_y^x : E_x \to E_y.$$

We have $\Pi_x^x = \mathrm{id}_{E_x}$ and $(\Pi_y^x)^{-1} = \Pi_x^y$.

We define the map $\Phi : \Omega \times [0,1] \to \Omega$, $\Phi(y,s) := \exp_x(s \cdot \exp_x^{-1}(y))$. Note that it is well defined and smooth since Ω is geodesically starshaped with respect to x.



Proposition 2.2.4. *Let Ω be timeoriented and geodesically starshaped with respect to $x \in \Omega$. Let P be a normally hyperbolic operator acting on $C^\infty(\Omega, E)$. Then there exist unique Hadamard coefficients V_x^k for P at x. They are given by*

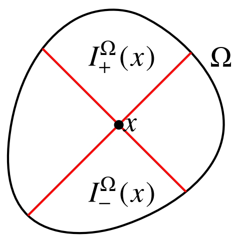
$$V_x^0(y) = \mu_x^{-1/2}(y) \Pi_y^x \tag{2.8}$$

and for $k \geq 1$

$$V_x^k(y) = -k\, \mu_x^{-1/2}(y)\, \Pi_y^x \int_0^1 \mu_x^{1/2}(\Phi(y,s)) s^{k-1}\, \Pi_x^{\Phi(y,s)}((PV_x^{k-1})\Phi(y,s))\, ds. \tag{2.9}$$

Proof. a) *Uniqueness.*

We put $\rho := \sqrt{|\Gamma_x|}$. We then have $\Gamma_x(y) = -\varepsilon\rho^2(y)$ where $\varepsilon = -1$ on $I_\pm^\Omega(x)$ and $\varepsilon = +1$ on $\Omega \setminus \left(J_+^\Omega(x) \cup J_-^\Omega(x)\right)$. We will derive the formulas for V_x^k on $\Omega \setminus \left(\partial J_+^\Omega(x) \cup \partial J_-^\Omega(x)\right)$. This is the region where ρ is smooth. By continuity, the formulas will then hold on all of Ω.



$\partial J_+^\Omega(x) \cup \partial J_-^\Omega(x)$

Using the identities

$$\tfrac{1}{2}\Box\Gamma_x - n = -\tfrac{1}{2}\partial_{\operatorname{grad}\Gamma_x} \log \mu_x = -\partial_{\operatorname{grad}\Gamma_x} \log(\mu_x^{1/2})$$

from Lemma 1.2.77.3 and

$$\partial_{\operatorname{grad}\Gamma_x}(\log \rho^k) \quad = \quad \frac{k}{2}\partial_{\operatorname{grad}\Gamma_x} \log \rho^2$$

$$
\begin{aligned}
&= \frac{k}{2}\partial_{\mathrm{grad}\,\Gamma_x}\log(-\varepsilon\Gamma_x)\\
&= \frac{k}{2}\frac{\langle \mathrm{grad}\,\Gamma_x, \mathrm{grad}\,(-\varepsilon\Gamma_x)\rangle}{-\varepsilon\Gamma_x}\\
&= -2k
\end{aligned}
$$

from Lemma 1.2.77.1 we find that the transport equation (2.7) is equivalent to

$$
\begin{aligned}
\nabla_{\mathrm{grad}\,\Gamma_x}\left(\mu_x^{1/2}\cdot\rho^k\cdot V_x^k\right) &= \mu_x^{1/2}\cdot\rho^k\nabla_{\mathrm{grad}\,\Gamma_x}V_x^k + \mu_x^{1/2}\cdot\rho^k\partial_{\mathrm{grad}\,\Gamma_x}\log(\mu_x^{1/2}\cdot\rho^k)V_x^k\\
&= \mu_x^{1/2}\cdot\rho^k\left(\nabla_{\mathrm{grad}\,\Gamma_x}V_x^k - \left(\tfrac{1}{2}\Box\Gamma_x - n + 2k\right)V_x^k\right)\\
&\overset{(2.7)}{=} \mu_x^{1/2}\cdot\rho^k\cdot 2k\cdot PV_x^{k-1}.
\end{aligned}
\tag{2.10}
$$

Now we want to reparametrize the geodesics starting at x such that Γ_x is the velocity vector field. Let $y\in\Omega$ and $\eta\in T_x\Omega$ such that $\exp_x(\eta)=y$. Set $c(t):=\exp_x(e^{2t}\cdot\eta)$ which yields a reparametrization of the geodesic β with $\beta(s)=\exp_x(s\cdot\eta)$.
By Lemma 1.2.77 (1)

$$
\begin{aligned}
\langle\dot c(t),\dot c(t)\rangle &= \langle 2e^{2t}\dot\beta(e^{2t}), 2e^{2t}\dot\beta(e^{2t})\rangle\\
&= 4e^{4t}\langle\dot\beta(e^{2t}),\dot\beta(e^{2t})\rangle\\
&= -4\,\Gamma_x\,(c(t))\\
&= \langle\mathrm{grad}\,\Gamma_x,\mathrm{grad}\,\Gamma_x\rangle.
\end{aligned}
$$

But we know that $\mathrm{grad}\,\Gamma_x$ and $\dot c(t)$ are parallel and point in the opposite directions. Therefore $\dot c(t)=-\mathrm{grad}\,\Gamma_x$.
For $k=0$ we now see, that (2.10) says $-\nabla_{\dot c(t)}(\mu_x^{1/2}V_x^0)=0$. Hence $\mu_x^{1/2}V_x^0$ is ∇-parallel along c. This is independent of the parametrization of c, hence

$$
\begin{aligned}
\left(\mu_x^{1/2}V_x^0\right)(y) &= \Pi_y^x\left(\mu_x^{1/2}V_x^0\right)(x)\\
&= \Pi_y^x\,\mathrm{id}_{E_x}\\
&= \Pi_y^x\,.
\end{aligned}
$$

This shows (2.8).
Next we determine V_x^k for $k\geq 1$. Let $y\in\Omega\setminus\left(\partial J_+^\Omega(x)\cup\partial J_-^\Omega(x)\right)$ be a point not on the light cone of x. Equation (2.10) says

$$
-\frac{\nabla}{dt}\left(\mu_x^{1/2}\cdot\rho^k\cdot V_x^k\right)=\mu_x^{1/2}\cdot\rho^k\cdot 2k\cdot PV_x^{k-1}.
$$

The relation between parallel transport and covariant derivative yields

$$
-\frac{d}{dt}\left(\Pi_x^{c(t)}\left(\mu_x^{1/2}\cdot\rho^k\cdot V_x^k\right)(c(t))\right)=\Pi_x^{c(t)}\left(\mu_x^{1/2}\cdot\rho^k\cdot 2k\cdot PV_x^{k-1}\right)(c(t))dt.
$$

By the fundamental theorem of calculus and now have

$$
-\,\Pi_x^y\left(\mu_x^{1/2}\cdot\rho^k\cdot V_x^k\right)(y)=\int_{-\infty}^{0}\Pi_x^{c(t)}\left(\mu_x^{1/2}\cdot\rho^k\cdot 2k\cdot PV_x^{k-1}\right)(c(t))dt.
\tag{2.11}
$$

Note that the boundary term at $t = -\infty$ vanishes because $\rho^k(x) = 0$. We compute

$$\begin{aligned}\rho^k(c(t)) &= |\Gamma_x(c(t))|^{k/2} \\ &= |\gamma(e^{2t}\eta)|^{k/2} \\ &= |e^{4t}\gamma(\eta)|^{k/2} \\ &= e^{2kt}|\gamma(\eta)|^{k/2}.\end{aligned}$$

Since $y \notin \left(\partial J_+^\Omega(x) \cup \partial J_-^\Omega(x)\right)$ we can divide (2.11) by $|\gamma(\eta)|^{k/2} \neq 0$. This yields

$$\begin{aligned}-\Pi_x^y\left(\mu_x^{1/2} \cdot e^{k\cdot 0} \cdot V_x^k\right)(y) &= 2k\int_{-\infty}^{0} \Pi_x^{c(t)}\left(\mu_x^{1/2} \cdot e^{2kt} \cdot PV_x^{k-1}\right)(c(t))dt \\ &= 2k\int_0^1 \Pi_x^{\Phi(y,s)}\left(\mu_x^{1/2} \cdot s^k \cdot PV_x^{k-1}\right)(\Phi(y,s))\frac{1}{2s}ds \\ &= k\int_0^1 \mu_x^{1/2}(\Phi(y,s)) \cdot s^{k-1} \cdot \Pi_x^{\Phi(y,s)} PV_x^{k-1}(\Phi(y,s))ds\end{aligned}$$

where we used the substitution $s = e^{2t}$ and the fact that the point $c(t)$ on the geodesic connecting x and y equals $\Phi(y, s)$. This yields (2.9).
b) *Existence.* To show existence we use formulas (2.8) and (2.9) as definitions. We observe that this defines smooth sections $V_x^k \in C^\infty(\Omega, E \otimes E_x^*)$. For all $k \geq 0$ we have to check Equation (2.10) from which (2.7) follows as we have already seen. Doing the calculations as in a) in reverse order shows this. $\square$

We have found formal fundamental solutions $\mathcal{R}_\pm(x)$ for P at fixed $x \in \Omega$.
Now we let x vary. Let $U \subset \Omega$ be an open subset such that Ω is geodesically starshaped with respect to every $x \in U$. This ensures that the Riesz distributions $R_\pm^\Omega(\alpha, x)$ are defined for all $x \in U$. We write $V_k(x, y) := V_x^k(y)$ for the Hadamard coefficients at x. Thus $V_k(x, y) \in E_x^* \otimes E_y = \mathrm{Hom}(E_x, E_y)$. The explicit formulas (2.8) and (2.9) show that the Hadamard coefficients V_k also depend smoothly on x, i. e.,

$$V_k \in C^\infty(U \times \Omega, E^* \boxtimes E).$$

We have formal fundamental solutions for P at all $x \in U$:

$$\mathcal{R}_\pm(x) = \sum_{k=0}^{\infty} V_k(x, \cdot)\, R_\pm^\Omega(2 + 2k, x).$$

The formulas for the Hadamard coefficients become particularly simple along the diagonal, i. e., for $x = y$. We have for any normally hyperbolic operator P

$$V_0(x, x) = \mu_x(x)^{-1/2}\Pi_x^x = \mathrm{id}_{E_x}.$$

For $k \geq 1$ we get

$$\begin{aligned} V_k(x,x) &= -k \underbrace{\mu_x^{-1/2}(x)}_{=1} \cdot 1 \cdot \underbrace{\Pi_x^x}_{=\mathrm{id}} \int_0^1 s^{k-1} \underbrace{\Pi_x^x}_{=\mathrm{id}} (P_{(2)} V_{k-1})(x,x) \mu_x^{-1/2}(x) ds \\ &= -(P_{(2)} V_{k-1})(x,x) \end{aligned}$$

where $P_{(2)}$ denotes the action of P on the second variable of V_{k-1}. Note that this does not give us a recursive formula to calculate $V_k(x,x)$ from $V_{k-1}(x,x)$. In order to do this, one needs to know the Hadamard coefficients for x and y independently.
We compute $V_1(x,x)$ for $P = \Box^\nabla + B$. By (2.9) and Lemma 2.1.9 we have

$$\begin{aligned} V_1(x,x) &= -(P_{(2)} V_0)(x,x) \\ &= -P(\mu_x^{-1/2} \Pi_\bullet^x)(x) \\ &= -\mu_x^{-1/2}(x) \cdot P(\Pi_\bullet^x)(x) + 2\nabla_{\underbrace{\mathrm{grad}\, \mu_x(x)}_{=0}} \Pi_\bullet^x(x) - (\Box \mu_x^{-1/2})(x) \cdot \mathrm{id}_{E_x} \\ &= -(\Box^\nabla + B)(\Pi_\bullet^x)(x) - (\Box \mu_x^{-1/2})(x) \cdot \mathrm{id}_{E_x} \\ &= -B|_x - (\Box \mu_x^{-1/2})(x) \cdot \mathrm{id}_{E_x}\,. \end{aligned}$$

From Corollary 1.2.75 we conclude

$$V_1(x,x) = \frac{\mathrm{scal}(x)}{6} \mathrm{id}_{E_x} - B|_x.$$

2.2.2 Approximate fundamental solutions

We want to make the series convergent by introducing certain cut-off functions. Since there are error terms produced by the cut-off functions the result is convergent but no longer solves the wave equation. We call it an approximate fundamental solution.
Assume that $\Omega' \subset M$ is a geodesically convex open subset. We then have the Hadamard coefficients $V_j \in C^\infty(\Omega' \times \Omega', E^* \boxtimes E)$ and for all $x \in \Omega'$ the formal fundamental solutions

$$\mathcal{R}_\pm(x) = \sum_{j=0}^{\infty} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x).$$

Fix an integer $N \geq \frac{n}{2}$ where n is the dimension of the manifold M. Then for all $j \geq N$ the distribution $R_\pm^{\Omega'}(2+2j,x)$ is a continuous function on Ω'. Hence we can split the formal fundamental solutions

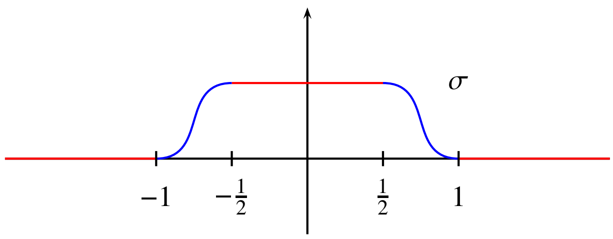
$$\mathcal{R}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$$

where $\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$ is a well-defined E_x^*-valued distribution in E over Ω' and $\sum_{j=N}^{\infty} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)$ is a formal sum of continuous sections, $V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) \in C^0(\Omega', E_x^* \otimes E)$ for $j \geq N$.

Using suitable cut-offs we will now replace the infinite formal part of the series by a convergent series. We need the following elementary lemma.

Lemma 2.2.5. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a smooth function vanishing outside $[-1, 1]$, such that $\sigma \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $0 \le \sigma \le 1$ everywhere. For every $l \in \mathbb{N}$ and every $\beta \ge l + 1$ there exists a constant $c(l, \beta)$ such that for all $0 < \varepsilon \le 1$ we have*

$$\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} \quad \le \quad \varepsilon \cdot c(l, \beta) \cdot \|\sigma\|_{C^l(\mathbb{R})}.$$



Proof. The generalized Leibniz rule yields

$$\begin{aligned}
&\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} \\
&\qquad \le \sum_{m=0}^{l} \binom{l}{m} \left\| \frac{1}{\varepsilon^m}\sigma^{(m)}(t/\varepsilon) \cdot \beta(\beta-1)\cdots(\beta-l+m+1)t^{\beta-l+m} \right\|_{C^0(\mathbb{R})} \\
&\qquad = \sum_{m=0}^{l} \binom{l}{m} \cdot \beta(\beta-1)\cdots(\beta-l+m+1)\varepsilon^{\beta-l} \left\| (t/\varepsilon)^{\beta-l+m}\sigma^{(m)}(t/\varepsilon) \right\|_{C^0(\mathbb{R})}.
\end{aligned}$$

Now $\sigma^{(m)}(t/\varepsilon)$ vanishes for $|t|/\varepsilon \ge 1$ and thus $\|(t/\varepsilon)^{\beta-m+l}\sigma^{(m)}(t/\varepsilon)\|_{C^0(\mathbb{R})} \le \|\sigma^{(m)}\|_{C^0(\mathbb{R})}$. Moreover, $\beta - l \ge 1$, hence $\varepsilon^{\beta-l} \le \varepsilon$. Therefore

$$\begin{aligned}
\left\| \frac{d^l}{dt^l}(\sigma(t/\varepsilon)t^\beta) \right\|_{C^0(\mathbb{R})} &\le \varepsilon \sum_{m=0}^{l} \binom{l}{m} \cdot \beta(\beta-1)\cdots(\beta-l+m+1) \left\| \sigma^{(m)} \right\|_{C^0(\mathbb{R})} \\
&\le \varepsilon\, c(l, \beta)\, \|\sigma\|_{C^l(\mathbb{R})}.
\end{aligned}$$

□

We define $\Gamma \in C^\infty(\Omega' \times \Omega', \mathbb{R})$ by $\Gamma(x, y) := \Gamma_x(y)$. Note that $\Gamma(x, y) = 0$ if and only if the geodesic joining x and y in Ω' is lightlike.
We now shrink Ω' slightly and replace it by a relatively compact open subset $\Omega \subset\subset \Omega'$. This will ensure that the Hadamard coefficients are bounded on $\overline{\Omega}$.

Lemma 2.2.6. *Let $\Omega \subset\subset \Omega'$ be a relatively compact open subset. Then there exists a sequence of $\varepsilon_j \in (0,1]$, $j \geq N$, such that for each $k \geq 0$ the series*

$$
\begin{aligned}
(x,y) \quad \mapsto \quad & \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y) \\
&= \begin{cases} \sum_{j=N+k}^{\infty} C(2+2j,n)\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,\Gamma(x,y)^{j+1-n/2} & \text{if } y \in J_{\pm}^{\Omega'}(x) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

converges in $C^k(\overline{\Omega}\times\overline{\Omega}, E^* \boxtimes E)$. In particular, the series

$$(x,y) \mapsto \sum_{j=N}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_{\pm}^{\Omega'}(2+2j,x)(y)$$

defines a continuous section over $\overline{\Omega}\times\overline{\Omega}$ and a smooth section over $(\overline{\Omega}\times\overline{\Omega}) \setminus \Gamma^{-1}(0)$.

Proof. For $j \geq N \geq \frac{n}{2}$ the exponent in $\Gamma(x,y)^{j+1-n/2}$ is positive. Therefore the piecewise definition of the j-th summand yields a continuous section over Ω'.
The factor $\sigma(\Gamma(x,y)/\varepsilon_j)$ vanishes whenever $\Gamma(x,y) \geq \varepsilon_j$. Hence for $j \geq N \geq \frac{n}{2}$ and $0 < \varepsilon_j \leq 1$

$$
\begin{aligned}
\Big\| (x,y) \mapsto \sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\, & R_{\pm}^{\Omega'}(2+2j,x)(y) \Big\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \\
&\leq \quad C(2+2j,n)\ \|V_j\|_{C^0(\overline{\Omega}\times\overline{\Omega})}\ \varepsilon_j^{j+1-n/2} \\
&\leq \quad C(2+2j,n)\ \|V_j\|_{C^0(\overline{\Omega}\times\overline{\Omega})}\ \varepsilon_j .
\end{aligned}
$$

Hence if we choose $\varepsilon_j \in (0,1]$ such that

$$C(2+2j,n)\ \|V_j\|_{C^0(\overline{\Omega}\times\overline{\Omega})}\ \varepsilon_j < 2^{-j},$$

then the series converges absolutely in the C^0-norm and therefore defines a continuous section.
For $k \geq 0$ and $j \geq N+k \geq \frac{n}{2}+k$ the function $\Gamma^{j+1-\frac{n}{2}}$ vanishes to $(k+1)$-st order along $\Gamma^{-1}(0)$. Thus the j-th summand in the series is of regularity C^k. Writing $\sigma_j(t) := \sigma(t/\varepsilon_j)t^{j+1-n/2}$ we know from Lemma 2.2.5 that

$$\|\sigma_j\|_{C^k(\mathbb{R})} \leq \varepsilon_j \cdot c_1(k,j,n) \cdot \|\sigma\|_{C^k(\mathbb{R})}$$

where here and henceforth $c_1, c_2, \ldots$ denote certain universal positive constants whose precise values are of no importance.

Using Lemmas 1.3.23 and 1.3.24 we obtain

$$
\begin{aligned}
&\left\|(x,y)\mapsto \sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\le\ C(2+2j,n)\|(\sigma_j\circ\Gamma)\cdot V_j\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\le\ c_2(k,j,n)\cdot\|\sigma_j\circ\Gamma\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\|V_j\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\le\ c_3(k,j,n)\cdot\|\sigma_j\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\dots,k}\|\Gamma\|^\ell_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\|V_j\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\le\ c_4(k,j,n)\cdot\varepsilon_j\cdot\|\sigma\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\dots,k}\|\Gamma\|^\ell_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\|V_j\|_{C^k(\overline{\Omega}\times\overline{\Omega})}.
\end{aligned}
$$

Hence if we add the (finitely many) conditions on ε_j that

$$c_4(k,j,n)\cdot\varepsilon_j\cdot\|V_j\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\le 2^{-j}$$

for all $k\le j-N$, then we have for fixed k

$$
\begin{aligned}
&\left\|(x,y)\mapsto \sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\le\ 2^{-j}\cdot\|\sigma\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\dots,k}\|\Gamma\|^\ell_{C^k(\overline{\Omega}\times\overline{\Omega})}
\end{aligned}
$$

for all $j\ge N+k$. Thus the series

$$(x,y)\mapsto\sum_{j=N+k}^{\infty}\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)$$

converges absolutely in $C^k(\overline{\Omega}\times\overline{\Omega},E^*\boxtimes E)$. All summands $\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)$ are smooth on $\overline{\Omega}\times\overline{\Omega}\setminus\Gamma^{-1}(0)$, thus

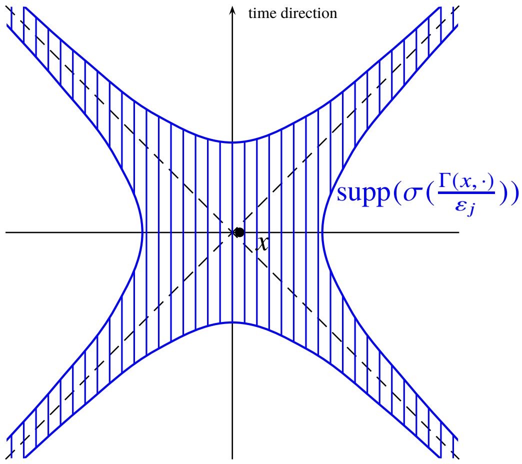
$$
\begin{aligned}
(x,y)\mapsto&\sum_{j=N}^{\infty}\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)\\
=&\sum_{j=N}^{N+k-1}\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)\\
&+\sum_{j=N+k}^{\infty}\sigma(\Gamma(x,y)/\varepsilon_j)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)
\end{aligned}
$$

is C^k for all k, hence smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\Gamma^{-1}(0)$. □

Define distributions $\widetilde{\mathcal{R}}_+(x)$ and $\widetilde{\mathcal{R}}_-(x)$ by

$$\widetilde{\mathcal{R}}_\pm(x):=\sum_{j=0}^{N-1}V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x).$$

The factor $\sigma(\Gamma(x,\cdot)/\varepsilon_j)$ in the infinite part does not vanish for $\Gamma(x,\cdot)<\varepsilon_j$. So the support of each summand in this series is given by points y close to the light cone.



By Lemma 2.2.6 and the properties of Riesz distributions we know that

$$\mathrm{supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset J_\pm^{\Omega'}(x) \text{ and} \tag{2.12}$$

$$\mathrm{sing\,supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset \partial J_\pm^{\Omega'}(x). \tag{2.13}$$

Moreover, $\mathrm{ord}(\widetilde{\mathcal{R}}_\pm(x)) \leq n+1$ because the infinite part of the series is continuous and hence of order 0 and for the finite part we have that in every summand α is positive.

Lemma 2.2.7. *The ε_j in Lemma 2.2.6 can be chosen such that in addition to the assertion in Lemma 2.2.6 we have on Ω*

$$P_{(2)}\widetilde{\mathcal{R}}_\pm(x) = \delta_x + K_\pm(x,\cdot) \tag{2.14}$$

with smooth $K_\pm \in C^\infty(\overline{\Omega}\times\overline{\Omega}, E^* \boxtimes E)$.

Proof. We apply P to the finite sum and the infinite sum separately. From properties (2.5) and (2.6) of the Hadamard coefficients we know

$$P_{(2)}\left(\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)\right) = \delta_x + (P_{(2)}V_{N-1}(x,\cdot))R_\pm^{\Omega'}(2N,x). \tag{2.15}$$

Moreover, by Lemma 1.3.22, we may interchange P with the infinite sum and we get

$$\begin{aligned}
&P_{(2)}\left(\sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)\right)\\
&= \sum_{j=N}^{\infty} P_{(2)}\left(\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)\right)\\
&= \sum_{j=N}^{\infty} \Big(\Box_{(2)}(\sigma(\Gamma(x,\cdot)/\varepsilon_j))V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x) - 2\nabla^{(2)}_{\mathrm{grad}_{(2)}\sigma(\Gamma(x,\cdot)/\varepsilon_j)}(V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x))
\end{aligned}$$

$$+\sigma(\Gamma(x,\cdot)/\varepsilon_j)P_{(2)}(V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x))\Big)$$

Here and in the following $\Box_{(2)}$, $\mathrm{grad}_{(2)}$, and $\nabla^{(2)}$ indicate that the operators are applied with respect to the y-variable just as for $P_{(2)}$. Abbreviating

$$\Sigma_1 := \sum_{j=N}^{\infty} \Box_{(2)}(\sigma(\Gamma(x,\cdot)/\varepsilon_j))V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x) \quad \text{and}$$

$$\Sigma_2 := -2\sum_{j=N}^{\infty} \nabla^{(2)}_{\mathrm{grad}_{(2)}\sigma(\Gamma(x,\cdot)/\varepsilon_j)}(V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x))$$

we have

$$\begin{aligned}
&P_{(2)}\left(\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)\right)\\
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)P_{(2)}(V_j(x,\cdot)R_\pm^{\Omega'}(2+2j,x))\\
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)\Big((P_{(2)}V_j(x,\cdot))R_\pm^{\Omega'}(2+2j,x)-2\nabla^{(2)}_{\mathrm{grad}_{(2)}R_\pm^{\Omega'}(2+2j,x)}V_j(x,\cdot)\\
&\qquad +V_j(x,\cdot)\Box_{(2)}R_\pm^{\Omega'}(2+2j,x)\Big).
\end{aligned}$$

Properties (2.5) and (2.6) of the Hadamard coefficients tell us

$$V_j(x,\cdot)\Box_{(2)}R_\pm^{\Omega'}(2+2j,x)-2\nabla^{(2)}_{\mathrm{grad}_{(2)}R_\pm^{\Omega'}(2+2j,x)}V_j(x,\cdot)=-P_{(2)}(V_{j-1}(x,\cdot)R_\pm^{\Omega'}(2+2j,x))$$

and hence

$$\begin{aligned}
&P_{(2)}\left(\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R_\pm^{\Omega'}(2+2j,x)\right)\\
&= \Sigma_1+\Sigma_2+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)\left((P_{(2)}V_j(x,\cdot))R_\pm^{\Omega'}(2+2j,x)-P_{(2)}V_{j-1}R_\pm^{\Omega'}(2j,x)\right)\\
&= \Sigma_1+\Sigma_2-\sigma(\Gamma(x,\cdot)/\varepsilon_N)P_{(2)}V_{N-1}R_\pm^{\Omega'}(2N,x)\\
&\qquad+\sum_{j=N}^{\infty}\left(\sigma(\Gamma(x,\cdot)/\varepsilon_j)-\sigma(\Gamma(x,\cdot)/\varepsilon_{j+1})\right)(P_{(2)}V_j(x,\cdot))R_\pm^{\Omega'}(2+2j,x).
\end{aligned}$$

Putting $\Sigma_3 := \sum_{j=N}^{\infty}\left(\sigma(\Gamma(x,\cdot)/\varepsilon_j)-\sigma(\Gamma(x,\cdot)/\varepsilon_{j+1})\right)(P_{(2)}V_j(x,\cdot))R_\pm^{\Omega'}(2+2j,x)$ and combining with (2.15) yields

$$\begin{aligned}
P_{(2)}\widetilde{\mathcal{R}}_\pm(x)-\delta_x &= (1-\sigma(\Gamma(x,\cdot)/\varepsilon_{N-1}))P_{(2)}V_{N-1}(x,\cdot)R_\pm^{\Omega'}(2N,x)+\Sigma_1+\Sigma_2+\Sigma_3\\
&=: K_\pm(x,\cdot)\,.
\end{aligned}\tag{2.16}$$

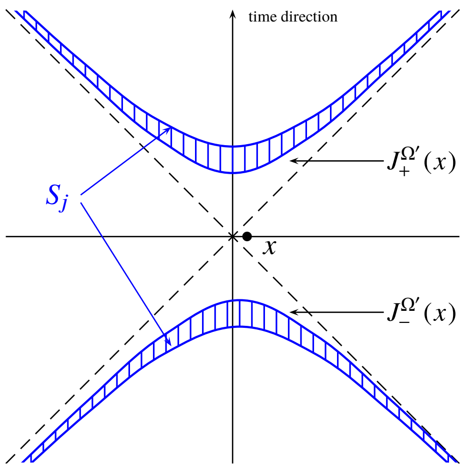
It remains to show that K_+ and K_- are smooth. Since

$$P_{(2)}V_{N-1}(x,y)R_\pm^{\Omega'}(2N,x)(y) = \begin{cases} C(2N,n)\,P_{(2)}V_{N-1}(x,y)\,\Gamma(x,y)^{N-n/2}, & \text{if } y \in J_\pm^{\Omega'}(x) \\ 0, & \text{otherwise} \end{cases}$$

is smooth on $(\Omega' \times \Omega') \setminus \Gamma^{-1}(0)$ and since $1 - \sigma(\Gamma(x,\cdot)/\varepsilon_j)$ vanishes on a neighborhood of $\Gamma^{-1}(0)$ we have that

$$(x,y) \mapsto (1 - \sigma(\Gamma(x,y)/\varepsilon_j)) \cdot P_{(2)}V_{N-1}(x,y)R_\pm^{\Omega'}(2N,x)(y)$$

is smooth. Similarly, the individual terms in the three infinite sums are smooth sections because $\sigma(\Gamma/\varepsilon_j) - \sigma(\Gamma/\varepsilon_{j+1})$, $\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$, and $\Box_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$ all vanish on a neighborhood of $\Gamma^{-1}(0)$. It remains to be shown that the three series in (2.16) converge in all C^k-norms.
We start with Σ_2. Let $S_j := \{(x,y) \in \Omega' \times \Omega' \mid \frac{\varepsilon_j}{2} \le |\Gamma(x,y)| \le \varepsilon_j\}$.



Section of S_j for fixed x

Since $\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})$ vanishes outside the "strips" S_j, there exist constants $c_1(k,n)$, $c_2(k,n)$ and $c_3(k,n,j)$ such that

$$\begin{aligned}
&\left\| \nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})} \left(V_j(\cdot,\cdot)\, R_\pm^{\Omega'}(2+2j,\cdot) \right) \right\|_{C^k(\overline{\Omega}\times\overline{\Omega})} \\
&= \left\| \nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma \circ \frac{\Gamma}{\varepsilon_j})} \left(V_j(\cdot,\cdot)\, R_\pm^{\Omega'}(2+2j,\cdot) \right) \right\|_{C^k(\overline{\Omega}\times\overline{\Omega}\cap S_j)} \\
&\le c_1(k,n) \cdot \left\| \sigma \circ \frac{\Gamma}{\varepsilon_j} \right\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)} \cdot \left\| V_j(\cdot,\cdot)\, R_\pm^{\Omega'}(2+2j,\cdot) \right\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)} \\
&\le c_2(k,n) \cdot \|\sigma\|_{C^{k+1}(\mathbb{R})} \cdot \max_{\ell=0,\ldots,k+1} \left\|\frac{\Gamma}{\varepsilon_j}\right\|^\ell_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)} \\
&\quad \cdot \|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)} \cdot \|R_\pm^{\Omega'}(2+2j,\cdot)\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}
\end{aligned}$$

$$
\begin{aligned}
&\leq c_2(k,n)\cdot\frac{1}{\varepsilon_j^{k+1}}\cdot\|\sigma\|_{C^{k+1}(\mathbb{R})}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\cdot\|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\|R_{\pm}^{\Omega'}(2+2j,\cdot)\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}\\
&\leq c_3(k,n,j)\cdot\frac{1}{\varepsilon_j^{k+1}}\cdot\|\sigma\|_{C^{k+1}(\mathbb{R})}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\cdot\|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\|\Gamma^{1+j-n/2}\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}.
\end{aligned}
$$

By Lemma 1.3.24 we have

$$
\begin{aligned}
&\|\Gamma^{1+j-n/2}\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}\\
&\quad\leq c_4(k)\cdot\|t\mapsto t^{1+j-n/2}\|_{C^{k+1}([\varepsilon_j/2,\varepsilon_j])}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}\\
&\quad\leq c_5(k,j,n)\cdot\varepsilon_j^{j-n/2-k}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega}\cap S_j)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|\nabla^{(2)}_{\mathrm{grad}_{(2)}(\sigma\circ\frac{\Gamma}{\varepsilon_j})}\left(V_j(\cdot,\cdot)\,R_{\pm}^{\Omega'}(2+2j,\cdot)\right)\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\leq c_6(k,j,n)\cdot\|\sigma\|_{C^{k+1}(\mathbb{R})}\cdot\left(\max_{\ell=0,\dots,k+1}\|\Gamma\|^{\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\right)^2\cdot\|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\varepsilon_j^{j-2k-n/2-1}\\
&\leq c_6(k,j,n)\cdot\|\sigma\|_{C^{k+1}(\mathbb{R})}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{2\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\varepsilon_j
\end{aligned}
$$

if $j\geq 2k+n/2+2$. Hence if we require the (finitely many) conditions

$$
c_6(k,j,n)\cdot\|V_j\|_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}\cdot\varepsilon_j\leq 2^{-j}
$$

on ε_j for all $k\leq j/2-n/4-1$, then almost all j-th terms of the series Σ_2 are bounded in the C^k-norm by $2^{-j}\cdot\|\sigma\|_{C^{k+1}(\mathbb{R})}\cdot\max_{\ell=0,\dots,k+1}\|\Gamma\|^{2\ell}_{C^{k+1}(\overline{\Omega}\times\overline{\Omega})}$. Thus Σ_2 converges in the C^k-norm for any k and defines a smooth section in $E^*\boxtimes E$ over $\overline{\Omega}\times\overline{\Omega}$.
The series Σ_1 is treated similarly. To examine Σ_3 we observe that for $j\geq k+\frac{n}{2}$

$$
\begin{aligned}
&\left\|\left((\sigma\circ\tfrac{\Gamma}{\varepsilon_j})-(\sigma\circ\tfrac{\Gamma}{\varepsilon_{j+1}})\right)\cdot(P_{(2)}V_j)\cdot R_{\pm}^{\Omega'}(2+2j,\cdot)\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\leq c_7(j,n)\cdot\left\|\left((\sigma\circ\tfrac{\Gamma}{\varepsilon_j})-(\sigma\circ\tfrac{\Gamma}{\varepsilon_{j+1}})\right)\cdot(P_{(2)}V_j)\cdot\Gamma^{1+j-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\leq c_8(k,j,n)\cdot\left\|\left((\sigma\circ\tfrac{\Gamma}{\varepsilon_j})-(\sigma\circ\tfrac{\Gamma}{\varepsilon_{j+1}})\right)\cdot\Gamma^{k+1}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\quad\cdot\left\|P_{(2)}V_j\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\left\|\Gamma^{j-k-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\
&\leq c_8(k,j,n)\cdot\left(\left\|(\sigma\circ\tfrac{\Gamma}{\varepsilon_j})\cdot\Gamma^{k+1}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}+\left\|(\sigma\circ\tfrac{\Gamma}{\varepsilon_{j+1}})\cdot\Gamma^{k+1}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\right)\\
&\quad\cdot\left\|P_{(2)}V_j\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\left\|\Gamma^{j-k-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}.
\end{aligned}
\tag{2.17}
$$

Putting $\sigma_j(t) := \sigma(t/\varepsilon_j)\cdot t^{k+1}$ we have $(\sigma\circ\frac{\Gamma}{\varepsilon_j})\cdot\Gamma^{k+1} = \sigma_j\circ\Gamma$. Hence by Lemmas 1.3.24 and 2.2.5

$$\begin{aligned}\left\|(\sigma\circ\tfrac{\Gamma}{\varepsilon_j})\cdot\Gamma^{k+1}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})} &= \left\|\sigma_j\circ\Gamma\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\ &\leq c_9(k,n)\cdot\left\|\sigma_j\right\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\ldots,k}\|\Gamma\|^{\ell}_{C^k(\overline{\Omega}\times\overline{\Omega})}\\ &\leq c_{10}(k,n)\cdot\varepsilon_j\cdot\|\sigma\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\ldots,k}\|\Gamma\|^{\ell}_{C^k(\overline{\Omega}\times\overline{\Omega})}.\end{aligned}$$

Plugging this into (2.17) yields

$$\begin{aligned}&\left\|\Big((\sigma\circ\tfrac{\Gamma}{\varepsilon_j})-(\sigma\circ\tfrac{\Gamma}{\varepsilon_{j+1}})\Big)\cdot(P_{(2)}V_j)\cdot R^{\Omega'}_{\pm}(2+2j,\cdot)\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\\ \leq\ & c_{11}(k,j,n)\cdot(\varepsilon_j+\varepsilon_{j+1})\cdot\|\sigma\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\ldots,k}\|\Gamma\|^{\ell}_{C^k(\overline{\Omega}\times\overline{\Omega})}\\ &\cdot\left\|P_{(2)}V_j\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\left\|\Gamma^{j-k-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}.\end{aligned}$$

Hence if we add the conditions on ε_j that

$$c_{11}(k,j,n)\cdot\varepsilon_j\cdot\left\|P_{(2)}V_j\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\left\|\Gamma^{j-k-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\leq 2^{-j-1}$$

for all $k\leq j-\frac{n}{2}$ and

$$c_{11}(k,j-1,n)\cdot\varepsilon_j\cdot\left\|P_{(2)}V_{j-1}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\left\|\Gamma^{j-1-k-n/2}\right\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\leq 2^{-j-2}$$

for all $k\leq j-1-\frac{n}{2}$, then we have that almost all j-th terms in Σ_3 are bounded in the C^k-norm by $2^{-j}\cdot\|\sigma\|_{C^k(\mathbb{R})}\cdot\max_{\ell=0,\ldots,k}\|\Gamma\|^{\ell}_{C^k(\overline{\Omega}\times\overline{\Omega})}$. Thus Σ_3 defines a smooth section as well. □

Lemma 2.2.8. *The ε_j in Lemmas 2.2.6 and 2.2.7 can be chosen such that in addition there is a constant $C>0$ so that*

$$|\widetilde{\mathcal{R}}_{\pm}(x)[\varphi]|\leq C\cdot\|\varphi\|_{C^{n+1}(\Omega)}$$

for all $x\in\overline{\Omega}$ and all $\varphi\in\mathcal{D}(\Omega,E^*)$. In particular, $\widetilde{\mathcal{R}}(x)$ is of order at most $n+1$. Moreover, for every fixed $\varphi\in\mathcal{D}(\Omega,E^*)$, the map $x\mapsto\widetilde{\mathcal{R}}_{\pm}(x)[\varphi]$ is a smooth section in E^*,

$$\widetilde{\mathcal{R}}_{\pm}(\cdot)[\varphi]\in C^{\infty}(\overline{\Omega},E^*).$$

We know already that for each $x\in\overline{\Omega}$ the distribution $\widetilde{\mathcal{R}}(x)$ is of order at most $n+1$. The point of the lemma is that the constant C in the estimate $|\widetilde{\mathcal{R}}_{\pm}(x)[\varphi]|\leq C\cdot\|\varphi\|_{C^{n+1}(\Omega)}$ can be chosen independently of x.

Proof. Recall the definition of $\widetilde{\mathcal{R}}_{\pm}(x)$,

$$\widetilde{\mathcal{R}}_{\pm}(x)=\sum_{j=0}^{N-1}V_j(x,\cdot)\,R^{\Omega'}_{\pm}(2+2j,x)+\sum_{j=N}^{\infty}\sigma(\Gamma(x,\cdot)/\varepsilon_j)V_j(x,\cdot)\,R^{\Omega'}_{\pm}(2+2j,x).$$

By Proposition 1.3.36 (10) there are constants $C_j > 0$ such that $|R_\pm^{\Omega'}(2+2j,x)[\varphi]| \le C_j \cdot \|\varphi\|_{C^{n+1}(\Omega)}$ for all φ and all $x \in \overline{\Omega}$. Thus there is a constant $C' > 0$ such that

$$\left|\sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]\right| \le C' \cdot \|\varphi\|_{C^{n+1}(\Omega)}$$

for all φ and all $x \in \overline{\Omega}$. The remainder term $\sum_{j=N}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) =: f(x,y)$ is a continuous section, hence

$$\begin{aligned}
\left|\sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]\right| &\le \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega}) \cdot \|\varphi\|_{C^0(\Omega)} \\
&\le \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega}) \cdot \|\varphi\|_{C^{n+1}(\Omega)}
\end{aligned}$$

for all φ and all $x \in \overline{\Omega}$. Therefore $C := C' + \|f\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \mathrm{vol}(\overline{\Omega})$ does the job.
To see smoothness in x we fix $k \ge 0$ and we write

$$\begin{aligned}
\widetilde{\mathcal{R}}_\pm(x)[\varphi] &= \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi] + \sum_{j=N}^{N+k-1} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi] \\
&\quad + \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi].
\end{aligned}$$

By Proposition 1.3.36 (11) the summands $V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]$ and $\sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]$ depend smoothly on x. By Lemma 2.2.6 the remainder $\sum_{j=N+k}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x)[\varphi]$ is C^k. Thus $x \mapsto \widetilde{\mathcal{R}}_\pm(x)[\varphi]$ is C^k for every k, hence smooth. □

Definition 2.2.9. If M is a timeoriented Lorentzian manifold, then we call a subset $S \subset M \times M$ **future-stretched** with respect to M if $y \in J_+^M(x)$ whenever $(x,y) \in S$. Analogously, we define **past-stretched** subsets.

We summarize the results obtained so far.

Proposition 2.2.10. *Let M be an n-dimensional timeoriented Lorentzian manifold and let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega' \subset M$ be a convex open subset. Fix an integer $N \ge \frac{n}{2}$ and fix a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfying $\sigma \equiv 1$ outside $[-1,1]$, $\sigma \equiv 0$ on $[-\frac{1}{2},\frac{1}{2}]$, and $0 \le \sigma \le 1$ everywhere.*
Then for every relatively compact open subset $\Omega \subset\subset \Omega'$ there exists a sequence $\varepsilon_j > 0$, $j \ge N$,

such that for every $x \in \overline{\Omega}$

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R^\Omega_\pm(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R^\Omega_\pm(2+2j,x)$$

defines a distribution on Ω *satisfying*

1. $\operatorname{supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset J^\Omega_\pm(x)$,
2. $\operatorname{sing\,supp}(\widetilde{\mathcal{R}}_\pm(x)) \subset C^\Omega_\pm(x)$,
3. $P_{(2)}\widetilde{\mathcal{R}}_\pm(x) = \delta_x + K_\pm(x,\cdot)$ *with smooth* $K_\pm \in C^\infty(\overline{\Omega}\times\overline{\Omega}, E^* \boxtimes E)$,
4. $\operatorname{supp}(K_+)$ *is future-stretched and* $\operatorname{supp}(K_-)$ *is past-stretched with respect to* Ω',
5. $\widetilde{\mathcal{R}}_\pm(x)[\varphi]$ *depends smoothly on* x *for every fixed* $\varphi \in \mathcal{D}(\Omega, E^*)$,
6. *there is a constant* $C > 0$ *such that* $|\widetilde{\mathcal{R}}_\pm(x)[\varphi]| \leq C \cdot \|\varphi\|_{C^{n+1}(\Omega)}$ *for all* $x \in \overline{\Omega}$ *and all* $\varphi \in \mathcal{D}(\Omega, E^*)$.

Proof. The only thing that remains to be shown is the statement (4). Recall from (2.16) that in the notation of the proof of Lemma 2.2.7

$$K_\pm(x,y) = (1 - \sigma(\Gamma(x,y)/\varepsilon_{N-1})) \cdot P_{(2)}V_{N-1}(x,y) \cdot R^{\Omega'}_\pm(2N,x)(y) + \Sigma_1 + \Sigma_2 + \Sigma_3.$$

The first term as well as all summands in the three infinite series Σ_1, Σ_2, and Σ_3 contain a factor $R^{\Omega'}_\pm(2j,x)(y)$ for some $j \geq N$. Hence if $K_+(x,y) \neq 0$, then $y \in \operatorname{supp}(R^\Omega_\pm(2j,x)) \subset J^\Omega_+(x)$. In other words, $\{(x,y) \in \Omega\times\Omega \,|\, K_+(x,y) \neq 0\}$ is future-stretched with respect to Ω'. Since Ω' is geodesically convex causal futures are closed. Hence $\operatorname{supp}(K_+) = \overline{\{(x,y) \in \Omega\times\Omega \,|\, K_+(x,y) \neq 0\}}$ is future-stretched with respect to Ω' as well. In the same way one sees that $\operatorname{supp}(K_-)$ is past-stretched. □

Definition 2.2.11. If the ε_j are chosen as in Proposition 2.2.10, then we call

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R^{\Omega'}_\pm(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R^{\Omega'}_\pm(2+2j,x)$$

an **approximate advanced** or **retarded fundamental solution**, respectively.

2.2.3 True fundamental solutions

Thirdly, we turn the approximate fundamental solution into a true one using certain integral operators.

We fix approximate fundamental solutions $\widetilde{\mathcal{R}}_\pm(x)$, i.e. we fix a sequence $\{\varepsilon_j\}_{j=N,N+1,\dots}$. For any smaller open subset $\Omega_1 \subset \Omega$ these same ε_j will still yield approximate fundamental solutions. We use the corresponding $K_\pm$ as an integral kernel to define an integral operator. Set for $u \in C^0(\Omega, E^*)$ and $x \in \Omega$

$$(\mathcal{K}_\pm u)(x) := \int_\Omega K_\pm(x,y)u(y)\ \mathrm{dV}(y). \tag{2.18}$$

Since $K_\pm$ is C^∞ so is $\mathcal{K}_\pm u$, i. e., $\mathcal{K}_\pm u \in C^\infty(\Omega, E^*)$.

Lemma 2.2.12. *Let $\Omega \subset\subset \Omega'$ be so small that*

$$\mathrm{vol}(\overline{\Omega}) \cdot \|K_\pm\|_{C^0(\overline{\Omega}\times\overline{\Omega})} < 1. \tag{2.19}$$

Then the following holds:

(a) The map

$$\mathrm{id} + \mathcal{K}_\pm : \quad C^k(\overline{\Omega}, E^*) \to C^k(\overline{\Omega}, E^*)$$

is an isomorphism with bounded inverse for all $k = 0, 1, 2, \dots$ and the inverse is given by the series

$$(\mathrm{id} + \mathcal{K}_\pm)^{-1} = \sum_{j=0}^\infty (-\mathcal{K}_\pm)^j$$

which converges in all C^k-operator norms.

(b) The operator $(\mathrm{id} + \mathcal{K}_+)^{-1} \circ \mathcal{K}_+$ has a smooth integral kernel with future-stretched support (with respect to $\overline{\Omega}$). The operator $(\mathrm{id} + \mathcal{K}_-)^{-1} \circ \mathcal{K}_-$ has a smooth integral kernel with past-stretched support (with respect to $\overline{\Omega}$).

Proof. *(a)* The operator $\mathcal{K}_\pm$ is bounded as an operator $C^0(\overline{\Omega}, E^*) \to C^k(\overline{\Omega}, E^*)$. Thus $\mathrm{id} + \mathcal{K}_\pm$ defines a bounded operator $C^k(\overline{\Omega}, E^*) \to C^k(\overline{\Omega}, E^*)$ for all k. Now

$$\begin{aligned} \|\mathcal{K}_\pm u\|_{C^0(\overline{\Omega})} &\leq \mathrm{vol}(\overline{\Omega}) \cdot \|K_\pm\|_{C^0(\overline{\Omega}\times\overline{\Omega})} \cdot \|u\|_{C^0(\overline{\Omega})} \\ &= (1-\eta) \cdot \|u\|_{C^0(\overline{\Omega})} \end{aligned}$$

where $\eta := 1 - \mathrm{vol}(\overline{\Omega}) \cdot \|K_\pm\|_{C^0(\overline{\Omega}\times\overline{\Omega})} > 0$. Hence the C^0-operator norm of $\mathcal{K}_\pm$ is less than 1 so that the Neumann series $\sum_{j=0}^\infty (-\mathcal{K}_\pm)^j$ converges in the C^0-operator norm and gives the inverse of $\mathrm{id} + \mathcal{K}_\pm$ on $C^0(\overline{\Omega}, E^*)$.
Next we replace the C^k-norm $\|\cdot\|_{C^k(\overline{\Omega})}$ on $C^k(\overline{\Omega}, E^*)$ as defined in (1.3.2) by the equivalent norm

$$|||u|||_{C^k(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + \frac{\eta}{2\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\overline{\Omega}\times\overline{\Omega})} + 1}\|u\|_{C^k(\overline{\Omega})}.$$

Then

$$
\begin{aligned}
&|||\mathcal{K}_\pm u|||_{C^k(\overline{\Omega})}\\
&= \|\mathcal{K}_\pm u\|_{C^0(\overline{\Omega})} + \frac{\eta}{2\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\overline{\Omega}\times\overline{\Omega})}+1}\|\mathcal{K}_\pm u\|_{C^k(\overline{\Omega})}\\
&\leq (1-\eta)\cdot\|u\|_{C^0(\overline{\Omega})} + \frac{\eta}{2\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\overline{\Omega}\times\overline{\Omega})}+1}\mathrm{vol}(\overline{\Omega})\|K_\pm\|_{C^k(\overline{\Omega}\times\overline{\Omega})}\|u\|_{C^0(\overline{\Omega})}\\
&\leq (1-\frac{\eta}{2})\|u\|_{C^0(\overline{\Omega})}\\
&\leq (1-\frac{\eta}{2})\ |||\ u\ |||_{C^k(\overline{\Omega})}\ .
\end{aligned}
$$

This shows that with respect to $|||\cdot|||_{C^k(\overline{\Omega})}$ the C^k-operator norm of $\mathcal{K}_\pm$ is less than 1. Thus the Neumann series $\sum_{j=0}^\infty(-\mathcal{K}_\pm)^j$ converges in all C^k-operator norms and $\mathrm{id}+\mathcal{K}_\pm$ is an isomorphism with bounded inverse on all $C^k(\overline{\Omega}, E^*)$.

(b) The operator $(\mathrm{id}+\mathcal{K}_+)^{-1}\circ\mathcal{K}_+ = -\sum\limits_{j=1}^{\infty}(-\mathcal{K}_+)^j$ has integral kernel $\sum\limits_{j=1}^{\infty}(-1)^j\,K_+^{(j)}(x,y)$, where

$$
K_\pm^{(j)}(x,y) := \int_\Omega\cdots\int_\Omega K_\pm(x,z_1)K_\pm(z_1,z_2)\cdots K_\pm(z_{j-1},y)\ \mathrm{dV}(z_1)\cdots\ \mathrm{dV}(z_{j-1}).
$$

By Proposition 2.2.10.4 the integral can be non-vanishing only if $z_1\in J_\pm(x)$, $z_2\in J_\pm(z_1),\dots,y\in J_\pm(z_{j-1})$ and hence $y\in J_\pm(x)$. Thus $\mathrm{supp}(K_\pm^{(j)})\subset\left\{(x,y)\in\overline{\Omega}\times\overline{\Omega}\mid y\in J_\pm^{\overline{\Omega}}(x)\right\}$ and

$$
\begin{aligned}
\|K_\pm^{(j)}\|_{C^k(\overline{\Omega}\times\overline{\Omega})} &\leq \|K_\pm\|^2_{C^k(\overline{\Omega}\times\overline{\Omega})}\cdot\mathrm{vol}(\overline{\Omega})^{j-1}\cdot\|K_\pm\|^{j-2}_{C^0(\overline{\Omega}\times\overline{\Omega})}\\
&\leq (1-\eta)^{j-2}\cdot\mathrm{vol}(\overline{\Omega})\cdot\|K_\pm\|^2_{C^k(\overline{\Omega}\times\overline{\Omega})}.
\end{aligned}
$$

Hence the series

$$
\sum_{j=1}^\infty(-1)^{j-1}K_\pm^{(j)}
$$

converges absolutely in all $C^k(\overline{\Omega}\times\overline{\Omega}, E^*\boxtimes E)$. Since this series yields the integral kernel of $(\mathrm{id}+\mathcal{K}_\pm)^{-1}\circ\mathcal{K}_\pm$ it is smooth and its support is contained in $\left\{(x,y)\in\overline{\Omega}\times\overline{\Omega}\mid y\in J_\pm^{\overline{\Omega}}(x)\right\}$. □

Corollary 2.2.13. *Let $\Omega\subset\subset\Omega'$ be as in Lemma 2.2.12. Then for each $u\in C^0(\overline{\Omega}, E)$*

$$
\mathrm{supp}((\mathrm{id}+\mathcal{K}_\pm)^{-1}u)\subset J_\mp^{\overline{\Omega}}(\mathrm{supp}(u)).
$$

Proof. From $u=(\mathrm{id}+\mathcal{K}_\pm)u-\mathcal{K}_\pm u$ it follows that

$$
(\mathrm{id}+\mathcal{K}_\pm)^{-1}u = u-(\mathrm{id}+\mathcal{K}_\pm)^{-1}\mathcal{K}_\pm u.
$$

So $(\mathrm{id}+\mathcal{K}_\pm)^{-1}u(x) \neq 0$ implies $u(x) \neq 0$ or $(\mathrm{id}+\mathcal{K}_\pm)^{-1}\mathcal{K}_\pm u(x) \neq 0$. Let $S_\pm$ be the integral kernel of $(\mathrm{id}+\mathcal{K}_\pm)^{-1}\mathcal{K}_\pm$, which has future-stretched resp. past streched support. Then $(\mathrm{id}+\mathcal{K}_\pm)^{-1}\mathcal{K}_\pm u(x) = \int S_\pm(x,y)u(y)dy$. The integrand vanishes at y unless $y \in \mathrm{supp}(u)$ and $y \in J_\pm(x)$. Since the integral is non-zero we have $\mathrm{supp}(u)\cap J_\pm(x) \neq \emptyset$. Hence $x \in J_\mp(\mathrm{supp}(u))$.□

The integral operator $\mathcal{K}_\pm$ now allows to construct true fundamental solutions. Fix $\varphi \in \mathcal{D}(\Omega, E^*)$. Then $x \mapsto \widetilde{\mathcal{R}}_\pm(x)[\varphi]$ defines a smooth section in E^* over $\overline{\Omega}$. Hence

$$F^\Omega_\pm(\cdot)[\varphi] := (\mathrm{id}+\mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) \tag{2.20}$$

defines a smooth section in E^*.

Lemma 2.2.14. *For each $x \in \Omega$ the map $\mathcal{D}(\Omega, E^*) \mapsto E^*_x$, $\varphi \mapsto F^\Omega_+(x)[\varphi]$, is an advanced fundamental solution at x on Ω and $\varphi \mapsto F^\Omega_-(x)[\varphi]$ is a retarded fundamental solution at x on Ω.*

Proof. We first check that $\varphi \mapsto F^\Omega_\pm(x)[\varphi]$ defines a distribution for any fixed $x \in \Omega$. Let $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega, E^*)$. Then $\varphi_m \to \varphi$ in $C^{n+1}(\Omega, E^*)$ and by the last point of Proposition 2.2.10 $\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi_m] \to \widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]$ in $C^0(\overline{\Omega}, E^*)$. Since $(\mathrm{id}+\mathcal{K}_\pm)^{-1}$ is bounded on C^0 we have $F^\Omega_\pm(\cdot)[\varphi_m] \to F^\Omega_\pm(\cdot)[\varphi]$ in C^0. In particular, $F^\Omega_\pm(x)[\varphi_m] \to F^\Omega_\pm(x)[\varphi]$.
Next we check that $F^\Omega_\pm(x)$ are fundamental solutions. We compute

$$\begin{aligned}
P_{(2)}F^\Omega_\pm(\cdot)[\varphi] &= F^\Omega_\pm(\cdot)[P^*\varphi] \\
&= (\mathrm{id}+\mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[P^*\varphi]) \\
&= (\mathrm{id}+\mathcal{K}_\pm)^{-1}(P_{(2)}\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) \\
&\overset{(2.14)}{=} (\mathrm{id}+\mathcal{K}_\pm)^{-1}(\varphi + \mathcal{K}_\pm\varphi) \\
&= \varphi.
\end{aligned}$$

Thus for fixed $x \in \Omega$,

$$PF^\Omega_\pm(x)[\varphi] = \varphi(x) = \delta_x[\varphi].$$

Finally, we want to show that $\mathrm{supp}(F^\Omega_\pm(x)) \subset J^\Omega_\pm(x)$.
We have

$$\begin{aligned}
\mathrm{supp}\left(F^\Omega_\pm(\cdot)[\varphi]\right) &= \mathrm{supp}\left((\mathrm{id}+\mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi])\right) \\
&\subset J^\Omega_\mp\left(\mathrm{supp}\left(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]\right)\right) \\
&\subset J^\Omega_\mp\left(J_\mp(\mathrm{supp}(\varphi)\right) \\
&= J^\Omega_\mp(\mathrm{supp}(\varphi)).
\end{aligned}$$

So for any $\varphi \in \mathcal{D}(\Omega, E^*)$ such that $\mathrm{supp}(\varphi) \cap J^\Omega_\pm(x) = \emptyset$ we find that $F^\Omega_\pm[\varphi] = 0$. □

We summarize the results.

Proposition 2.2.15. *Let M be a timeoriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega \subset\subset M$ be a relatively compact causal domain. Suppose that Ω is sufficiently small in the sense that (2.19) holds.*
Then for each $x \in \Omega$

1. *the distributions $F_+^\Omega(x)$ and $F_-^\Omega(x)$ defined in (2.20) are fundamental solutions for P at x over Ω,*
2. $\mathrm{supp}(F_\pm^\Omega(x)) \subset J_\pm^\Omega(x)$,
3. *for each $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x' \mapsto F_\pm^\Omega(x')[\varphi]$ are smooth sections in E^* over Ω.* □

Corollary 2.2.16. *Let M be a timeoriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*
Then each point in M possesses an arbitrarily small causal neighborhood Ω such that for each $x \in \Omega$ there exist fundamental solutions $F_\pm^\Omega(x)$ for P over Ω at x. They satisfy

1. $\mathrm{supp}(F_\pm^\Omega(x)) \subset J_\pm^\Omega(x)$,
2. *for each $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x \mapsto F_\pm^\Omega(x)[\varphi]$ are smooth sections in E^*.* □

2.2.4 The formal fundamental solution is asymptotic

Finally, we show that the formal fundamental solution constructed in Section 2.2.1 is asymptotic to the true fundamental solution. This implies that the singularity structure of the fundamental solution is completely determined by the Hadamard coefficients which are in turn determined by the geometry of the manifold and the coefficients of the operator.
Let M be a timeoriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega' \subset M$ be a convex domain and let $\Omega \subset \Omega'$ be a relatively compact causal domain with $\overline{\Omega} \subset \Omega'$. We assume that Ω is so small that Corollary 2.2.16 applies. Using Riesz distributions and Hadamard coefficients we have constructed the formal fundamental solutions at $x \in \Omega$

$$\mathcal{R}_\pm(x) = \sum_{j=0}^{\infty} V_j(x, \cdot)\, R_\pm^{\Omega'}(2 + 2j, x),$$

the approximate fundamental solutions

$$\widetilde{\mathcal{R}}_\pm(x) = \sum_{j=0}^{N-1} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x) + \sum_{j=N}^{\infty} \sigma(\Gamma(x,\cdot)/\varepsilon_j) V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x),$$

where $N \geq \frac{n}{2}$ is fixed, and the true fundamental solutions $F_\pm^\Omega(x)$,

$$F_\pm^\Omega(\cdot)[\varphi] = (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]).$$

The purpose of this section is to show that, in a suitable sense, the formal fundamental solution is an asymptotic expansion of the true fundamental solution. For $k \geq 0$ we define the **truncated formal fundamental solution**

$$\mathcal{R}_\pm^{N+k}(x) := \sum_{j=0}^{N-1+k} V_j(x,\cdot)\, R_\pm^{\Omega'}(2+2j,x).$$

Hence we cut the formal fundamental solution at the $(N+k)$-th term. The truncated formal fundamental solution is a well-defined distribution on Ω', $\mathcal{R}_\pm^{N+k}(x) \in \mathcal{D}'(\Omega', E, E_x^*)$. We will show that the true fundamental solution coincides with the truncated formal fundamental solution up to an error term which is very regular along the light cone. The larger k is, the more regular is the error term.

Proposition 2.2.17. *For every $k \in \mathbb{N}$ and every $x \in \Omega$ the difference of distributions $F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)$ is a C^k-section in E. In fact,*

$$(x,y) \mapsto \left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y)$$

is of regularity C^k on $\Omega \times \Omega$.

Proof. We write

$$\left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y) = \left(F_\pm^\Omega(x) - \widetilde{\mathcal{R}}_\pm(x)\right)(y) + \left(\widetilde{\mathcal{R}}_\pm(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y)$$

and we show that $\left(\widetilde{\mathcal{R}}_\pm(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y)$ and $\left(F_\pm^\Omega(x) - \widetilde{\mathcal{R}}_\pm(x)\right)(y)$ are both C^k in (x,y). Now

$$\begin{aligned}\left(\widetilde{\mathcal{R}}_\pm(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y) &= \sum_{j=N}^{N+k-1} \left(\sigma(\Gamma(x,y)/\varepsilon_j) - 1\right) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y) \\ &\quad + \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y).\end{aligned}$$

From Lemma 2.2.6 we know that the infinite part $(x,y) \mapsto \sum_{j=N+k}^{\infty} \sigma(\Gamma(x,y)/\varepsilon_j) V_j(x,y)\, R_\pm^{\Omega'}(2+2j,x)(y)$ is C^k. The finite part $(x,y) \mapsto$

$\sum_{j=N}^{N+k-1}\left(\sigma(\Gamma(x,y)/\varepsilon_j)-1\right)V_j(x,y)\,R_\pm^{\Omega'}(2+2j,x)(y)$ is actually smooth since $\sigma(\Gamma/\varepsilon_j)-1$ vanishes on a neighborhood of $\Gamma^{-1}(0)$ which is precisely the locus where $(x,y)\mapsto R_\pm^{\Omega'}(2+2j,x)(y)$ is nonsmooth. Furthermore,

$$\begin{aligned} F_\pm^\Omega(\cdot)[\varphi]-\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi] &= \left((\mathrm{id}+\mathcal{K}_\pm)^{-1}-\mathrm{id}\right)(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) \\ &= -\left((\mathrm{id}+\mathcal{K}_\pm)^{-1}\circ\mathcal{K}_\pm\right)(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]). \end{aligned}$$

By Lemma 2.2.12 the operator $-\left((\mathrm{id}+\mathcal{K}_\pm)^{-1}\circ\mathcal{K}_\pm\right)$ has a smooth integral kernel $L_\pm(x,y)$ whose support is future or past-stretched respectively. Hence

$$\begin{aligned} &F_\pm^\Omega(x)[\varphi]-\widetilde{\mathcal{R}}_\pm(x)[\varphi] \\ &= \int_{\overline{\Omega}} L_\pm(x,y)\widetilde{\mathcal{R}}_\pm(y)[\varphi]\,\mathrm{dV}(y) \\ &= \sum_{j=0}^{N-1}\int_{\overline{\Omega}} L_\pm(x,y)V_j(y,\cdot)\,R_\pm^{\Omega'}(2+2j,y)[\varphi]\,\mathrm{dV}(y) \\ &\quad+\sum_{j=N}^{N+k-1}\int_{\overline{\Omega}} L_\pm(x,y)\sigma(\Gamma(y,\cdot)/\varepsilon_j)V_j(y,\cdot)\,R_\pm^{\Omega'}(2+2j,y)[\varphi]\,\mathrm{dV}(y) \\ &\quad+\int_{\overline{\Omega}\times\overline{\Omega}} L_\pm(x,y)f(y,z)\varphi(z)\,\mathrm{dV}(z)\,\mathrm{dV}(y) \end{aligned}$$

where $f(y,z)=\sum_{j=N+k}^{\infty}\sigma(\Gamma(y,z)/\varepsilon_j)V_j(y,z)\,R_\pm^{\Omega'}(2+2j,y)(z)$ is C^k by Lemma 2.2.6. Thus $(x,z)\mapsto\int_{\overline{\Omega}}L_\pm(x,y)f(y,z)\,\mathrm{dV}(y)$ is a C^k-section. Write $\tilde{V}_j(y,z):=V_j(y,z)$ if $j\le N-1$ and $\tilde{V}_j(y,z):=\sigma(\Gamma(y,z)/\varepsilon_j)V_j(y,z)$ if $j\ge N$. It follows from Lemma 1.3.38

$$\begin{aligned} &\int_{\overline{\Omega}} L_\pm(x,y)\tilde{V}_j(y,\cdot)\,R_\pm^{\Omega'}(2+2j,y)[\varphi]\,\mathrm{dV}(y) \\ &= \int_{\overline{\Omega}} R_\pm^{\Omega'}(2+2j,y)[z\mapsto L_\pm(x,y)\tilde{V}_j(y,z)\varphi(z)]\,\mathrm{dV}(y) \\ &= \int_{\overline{\Omega}} R_\mp^{\Omega'}(2+2j,z)[y\mapsto L_\pm(x,y)\tilde{V}_j(y,z)\varphi(z)]\,\mathrm{dV}(z) \\ &= \int_{\overline{\Omega}} R_\mp^{\Omega'}(2+2j,z)[y\mapsto L_\pm(x,y)\tilde{V}_j(y,z)]\varphi(z)\,\mathrm{dV}(z) \\ &= \int_{\overline{\Omega}} W_j(x,z)\varphi(z)\,\mathrm{dV}(z) \end{aligned}$$

where $W_j(x,z)=R_\mp^{\Omega'}(2+2j,z)[y\mapsto L_\pm(x,y)\tilde{V}_j(y,z)]$ is smooth in (x,z) by Proposition 1.3.36 (11). Hence

$$\left(F_\pm^\Omega(x)-\widetilde{\mathcal{R}}_\pm(x)\right)(z)=\sum_{j=0}^{N+k-1}W_j(x,z)+\int_{\overline{\Omega}}L_\pm(x,y)f(y,z)\,\mathrm{dV}(y)$$

is C^k in (x,z). ◻

The following theorem tells us that the formal fundamental solutions are asymptotic expansions of the true fundamental solutions near the light cone.

Theorem 2.2.18. *Let M be a timeoriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E. Let $\Omega \subset M$ be a relatively compact causal domain and let $x \in \Omega$. Let $F^{\Omega}_{\pm}$ denote the fundamental solutions of P at x and $\mathcal{R}^{N+k}_{\pm}(x)$ the truncated formal fundamental solutions.*
Then for each $k \in \mathbb{N}$ there exists a constant C_k such that

$$\left\| \left(F^{\Omega}_{\pm}(x) - \mathcal{R}^{N+k}_{\pm}(x) \right)(y) \right\| \leq C_k \cdot |\Gamma(x,y)|^k$$

for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

Here $\|\cdot\|$ denotes an auxiliary norm on $E^* \boxtimes E$. The proof requires some preparation.

Lemma 2.2.19. *Let M be a smooth manifold. Let $H_1, H_2 \subset M$ be two smooth hypersurfaces globally defined by the equations $\varphi_1 = 0$ and $\varphi_2 = 0$ respectively, where $\varphi_1, \varphi_2 : M \to \mathbb{R}$ are smooth functions on M satisfying $d_x\varphi_i \neq 0$ for every $x \in H_i$, $i = 1, 2$. We assume that H_1 and H_2 intersect transversally.*
Let $f : M \to \mathbb{R}$ be a C^k-function on M, $k \in \mathbb{N}$. Let $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 \leq k$. We assume that f vanishes to order k_i along H_i, i. e., in local coordinates $\frac{\partial^{|\alpha|} f}{\partial x^{\alpha}}(x) = 0$ for every $x \in H_i$ and every multi-index α with $|\alpha| \leq k_i - 1$.
Then there exists a continuous function $F : M \to \mathbb{R}$ such that

$$f = \varphi_1^{k_1} \varphi_2^{k_2} F.$$

Proof of Lemma 2.2.19.. We first prove the existence of a C^{k-k_1}-function $F_1 : M \to \mathbb{R}$ such that

$$f = \varphi_1^{k_1} F_1.$$

This is equivalent to saying that the function $f/\varphi_1^{k_1}$ being well-defined and C^k on $M \setminus H_1$ extends to a C^{k-k_1}-function F_1 on M. Since it suffices to prove this locally, we introduce local coordinates $x^1, \ldots, x^n$ so that $\varphi_1(x) = x^1$. Hence in this local chart $H_1 = \{x^1 = 0\}$.
Since $f(0, x^2, \ldots, x^n) = \frac{\partial^j f}{\partial (x^1)^j}(0, x^2, \ldots, x^n) = 0$ for any $(x^2, \ldots, x^n)$ and $j \leq k_1 - 1$ we obtain from the Taylor expansion of f in the x^1-direction to the order $k_1 - 1$ with integral remainder term

$$f(x^1, x^2, \ldots, x^n) = \int_0^{x^1} \frac{(x^1 - t)^{k_1 - 1}}{(k_1 - 1)!} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(t, x^2, \ldots, x^n) dt.$$

In particular, for $x^1 \neq 0$

$$\begin{aligned} f(x^1, x^2, \dots, x^n) &= (x^1)^{k_1-1} \int_0^{x^1} \frac{1}{(k_1-1)!} \left(\frac{x^1 - t}{x^1}\right)^{k_1-1} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(t, x^2, \dots, x^n)dt \\ &= \frac{(x^1)^{k_1-1}}{(k_1-1)!} \int_0^1 (1-u)^{k_1-1} x^1 \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(x^1 u, x^2, \dots, x^n)du \\ &= \frac{(x^1)^{k_1}}{(k_1-1)!} \int_0^1 (1-u)^{k_1-1} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(x^1 u, x^2, \dots, x^n)du. \end{aligned}$$

Now $F_1(x^1, \dots, x^n) := \frac{1}{(k_1-1)!} \int_0^1 (1-u)^{k_1-1} \frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}(x^1 u, x^2, \dots, x^n)du$ yields a C^{k-k_1}-function because $\frac{\partial^{k_1} f}{\partial (x^1)^{k_1}}$ is C^{k-k_1}. Moreover, we have

$$f = (x^1)^{k_1} \cdot F_1 = \varphi^{k_1} \cdot F_1.$$

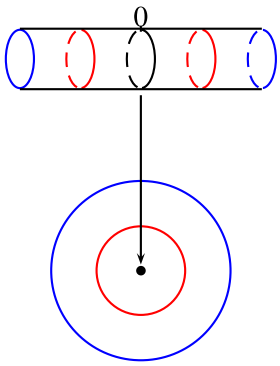
On $M \setminus H_1$ we have $F_1 = f/\varphi_1^{k_1}$ and so F_1 vanishes to the order k_2 on $H_2 \setminus H_1$ because f does. Since H_1 and H_2 intersect transversally the subset $H_2 \setminus H_1$ is dense in H_2. Therefore the function F_1 vanishes to the order k_2 on all of H_2. Applying the considerations above to F_1 yields a $C^{k-k_1-k_2}$-function $F : M \to \mathbb{R}$ such that $F_1 = \varphi_2^{k_2} \cdot F$. This concludes the proof. □

Lemma 2.2.20. *Let $f : \mathbb{R}^n \to \mathbb{R}$ a C^{3k+1}-function. We equip $\mathbb{R}^n$ with its standard Minkowski product $\langle \cdot, \cdot \rangle$ and we assume that f vanishes on all spacelike vectors.*
Then there exists a continuous function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$f = h \cdot \gamma^k$$

where $\gamma(x) = -\langle x, x \rangle$.

Proof of Lemma 2.2.20. The problem here is that the hypersurface $\{\gamma = 0\}$ is the light cone which contains 0 as a singular point so that Lemma 2.2.19 does not apply directly. We will get around this difficulty by resolving the singularity.
Let $\pi : M := \mathbb{R} \times S^{n-1} \to \mathbb{R}^n$ be the map defined by $\pi(t, x) := tx$. It is smooth on $M = \mathbb{R} \times S^{n-1}$ and outside $\pi^{-1}(\{0\}) = \{0\} \times S^{n-1}$ it is a two-fold covering of $\mathbb{R}^n \setminus \{0\}$.



The function $\widehat{f} := f \circ \pi : M \to \mathbb{R}$ is C^{3k+1} since f is.
Consider the functions $\widehat{\gamma} : M \to \mathbb{R}$, $\widehat{\gamma}(t,x) := \gamma(x)$, and $\pi_1 : M \to \mathbb{R}$, $\pi_1(t,x) := t$. These functions are smooth and have only regular points on M. For $\widehat{\gamma}$ this follows from $d_x\gamma \neq 0$ for every $x \in S^{n-1}$. Therefore $\widehat{C}(0) := \widehat{\gamma}^{-1}(\{0\})$ and $\{0\} \times S^{n-1} = \pi_1^{-1}(\{0\})$ are smooth embedded hypersurfaces. Since the differentials of $\widehat{\gamma}$ and of π_1 are linearly independent the hypersurfaces intersect transversally. Furthermore, one obviously has $\pi(\widehat{C}(0)) = C(0)$ and $\pi(\{0\}\times S^{n-1}) = \{0\}$. Since f is C^{3k+1} and vanishes on all spacelike vectors f vanishes to the order $3k+2$ along $C(0)$ (and in particular at 0). Hence $\widehat{f}$ vanishes to the order $3k+2$ along $\widehat{C}(0)$ and along $\{0\} \times S^{n-1}$. Applying Lemma 2.2.19 to $\widehat{f}$, $\varphi_1 := \pi_1$ and $\varphi_2 := \widehat{\gamma}$, with $k_1 := 2k+1$ and $k_2 := k$, yields a continuous function $\widehat{F} : \mathbb{R} \times S^{n-1} \to \mathbb{R}$ such that

$$\widehat{f} = \pi_1^{2k+1} \cdot \widehat{\gamma}^k \cdot \widehat{F}. \tag{2.21}$$

For $y \in \mathbb{R}^n$ we set

$$h(y) := \begin{cases} \|y\| \cdot \widehat{F}(\|y\|, \frac{y}{\|y\|}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0, \end{cases}$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^n$. The function h is obviously continuous on $\mathbb{R}^n$. It remains to show $f = \gamma^k \cdot h$. For $y \in \mathbb{R}^n \setminus \{0\}$ we have

$$\begin{aligned} f(y) &= f\left(\|y\| \cdot \frac{y}{\|y\|}\right) \\ &= \widehat{f}\left(\|y\|, \frac{y}{\|y\|}\right) \\ &\overset{(2.21)}{=} \|y\|^{2k+1} \cdot \gamma\left(\frac{y}{\|y\|}\right)^k \cdot \widehat{F}\left(\|y\|, \frac{y}{\|y\|}\right) \\ &= \|y\|^{2k} \cdot \gamma\left(\frac{y}{\|y\|}\right)^k \cdot h(y) \\ &= \gamma(y)^k \cdot h(y). \end{aligned}$$

For $y = 0$ the equation $f(y) = \gamma(y)^k \cdot h(y)$ holds trivially. $\square$

Proof of Theorem 2.2.18. Repeatedly using Proposition 1.3.36 (3) we find constants C'_j such that

$$
\begin{aligned}
&\left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y) \\
&= \left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\right)(y) + \sum_{j=N+k}^{N+3k} V_j(x,y)\cdot R_\pm^{\Omega'}(2+2j,x)(y) \\
&= \left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\right)(y) + \sum_{j=N+k}^{N+3k} V_j(x,y)\cdot C'_j\cdot\Gamma(x,y)^k\cdot R_\pm^{\Omega'}(2+2(j-k),x)(y).
\end{aligned}
$$

Now $h_j(x,y) := C'_j \cdot V_j(x,y)\cdot R_\pm^{\Omega'}(2+2(j-k),x)(y)$ is continuous since $2+2(j-k) \geq 2+2N \geq 2+n > n$. By Proposition 2.2.17 the section $(x,y)\mapsto \left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\right)(y)$ is of regularity C^{3k+1}. Moreover, we know $\operatorname{supp}(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)) \subset J_\pm^\Omega(x)$. Hence we may apply Lemma 2.2.20 in normal coordinates and we obtain a continuous section h such that

$$\left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+3k+1}(x)\right)(y) = \Gamma(x,y)^k\cdot h(x,y).$$

This shows

$$\left(F_\pm^\Omega(x) - \mathcal{R}_\pm^{N+k}(x)\right)(y) = \left(h(x,y) + \sum_{j=N+k}^{N+3k} h_j(x,y)\right)\Gamma(x,y)^k.$$

Now $C_k := \|h + \sum_{j=N+k}^{N+3k} h_j\|_{C^0(\overline{\Omega}\times\overline{\Omega})}$ does the job. □

2.3 Solving the inhomogeneous equation on small domains

We want to solve the inhomogeneous equation $Pu = v$ for given v with small support.
Let $\Omega \subset M$ satisfy the hypotheses of Lemma 2.2.12. In particular, Ω is relatively compact, causal, and has "small volume". Such domains will be referred to as **RCCSV** (for "Relatively Compact Causal with Small Volume"). Note that each point in a Lorentzian manifold possesses RCCSV-neighborhoods.
Let $F_\pm^\Omega(x)$ be the corresponding fundamental solutions for P at $x \in \Omega$ over Ω. Recall that for $\varphi \in \mathcal{D}(\Omega, E^*)$ the maps $x \mapsto F_\pm^\Omega(x)[\varphi]$ are smooth sections in E^*. Using the natural pairing $E_x^* \otimes E_x \to \mathbb{K}$, $\ell\otimes e \mapsto \ell\cdot e$, we obtain a smooth $\mathbb{K}$-valued function $x \mapsto F_\pm^\Omega(x)[\varphi]\cdot v(x)$ with compact support.
We put

$$u_\pm[\varphi] := \int_\Omega F_\pm^\Omega(x)[\varphi]\cdot v(x)\,\mathrm{dV}(x). \tag{2.22}$$

Lemma 2.3.1. *The $u_\pm$ defined in (2.22) are distributions satisfying*

$$Pu_\pm = v$$

and

$$\mathrm{supp}(u_\pm) \subset J_\pm^\Omega(\mathrm{supp}(v)).$$

Proof. *(a)* We show that $u_\pm$ are distributions. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. By Lemma 2.2.8 and (2.20):

$$|u_\pm[\varphi]| \le \mathrm{vol}(\Omega) \cdot \max_{x \in \overline{\Omega}} |F_\pm(x)[\varphi]| \cdot \|v\|_{C^0} \le \quad C' \cdot \|\varphi\|_{C^{n+1}(\Omega)} .$$

This proves that $u_\pm$ depend continuously on φ with respect to the C^{n+1}-norm and hence also with respect to the topology of $\mathcal{D}(\Omega, E^*)$.
(b) We show $Pu_\pm = v$. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. We compute

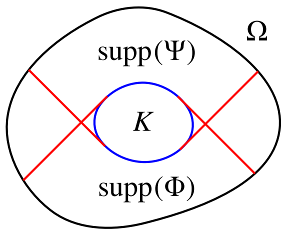
$$\begin{aligned}
Pu_\pm[\varphi] &= u_\pm[P^*\varphi] \\
&= \int_\Omega F_\pm^\Omega(x)[P^*\varphi] \cdot v(x) \, \mathrm{dV}(x) \\
&= \int_\Omega \underbrace{P_{(2)} F_\pm^\Omega(x)}_{=\delta_x}[\varphi] \cdot v(x) \, \mathrm{dV}(x) \\
&= \int_\Omega \varphi(x) \cdot v(x) \, \mathrm{dV}(x) \\
&= v[\varphi].
\end{aligned}$$

(c) It remains to show the assertions about the supports of $u_\pm$. Let $\varphi \in \mathcal{D}(\Omega, E^*)$ and assume $\mathrm{supp}(\varphi) \cap J_\pm^\Omega(\mathrm{supp}(v)) = \emptyset$. In case of J_+ this means that there is no future-directed curve starting in $\mathrm{supp}(v)$ and ending in $\mathrm{supp}(\varphi)$. In other words, there is no past-directed curve starting in $\mathrm{supp}(\varphi)$ and ending in $\mathrm{supp}(v)$. Hence $\mathrm{supp}(v) \cap J_\mp^\Omega(\mathrm{supp}(\varphi)) = \emptyset$. Since $J_\mp^\Omega(\mathrm{supp}(\varphi))$ contains the support of $x \mapsto F_\pm^\Omega(x)[\varphi]$ we have $\mathrm{supp}(v) \cap \mathrm{supp}(F_\pm^\Omega(\cdot)[\varphi]) = \emptyset$. Hence the integrand in (2.22) vanishes identically and therefore $u_\pm[\varphi] = 0$. This proves $\mathrm{supp}(u_\pm) \subset J_\pm^\Omega(\mathrm{supp}(v))$. □

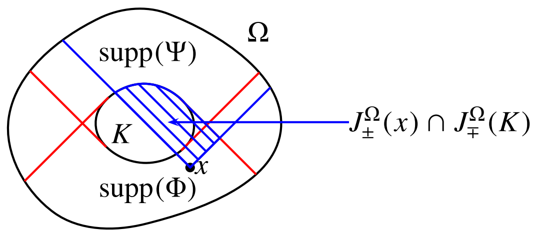
Next we show that the distributions u_+ and u_- are actually smooth sections. For this, we need the following lemma.

Lemma 2.3.2. *Let $\Omega \subset M$ be an RCCSV-domain. Let $K \subset \Omega$ be a compact subset. Let $V \in C^\infty(\overline{\Omega} \times \overline{\Omega}, E^* \boxtimes E)$. Let $\Phi \in C^{n+1}(\overline{\Omega}, E^*)$ and $\Psi \in C^{n+1}(\overline{\Omega}, E)$ be such that* $\mathrm{supp}(\Phi) \subset J_\mp^\Omega(K)$ *and* $\mathrm{supp}(\Psi) \subset J_\pm^\Omega(K)$.
Then for all $j \ge 0$

$$\int_\Omega \left(V(x,\cdot) R_\pm^{\Omega'}(2+2j, x)\right)[\Phi] \cdot \Psi(x) \, \mathrm{dV}(x) = \int_\Omega \Phi(y) \cdot \left(V(\cdot, y) R_\mp^{\Omega'}(2+2j, y)\right)[\Psi] \, \mathrm{dV}(y).$$



Proof. Since Ω is a causal domain and therefore globally hyperbolic, we know that $J_\pm^\Omega(x) \cap J_\mp^\Omega(K)$ is compact, see Proposition 1.2.56.



Hence $\mathrm{supp}(R_\pm^{\Omega'}(2+2j,x)) \cap \mathrm{supp}(\Phi) \subset J_\pm^\Omega(x) \cap J_\mp^\Omega(K)$ is compact. Since the distribution $R_\pm^{\Omega'}(2+2j,x)$ is of order $\le n+1$ we may apply $V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)$ to Φ. By Proposition 1.3.36.12 the section $x \mapsto V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)[\Phi]$ is continuous. Hence the left hand side is well defined. Similarly, the integral on the right hand side is well-defined. By Lemma 1.3.37

$$
\begin{aligned}
&\int_\Omega \left(V(x,\cdot)R_\pm^{\Omega'}(2+2j,x)\right)[\Phi] \cdot \Psi(x)\ \mathrm{dV}(x) \\
&\qquad = \int_\Omega R_\pm^{\Omega'}(2+2j,x)[y \mapsto V(x,y)^*\Phi(y)] \cdot \Psi(x)\ \mathrm{dV}(x) \\
&\qquad = \int_\Omega R_\pm^{\Omega'}(2+2j,x)[y \mapsto \Phi(y)V(x,y)\Psi(x)]\ \mathrm{dV}(x) \\
&\qquad = \int_\Omega R_\mp^{\Omega'}(2+2j,y)[x \mapsto \Phi(y)V(x,y)\Psi(x)]\ \mathrm{dV}(y) \\
&\qquad = \int_\Omega \Phi(y) \cdot \left(V(\cdot,y)R_\mp^{\Omega}(2+2j,y)[\Psi]\right)\ \mathrm{dV}(y).
\end{aligned}
$$
□

Lemma 2.3.3. *Let $\Omega \subset M$ be an RCCSV-domain.*
Then the distributions $u_\pm$ defined in (2.22) are smooth sections in E, i. e., $u_\pm \in C^\infty(\Omega, E)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega, E^*)$. Put $K := \operatorname{supp}(\varphi) \cup \operatorname{supp}(v)$. Let $L_\pm \in C^\infty(\overline{\Omega} \times \overline{\Omega}, E^* \boxtimes E)$ be the smooth integral kernel of $(\mathrm{id} + \mathcal{K}_\pm)^{-1} \circ \mathcal{K}_\pm$. We recall from (2.20)

$$F^\Omega_\pm(\cdot)[\varphi] = (\mathrm{id} + \mathcal{K}_\pm)^{-1}(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]) = \widetilde{\mathcal{R}}_\pm(\cdot)[\varphi] - (\mathrm{id} + \mathcal{K}_\pm)^{-1}\mathcal{K}_\pm(\widetilde{\mathcal{R}}_\pm(\cdot)[\varphi]).$$

Therefore

$$\begin{aligned}
u_\pm[\varphi] &= \int_\Omega F^\Omega_\pm(x)[\varphi] \cdot v(x)\ \mathrm{dV}(x) \\
&= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot v(x)\ \mathrm{dV}(x) - \int_\Omega \int_\Omega L_\pm(y,x) \cdot \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot v(y)\ \mathrm{dV}(x)\ \mathrm{dV}(y) \\
&= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot w(x)\ \mathrm{dV}(x) \\
&= \sum_{j=0}^\infty \int_\Omega \tilde{V}_j(x,\cdot) R^{\Omega'}_\pm(2+2j, x)[\varphi] \cdot w(x)\ \mathrm{dV}(x)
\end{aligned}$$

where we again wrote $\tilde{V}_j(y,z) = V_j(y,z)$ if $j \le N-1$ and $\tilde{V}_j(y,z) = \sigma(\Gamma(y,z)/\varepsilon_j)V_j(y,z)$ if $j \ge N$ and $w(x) := v(x) - \int_\Omega v(y) \cdot L_\pm(y,x)\ \mathrm{dV}(y) \in E_x$. Note that $w \in C^\infty(\overline{\Omega}, E)$.
By Lemma 2.2.12, $\operatorname{supp}(L_\pm) \subset \{(y,x) \in \overline{\Omega} \times \overline{\Omega} \,|\, x \in J^{\overline{\Omega}}_\pm(y)\}$. Hence $\operatorname{supp}(w) \subset J^{\overline{\Omega}}_\pm(\operatorname{supp}(v))$.
We may therefore apply Lemma 2.3.2 with $\Phi = \varphi$ and $\Psi = w$.
For the j-th summand we then find

$$\int_\Omega \tilde{V}_j(x,\cdot) R^{\Omega'}_\pm(2+2j, x)[\varphi] \cdot w(x)\ \mathrm{dV}(x) = \int_\Omega \varphi(y) \tilde{V}_j(\cdot, y) R^{\Omega'}_\mp(2+2j, y)[w]\ \mathrm{dV}(y)$$

Summation over j yields

$$\begin{aligned}
u_\pm[\varphi] &= \int_\Omega \widetilde{\mathcal{R}}_\pm(x)[\varphi] \cdot w(x)\ \mathrm{dV}(x) \\
&= \sum_{j=0}^\infty \int_\Omega \varphi(y) \tilde{V}_j(\cdot, y) R^{\Omega'}_\mp(2+2j, y)[w]\ \mathrm{dV}(y).
\end{aligned}$$

Thus

$$u_\pm(y) = \sum_{j=0}^\infty \left(\tilde{V}_j(\cdot, y) R^{\Omega'}_\mp(2+2j, y)\right)[w].$$

Proposition 1.3.36.11 shows that all summands are smooth in y. By the choice of the ε_j the series converges in all C^k-norms. Hence $u_\pm$ is smooth. $\square$

We summarize

Theorem 2.3.4. *Let M be a timeoriented Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\Omega \subset M$ be an RCCSV-domain. Then for each $v \in \mathcal{D}(\Omega, E)$ there exist $u_\pm \in C^\infty(\Omega, E)$ satisfying*

1. $\int_\Omega \phi(x) \cdot u_\pm(x)\ \mathrm{dV} = \int_\Omega F_\pm^\Omega(x)[\varphi] \cdot v(x)\ \mathrm{dV}$ *for each $\phi \in \mathcal{D}(\Omega, E^*)$,*
2. $Pu_\pm = v$,
3. $\mathrm{supp}(u_\pm) \subset J_\pm^\Omega(\mathrm{supp}(v))$. □

2.4 The Cauchy problem

Next we prove existence of solutions to the Cauchy problem on small domains. Let $\Omega \subset M$ be an RCCSV-domain. Since causal domains are contained in convex domains by definition and convex domains are contractible, the vector bundle E is trivial over any RCCSV-domain Ω.

Proposition 2.4.1. *Let M be a timeoriented Lorentzian manifold and let $S \subset M$ be a spacelike smooth hypersurface. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S and let ∇ be the P-compatible connection.*
Then for each RCCSV-domain $\Omega \subset M$ such that $S \cap \Omega$ is a (spacelike) Cauchy hypersurface in Ω, the following holds:
For every $u_0, u_1 \in \mathcal{D}(S \cap \Omega, E)$ and for every $f \in \mathcal{D}(\Omega, E)$ there exists a solution $u \in C^\infty(\Omega, E)$ of the Cauchy problem

$$\begin{cases} Pu &= f & \text{on } \Omega, \\ u &= u_0 & \text{along } S \cap \Omega, \\ \nabla_{\mathfrak{n}} u &= u_1 & \text{along } S \cap \Omega. \end{cases}$$

Moreover, $\mathrm{supp}(u) \subset J^M(K)$ *where* $K = \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

Proof. Since causal domains are globally hyperbolic we may apply Theorem 1.2.53 and find an isometry $\Omega = \mathbb{R} \times (S \cap \Omega)$ where the metric takes the form $-N^2 dt^2 + g_t$. Here N is smooth, each $\{t\} \times (S \cap \Omega)$ is a smooth spacelike Cauchy hypersurface in Ω, and $(S \cap \Omega) = t^{-1}(0)$. Note that the future-directed unit normal vector field $\mathfrak{n}$ along $\{t\} \times (S \cap \Omega)$ is given by $\mathfrak{n} = -\frac{1}{N}\mathrm{grad}\, t$.
We trivialize the bundle E over Ω and identify sections in E with $\mathbb{K}^r$-valued functions where r is the rank of E.
Assume for a moment that u were a solution to the Cauchy problem of the form

$$u(t, x) = \sum_{j=0}^{\infty} t^j \tilde{u}_j(x)$$

where $x \in S \cap \Omega$. On $S \cap \Omega$ we have that $\tilde{u}_0 = u_0$, $\tilde{u}_1 = -Nu_1$. Write $P = \frac{1}{N^2}\frac{\partial^2}{\partial t^2} + Y$ where Y is a differential operator containing t-derivatives only up to order 1. Equation

$$f = Pu = \left(\frac{1}{N^2}\frac{\partial^2}{\partial t^2} + Y\right) u = \frac{1}{N^2}\sum_{j=2}^{\infty} j(j-1)t^{j-2}\tilde{u}_j + Yu \tag{2.23}$$

evaluated at $t = 0$ gives

$$2N^{-2}(0,x)\tilde{u}_2(x) = -Y(\tilde{u}_0 + t\tilde{u}_1)(0,x) + f(0,x)$$

for every $x \in S \cap \Omega$. Thus $\tilde{u}_2$ is determined by $\tilde{u}_0$, $\tilde{u}_1$, and $f|_S$. Differentiating (2.23) with respect to $\frac{\partial}{\partial t}$ and repeating the procedure shows that each $\tilde{u}_j$ is recursively determined by $\tilde{u}_0, \ldots, \tilde{u}_{j-1}$ and the normal derivatives of f along S.
Now we drop the assumption that we have a t-power series u solving the problem but we *define* the $\tilde{u}_j$, $j \geq 2$, by these recursive relations. In general, the so constructed series will be nonconvergent and we will now use a cut-off function σ as before to make it convergent.
Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\sigma|_{[-1/2,1/2]} \equiv 1$ and $\sigma \equiv 0$ outside $[-1,1]$. We claim that we can find a sequence of $\varepsilon_j \in (0,1)$ such that

$$\hat{u}(t,x) := \sum_{j=0}^{\infty} \sigma(t/\varepsilon_j)t^j\tilde{u}_j(x) \tag{2.24}$$

defines a smooth section that can be differentiated termwise.
By Lemma 1.3.23 we have for $j > k$

$$\|(t,x) \mapsto \sigma(t/\varepsilon_j)t^j\tilde{u}_j(x)\|_{C^k(\Omega)} \leq c(k) \cdot \left\|t \mapsto \sigma(t/\varepsilon_j)t^j\right\|_{C^k(\mathbb{R})} \cdot \|\tilde{u}_j\|_{C^k(S)}.$$

Here and in the following $c(k)$, $c'(k,j)$, and $c''(k,j)$ denote universal constants depending only on k and j. By Lemma 2.2.5 we have for $l \leq k$ and $0 < \varepsilon_j \leq 1$

$$\left\|\frac{d^l}{dt^l}(\sigma(t/\varepsilon_j)t^j)\right\|_{C^0(\mathbb{R})} \quad \leq \quad \varepsilon_j\, c'(l,j)\, \|\sigma\|_{C^l(\mathbb{R})},$$

thus, taking the maximum

$$\|(t,x) \mapsto \sigma(t/\varepsilon_j)t^j\tilde{u}_j(x)\|_{C^k(\Omega)} \leq \varepsilon_j\, c''(k,j)\, \|\sigma\|_{C^k(\mathbb{R})}\, \|\tilde{u}_j\|_{C^k(S)}.$$

Now we choose $0 < \varepsilon_j \leq 1$ so that $\varepsilon_j\, c''(k,j)\, \|\sigma\|_{C^k(\mathbb{R})}\, \|\tilde{u}_j\|_{C^k(S)} \leq 2^{-j}$ for all $k < j$. This can be done since here we have only finitely many conditions on each ε_j. Then the series (2.24) defining $\hat{u}$ converges absolutely in the C^k-norm for all k. Hence $\hat{u}$ is a smooth section with compact support and can be differentiated termwise.
From the construction we have that $\mathrm{supp}(\tilde{u}_j) \subset \mathrm{supp}(\tilde{u}_0) \cup \mathrm{supp}(\tilde{u}_1) \cup (\mathrm{supp}(f) \cap S)$ for all j. Since $\mathrm{supp}(\tilde{u}_0) \cup \mathrm{supp}(\tilde{u}_1) \cup (\mathrm{supp}(f) \cap S) \subset S \cap K$, we see that $\mathrm{supp}(\hat{u}) \subset \mathbb{R} \times (S \cap K)$.
Applying P to $\hat{u}$ will no longer give f because of errors introduced by the cut-off-function σ. We have to correct this in the following.

First we see that $\hat{u}$ still fulfills the initial conditions. This is because $\sigma \equiv 1$ on a neighborhood of $\{t = 0\}$, hence at $t = 0$ the cut-off is irrelevant.
By the choice of the $\tilde{u}_j$ the section $P\hat{u} - f$ vanishes to infinite order along S. Therefore

$$w(t,x) := \begin{cases} (P\hat{u} - f)(t,x), & \text{on } \{t \geq 0\} = J_+(S \cap \Omega), \\ 0, & \text{on } \{t < 0\}, \end{cases}$$

defines a smooth section. Moreover, $\mathrm{supp}(w) \subset (S \cap K) \times [0,1]$ and hence has compact support. By Theorem 2.3.4 we can solve the equation $P\tilde{u} = w$ with a smooth section $\tilde{u}$ and $\mathrm{supp}(\tilde{u}) \subset J_+^\Omega(\mathrm{supp}(w)) \subset J_+^\Omega(S \cap \Omega) \cap J_+^\Omega(K)$. Now $u_+ := \hat{u} - \tilde{u}$ is a smooth section such that $Pu_+ = P\hat{u} - P\tilde{u} = w + f - w = f$ on $J_+^\Omega(S \cap \Omega)$.
Since $\tilde{u} = 0$ on $I_-^\Omega(S)$ the section u_+ coincides with $\hat{u}$ to infinite order along S. In particular, $u_+|_S = \tilde{u}|_S = u_0$ and $\nabla_\mathfrak{n} u_+ = \nabla_\mathfrak{n} \tilde{u} = u_1$. Moreover, $\mathrm{supp}(u_+) \subset \mathrm{supp}(\hat{u}) \cup \mathrm{supp}(\tilde{u}) \subset J^M(K)$. Thus u_+ has all the required properties on $J_+^M(S)$.
Similarly, one constructs u_- on $J_-^M(S)$. Since both u_+ and u_- coincide to infinite order with $\hat{u}$ along S we obtain the smooth solution u with $\mathrm{supp}(u) \subset J^\Omega(K)$ by setting

$$u(t,x) := \begin{cases} u_+(t,x), & \text{if } t \geq 0, \\ u_-(t,x), & \text{if } t \leq 0. \end{cases} \qquad \square$$

2.5 Exercises

2.5.1. Let M be a Lorentzian manifold and $E \to M$ a vector bundle with a fiberwise inner product $\langle \cdot, \cdot \rangle$. Let P be a normally hyperbolic operator acting on sections of E. Write $P = \Box^\nabla + B$ for the P-compatible connection ∇.
Show that P is formally selfadjoint if and only if ∇ is metric w.r.t. $\langle \cdot, \cdot \rangle$ and B is pointwise selfadjoint.

2.5.2. Let X be a smooth vector field and V a smooth function on a Lorentzian manifold. We consider the normally hyperbolic operator $P = \Box + \partial_X + V$ acting on functions.
Determine the P-compatible connection ∇ and the potential B in $P = \Box^\nabla + B$.

2.5.3. Compute the Hadamard coefficients of the Klein-Gordon operator $P = \Box + m^2$, $m \geq 0$, on Minkowski space.

2.5.4. Let (X, h) be a Riemannian manifold and equip $M = \mathbb{R} \times X$ with the Lorentzian metric $g = -dt^2 + h$ where t denotes the standard coordinate in $\mathbb{R}$.

a) Show that if $x^1, \ldots, x^{n-1}$ are normal coordinates around $\hat{x}$ on X, then $t, x^1, \ldots, x^{n-1}$ are normal coordinates around $(0, \hat{x})$ on M.

b) What is the relation between the volume distortion functions $\mu_{\hat{x}}$ on X and $\mu_{(0,\hat{x})}$ on M?

2.5.5. In addition to the notation in Exercise 2.5.4, let E be a vector bundle over X. The bundle pulled back to M via the projection $M \to X$ is also denoted by E.

a) Show that a differential operator $L \in \mathscr{D}\!\mathit{iff}_2(E,E)$ on X is of Laplace type if and only if $P := \frac{\partial^2}{\partial t^2} + L$ is normally hyperbolic on M.

b) Show that the Hadamard coefficients of P are parallel along the curves $c(t) = (t, \hat{x})$ where $\hat{x} \in X$ is constant.

2.5.6. Show that the distributions $(\frac{\partial}{\partial x^1})^k R_\pm(2)$ on Minkowski space have order $k-2$ at least, $k = 2, 3, \ldots$.

2.5.7. Define the derivative of the Riesz distributions with respect to α by

$$\frac{\partial R_\pm}{\partial \alpha}(\alpha_0)[\varphi] := \frac{\partial}{\partial \alpha}\Big|_{\alpha=\alpha_0} (R_\pm(\alpha)[\varphi]) .$$

a) Show that $u = \gamma \cdot (\frac{\partial R_+}{\partial \alpha}(0) - \frac{\partial R_-}{\partial \alpha}(0))$ is a nontrivial solution of $\Box u = 0$ on n-dimensional Minkowski space, provided $n \geq 3$.

b) What if $n = 2$?

2.5.8. Show that the formal fundamental solution

$$\mathcal{R}_\pm = \sum_{j=0}^{\infty} (-1)^j m^{2j} R_\pm(2+2j)$$

of the Klein-Gordon operator $P = \Box + m^2$ on Minkowski space M (cf. Exercise 2.5.3) converges in $\mathcal{D}'(M)$ and hence defines a true fundamental solution at $x = 0$.
Hint: Show that for sufficiently large N the series $\sum_{j=N}^{\infty} (-1)^j m^{2j} R_\pm(2+2j)$ converges uniformly on compact subsets of M.

2.5.9. Show that the ε_j in Lemma 2.2.7 can be chosen such that the sum

$$\Sigma_1 = \sum_{j=N}^{\infty} \Box_{(2)}(\sigma(\Gamma/\varepsilon_j))\ V_j\ R_\pm^{\Omega'}(2+2j, \cdot)$$

converges absolutely in $C^k(\overline{\Omega} \times \overline{\Omega}, E^* \boxtimes E)$ for all $k \in \mathbb{N}$.

2.5.10. Show that every C^2-solution u of $\Box u = 0$ on the 2-dimensional Minkowski space is uniquely determined by u_0 and u_1 where $u_0(x) = u(0,x)$ and $u_1(x) = \frac{\partial u}{\partial t}(0,x)$.
Hint: Observe $\Box = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})$.

2.5.11. Consider the situation in Proposition 2.4.1 except that ∇ is an arbitrary connection on the vector bundle on which P acts.

a) Show that Proposition 2.4.1 still holds.

b) Show that if uniqueness of u holds for the P-compatible connection then it also holds for ∇.

2.5.12. Let u_0, u_1, and f be as in Proposition 2.4.1 and let u be the solution of the Cauchy problem constructed in the proof. The statement on the support of u can be improved to

$$\mathrm{supp}(u) \subset J^{\Omega}(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1)) \cup J^{\Omega}_{+}(\mathrm{supp}(f) \cap J^{\Omega}_{+}(S \cap \Omega)) \cup J^{\Omega}_{-}(\mathrm{supp}(f) \cap J^{\Omega}_{-}(S \cap \Omega)) \,.$$

a) Illustrate by example or drawing that this improves the statement in Proposition 2.4.1.

b) Explain how the proof of Proposition 2.4.1 needs to be refined in order to yield this version of the support estimate.

3 Linear wave equations - global theory

Our aim in the next chapter is to study fundamental solutions, solutions to inhomogeneous equations and the Cauchy problem on arbitrary globally hyperbolic manifolds.

3.1 Uniqueness of the fundamental solution

To motivate the line of the argument in the following we first give an incorrect proof of the following false statement:
Let P be a normally hyperbolic operator on a Lorentzian manifold M. Then every solution of $Pu = 0$ is trivial, i.e., $u = 0$.

Incorrect proof. Let $x \in M$. We choose an RCCSV-neighboorhood Ω of x and we want to show $u[\phi] = 0$ for all test sections $\phi \in C_c^\infty(\Omega, E^*)$. By Theorem 2.3.4 we can solve $P^*\psi = \phi$ in Ω. Now we compute

$$u[\phi] = u[P^*\psi] \overset{(*)}{=} \underbrace{Pu}_{=0}[\psi] = 0.$$

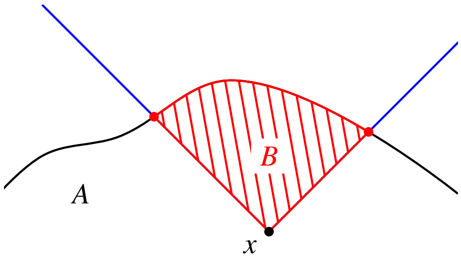
Hence $u = 0$ on Ω and since x was arbitrary $u = 0$ on M. □

We know that the statement we just "proved" is false. For instance, constant functions $u \equiv c$ satisfy $\Box u = 0$ without being trivial. Where did the proof fail?
The problem is that equation $(*)$ is not justified because ψ does not have compact support and hence is not a test section. Nevertheless, the argument can be rectified under suitable assumptions on $\mathrm{supp}(u)$. We show uniqueness of solutions to the wave equation with future or past-compact support.

Theorem 3.1.1. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections of a vector bundle E over M.*
Then any distribution $u \in \mathcal{D}'(M, E)$ with past or future compact support solving the equation $Pu = 0$ must vanish identically on M,

$$u \equiv 0.$$

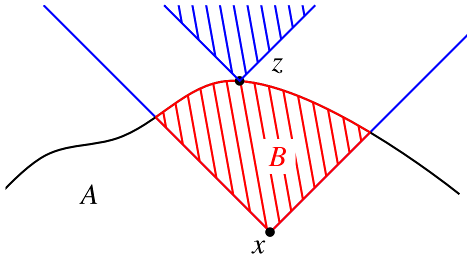
Proof. Without loss of generality let $A := \mathrm{supp}(u)$ be future compact. We will show that A is empty. Assume the contrary and consider some $x \in A$. Then the set $B := J_+^M(x) \cap A$ is compact. By Proposition 1.2.60 the map $y \mapsto \tau(x, y)$ is finite and continuous on a globally hyperbolic manifold. Therefore $y \mapsto \tau(x, y)$ attains its maximum on B at some point $z \in B$.



On a globally hyperbolic manifold the relation "$\leq$" is closed. Moreover, the implication "$u \leq v$ and $v \leq u \Rightarrow u = v$" holds because there are no causal loops. The relation "$\leq$" now turns B into a partially ordered set.

We check that Zorn's lemma can be applied to $(B, \leq)$. Let B' be a totally ordered subset of B. Choose[1] a countable dense subset $B'' \subset B'$. Then B'' is totally ordered as well and can be written as $B'' = \{\zeta_1, \zeta_2, \zeta_3, \ldots\}$. Let z_i be the largest element in $\{\zeta_1 \ldots, \zeta_i\}$. This yields a monotonically increasing sequence $(z_i)_i$ which eventually becomes at least as large as any given $\zeta \in B''$.

By compactness of B a subsequence of $(z_i)_i$ converges to some $z' \in B$ as $i \to \infty$. Since the relation "$\leq$" is closed one easily sees that z' is an upper bound for B''. Since $B'' \subset B'$ is dense and "$\leq$" is closed, z' is also an upper bound for B'. Hence Zorn's lemma applies and yields a maximal element $z_0 \in B$. Replacing z by z_0 we may therefore assume that $\tau(y, \cdot)$ attains its maximum at z and that $A \cap J_+^M(z) = \{z\}$.



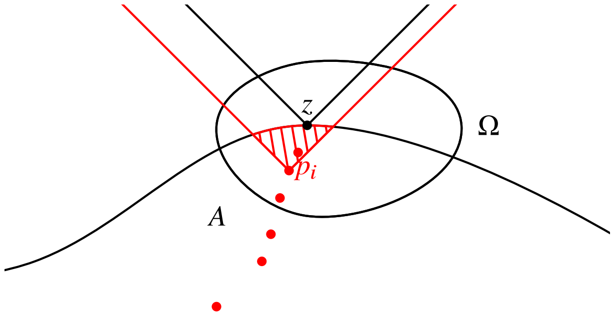
We fix an RCCSV-neighborhood $\Omega \subset M$ of z. Let p_i be a sequence of points such that $p_i \to z$.

Claim: For i sufficiently large we have $J_+^M(p_i) \cap A \subset \Omega$.

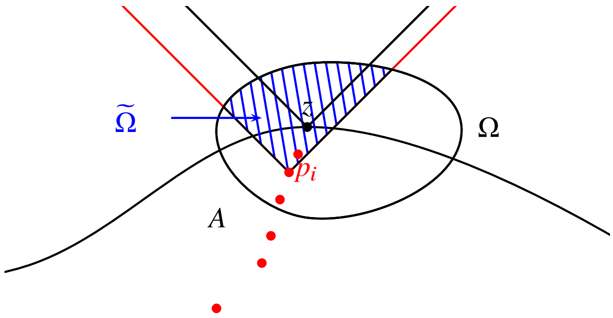
Suppose the contrary. Then there is for each i a point $q_i \in J_+^M(p_i) \cap A$ such that $q_i \notin \Omega$. Since $q_i \in J_+^M(x) \cap A$ for all i and $J_+^M(x) \cap A$ is compact we have, after passing to a subsequence, that $q_i \to q \in J_+^M(x) \cap A$. From $q_i \geq p_i$, $q_i \to q$, $p_i \to z$, and the fact that "$\leq$" is closed we

[1] Every (infinite) subset of a manifold has a countable dense subset. This follows from existence of a countable basis of the topology.

conclude $q \geq z$. Thus $q \in J_+^M(z) \cap A$, hence $q = z$. On the other hand, $q \notin \Omega$ since all $q_i \notin \Omega$, a contradiction. ✓



For $p_i \in I_-^M(z)$ with $J_+^M(p_i) \cap A \subset \Omega$ we put $\widetilde{\Omega} := \Omega \cap I_+^M(p_i)$ and note that $\widetilde{\Omega}$ is an open neighborhood of z.



We choose a cut-off function $\eta \in \mathcal{D}(\Omega, \mathbb{R})$ such that $\eta|_{J_+^M(p_i) \cap A} \equiv 1$.
Now we consider some arbitrary $\varphi \in \mathcal{D}(\widetilde{\Omega}, E^*)$. We will show that $u[\varphi] = 0$. This then proves that $u|_{\widetilde{\Omega}} = 0$, in particular, $z \notin A = \mathrm{supp}(u)$, the desired contradiction.
By the choice of Ω we can solve the inhomogeneous equation $P^*\psi = \varphi$ on Ω with $\psi \in C^\infty(\Omega, E^*)$ and $\mathrm{supp}(\psi) \subset J_+^\Omega(\mathrm{supp}(\varphi)) \subset J_+^M(p_i) \cap \Omega$. Then $\mathrm{supp}(u) \cap \mathrm{supp}(\psi) \subset A \cap J_+^M(p_i) \cap \Omega = A \cap J_+^M(p_i)$. Hence $\eta|_{\mathrm{supp}(u) \cap \mathrm{supp}(\psi)} = 1$. Thus

$$u[\varphi] = u[P^*\psi] = u[P^*(\eta\psi)] = (Pu)[\eta\psi] = 0. \qquad \square$$

Corollary 3.1.2. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*

Then for every $x \in M$ there exists at most one advanced and at most one retarded fundamental solution for P at x.

Proof. Let F_1 and F_2 be two advanced fundamental solutions at x. Then $u = F_1 - F_2$ is a solution for $Pu = 0$. Since F_1 and F_2 are advanced solutions we know that $\mathrm{supp}(u) \subset \mathrm{supp}(F_1) \cup \mathrm{supp}(F_2) \subset J_+(x)$. On a globally hyperbolic manifold $J_+(x)$ is past compact. Then Theorem 3.1.1 shows that $u = 0$ and hence $F_1 = F_2$. □

3.2 The Cauchy problem

We start by identifying the divergence term that appears when one compares the operator P with its formal adjoint P.

Lemma 3.2.1. *Let E be a vector bundle over the timeoriented Lorentzian manifold M. Let P be a normally hyperbolic operator acting on sections in E. Let ∇ be the P-compatible connection on E.*
Then for every $\psi \in C^\infty(M, E^*)$ and $v \in C^\infty(M, E)$,

$$\psi \cdot (Pv) - (P^*\psi) \cdot v = \mathrm{div}\,(W),$$

where the vector field $W \in C^\infty(M, TM \otimes_{\mathbb{R}} \mathbb{K})$ is characterized by

$$\langle W, X\rangle = (\nabla_X \psi) \cdot v - \psi \cdot (\nabla_X v)$$

for all $X \in C^\infty(M, TM)$.

Proof. The Levi-Civita connection on TM and the P-compatible connection ∇ on E induce connections on $T^*M \otimes E$ and on $T^*M \otimes E^*$ which we also denote by ∇ for simplicity. We define a linear differential operator $L : C^\infty(M, T^*M \otimes E^*) \to C^\infty(M, E^*)$ of first order by

$$Ls := -\sum_{j=1}^{n} \epsilon_j (\nabla_{e_j} s)(e_j)$$

where $e_1, \ldots, e_n$ is a local Lorentz orthonormal frame of TM and $\epsilon_j = \langle e_j, e_j\rangle$. It is easily checked that this definition does not depend on the choice of orthonormal frame. Write $e_1^*, \ldots, e_n^*$ for the dual frame of T^*M. The metric $\langle \cdot, \cdot \rangle$ on TM and the natural pairing $E^* \otimes E \to \mathbb{K}$, $\psi \otimes v \mapsto \psi \cdot v$, induce a pairing $(T^*M \otimes E^*) \otimes (T^*M \otimes E) \to \mathbb{K}$ which we again denote by $\langle \cdot, \cdot \rangle$. For all $\psi \in C^\infty(M, E^*)$ and $s \in C^\infty(M, T^*M \otimes E)$ we obtain

$$\langle \nabla\psi, s\rangle \;=\; \sum_{j,k=1}^{n} \langle e_j^* \otimes \nabla_{e_j}\psi\,,\, e_k^* \otimes s(e_k)\rangle$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{n} \langle e_j^*, e_k^* \rangle \cdot (\nabla_{e_j}\psi) \cdot s(e_k) \\
&= \sum_{j=1}^{n} \varepsilon_j (\nabla_{e_j}\psi) \cdot s(e_j) \\
&= \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot (\nabla_{e_j} s)(e_j) - \psi \cdot s(\nabla_{e_j} e_j)\right) \\
&= \psi \cdot (Ls) + \sum_{j=1}^{n} \varepsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot s(\nabla_{e_j} e_j)\right). \qquad (3.1)
\end{aligned}
$$

Let V_1 be the unique $\mathbb{K}$-valued vector field characterized by $\langle V_1, X \rangle = \psi \cdot s(X)$ for every $X \in C^\infty(M, TM)$. Then

$$
\begin{aligned}
\mathrm{div}\,(V_1) &= \sum_{j=1}^{n} \epsilon_j \langle \nabla_{e_j} V_1, e_j \rangle \\
&= \sum_{j=1}^{n} \epsilon_j \left(\partial_{e_j}\langle V_1, e_j \rangle - \langle V_1, \nabla_{e_j} e_j \rangle\right) \\
&= \sum_{j=1}^{n} \epsilon_j \left(\partial_{e_j}(\psi \cdot s(e_j)) - \psi \cdot s(\nabla_{e_j} e_j)\right).
\end{aligned}
$$

Plugging this into (3.1) yields

$$\langle \nabla\psi, s \rangle = \psi \cdot Ls + \mathrm{div}\,(V_1).$$

In particular, if $v \in C^\infty(M, E)$ we get for $s := \nabla v \in C^\infty(M, T^*M \otimes E)$

$$\langle \nabla\psi, \nabla v \rangle = \psi \cdot L\nabla v + \mathrm{div}\,(V_1) = \psi \cdot \Box^\nabla v + \mathrm{div}\,(V_1),$$

hence

$$\psi \cdot \Box^\nabla v = \langle \nabla\psi, \nabla v \rangle - \mathrm{div}\,(V_1) \qquad (3.2)$$

where $\langle V_1, X \rangle = \psi \cdot \nabla_X v$ for all $X \in C^\infty(M, TM)$. Similarly, by interchanging the role of ψ and v we obtain

$$(\Box^\nabla \psi) \cdot v = \langle \nabla\psi, \nabla v \rangle - \mathrm{div}\,(V_2)$$

where V_2 is the vector field characterized by $\langle V_2, X \rangle = (\nabla_X \psi) \cdot v$ for all $X \in C^\infty(M, TM)$. Thus comparing leads to

$$\psi \cdot \Box^\nabla v - (\Box^\nabla \psi) \cdot v = \mathrm{div}\,(W) \qquad (3.3)$$

where $W = V_2 - V_1$. Since ∇ is the P-compatible connection on E we have $P = \Box^\nabla + B$ for some $B \in C^\infty(M, \mathrm{End}(E))$, see Lemma 2.1.7. Thus

$$\psi \cdot Pv = \psi \cdot \Box^\nabla v + \psi \cdot Bv = (\Box^\nabla \psi) \cdot v + \mathrm{div}\,(W) + (B^*\psi) \cdot v.$$

If ψ or v has compact support, then we can integrate $\psi \cdot Pv$ and the divergence term vanishes:

$$\int_M \psi \cdot Pv \ \mathrm{dV} = \int_M \Big((\Box^\nabla \psi) \cdot v + (B^* \psi) \cdot v \Big) \ \mathrm{dV}.$$

Therefore $\Box^\nabla + B^* = P^*$ and $\psi \cdot Pv - P^*\psi \cdot v = \mathrm{div}\,(W)$ as claimed. ◻

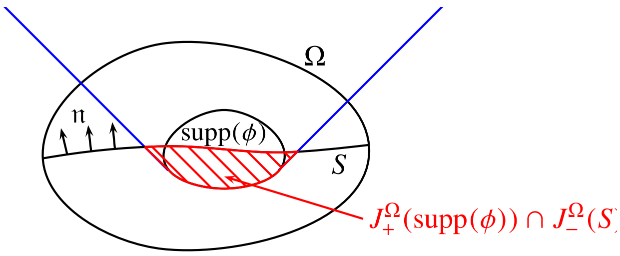
This now yields a local formula allowing us to control a solution of $Pu = 0$ in terms of its Cauchy data.

Lemma 3.2.2. *Let E be a vector bundle over a timeoriented Lorentzian manifold M and let P be a normally hyperbolic operator acting on sections in E. Let ∇ be the P-compatible connection on E. Let $\Omega \subset M$ be an RCCSV-domain. Let S be a smooth spacelike Cauchy hypersurface in Ω. Denote by $\mathfrak{n}$ the future directed (timelike) unit normal vector field along S. For every $x \in \Omega$ let $F^\Omega_\pm(x)$ be the fundamental solution for P^* at x with support in $J^\Omega_\pm(x)$ constructed in Proposition 2.2.15. Define $F^\Omega[\varphi] := F^\Omega_+(\cdot)[\varphi] - F^\Omega_-(\cdot)[\varphi] \in C^\infty(\Omega, E^*)$.*
For every smooth solution $u \in C^\infty(\Omega, E)$ of $Pu = 0$ on Ω

$$u[\varphi] = \int_S \Big((\nabla_{\mathfrak{n}}(F^\Omega[\varphi])) \cdot u_0 - (F^\Omega[\varphi]) \cdot u_1 \Big) \mathrm{dA},$$

where $u_0 := u_{|S}$ and $u_1 := \nabla_{\mathfrak{n}} u$.

Proof. Fix $\varphi \in \mathcal{D}(\Omega, E^*)$. We consider the distribution ψ defined by $\psi[w] := \int_\Omega \varphi(x) \cdot F^\Omega_+(x)[w] \ \mathrm{dV}$ for every $w \in \mathcal{D}(\Omega, E)$. By Theorem 2.3.4 we know that $\psi \in C^\infty(\Omega, E^*)$, has its support contained in $J^\Omega_+(\mathrm{supp}(\varphi))$ and satisfies $P^*\psi = \varphi$.
Let W be the vector field from Lemma 3.2.1 with u instead of v.



Since by Proposition 1.2.56 the subset $J^\Omega_+(\mathrm{supp}(\varphi)) \cap J^\Omega_-(S)$ of Ω is compact, Theorem 1.2.72 applies to $D := I^\Omega_-(S)$ and the vector field W:

$$\begin{aligned} \int_D ((P^*\psi) \cdot u - \psi \cdot (Pu)) \ \mathrm{dV} &= -\int_D \mathrm{div}\,(W) \ \mathrm{dV} \\ &= -\underbrace{\langle \mathfrak{n}, \mathfrak{n} \rangle}_{=-1} \int_{\partial D} \langle W, \mathfrak{n} \rangle \mathrm{dA} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial S} \langle W, \mathfrak{n} \rangle \mathrm{dA} \\
&= \int_S \left((\nabla_{\mathfrak{n}} \psi) \cdot u - \psi \cdot (\nabla_{\mathfrak{n}} u) \right) \mathrm{dA}.
\end{aligned}
$$

On the other hand,

$$
\int_D ((P^*\psi) \cdot u - \psi \cdot (Pu)) \ \mathrm{dV} = \int_{I_-^\Omega(S)} ((\underbrace{P^*\psi}_{=\varphi}) \cdot u - \psi \cdot (\underbrace{Pu}_{=0})) \ \mathrm{dV} = \int_{I_-^\Omega(S)} \varphi \cdot u \ \mathrm{dV}.
$$

Thus

$$
\int_{I_-^\Omega(S)} \varphi \cdot u \ \mathrm{dV} = \int_S ((\nabla_{\mathfrak{n}} \psi) \cdot u - \psi \cdot (\nabla_{\mathfrak{n}} u)) \, \mathrm{dA}. \tag{3.4}
$$

Similarly, using $D = I_+^\Omega(S)$ and $\psi'[w] := \int_\Omega \varphi(x) \cdot F_-^\Omega(x)[w] \ \mathrm{dV}$ for any $w \in \mathcal{D}(\Omega, E)$ one gets

$$
\int_{I_+^\Omega(S)} \varphi \cdot u \ \mathrm{dV} = \int_S (\psi' \cdot (\nabla_{\mathfrak{n}} u) - (\nabla_{\mathfrak{n}} \psi') \cdot u) \, \mathrm{dA}. \tag{3.5}
$$

The different sign is caused by the fact that $\mathfrak{n}$ is the *interior* unit normal to $I_+^\Omega(S)$. Adding (3.4) and (3.5) we get

$$
\int_\Omega \varphi \cdot u \ \mathrm{dV} = \int_S ((\nabla_{\mathfrak{n}}(\psi - \psi')) \cdot u - (\psi - \psi') \cdot (\nabla_{\mathfrak{n}} u)) \, \mathrm{dA}
$$

which is the desired result. □

Corollary 3.2.3. *Let Ω, u, u_0, and u_1 be as in Lemma 3.2.2. Then*

$$
\operatorname{supp}(u) \subset J^\Omega(K)
$$

where $K = \operatorname{supp}(u_0) \cup \operatorname{supp}(u_1)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega, E^*)$ with $\operatorname{supp}(\varphi) \cap J^\Omega(K) = \emptyset$. We will show that $u[\varphi] = 0$. Since $\operatorname{supp}(\varphi) \cap J^\Omega(K) = \emptyset$ implies $J^\Omega(\operatorname{supp}\varphi) \cap K = \emptyset$ and $\operatorname{supp}(F^\Omega[\varphi]) \cap K \subset J^\Omega(\operatorname{supp}\varphi) \cap K$ we find that $\operatorname{supp}(F^\Omega[\varphi]) \cap K = \emptyset$. By Lemma 3.2.2 $u[\varphi] = \int_S \left((\nabla_{\mathfrak{n}}(F^\Omega[\varphi])) \cdot u_0 - (F^\Omega[\varphi]) \cdot u_1 \right) \mathrm{dA}$. But the support of the integrand of the right hand side is given exactly by the empty set $\operatorname{supp}(F^\Omega[\varphi]) \cap K$. Hence $u[\varphi] = 0$. □

These local considerations already suffice to establish uniqueness of solutions to the Cauchy problem on general globally hyperbolic manifolds.

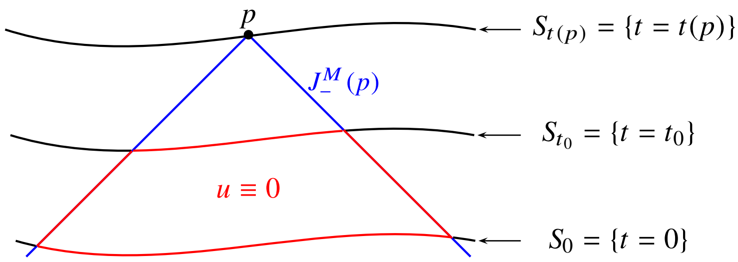
Corollary 3.2.4. *Let E be a vector bundle over a globally hyperbolic Lorentzian manifold M. Let ∇ be a connection on E and let $P = \Box^\nabla + B$ be a normally hyperbolic operator acting on sections in E. Let S be a smooth spacelike Cauchy hypersurface in M, and let $\mathfrak{n}$ be the future directed (timelike) unit normal vector field along S.*
If $u \in C^\infty(M, E)$ solves

$$\begin{cases} Pu &= 0 \quad \text{on } M, \\ u &= 0 \quad \text{along } S, \\ \nabla_{\mathfrak{n}} u &= 0 \quad \text{along } S, \end{cases}$$

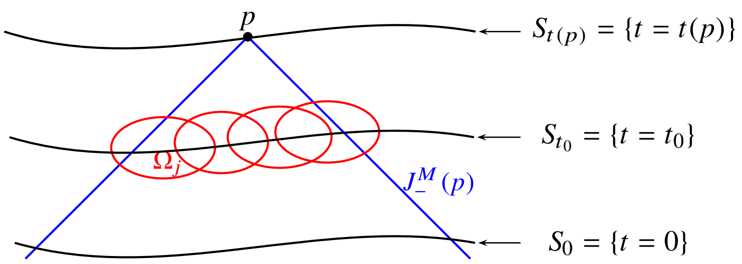
then $u = 0$ on M.

Proof. By Theorem 1.2.53 there is a Cauchy time function $t : M \to \mathbb{R}$ and a foliation of M by spacelike smooth Cauchy hypersurfaces such that $S = t^{-1}(0)$. Extend $\mathfrak{n}$ smoothly to all of M such that $\mathfrak{n}_{|S_t}$ is the unit future directed (timelike) normal vector field on S_t for every $t \in \mathbb{R}$. Let $p \in M$. We show that $u(p) = 0$.
Without loss of generality let $t(p) > 0$ and let p be in the chronological future of S. Set

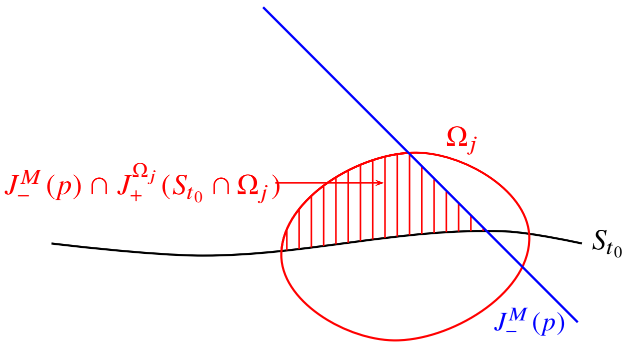
$$t_0 := \sup\left\{\tau \in [0, t(p)] \,\middle|\, u \text{ vanishes on } J_-^M(p) \cap \{0 \le t \le \tau\}\right\}.$$



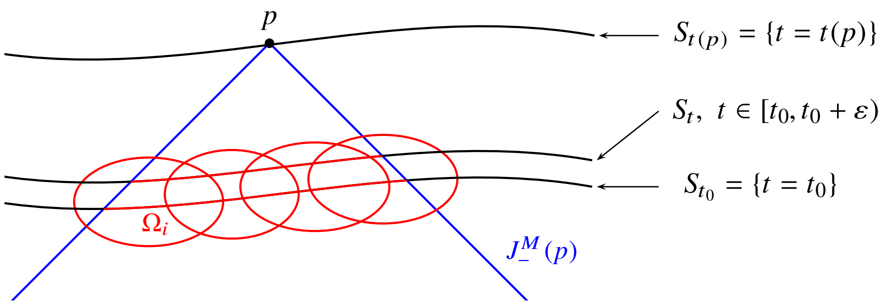
We will show that $t_0 = t(p)$ which implies in particular $u(p) = 0$.
The initial data on $S_{t_0} \cap J_-^M(p)$ vanishes, i.e. for $u_0 := u_{|S_{t_0}}$ and $u_1 := (\nabla_{\mathfrak{n}} u)_{|S_{t_0}}$, we have that $u_0 = 0$ and $u_1 = 0$ on $S_{t_0} \cap J_-^M(p)$ because $u \equiv 0$ on $J_-^M(p) \cap \{0 \le t \le t_0\}$.
For each $x \in J_-^M(p) \cap S_{t_0}$ we may choose an RCCSV-neighborhood Ω of x such that $S_{t_0} \cap \Omega$ is a Cauchy hypersurface of Ω.
By Proposition 1.2.56 the intersection $S_{t_0} \cap J_-^M(p)$ is compact. Hence it can be covered by finitely many open subsets Ω_i, $1 \le i \le N$, satisfying these conditions.



By assumption $u_{|\Omega_j}$ is a solution for the Cauchy problem $Pu = 0$ with certain initial data on $\Omega_j \cap S_{t_0}$. We know, that the initial data can be nonvanishing only outside $S_{t_0} \cap J_-^M(p)$. Hence Corollary 3.2.3 implies $\mathrm{supp}(u_{|\Omega_j}) \cap J_-^M(p) \subset J^{\Omega_j}(\mathrm{supp}(u) \cap S \cap \Omega_j) \cap J_-^M(p) = \emptyset$.



This implies that u vanishes identically on $(\Omega_1 \cup \cdots \cup \Omega_N) \cap J_-^M(p)$.
Since $(\Omega_1 \cup \cdots \cup \Omega_N) \cap J_-^M(p)$ is an open neighborhood of the compact set $S_{t_0} \cap J_-^M(p)$ in $J_-^M(p)$ there exists an $\varepsilon > 0$ such that $S_t \cap J_-^M(p) \subset \Omega_1 \cup \cdots \cup \Omega_N$ for every $t \in [t_0, t_0 + \varepsilon)$.



Hence u vanishes on $S_t \cap J_-^M(p)$ for all $t \in [t_0, t_0 + \varepsilon)$. This is a contradiction to the maximality of t_0 unless $t_0 = t(p)$. □

In order to show existence of solutions to the Cauchy problem on globally hyperbolic manifolds we need some preparation. Let M be globally hyperbolic. We consider a Cauchy time function $t : M \to \mathbb{R}$. W.l.o.g. let t be surjective so that $M = \mathbb{R}\times S$ and the metric is of the form $-N^2 dt^2 + g_t$. In particular, M is foliated by the smooth spacelike Cauchy hypersurfaces $\{t_0\} \times S =: S_{t_0}$ where $t_0 \in \mathbb{R}$.

Let $p \in M$. For any $r > 0$ we denote by $B_r(p)$ the open ball of radius r about p in the Riemannian manifold $S_t(p)$ with respect to the Riemannian metric $g_{t(p)}$ on $S_t(p)$, i.e.

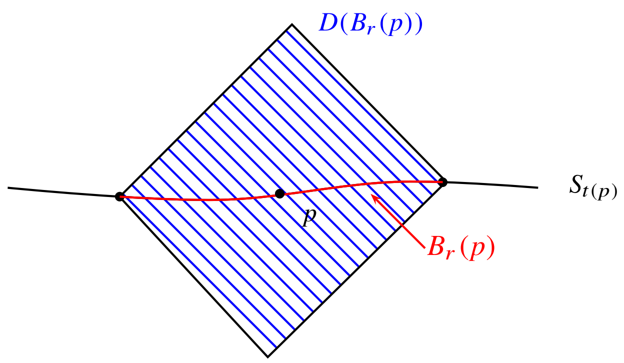
$$B_r(p) := \{q \in S_{t(p)} \mid \mathrm{dist}^{g_{t(p)}}(p,q) < r\} .$$

Then $B_r(p)$ is open as a subset of $S_t(p)$ but not as a subset of M.

Recall from the exercises that $D(A)$ denotes the Cauchy development[2] of a subset A of M.

We define the function $\rho : M \to (0,\infty]$ by

$$\rho(p) := \sup\{r > 0 \mid D(B_r(p)) \text{ is an RCCSV domain}\}.$$



Lemma 3.2.5. *The function ρ is lower semi-continuous on M.*

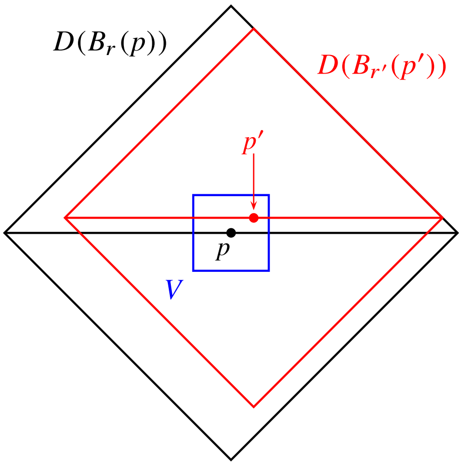
Proof. First note that ρ is well defined since every point has a RCCSV-neighborhood. Let $\epsilon > 0$. Let $p \in M$ and $r > 0$ be such that $\rho(p) > r$ and $r > \rho(p) - \frac{\epsilon}{2}$. We want to show $\rho(p') > r - \epsilon$ for all p' in a neighborhood of p.

For any point $p' \in D(B_r(p))$ consider

$$\lambda(p') := \sup\{r' > 0 \mid B_{r'}(p') \subset D(B_r(p))\}.$$

Claim: There exists a neighborhood V of p such that for every $p' \in V$ one has $\lambda(p') > r - \epsilon$.

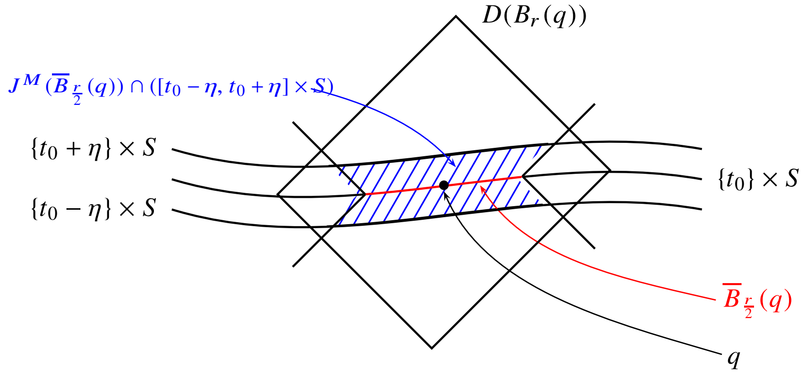
[2]The **Cauchy development** of a subset S of a timeoriented Lorentzian manifold M is the set $D(S)$ of points of M through which every inextendible causal curve in M meets S.



Let us assume the claim for a moment. Let $p' \in V$. Pick r' with $r - \epsilon < r' < \lambda(p')$. Hence $B_{r'}(p') \subset D(B_r(p))$. It follows from the definition of the Cauchy development that $D(B_{r'}(p')) \subset D(B_r(p))$. Since $D(B_r(p))$ is RCCSV the subset $D(B_{r'}(p'))$ is RCCSV as well. Thus $\rho(p') \geq r' > r - \epsilon \geq \rho(p) - \frac{3}{2}\epsilon$. This then concludes the proof.
It remains to show the claim. Assume the claim is false. Then there is a sequence $(p_i)_i$ of points in M converging to p such that $\lambda(p_i) \leq r - \epsilon$ for all i. Hence for $r_0 := r - \epsilon/2$ we have $B_{r_0}(p_i) \not\subset D(B_r(p))$. Choose $x_i \in B_{r_0}(p_i) \setminus D(B_r(p))$.
The closed set $\overline{B}_r(p)$ is contained in the compact set $\overline{D}(B_r(p))$ and therefore compact itself. Thus $[-1,1] \times \overline{B}_r(p)$ is compact. For i sufficiently large $B_{r_0}(p_i) \subset [-1,1] \times \overline{B}_r(p)$ and therefore $x_i \in [-1,1] \times \overline{B}_r(p)$. We pass to a convergent subsequence $x_i \to x$. Since $p_i \to p$ and $x_i \in \overline{B}_{r_0}(p_i)$ we have $x \in \overline{B}_{r_0}(p)$. Hence $x \in B_r(p)$. Since $D(B_r(p))$ is an open neighborhood of x we must have $x_i \in D(B_r(p))$ for sufficiently large i. This contradicts the choice of the x_i. $\square$

For every $r > 0$ and $q \in M = \mathbb{R} \times S$ consider

$$\theta_r(q) := \sup\{\eta > 0 \mid J^M(\overline{B}_{r/2}(q)) \cap ([t_0 - \eta, t_0 + \eta] \times S) \subset D(B_r(q))\}.$$



Remark 3.2.6. There exist $\eta > 0$ with $J^M(\overline{B}_{r/2}(q)) \cap ([t_0-\eta, t_0+\eta] \times S) \subset D(B_r(q))$. Hence $\theta_r(q) > 0$.
One can see this as follows. If no such η existed, then there would be points $x_i \in J^M(\overline{B}_{r/2}(q)) \cap ([t_0 - \frac{1}{i}, t_0 + \frac{1}{i}] \times S)$ but $x_i \notin D(B_r(q))$, $i \in \mathbb{N}$. All x_i lie in the compact set $J^M(\overline{B}_{r/2}(q)) \cap ([t_0-1, t_0+1] \times S)$. Hence we may pass to a convergent subsequence $x_i \to x$. Then $x \in J^M(\overline{B}_{r/2}(q)) \cap (\{t_0\} \times S) = \overline{B}_{r/2}(q)$. Since $D(B_r(q))$ is an open neighborhood of $\overline{B}_{r/2}(q)$ we must have $x_i \in D(B_r(q_0))$ for sufficiently large i in contradiction to the choice of the x_i.

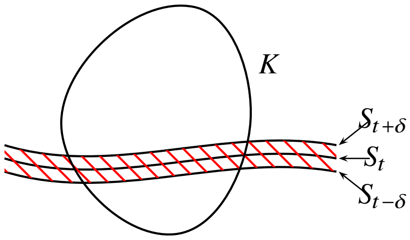
Lemma 3.2.7. *The function $\theta_r : M \to (0, \infty]$ is lower semi-continuous.*

Proof. Fix $q \in M$. Let $\epsilon > 0$. We need to find a neighborhood U of q such that for all $q' \in U$ we have $\theta_r(q') \geq \theta_r(q) - \epsilon$.
Put $\eta := \theta_r(q)$ and choose let $t_0 = t(q)$. Assume no such neighborhood U exists. Then there is a sequence $(q_i)_i$ in M such that $q_i \to q$ and $\theta_r(q_i) < \eta - \epsilon$ for all i. We know that $J^M(\overline{B}_{r/2}(q_i)) \cap ([t_i - \eta + \epsilon, t_i + \eta - \epsilon] \times S) \not\subset D(B_r(q_i))$. Hence we can choose $x_i \in J^M(\overline{B}_{r/2}(q_i)) \cap ([t_i - \eta + \epsilon, t_i + \eta - \epsilon] \times S)$ but $x_i \notin D(B_r(q_i))$.
But this implies that $x_i \to x$ with $x \in J^M(\overline{B}_{r/2}(q)) \cap ([t_i - \eta + \epsilon, t_i + \eta - \epsilon] \times S)$ and $t(x) = t(q)$ for reasons of continuity. We obtain $x \in \overline{B}_{\frac{r}{2}}(q) \subset D(\overline{B}_{\frac{3r}{2}}(q))$. Since $x_i \notin D(B_r(q_i))$ we get that $x_i \notin D(B_{\frac{3r}{2}}(q))$ for all $i > 0$. But contradicts $x \in \overline{B}_{\frac{r}{2}}(q) \subset D(\overline{B}_{\frac{3r}{2}}(q))$.

We are now ready to reach a first global existence result. We first globalize in the spatial direction.

Lemma 3.2.8. *For each compact subset $K \subset M$ there exists $\delta > 0$ such that for each $t \in \mathbb{R}$ and any $u_0, u_1 \in \mathcal{D}(S_t, E)$ with $\mathrm{supp}(u_j) \subset K$, $j = 1, 2$, there is a smooth solution u of $Pu = 0$ defined on $(t-\delta, t+\delta) \times S$ satisfying $u|_{S_t} = u_0$ and $\nabla_{\mathfrak{n}} u|_{S_t} = u_1$. Moreover,*

$\operatorname{supp}(u) \subset J^M(K \cap S_t)$.



Proof. By Lemma 3.2.5 the function ρ admits a minimum on the compact set K. Hence there is a constant $r_0 > 0$ such that $\rho(q) > 2r_0$ for all $q \in K$. Likewise, by Lemma 3.2.7 the function θ_{2r_0} admits a minimum on the compact set K. Therefore we can choose $\delta > 0$ such that $\theta_{2r_0}(q) > \delta$ for all $q \in K$.
Now fix $t \in \mathbb{R}$. Cover the compact set $S_t \cap K$ by finitely many balls $B_{r_0}(q_1), \ldots, B_{r_0}(q_N)$, $q_j \in S_t \cap K$.
Let $u_0, u_1 \in \mathcal{D}(S_t, E)$ with $\operatorname{supp}(u_j) \subset K$. Using a partition of unity write $u_0 = u_{0,1} + \ldots + u_{0,N}$ with $\operatorname{supp}(u_{0,j}) \subset B_{r_0}(q_j)$ and similarly $u_1 = u_{1,1} + \ldots + u_{1,N}$. The set $D(B_{2r_0}(q_j))$ is RCCSV. By Proposition 2.4.1 we can find a solution w_j of $Pw_j = 0$ on $D(B_{2r_0}(q_j))$ with $w_j|_{S_t} = u_{0,j}$ and $\nabla_{\mathfrak{n}} w_j|_{S_t} = u_{1,j}$. Moreover, $\operatorname{supp}(w_j) \subset J^M(B_{r_0}(q_j))$. From $J^M(B_{r_0}(q_j)) \cap (t-\delta, t+\delta) \times S \subset D(B_{2r_0}(q_j))$ we see that w_j is defined on $J^M(B_{r_0}(q_j)) \cap (t-\delta, t+\delta) \times S$.
Extend w_j smoothly by zero to all of $(t-\delta, t+\delta) \times S$. Now $u := w_1 + \ldots + w_N$ is a solution defined on $(t-\delta, t+\delta) \times S$ as required. $\square$

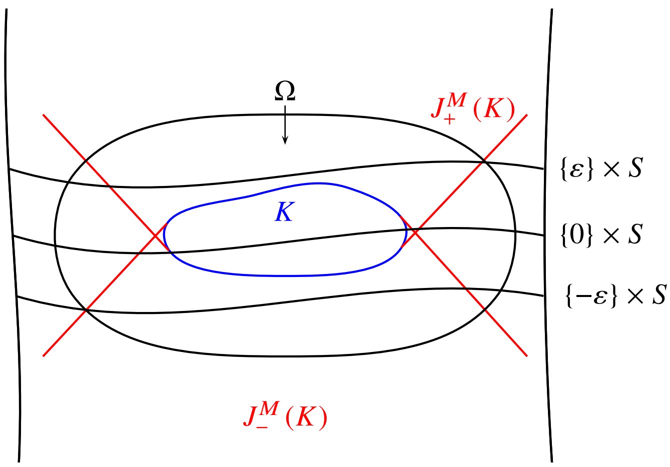
Now we are ready for the main theorem of this section.

Theorem 3.2.9. *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a spacelike Cauchy hypersurface. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a normally hyperbolic operator acting on sections in E.*
Then for each $u_0, u_1 \in \mathcal{D}(S, E)$ and for each $f \in \mathcal{D}(M, E)$ there exists a unique $u \in C^\infty(M, E)$ satisfying

$$\begin{cases} Pu &= f \quad \text{on } M, \\ u &= u_0 \quad \text{along } S, \\ \nabla_{\mathfrak{n}} u &= u_1 \quad \text{along } S, \end{cases}$$

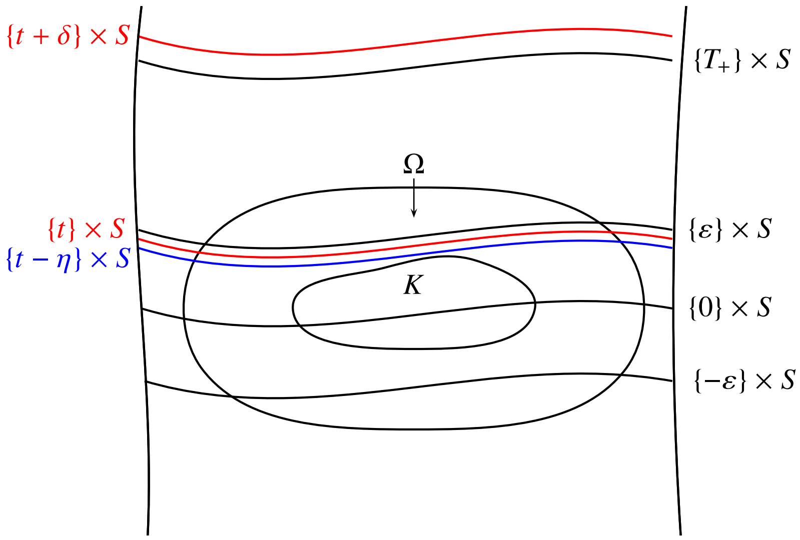
Moreover, $\operatorname{supp}(u) \subset J^M(K)$ *where* $K = \operatorname{supp}(u_0) \cup \operatorname{supp}(u_1) \cup \operatorname{supp}(f)$.

Proof. Uniqueness of the solution follows directly from Corollary 3.2.4.
The existence proof is done in two steps. First we use the local result for RCCSV-domains to obtain a solution on a spacelike strip $(-\varepsilon,\varepsilon)\times S$. In the second step we use Lemma 3.2.8 to show that this solution can be extended in time direction to all times t.
Step 1: We may w.l.o.g. assume that $K \subset \Omega$ where Ω is an RCCSV domain: Namely, let $u_0, u_1 \in \mathcal{D}(S,E)$ and $f \in \mathcal{D}(M,E)$. Using a partition of unity $(\chi_j)_{j=1,\dots,m}$ we can write $u_0 = u_{0,1} + \dots + u_{0,m}$, $u_1 = u_{1,1} + \dots + u_{1,m}$ and $f = f_1 + \dots + f_m$ where $u_{0,j} = \chi_j u_0$, $u_{1,j} = \chi_j u_1$, and $f_j = \chi_j f$. We may assume that each χ_j (and hence each $u_{i,j}$ and f_j) have support in an RCCSV-domain Ω_j. If we can solve the Cauchy problem on M for the data $(u_{0,j}, u_{1,j}, f_j)$, then we can add these solutions to obtain one for u_0, u_1, and f. Hence we can without loss of generality assume that there is an Ω as in Proposition 2.4.1 such that $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f) \subset \Omega$.
By Theorem 1.2.53 the spacetime M is isometric to $\mathbb{R}\times S$ with a Lorentzian metric of the form $-N^2dt^2 + g_t$ where S corresponds to $\{0\}\times S$, and each $S_t := \{t\}\times S$ is a spacelike Cauchy hypersurface in M. Let u be the solution on Ω as asserted by Proposition 2.4.1. In particular, $\mathrm{supp}(u) \subset J^M(K)$. By choosing the partition of unity $(\chi_j)_j$ appropriately we can assume that K is so small that there exists an $\varepsilon > 0$ such that $((-\varepsilon,\varepsilon)\times S)\cap J^M(K) \subset \Omega$ and $K \subset (-\varepsilon,\varepsilon)\times S$.



Hence we can extend u by 0 to a smooth solution on all of $(-\varepsilon,\varepsilon)\times S$.
Step 2: Let u_i be an extension of u to a smooth solution on $(-\varepsilon, T_i)\times S$ with support contained in $J^M(K)$. First we see that if we have two extensions u_1 and u_2 for $T_1 < T_2$, then the restriction of u_2 to $(-\varepsilon, T_1)\times S$ must coincide with u_1. This follows by uniqueness since both solve $Pu_i = f$ on $(-\varepsilon, T_1)\times S$ and have the same initial data for a $t < T_1$ Note here that Corollary 3.2.4 applies because $(-\epsilon, T_1)\times S$ is a globally hyperbolic manifold in its own right.
Now let T_+ be the supremum of all T for which u can be extended to a smooth solution on $(-\varepsilon, T)\times S$ with support contained in $J^M(K)$. If we show $T_+ = \infty$ we obtain a solution on $(-\varepsilon,\infty)\times S$. Similarly considering the corresponding infimum T_- then yields a solution on all of $M = \mathbb{R}\times S$.
Assume that $T_+ < +\infty$. Put $\hat{K} := ([-\epsilon, T_+]\times S)\cap J^M(K)$. By Proposition 1.2.56 $\hat{K}$ is compact.

Apply Lemma 3.2.8 to $\hat{K}$ and get $\delta > 0$ such that for each $t \in \mathbb{R}$ there is a smooth solution w of $Pw = 0$ defined on $(t - \delta, t + \delta) \times S$ satisfying $w|_{S_t} = u|_{S_t}$ and $\nabla_{\mathfrak{n}} w|_{S_t} = \nabla_{\mathfrak{n}} u|_{S_t}$. Fix $t < T_+$ such that $T_+ - t < \delta$ and still $K \subset (-\epsilon, t) \times S$.



On $(t - \eta, t + \delta) \times S$ the section f vanishes with $\eta > 0$ small enough.
Thus w coincides with u on $(t-\eta, t)\times S$. Here again, Corollary 3.2.4 applies because $(t-\eta, t+\delta)\times S$ is a globally hyperbolic manifold in its own right. Hence w extends the solution u smoothly to $(-\varepsilon, t + \delta) \times S$. The support of this extension is still contained in $J^M(K)$ because

$$\mathrm{supp}\,(w|_{[t,t+\delta)\times S}) \subset J_+^M\,(\mathrm{supp}(u|_{S_t}) \cup \mathrm{supp}(\nabla_{\mathfrak{n}} u|_{S_t})) \subset J_+^M(\hat{K} \cap S_t) \subset J_+^M(J_+^M(K)) = J_+^M(K).$$

Since $T_+ < t + \delta$ this contradicts the maximality of T_+. Therefore $T_+ = +\infty$. Similarly, one sees $T_- = -\infty$ which concludes the proof. □

Next we see that the solution to the Cauchy problem depends continuously on the data.

Theorem 3.2.10. *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a spacelike Cauchy hypersurface. Let $\mathfrak{n}$ be the future directed timelike unit normal field along S. Let E be vector bundle over M and let P be a normally hyperbolic operator acting on sections in E.*

Then the map $\Phi : \mathcal{D}(M,E) \oplus \mathcal{D}(S,E) \oplus \mathcal{D}(S,E) \to C^\infty(M,E)$ sending (f, u_0, u_1) to the unique solution u of the Cauchy problem $Pu = f$, $u|_S = u|_0$, $\nabla_{\mathfrak{n}} u = u_1$ is linear continuous.

Proof. First we look at the opposite direction and see that the map

$$\begin{aligned}\mathcal{P} : C^\infty(M,E) &\to C^\infty(M,E) \oplus C^\infty(S,E) \oplus C^\infty(S,E),\\ u &\mapsto (Pu, u|_S, \nabla_{\mathfrak{n}} u),\end{aligned}$$

is obviously linear and continuous. We want to use the open mapping theorem but for this we need a linear bijection between Fréchet spaces. To obtain this we first fix a compact subset $K \subset M$ and set

$$\begin{aligned}\mathcal{D}_K(M,E) &:= \{f \in \mathcal{D}(M,E) \mid \operatorname{supp}(f) \subset K\},\\ \mathcal{D}_K(S,E) &:= \{v \in \mathcal{D}(S,E) \mid \operatorname{supp}(v) \subset K \cap S\}, \text{ and}\\ \mathcal{V}_K &:= \mathcal{P}^{-1}(\mathcal{D}_K(M,E) \oplus \mathcal{D}_K(S,E) \oplus \mathcal{D}_K(S,E)).\end{aligned}$$

Since $\mathcal{D}_K(M,E) \subset C^\infty(M,E)$ and $\mathcal{D}_K(S,E) \subset C^\infty(S,E)$ are closed subsets they are Fréchet spaces and therefore $\mathcal{D}_K(M,E) \oplus \mathcal{D}_K(S,E) \oplus \mathcal{D}_K(S,E)$ is a Fréchet space as well. Hence $\mathcal{V}_K$ is a Fréchet space as the preimage under the continuous map $\mathcal{P}$. Thus $\mathcal{P} : \mathcal{V}_K \to \mathcal{D}_K(M,E) \oplus \mathcal{D}_K(S,E) \oplus \mathcal{D}_K(S,E)$ is a linear, continuous and bijective map between Fréchet spaces. By the open mapping theorem [14, Thm. V.6, p. 132] the inverse mapping $\mathcal{P}^{-1} : \mathcal{D}_K(M,E) \oplus \mathcal{D}_K(S,E) \oplus \mathcal{D}_K(S,E) \to \mathcal{V}_K \subset C^\infty(M,E)$ is continuous as well.
Now let $(f_j, u_{0,j}, u_{1,j}) \to (f, u_0, u_1)$ in $\mathcal{D}(M,E) \oplus \mathcal{D}(S,E) \oplus \mathcal{D}(S,E)$. Then we can choose a compact subset $K \subset M$ such that $(f_j, u_{0,j}, u_{1,j}) \to (f, u_0, u_1)$ in $\mathcal{D}_K(M,E) \oplus \mathcal{D}_K(S,E) \oplus \mathcal{D}_K(S,E)$ (up to finitely many members of the sequence). Hence we see

$$\Phi(f_j, u_{0,j}, u_{1,j}) = \mathcal{P}^{-1}(f_j, u_{0,j}, u_{1,j}) \to \mathcal{P}^{-1}(f, u_0, u_1) = \Phi(f, u_0, u_1)$$

which yields the continuity of Φ. □

3.3 Fundamental solutions

Using the knowledge about the Cauchy problem which we obtained in the previous section it is now not hard to find global fundamental solutions on a globally hyperbolic manifold.

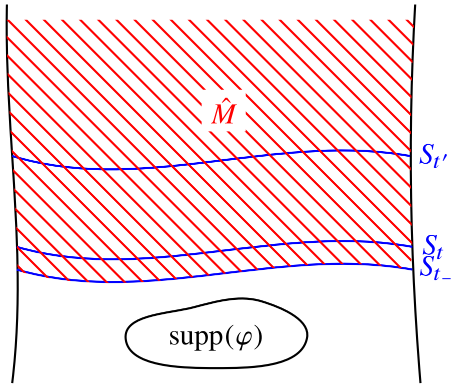
Theorem 3.3.1. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*
Then for every $x \in M$ there is exactly one fundamental solution $F_+(x)$ for P at x with past compact support and exactly one fundamental solution $F_-(x)$ for P at x with future compact support. They satisfy

1. $\operatorname{supp}(F_\pm(x)) \subset J^M_\pm(x)$,

2. *for each $\varphi \in \mathcal{D}(M, E^*)$ the maps $x \mapsto F_\pm(x)[\varphi]$ are smooth sections in E^* satisfying the differential equation $P^*(F_\pm(\cdot)[\varphi]) = \varphi$.*

Proof. Uniqueness of the fundamental solutions is a consequence of Corollary 3.1.2. To show existence fix a foliation of M by spacelike Cauchy hypersurfaces S_t, $t \in \mathbb{R}$ as in Theorem 1.2.45. Let $\mathfrak{n}$ be the future directed unit normal field along the leaves S_t. Let $\varphi \in \mathcal{D}(M, E^*)$. Choose t so large that $\mathrm{supp}(\varphi) \subset I_-^M(S_t)$. By Theorem 3.2.9 there exists a unique $\chi_\varphi \in C^\infty(M, E^*)$ such that $P^*\chi_\varphi = \varphi$ and $\chi_\varphi|_{S_t} = (\nabla_{\mathfrak{n}}\chi_\varphi)|_{S_t} = 0$. By Theorem 3.2.10 χ_φ depends continuously on φ.

We check that χ_φ does not depend on the choice of t. Let $t < t'$ be such that $\mathrm{supp}(\varphi) \subset I_-^M(S_t) \subset I_-^M(S_{t'})$. Let χ_φ and χ'_φ be the corresponding solutions. Choose $t_- < t$ so that still $\mathrm{supp}(\varphi) \subset I_-^M(S_{t_-})$. The open subset $\hat{M} := \bigcup_{\tau > t_-} S_\tau \subset M$ is a globally hyperbolic Lorentzian manifold itself. Now χ'_φ satisfies $P^*\chi'_\varphi = 0$ on $\hat{M}$ with vanishing Cauchy data on $S_{t'}$. By Corollary 3.2.4 $\chi'_\varphi = 0$ on $\hat{M}$. In particular, χ'_φ has vanishing Cauchy data on S_t as well. Thus $\chi_\varphi - \chi'_\varphi$ has vanishing Cauchy data on S_t and solves $P^*(\chi_\varphi - \chi'_\varphi) = 0$ on all of M. Again by Corollary 3.2.4 we conclude $\chi_\varphi - \chi'_\varphi = 0$ on M.



Fix $x \in M$. We define $F_+(x)$ as the composition

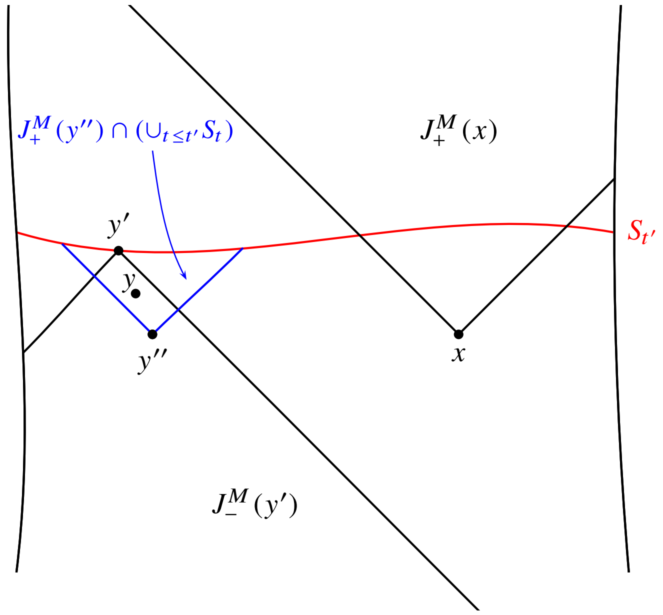
$$\mathcal{D}(M, E^*) \to C^\infty(M, E^*) \to E_x^*,$$
$$\varphi \mapsto \chi_\varphi \mapsto \chi_\varphi(x).$$

We already noted that χ_φ depends continuously on φ. The evaluation map $C^\infty(M, E) \to E_x$ is continuous too, hence the map $\mathcal{D}(M, E^*) \to E_x^*$, $\varphi \mapsto \chi_\varphi(x)$, is also continuous. Thus $F_+(x)$ defines a distribution. By definition $P^*(F_+(\cdot)[\varphi]) = P^*\chi_\varphi = \varphi$.
Now $P^*\chi_{P^*\varphi} = P^*\varphi$, hence $P^*(\chi_{P^*\varphi} - \varphi) = 0$. Since both $\chi_{P^*\varphi}$ and φ vanish along S_t we conclude from Corollary 3.2.4 $\chi_{P^*\varphi} = \varphi$. Thus

$$(PF_+(x))[\varphi] = F_+(x)[P^*\varphi] = \chi_{P^*\varphi}(x) = \varphi(x) = \delta_x[\varphi].$$

Hence $F_+(x)$ is a fundamental solution of P at x.
It remains to show $\mathrm{supp}(F_+(x)) \subset J_+^M(x)$. Let $y \in M \setminus J_+^M(x)$. We have to construct a neighborhood of y such that for each test section $\varphi \in \mathcal{D}(M, E^*)$ whose support is contained in

this neighborhood we have $F_+(x)[\varphi] = \chi_\varphi(x) = 0$. Since M is globally hyperbolic $J_+^M(x)$ is closed and therefore $J_+^M(x) \cap J_-^M(y') = \emptyset$ for all y' sufficiently close to y. We choose $y' \in I_+^M(y)$ and $y'' \in I_-^M(y)$ so close that $J_+^M(x) \cap J_-^M(y') = \emptyset$ and $\left(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\right) \cap J_+^M(x) = \emptyset$ where $t' \in \mathbb{R}$ is such that $y' \in S_{t'}$.



Now $K := J_-^M(y') \cap J_+^M(y'')$ is a compact neighborhood of y. Let $\varphi \in \mathcal{D}(M, E^*)$ be such that $\mathrm{supp}(\varphi) \subset K$. By Theorem 3.2.9 $\mathrm{supp}(\chi_\varphi) \subset J_+^M(K) \cup J_-^M(K) \subset J_+^M(y'') \cup J_-^M(y')$. By the independence of χ_φ of the choice of $t > t'$ we have that χ_φ vanishes on $\bigcup_{t > t'} S_t$. Hence $\mathrm{supp}(\chi_\varphi) \subset \left(J_+^M(y'') \cap \bigcup_{t \le t'} S_t\right) \cup J_-^M(y')$ and is therefore disjoint from $J_+^M(x)$. Thus $F_+(x)[\varphi] = \chi_\varphi(x) = 0$ as required. □

3.4 Green's operators

Now we want to find "solution operators" for a given normally hyperbolic operator P. More precisely, we want to find operators which are inverses of P when restricted to suitable spaces of sections. We will see that existence of such operators is basically equivalent to the existence of fundamental solutions.

Definition 3.4.1. Let M be a timeoriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. A linear map $G_+ : \mathcal{D}(M, E) \to C^\infty(M, E)$ satisfying

(i) $P \circ G_+ = \mathrm{id}_{\mathcal{D}(M,E)}$,

(ii) $G_+ \circ P|_{\mathcal{D}(M,E)} = \mathrm{id}_{\mathcal{D}(M,E)}$,

(iii) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M,E)$,

is called an **advanced Green's operator for** P. Similarly, a linear map $G_- : \mathcal{D}(M,E) \to C^\infty(M,E)$ satisfying (i), (ii), and

(iii') $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathcal{D}(M,E)$

instead of (iii) is called a **retarded Green's operator for** P.

Fundamental solutions and Green's operators are closely related.

Proposition 3.4.2. *Let M be a timeoriented connected Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*
If $F_\pm(x)$ is a family of advanced or retarded fundamental solutions for the adjoint operator P^* and if $F_\pm(x)$ depend smoothly on x in the sense that $x \mapsto F_\pm(x)[\varphi]$ is smooth for each test section φ and satisfies the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$, then

$$(G_\pm\varphi)(x) := F_\mp(x)[\varphi] \tag{3.6}$$

defines advanced or retarded Green's operators for P respectively. Conversely, given Green's operators $G_\pm$ for P, then (3.6) defines fundamental solutions for P^* depending smoothly on x and satisfying $P(F_\pm(\cdot)[\varphi]) = \varphi$ for each test section φ.

Proof. Let $F_\pm(x)$ be a family of advanced and retarded fundamental solutions for the adjoint operator P^* respectively. Let $F_\pm(x)$ depend smoothly on x and suppose the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$ holds. By definition we have

$$P(G_\pm\varphi) = P(F_\mp(\cdot)[\varphi]) = \varphi$$

thus showing (i). Assertion (ii) follows from the fact that the $F_\pm(x)$ are fundamental solutions,

$$G_\pm(P\varphi)(x) = F_\mp(x)[P\varphi] = P^*F_\mp(x)[\varphi] = \delta_x[\varphi] = \varphi(x).$$

To show (iii) let $x \in M$ such that $(G_+\varphi)(x) \neq 0$. Since $\mathrm{supp}(F_-(x)) \subset J_-^M(x)$ the support of φ must hit $J_-^M(x)$. Hence $x \in J_+^M(\mathrm{supp}(\varphi))$ and therefore $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$. The argument for G_- is analogous.
The converse is similar. □

Theorem 3.3.1 immediately yields

Corollary 3.4.3. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M.*
Then there exist unique advanced and retarded Green's operators $G_\pm : \mathcal{D}(M,E) \to C^\infty(M,E)$ for P. □

Lemma 3.4.4. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a normally hyperbolic operator acting on sections in a vector bundle E over M. Let $G_\pm$ be the Green's operators for P and $G^*_\pm$ the Green's operators for the adjoint operator P^*. Then*

$$\int_M (G^*_\pm \varphi) \cdot \psi \ \mathrm{dV} = \int_M \varphi \cdot (G_\mp \psi) \ \mathrm{dV} \tag{3.7}$$

holds for all $\varphi \in \mathcal{D}(M,E^*)$ and $\psi \in \mathcal{D}(M,E)$.

Proof. For the Green's operators we have $PG_\pm = \mathrm{id}_{\mathcal{D}(M,E)}$ and $P^*G^*_\pm = \mathrm{id}_{\mathcal{D}(M,E^*)}$ and hence

$$\begin{aligned} \int_M (G^*_\pm \varphi) \cdot \psi \ \mathrm{dV} &= \int_M (G^*_\pm \varphi) \cdot (PG_\mp \psi) \ \mathrm{dV} \\ &= \int_M (P^* G^*_\pm \varphi) \cdot (G_\mp \psi) \ \mathrm{dV} \\ &= \int_M \varphi \cdot (G_\mp \psi) \ \mathrm{dV}. \end{aligned}$$

Notice that $\mathrm{supp}(G_\pm \phi) \cap \mathrm{supp}(G_\mp \psi) \subset J^M_\pm(\mathrm{supp}(\phi)) \cap J^M_\mp(\mathrm{supp}(\psi))$ is compact in a globally hyperbolic manifold so that the partial integration in the second equation is justified. □

3.5 Support systems

In the following, let M always be globally hyperbolic and $E \to M$ be a $\mathbb{K}$-vector bundle with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
For a closed subset $A \subset M$ denote by $C^\infty_A(M,E)$ the space of all smooth sections f of E with $\mathrm{supp} f \subset A$. Then $C^\infty_A(M,E)$ is a closed subspace of $C^\infty(M,E)$. Moreover, if $A_1 \subset A_2$ then $C^\infty_{A_1}(M,E)$ is a closed subspace of $C^\infty_{A_2}(M,E)$.
We denote by C_M the set of all closed subsets of M.

Definition 3.5.1. A subset $\mathcal{A} \subset C_M$ is called a **support system** on M if the following holds:

(i) For any $A, A' \in \mathcal{A}$ we have $A \cup A' \in \mathcal{A}$;

(ii) For any $A \in \mathcal{A}$ there is an $A' \in \mathcal{A}$ such that A is contained in the interior of A';

(iii) If $A \in \mathcal{A}$ and $A' \subset A$ is a closed subset, then $A' \in \mathcal{A}$.

The first condition implies that $\mathcal{A}$ is a directed system with respect to inclusion. The third condition is harmless; if $\mathcal{A}$ satisfies (i) and (ii), then adding all closed subsets of the members of $\mathcal{A}$ to $\mathcal{A}$ will give a support system.
Given a support system on M we define

$$C^\infty_{\mathcal{A}}(M, E) := \bigcup_{A \in \mathcal{A}} C^\infty_A(M, E).$$

Due to (i) $C^\infty_{\mathcal{A}}(M, E)$ is a subspace of $C^\infty(M, E)$. The topology on $C^\infty_{\mathcal{A}}(M, E)$ is induced by its open convex subsets where a convex subset $O \subset C^\infty_{\mathcal{A}}(M, E)$ is open by definition if and only if $O \cap C^\infty_A(M, E)$ is open for all $A \in \mathcal{A}$. Note that $C^\infty_{\mathcal{A}}(M, E)$ is not a closed subspace of $C^\infty(M, E)$ in general.

Definition 3.5.2. We call a support system **essentially countable** if there is a sequence $A_1, A_2, A_3, \ldots \in \mathcal{A}$ such that each $A_j \subset A_{j+1}$ and for any $A \in \mathcal{A}$ there exists a j with $A \subset A_j$. Such a sequence $A_1 \subset A_2 \subset A_3 \subset \cdots$ is called a **basic chain** of $\mathcal{A}$.

Lemma 3.5.3. *Let $\mathcal{A}$ be an essentially countable support system on M. If $V \subset C^\infty_{\mathcal{A}}(M, E)$ is a bounded subset*[3] *then there exists an $A \in \mathcal{A}$ such that $V \subset C^\infty_A(M, E)$. In particular, for any convergent sequence $f_j \in C^\infty_{\mathcal{A}}(M, E)$ there exists an $A \in \mathcal{A}$ such that $f_j \in C^\infty_A(M, E)$ for all j.*

This shows that a sequence (f_j) converges in $C^\infty_{\mathcal{A}}(M, E)$ if and only if there exists an $A \in \mathcal{A}$ such that $f_j \in C^\infty_A(M, E)$ for all j and (f_j) converges in $C^\infty_A(M, E)$.

Proof of Lemma 3.5.3. Consider a basic chain $A_1 \subset A_2 \subset A_3 \subset \ldots$. Let $V \subset C^\infty_{\mathcal{A}}(M, E)$ be a subset not contained in any $C^\infty_{A_j}(M, E)$. We have to show that V is not bounded. Pick points $x_j \in M \setminus A_j$ and sections $f_j \in V$ with $f_j(x_j) \neq 0$. Define the convex set

$$W := \left\{ f \in C^\infty_{\mathcal{A}}(M, E) \,\middle|\, |f(x_j)| < \frac{|f_j(x_j)|}{j} \text{ for all } j \right\}.$$

Each $A \in \mathcal{A}$ contains only finitely many x_j. Thus $W \cap C^\infty_A(M, E) = \{f \in C^\infty_A(M, E) \mid \|f\|_{\{x_j\},0} < |f_j(x_j)|/j\}$ is open in $C^\infty_A(M, E)$. Therefore W is an open neighborhood of 0 in $C^\infty_{\mathcal{A}}(M, E)$.
For any $T > 0$ we have $T \cdot W = \{f \in C^\infty_{\mathcal{A}}(M, E) \mid |f(x_j)| < \frac{T}{j}|f_j(x_j)| \text{ for all } j\}$ and hence $f_j \notin TW$ for $j > T$. Thus V is not contained in any TW and is therefore not bounded. □

[3] A subset V is bounded if for any open neighborhood U of 0 there exists a $T > 0$ such that $V \subset T \cdot U$

Example 3.5.4. The system $\mathcal{A} = C_M$ of all closed subsets is an essentially countable support system on M. A basic chain is given by the constant sequence $M \subset M \subset M \subset \cdots$. Clearly, $C^\infty_{C_M}(M, E) = C^\infty(M, E)$.

Example 3.5.5. Let $\mathcal{A} = c$ where c is the set of all compact subsets of M. A basic chain can be constructed as follows: Provide M with a complete Riemannian metric γ. Fix a point $x \in M$. Now let A_j be the closed ball centered at x with radius j with respect to γ.
Then $C^\infty_c(M, E)$ is the space of compactly supported smooth sections.

Example 3.5.6. Let $\mathcal{A} = sc$ be the set of all spatially compact subsets of M. If $K_1 \subset K_2 \subset K_3 \subset \cdots$ is a basic chain of c, then $J(K_1) \subset J(K_2) \subset J(K_3) \subset \cdots$ is a basic chain of sc. Hence sc is essentially countable.
Now $C^\infty_{sc}(M, E)$ is the space of smooth sections with spatially compact support.

Example 3.5.7. Let $\mathcal{A} = spc$ be the set of all strictly past compact subsets of M. As in the previous example we see that spc is essentially countable. Now $C^\infty_{spc}(M, E)$ is the space of smooth sections with strictly past-compact support.
Similarly, one can define the space $C^\infty_{sfc}(M, E)$ of smooth sections with strictly future-compact support.

Example 3.5.8. Let $\mathcal{A} = pc$ be the set of all past-compact subsets. If M is spatially compact then $pc = spc$ by 1.2.69 but in general pc is strictly larger than spc. We obtain the space $C^\infty_{pc}(M, E)$ of smooth sections with past-compact support.
In general, the support system pc is not essentially countable. The following example was communicated to me by Miguel Sánchez. Let M be the $(1 + 1)$-dimensional Minkowski space. Let $A_1 \subset A_2 \subset A_3 \subset \cdots \subset M$ be a chain of past-compact subsets. Look at the "future-diverging" sequence of points $(n, 0) \in M$ and choose points[4] $p_n \in M \setminus (A_n \cup J_-(n, 0))$. By construction, $A := \{p_1, p_2, p_3, \ldots\}$ is not contained in any A_n but A is past compact. Namely, let $x \in M$. Then there exists an n such that $x \in J_-(n, 0)$. Now $J_-(x) \cap A \subset J_-(n, 0) \cap A$ is finite and hence compact. Thus no chain in pc captures all elements of pc, so pc is not essentially countable.

Example 3.5.9. A similar discussion as in the previous example yields the space $C^\infty_{fc}(M, E)$ of smooth sections with future-compact support and the space $C^\infty_{tc}(M, E)$ of smooth sections with temporally-compact support. Both support systems are not essentially countable in general. But again, if M is spatially compact, they are because then $fc = sfc$ and $tc = c$ by 1.2.70.

If $\mathcal{A} \subset \mathcal{A}'$, then $C^\infty_{\mathcal{A}}(M, E) \subset C^\infty_{\mathcal{A}'}(M, E)$ and the inclusion map is continuous. Hence we obtain the following diagram of continuously embedded spaces:

[4] Note that $M \setminus I^-(n, 0)$ is not past compact so that $A_n \cup J_-(n, 0)$ cannot be all of M.

$$\begin{tikzcd}
 & C^\infty_{spc}(M,E) \arrow[r, hook] \arrow[dr, hook] & C^\infty_{pc}(M,E) \arrow[dr, hook] & \\
C^\infty_c(M,E) \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & C^\infty_{tc}(M,E) \arrow[ur, hook] \arrow[dr, hook] & C^\infty_{sc}(M,E) \arrow[r, hook] & C^\infty(M,E) \\
 & C^\infty_{sfc}(M,E) \arrow[r, hook] \arrow[ur, hook] & C^\infty_{fc}(M,E) \arrow[ur, hook] &
\end{tikzcd}$$

Remark 3.5.10. Concerning the continuity of the maps, let X be a locally convex topological vector space. Then a linear map $f : C^\infty_{\mathcal{A}}(M,E) \to X$ is continuous if and only if the maps $f|_{C^\infty_A(M,E)}$ are continuous for all $A \in \mathcal{A}$.
Namely, f is continuous if and only if $f^{-1}(O)$ is open in $C^\infty_{\mathcal{A}}(M,E)$ for all open convex neighborhoods O at 0 in X. But this is equivalent to the fact that for all O the set $f^{-1}(O) \cap C^\infty_A(M,E)$ is open in $C^\infty_A(M,E)$ for all $A \in \mathcal{A}$. This just means that $f|_{C^\infty_A(M,E)}$ is continuous for all $A \in \mathcal{A}$.
In particular, choosing $f = \mathrm{id}$ shows that the embedding $C^\infty_A(M,E) \hookrightarrow C^\infty_{\mathcal{A}}(M,E)$ is continuous for every $A \in \mathcal{A}$.

Moreover, all embeddings in the diagram have dense image. Namely, we have

Lemma 3.5.11. *Let $\mathcal{A}$ be a support system on M such that $c \subset \mathcal{A}$, i.e., each compact set is contained in $\mathcal{A}$. Then $C^\infty_c(M,E)$ is a dense subspace of $C^\infty_{\mathcal{A}}(M,E)$.*

Proof. Let $f \in C^\infty_{\mathcal{A}}(M,E)$ and let O be a convex open neighborhood of f in $C^\infty_{\mathcal{A}}(M,E)$. Let $A \in \mathcal{A}$ with $f \in C^\infty_A(M,E)$. Since $O \cap C^\infty_A(M,E)$ is open in $C^\infty_A(M,E)$ there exists an $\epsilon > 0$ and a seminorm $\|\cdot\|_{K,m}$ such that

$$\{g \in C^\infty_A(M,E) \mid \|f - g\|_{K,m} < \epsilon\} \subset O \cap C^\infty_A(M,E).$$

Pick a cutoff function $\chi \in C^\infty_c(M,\mathbb{R})$ with $\chi \equiv 1$ on K. Then for $g := \chi \cdot f \in C^\infty_c(M,E)$ we find that $\|f - g\|_{K,m} = \|\chi \cdot f - f\|_{K,m} = 0$. Thus $g \in O \cap C^\infty_A(M,E)$. □

Definition 3.5.12. Two support systems $\mathcal{A}$ and $\mathcal{B}$ be on M are said to be **in duality** if for any $C \in \mathcal{C}_M$:

(i) $C \in \mathcal{A}$ if and only if $C \cap B$ is compact for all $B \in \mathcal{B}$;

(ii) $C \in \mathcal{B}$ if and only if $C \cap A$ is compact for all $A \in \mathcal{A}$.

Example 3.5.13. Here are some examples of support systems $\mathcal{A}$ and $\mathcal{B}$ in duality. The last column contains a justification of this fact.

$\mathcal{A}$	$\mathcal{B}$	why?
C_M	c	obvious
pc	sfc	Lemma 1.2.71 (i) and (v)
fc	spc	Lemma 1.2.71 (ii) and (iv)
tc	sc	Lemma 1.2.71 (iii) and (vi)

Support systems in duality

Now we turn to distributional sections with support in a support system.

Lemma 3.5.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two support systems on M in duality. Then a distributional section $f \in \mathcal{D}'(M,E)$ has support contained in $\mathcal{A}$ if and only if f extends to a continuous linear functional on $C^\infty_{\mathcal{B}}(M,E^*)$.*

Proof. a) Suppose first that $\mathrm{supp} f \in \mathcal{A}$. Let $B \in \mathcal{B}$. Since $\mathrm{supp} f \cap B$ is compact there is a cutoff function $\chi \in C^\infty_c(M,\mathbb{R})$ with $\chi \equiv 1$ on a neighborhood of $\mathrm{supp} f \cap B$. We extend f to a linear functional F_B on $C^\infty_B(M,E^*)$ by

$$F_B[\phi] := f[\chi\phi].$$

This extension is independent of the choice of χ because for another choice χ', f and $\chi\phi - \chi'\phi$ have disjoint supports. If $\phi_j \to 0$ in $C^\infty_B(M,E^*)$, then $\chi\phi_j \to 0$ in $C^\infty_c(M,E^*)$ and hence $F_B[\phi_j] = f[\chi\phi_j] \to 0$. Thus F_B is continuous.
Doing this for every $B \in \mathcal{B}$ we obtain an extension F of f to a linear functional on $C^\infty_{\mathcal{B}}(M,E^*)$ with F_B being the restriction of F to $C^\infty_B(M,E^*)$. Continuity of F holds because each F_B is continuous.
b) Conversely, assume that f extends to a continuous linear functional F on $C^\infty_{\mathcal{B}}(M,E^*)$. We check that $\mathrm{supp} f \in \mathcal{A}$ by showing that $\mathrm{supp} f \cap B$ is compact for every $B \in \mathcal{B}$.
Let $B \in \mathcal{B}$. Choose $B' \in \mathcal{B}$ such that B is contained in the interior of B'. Since the restriction $F_{B'}$ of F to $C^\infty_{B'}(M,E^*)$ is linear and continuous, there exists a seminorm $\|\cdot\|_{K,m}$ and a constant $C > 0$ such that

$$|F_{B'}[\phi]| \leq C \cdot \|\phi\|_{K,m}$$

for all $\phi \in C^\infty_{B'}(M,E^*)$. In particular, $F_{B'}[\phi] = 0$ if $\mathrm{supp}(\phi)$ and K are disjoint.
Claim: $B \cap (M \setminus K) \subset M \setminus \mathrm{supp}(F)$.
Namely, let $x \in B \cap (M \setminus K)$. Then x lies in the interior of B'. Hence there is an open neighborhood U of x entirely contained in B'. Since $x \notin K$ we may assume that U and K are disjoint. Now we know that for all $\phi \in C^\infty_c(M,E^*)$ with $\mathrm{supp}(\phi) \subset U$ we have $F[\phi] = 0$. Thus $x \notin \mathrm{supp}(F)$. ✓
The claim implies $\mathrm{supp}(F) \subset (M \setminus B) \cup K$ and hence $\mathrm{supp}(F) \cap B \subset K$. Therefore the intersection $\mathrm{supp}(F) \cap B$ is compact. □

Notation 3.5.15. Let $A \subset M$ be a closed subset and $\mathcal{A}$ be a support system on M. We define

$$\mathcal{D}'_A(M, E) := \{f \in \mathcal{D}'(M, E) \mid \operatorname{supp}(f) \subset A\}$$

and

$$\mathcal{D}'_{\mathcal{A}}(M, E) := \bigcup_{A \in \mathcal{A}} \mathcal{D}'_A(M, E)$$

For $\mathcal{A}$ being one of the support systems C, pc, fc, tc, sc, spc, sfc, or c we equip the spaces $\mathcal{D}'_{\mathcal{A}}(M, E)$ with the weak*-topology. This means that a sequence $f_j \in \mathcal{D}'_{\mathcal{A}}(M, E)$ converges if and only if $f_j[\phi]$ converges for every fixed $\phi \in C^\infty_{\mathcal{B}}(M, E^*)$, where $\mathcal{B}$ is the dual support system as in the table.
Note that if $\mathcal{A} \subset \mathcal{B}$ for two support systems $\mathcal{A}$ and $\mathcal{B}$ we obtain a continuous embedding

$$C^\infty_{\mathcal{A}}(M, E^*) \hookrightarrow C^\infty_{\mathcal{B}}(M, E^*).$$

For the dual spaces we obtain a continuous linear embedding

$$\mathcal{D}'_{\mathcal{B}}(M, E) \hookleftarrow \mathcal{D}'_{\mathcal{A}}(M, E).$$

Now Dualizing the diagram for smooth sections, our list of support systems in duality and Lemma 3.5.14 yield the following diagram of continuous embeddings of several spaces of distributions, characterized by different support properties:

$$\begin{tikzcd}
 & \mathcal{D}'_{fc}(M,E) \arrow[dl] & \mathcal{D}'_{sfc}(M,E) \arrow[l, hook'] \arrow[dl, hook'] & \\
\mathcal{D}'(M,E) & \mathcal{D}'_{sc}(M,E) \arrow[l, hook'] & \mathcal{D}'_{tc}(M,E) \arrow[ul, hook'] \arrow[dl, hook'] & \mathcal{D}'_{c}(M,E) \arrow[l, hook'] \arrow[ul, hook'] \arrow[dl, hook'] \\
 & \mathcal{D}'_{pc}(M,E) \arrow[ul, hook'] & \mathcal{D}'_{spc}(M,E) \arrow[l, hook'] \arrow[ul, hook'] &
\end{tikzcd}$$

Lemma 3.5.16. *Let $\mathcal{A}$ be one of the support systems C, pc, fc, tc, sc, spc, sfc, or c. Then $C^\infty_c(M, E)$ is a dense subspace of $\mathcal{D}'_{\mathcal{A}}(M, E)$.*

Proof. Let $\mathcal{B}$ be the dual support system to $\mathcal{A}$ as in the table. Let $u \in \mathcal{D}'_{\mathcal{A}}(M, E)$. Put $A := \operatorname{supp}(u)$, hence $u \in \mathcal{D}'_A(M, E)$. It is well known that $C^\infty_c(M, E)$ is dense in $\mathcal{D}'(M, E)$. Hence there is a sequence $u_j \in C^\infty_c(M, E)$ with $u_j \to u$ in $\mathcal{D}'(M, E)$.
Choose $A' \in \mathcal{A}$ such that A is contained in the interior of A'. Let $\chi \in C^\infty(M, \mathbb{R})$ be a function such that $\chi \equiv 1$ on A and $\operatorname{supp}\chi \subset A'$.

Let $\phi \in C^\infty_B(M, E^*)$ where $B \in \mathcal{B}$. Since $A' \cap B$ is compact, the section $\chi\phi$ has compact support. Therefore

$$(\chi u_j)[\phi] = u_j[\chi\phi] \to u[\chi\phi] = (\chi u)[\phi] = u[\phi].$$

Thus the compactly supported sections χu_j converge to u in $\mathcal{D}'_{\mathcal{A}}(M, E)$. □

3.6 Green-hyperbolic operators

We will now enlarge the class of differential operator significantly, from normally hyperbolic operator to Green-hyperbolic operators Let $E_1, E_2 \to M$ be vector bundles over a globally hyperbolic manifold. Let $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ be a linear differential operator.

Definition 3.6.1. An **advanced Green's operator** of P is a linear map $G_+ : C^\infty_c(M, E_2) \to C^\infty(M, E_1)$ such that

(i) $G_+ P f = f$ for all $f \in C^\infty_c(M, E_1)$;

(ii) $P G_+ f = f$ for all $f \in C^\infty_c(M, E_2)$;

(iii) $\mathrm{supp}(G_+ f) \subset J_+(\mathrm{supp} f)$ for all $f \in C^\infty_c(M, E_2)$.

A linear map $G_- : C^\infty_c(M, E_2) \to C^\infty(M, E_1)$ is called a **retarded Green's operator** of P if (i), (ii) hold and

(iii)' $\mathrm{supp}(G_- f) \subset J_-(\mathrm{supp} f)$ holds for every $f \in C^\infty_c(M, E_2)$.

Definition 3.6.2. The operator P is called **Green hyperbolic** if P and P^* have advanced and retarded Green's operators.

Example 3.6.3. Normally hyperbolic operators are Green hyperbolic by Corollary 3.4.3. Note here that the formal dual of a normally hyperbolic operator is again normally hyperbolic.

Example 3.6.4. Let $E = T^*M$ and $m > 0$. Then $P = \delta d + m^2$ is the **Proca operator**. The Proca operator is not normally hyperbolic but it is Green hyperbolic. To see this, we look at $\tilde{P} := d\delta + \delta d + m^2$, which is a normally hyperbolic operator and hence has Green's operators $\tilde{G}_\pm$. We set $Q := m^{-2} d\delta + \mathrm{id}$ and check that $G_\pm := Q \circ \tilde{G}_\pm$ are Green's operators of P. First note that $PQ = QP = \tilde{P}$. In particular $Q\tilde{P} = QPQ = \tilde{P}Q$, so Q commutes with $\tilde{P}$. Hence it also commutes with the $\tilde{G}_\pm$ (Exercise). Now we calculate

$$G_\pm P = Q\tilde{G}_\pm P = \tilde{G}_\pm QP = \tilde{G}_\pm \tilde{P} = \mathrm{id}$$

and

$$PG_\pm = PQ\tilde{G}_\pm = \tilde{P}\tilde{G}_\pm = \mathrm{id}.$$

Since the differential operator Q does not enlarge the supports, the support properties of $\tilde{G}_\pm$ directly pass to $G_\pm$. Hence the $G_\pm$ are Green's operators of P. Similarly, one gets Green's operators for P^*.

Green hyperbolicity persists under restriction to suitable subregions of the manifold M.

Lemma 3.6.5. *Let M be globally hyperbolic and let $N \subset M$ be an open subset which is causally compatible and globally hyperbolic. Then the restriction of P to N is again Green hyperbolic.*

Proof. We construct an advanced Green's operator for the restriction $P|_N$ of P to N. The construction of the retarded Green's operator and the ones for P^* are analogous. Denote by $\mathrm{ext} : C_c^\infty(N, E_2|_N) \to C_c^\infty(M, E_2)$ the extension-by-zero operator and by $\mathrm{res} : C^\infty(M, E_1) \to C^\infty(N, E_1|_N)$ the restriction operator. Let $G_+ : C_c^\infty(M, E_2) \to C^\infty(M, E_1)$ be the advanced Green's operator of P. We claim that

$$G_+^N := \mathrm{res} \circ G_+ \circ \mathrm{ext} : \quad C_c^\infty(N, E_2|_N) \to C^\infty(N, E_1|_N)$$

is an advanced Green's operator of $P|_N$. Since differential operators commute with restrictions and extensions we easily check for $f \in C_c^\infty(N, E_i|_N)$:

$$P|_N(G_+^N f) = \mathrm{res} \circ P \circ G_+ \circ \mathrm{ext} f = \mathrm{res} \circ \mathrm{ext} f = f$$

and

$$G_+^N(P|_N f) = \mathrm{res} \circ G_+ \circ \mathrm{ext} \circ \mathrm{res} \circ P \circ \mathrm{ext} f = \mathrm{res} \circ G_+ \circ P \circ \mathrm{ext} f = \mathrm{res} \circ \mathrm{ext} f = f.$$

This shows (i) and (ii) in Definition 3.6.1. As to (iii) we see

$$\begin{aligned} \mathrm{supp}(G_+^N f) &= \mathrm{supp}(\mathrm{res} \circ G_+ \circ \mathrm{ext} f) = \mathrm{supp}(G_+ \circ \mathrm{ext} f) \cap N \\ &\subset J_+^M(\mathrm{supp}(\mathrm{ext} f)) \cap N = J_+^M(\mathrm{supp} f) \cap N = J_+^N(\mathrm{supp} f). \end{aligned}$$

In the last equality we used that N is causally compatible. □

Definition 3.6.6. Let $G_\pm$ be advanced and retarded Green's operators of P. Then

$$G := G_+ - G_- : C_c^\infty(M, E_2) \to C^\infty(M, E_1)$$

is called the **causal propagator**.

From the support properties for the advanced and retarded Green's operators ((iii) and (iii)' in Definition 3.6.1), namely

$$\begin{aligned}\operatorname{supp}(G_+ f) &\subset J_+(\operatorname{supp} f)\\ \operatorname{supp}(G_- f) &\subset J_-(\operatorname{supp} f)\end{aligned}$$

for all $f \in C_c^\infty(M, E_2)$ we see that the Green's operators of P give rise to linear maps

$$\begin{aligned}G_+ : C_c^\infty(M, E_2) &\to C_{spc}^\infty(M, E_1),\\ G_- : C_c^\infty(M, E_2) &\to C_{sfc}^\infty(M, E_1),\\ G : C_c^\infty(M, E_2) &\to C_{sc}^\infty(M, E_1).\end{aligned}$$

This motivates the following extensions:

Theorem 3.6.7. *There are unique linear extensions*

$$\overline{G}_+ : C_{pc}^\infty(M, E_2) \to C_{pc}^\infty(M, E_1) \quad \text{and} \quad \overline{G}_- : C_{fc}^\infty(M, E_2) \to C_{fc}^\infty(M, E_1)$$

of G_+ and G_- respectively, such that

(i) $\overline{G}_+ P f = f$ *for all* $f \in C_{pc}^\infty(M, E_1)$;

(ii) $P\overline{G}_+ f = f$ *for all* $f \in C_{pc}^\infty(M, E_2)$;

(iii) $\operatorname{supp}(\overline{G}_+ f) \subset J_+(\operatorname{supp} f)$ *for all* $f \in C_{pc}^\infty(M, E_2)$;

and similarly for $\overline{G}_-$.

Proof. We only consider $\overline{G}_+$, the proof for $\overline{G}_-$ being analogous.
a) Let $f \in C_{pc}^\infty(M.E)$. Given $x \in M$ we define $(\overline{G}_+ f)(x)$ as follows: Since $J_-(x) \cap \operatorname{supp} f$ is compact we can choose a cutoff function $\chi \in C_c^\infty(M, \mathbb{R})$ with $\chi \equiv 1$ on a neighborhood of $J_-(x) \cap \operatorname{supp} f$. Now we put

$$(\overline{G}_+ f)(x) := (G_+(\chi f))(x). \tag{3.8}$$

b) The definition in (3.8) is independent of the choice of χ. Namely, let χ' be another such cutoff function. It suffices to show $x \notin \operatorname{supp}(G_+((\chi - \chi')f))$. If $x \in \operatorname{supp}(G_+((\chi - \chi')f)) \subset J_+(\operatorname{supp}((\chi - \chi')f))$ then there would be a causal curve from $\operatorname{supp}((\chi - \chi')f)$ to x. Hence $\operatorname{supp}((\chi - \chi')f) \cap J_-(x)$ would be nonempty. On the other hand,

$$\begin{aligned}\operatorname{supp}((\chi - \chi')f) \cap J_-(x) &= \operatorname{supp}(\chi - \chi') \cap \operatorname{supp} f \cap J_-(x)\\ &\subset \operatorname{supp}(\chi - \chi') \cap \{\chi \equiv \chi' \equiv 1\}\\ &= \emptyset,\end{aligned}$$

a contradiction.

c) The section $\overline{G}_+ f$ is smooth. Namely, a cutoff function χ for $x \in M$ also works for all $x' \in J_-(x)$ simply because $J_-(x') \subset J_-(x)$. In particular, on the open set $I^-(x)$ we have $\overline{G}_+ f = G_+(\chi f)$ for a fixed χ. Hence $\overline{G}_+ f$ is smooth on $I^-(x)$. Since any point in M is contained in $I^-(x)$ for some x, $\overline{G}_+ f$ is smooth on M.
d) The operator $\overline{G}_+$ is linear. The only issue here is additivity. Let $f_1, f_2 \in C^\infty_{pc}(M, E_2)$. Then $\mathrm{supp}(f_1) \cap J_-(x)$ and $\mathrm{supp}(f_2) \cap J_-(x)$ are both compact and we may choose the cutoff function χ such that $\chi \equiv 1$ on neighborhoods of both $\mathrm{supp}(f_1) \cap J_-(x)$ and $\mathrm{supp}(f_2) \cap J_-(x)$. Then $\chi \equiv 1$ on a neighborhood of $\mathrm{supp}(f_1 + f_2) \cap J_-(x)$ and we get

$$\begin{aligned}(\overline{G}_+(f_1 + f_2))(x) &= (G_+(\chi f_1 + \chi f_2))(x) \\ &= (G_+(\chi f_1)(x) + (G_+(\chi f_2))(x) \\ &= (\overline{G}_+ f_1)(x) + (\overline{G}_+ f_2))(x).\end{aligned}$$

e) For $\overline{G}_+$, properties (i), (ii) and (iii) hold: Let $x \in M$ and χ a cutoff function which is identically $\equiv 1$ on a neighborhood of $\mathrm{supp} f \cap J_-(x)$. In particular, we may choose $\chi \equiv 1$ on a neighborhood of x. Then

$$(P\overline{G}_+ f)(x) = (PG_+(\chi f))(x) = (\chi f)(x) = f(x).$$

This shows (ii). Moreover,

$$\begin{aligned}(\overline{G}_+ P f)(x) &= (G_+(\chi \cdot P f))(x) \\ &= (G_+ P(\chi f))(x) + (G_+([\chi, P] f))(x) \\ &= f(x) + (G_+([\chi, P] f))(x).\end{aligned}$$

In order to prove (i) we have to show $x \notin \mathrm{supp}(G_+([\chi, P] f))$. The coefficients of the differential operator $[\chi, P]$ vanish where $\chi \equiv 1$, hence in particular on $\mathrm{supp} f \cap J_-(x)$. Now we find

$$\begin{aligned}\mathrm{supp}(G_+([\chi, P] f)) &\subset J_+(\mathrm{supp}([\chi, P] f)) \\ &\subset J_+(\mathrm{supp} f \setminus J_-(x)) \\ &\subset J_+(\mathrm{supp} f) \setminus \{x\}\end{aligned}$$

and therefore $x \notin \mathrm{supp}(G_+([\chi, P] f))$.
As to (iii) we see for $f \in C^\infty_{pc}(M, E_2)$

$$\mathrm{supp}(\overline{G}_+ f) \subset \bigcup_\chi \mathrm{supp}(G_+(\chi f)) \subset \bigcup_\chi J_+(\mathrm{supp}(\chi f)) \subset J_+(\mathrm{supp} f).$$

Here the union is taken over all $\chi \in C^\infty_c(M, \mathbb{R})$.
f) Since the causal future of a past-compact set is again past compact, (iii) shows that $\overline{G}_+$ maps sections with past-compact support to sections with past-compact support.
g) Now (i) and (ii) show that P considered as an operator $C^\infty_{pc}(M, E_1) \to C^\infty_{pc}(M, E_2)$ is bijective and that $\overline{G}_+$ is its inverse, i.e.

$$\overline{G}_+ = \left(P|_{C^\infty_{pc}(M,E_1)} : C^\infty_{pc}(M, E_1) \to C^\infty_{pc}(M, E_2)\right)^{-1}.$$

In particular, $\overline{G}_+$ is uniquely determined. $\square$

Corollary 3.6.8. *There are no nontrivial solutions $f \in C^\infty(M, E_1)$ of the differential equation $Pf = 0$ with past-compact or future-compact support. For any $g \in C^\infty_{pc}(M, E_2)$ or $g \in C^\infty_{fc}(M, E_2)$ there exists a unique $f \in C^\infty(M, E_1)$ solving $Pf = g$ and such that $\mathrm{supp}(f) \subset J_+(\mathrm{supp}(g))$ or $\mathrm{supp}(f) \subset J_-(\mathrm{supp}(g))$, respectively.*

Since the causal future of a strictly past-compact set is again strictly past compact we can restrict $\overline{G}_+$ to smooth sections with strictly past-compact support and we get

Corollary 3.6.9. *There are unique linear extensions*

$$\tilde{G}_+ : C^\infty_{spc}(M, E_2) \to C^\infty_{spc}(M, E_1) \quad \text{and} \quad \tilde{G}_- : C^\infty_{sfc}(M, E_2) \to C^\infty_{sfc}(M, E_1)$$

of G_+ and G_- respectively, such that

(i) $\tilde{G}_+ Pf = f$ *for all* $f \in C^\infty_{spc}(M, E_1)$;

(ii) $P\tilde{G}_+ f = f$ *for all* $f \in C^\infty_{spc}(M, E_2)$;

(iii) $\mathrm{supp}(\tilde{G}_+ f) \subset J_+(\mathrm{supp} f)$ *for all* $f \in C^\infty_{spc}(M, E_2)$;

and similarly for $\tilde{G}_-$.

Corollary 3.6.10. *The Green's operators $G_\pm : C^\infty_c(M, E_2) \to C^\infty(M, E_1)$ of a Green-hyperbolic operator P are unique.*

Proof. The advanced Green's operator G_+ is a restriction of the operator $\overline{G}_+$ which is uniquely determined by P (as the inverse of $P : C^\infty_{pc}(M, E) \to C^\infty_{pc}(M, E)$). In other words, we obtain the advanced Green's operator G_+ of P by composing the following maps

$$C^\infty_c(M, E_2) \hookrightarrow C^\infty_{pc}(M, E_2) \xrightarrow{P^{-1}} C^\infty_{pc}(M, E_1) \hookrightarrow C^\infty(M, E_1).$$

Similar arguments show uniqueness of G_-. □

Corollary 3.6.11. *Let $P_1 : C^\infty(M, E_1) \to C^\infty(M, E_2)$ and $P_2 : C^\infty(M, E_2) \to C^\infty(M, E_3)$ be Green hyperbolic. Then $P_2 \circ P_1 : C^\infty(M, E_1) \to C^\infty(M, E_3)$ is Green hyperbolic.*

Proof. Denote the Green's operators of P_i by $G^i_\pm$. We obtain an advanced Green's operator of

$P_2 \circ P_1$ by composing the following maps:

$$C_c^\infty(M, E_3) \hookrightarrow C_{pc}^\infty(M, E_3) \xrightarrow{\overline{G}_+^2} C_{pc}^\infty(M, E_2) \xrightarrow{\overline{G}_+^1} C_{pc}^\infty(M, E_2) \hookrightarrow C^\infty(M, E_1)$$

and similarly for the retarded Green's operator and the dual operator. □

There is a very useful partial inverse to Corollary 3.6.11.

Corollary 3.6.12. *Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a differential operator such that P^2 is Green hyperbolic. Then P itself is Green hyperbolic.*

Proof. Theorem 3.6.7 applied to P^2 tells us that P^2 maps $C_{pc}^\infty(M, E)$ bijectively onto itself. Hence P itself also maps $C_{pc}^\infty(M, E)$ bijectively onto itself. Let G_+ denote the composition $C_c^\infty(M, E) \hookrightarrow C_{pc}^\infty(M, E) \xrightarrow{P^{-1}} C_{pc}^\infty(M, E) \hookrightarrow C^\infty(M, E)$. Then G_+ obviously satisfies (i) and (ii) in Definition 3.6.1.

As to (iii), let $f \in C_c^\infty(M, E)$. Put $A := J_+(\mathrm{supp} f) \in pc$. Again by Theorem 3.6.7, P^2 maps $C_A^\infty(M, E)$ bijectively onto itself. Hence so does P which implies that G_+ maps $C_A^\infty(M, E)$ bijectively onto itself. In particular, $\mathrm{supp}(G_+ f) \subset A = J_+(\mathrm{supp} f)$.

The arguments for G_- and for P^* are analogous. □

Definition 3.6.13. A differential operator $P \in \mathscr{Diff}_1(E, E)$ of first order is said to be of **Dirac type** if P^2 is normally hyperbolic.

Remark 3.6.14. Since normally hyperbolic operators are Green hyperbolic, Corollary 3.6.12 tells us that Dirac-type operators are Green hyperbolic too.

The direct sum of two Green-hyperbolic operators is again Green hyperbolic.

Lemma 3.6.15. *Let $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ and $Q : C^\infty(M, E_1') \to C^\infty(M, E_2')$ be Green hyperbolic. Then the operator*

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} : C^\infty(M, E_1 \oplus E_1') \to C^\infty(M, E_2 \oplus E_2')$$

is also Green hyperbolic.

Proof. If $G_\pm$ and $G'_\pm$ are the Green's operators for P and Q respectively, then $\begin{pmatrix} G_\pm & 0 \\ 0 & G'_\pm \end{pmatrix}$ yields Green's operators for $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$. □

Remark 3.6.16. The simple construction in Lemma 3.6.15 shows that Green hyperbolicity cannot be read off the principal sympbol of the operator. For instance, P could be a normally hyperbolic operator and Q a Dirac-type operator. Then the total Green-hyperbolic operator in Lemma 3.6.15 is of second order and the principal symbol does not see Q and therefore cannot recognize Q as a Green hyperbolic operator.
For similar reasons, it is not clear how to characterize Green hyperbolicity in terms of well-posedness of a Cauchy problem in general.

Now we get the following variation of Corollary 3.6.12 for operators acting on sections of two different bundles:

Corollary 3.6.17. *Let $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ be a differential operator and let E_1 and E_2 carry nondegenerate (but possibly indefinite) fiber metrics. Let $P^t : C^\infty(M, E_2) \to C^\infty(M, E_1)$ be the formally adjoint operator.*
If $P^t P$ and PP^t are Green hyperbolic, then P and P^t are Green hyperbolic too.

Proof. Consider the operator $\mathcal{P} : C^\infty(M, E_1 \oplus E_2) \to C^\infty(M, E_1 \oplus E_2)$ defined by

$$\mathcal{P} = \begin{pmatrix} 0 & P^t \\ P & 0 \end{pmatrix}.$$

Since $P^t P$ and PP^t are Green hyperbolic so is

$$\mathcal{P}^2 = \begin{pmatrix} P^t P & 0 \\ 0 & PP^t \end{pmatrix}.$$

By Corollary 3.6.12, $\mathcal{P}$ is Green hyperbolic. Let

$$\mathcal{G}_\pm = \begin{pmatrix} G^{11}_\pm & G^{21}_\pm \\ G^{12}_\pm & G^{22}_\pm \end{pmatrix}$$

be the Green's operators of $\mathcal{P}$. Then one easily sees that $G^{21}_\pm$ are Green's operators for P and $G^{12}_\pm$ for P^t. □

Example 3.6.18. Consider the classical Dirac operator acting on sections of the spinor bundle $E = \Sigma M$ (see [4] for details). If M is even dimensional, then the spinor bundle splits into "chirality subbundles" $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. Then the Dirac operator interchanges these bundles and is given by

$$\begin{pmatrix} 0 & D^t \\ D & 0 \end{pmatrix}.$$

with operators $D : C^\infty(M, \Sigma^+ M) \to C^\infty(M, \Sigma^- M)$ and $D^t : C^\infty(M, \Sigma^- M) \to C^\infty(M, \Sigma^+ M)$. By Corollary 3.6.17, they are Green hyperbolic too.

Corollary 3.6.19. *The Green's operators $G_\pm : C_c^\infty(M, E_2) \to C^\infty(M, E_1)$ as well as the extensions*

$$\tilde{G}_+ : C_{spc}^\infty(M, E_2) \to C_{spc}^\infty(M, E_1), \quad \tilde{G}_- : C_{sfc}^\infty(M, E_2) \to C_{sfc}^\infty(M, E_1),$$
$$\overline{G}_+ : C_{pc}^\infty(M, E_2) \to C_{pc}^\infty(M, E_1), \quad \overline{G}_- : C_{fc}^\infty(M, E_2) \to C_{fc}^\infty(M, E_1)$$

are continuous.

Proof. a) $\overline{G}_+$ is continuous:
The operator $\overline{G}_+ : C_{pc}^\infty(M, E_2) \to C_{pc}^\infty(M, E_1)$ is the inverse of P when considered as an operator $C_{pc}^\infty(M, E_1) \to C_{pc}^\infty(M, E_2)$. If $A \in pc$, then also $J_+(A) \in pc$. Now $\overline{G}_+$ maps sections with support in $J_+(A)$ to sections with support in $J_+(J_+(A)) = J_+(A)$. Hence P yields a bijective linear operator $C_{J_+(A)}^\infty(M, E_1) \to C_{J_+(A)}^\infty(M, E_2)$ with inverse given by the restriction of $\overline{G}_+$ to $C_{J_+(A)}^\infty(M, E_2)$. By the open mapping theorem for Fréchet spaces $\overline{G}_+$ is continuous as a map $C_{J_+(A)}^\infty(M, E_2) \to C_{J_+(A)}^\infty(M, E_1)$. But now the operator $\overline{G}_+$ as a map $C_A^\infty(M, E_2) \to C_{pc}^\infty(M, E_1)$ is given by the following composition of continuous maps:

$$C_A^\infty(M, E_2) \hookrightarrow C_{J_+(A)}^\infty(M, E_2) \xrightarrow{\overline{G}_+|_{C_{J_+(A)}^\infty(M,E_2)}} C_{J_+(A)}^\infty(M, E_1) \hookrightarrow C_{pc}^\infty(M, E_1)$$

where we have the continuous embeddings $C_A^\infty(M, E_2) \subset C_{J_+(A)}^\infty(M, E_2)$ and $C_{J_+(A)}^\infty(M, E_1) \subset C_{pc}^\infty(M, E_1)$. Hence $\overline{G}_+$ as a map $C_A^\infty(M, E_2) \to C_{pc}^\infty(M, E_1)$ is continuous. Since this holds for any $A \in pc$, we conclude that $\overline{G}_+ : C_{pc}^\infty(M, E_2) \to C_{pc}^\infty(M, E_1)$ is continuous.
b) A similar argument shows that $\tilde{G}_+ : C_{spc}^\infty(M, E_2) \to C_{spc}^\infty(M, E_1)$ is continuous.
c) G_+ is continuous:
Using the continuous embeddings $C_c^\infty(M, E_2) \subset C_{spc}^\infty(M, E_2)$ and $C_{spc}^\infty(M, E_1) \subset C^\infty(M, E_1)$ we see that the Green's operator G_+ is given by the following composition:

$$C_c^\infty(M, E_2) \hookrightarrow C_{pc}^\infty(M, E_2) \xrightarrow{\tilde{G}_+} C_{pc}^\infty(M, E_1) \hookrightarrow C^\infty(M, E_1)$$

d) The same reasoning proves the claim for G_-, $\tilde{G}_-$, and $\overline{G}_-$. □

Next we show that the Green's operators of the dual operator are essentially the duals of the Green's operators. The roles of "advanced" and "retarded" get interchanged.

Lemma 3.6.20. *Let $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ be Green hyperbolic. Denote the Green's operators of P by $G_\pm$ and the ones of P^* by $G^*_\pm$. Then*

$$\int_M \langle \tilde{G}^*_- \varphi, f \rangle \, \mathrm{dV} = \int_M \langle \varphi, \overline{G}_+ f \rangle \, \mathrm{dV}$$

holds for all $\varphi \in C^\infty_{sfc}(M, E_1^*)$ and $f \in C^\infty_{pc}(M, E_2)$. Similarly,

$$\int_M \langle \tilde{G}^*_+ \varphi, f \rangle \, \mathrm{dV} = \int_M \langle \varphi, \overline{G}_- f \rangle \, \mathrm{dV}$$

holds for all $\varphi \in C^\infty_{spc}(M, E_1^*)$ and $f \in C^\infty_{fc}(M, E_2)$.

Proof. By (ii) in Theorem 3.6.7 we have

$$\begin{aligned}
\int_M \langle \tilde{G}^*_- \varphi, f \rangle \, \mathrm{dV} &= \int_M \langle \tilde{G}^*_- \varphi, P(\overline{G}_+ f) \rangle \, \mathrm{dV} \\
&= \int_M \langle P^*(\tilde{G}^*_- \varphi), \overline{G}_+ f \rangle \, \mathrm{dV} \\
&= \int_M \langle \varphi, \overline{G}_+ f \rangle \, \mathrm{dV}.
\end{aligned}$$

The integration by parts is justified because the intersection $\mathrm{supp}(\tilde{G}^*_- \varphi) \cap \mathrm{supp}(\overline{G}_+ f)$ of a strictly future-compact set and a past-compact set is compact. The second assertion is analogous. □

The fact that a Green-hyperbolic operator P is an isomorphism on smooth sections with past-compact support can be expressed by saying that

$$\{0\} \to C^\infty_{pc}(M, E_1) \xrightarrow{P} C^\infty_{pc}(M, E_2) \to \{0\}$$

is an exact sequence. Similarly ,

$$\begin{aligned}
\{0\} \to C^\infty_{fc}(M, E_1) &\xrightarrow{P} C^\infty_{fc}(M, E_2) \to \{0\}, \\
\{0\} \to C^\infty_{spc}(M, E_1) &\xrightarrow{P} C^\infty_{spc}(M, E_2) \to \{0\}, \\
\{0\} \to C^\infty_{sfc}(M, E_1) &\xrightarrow{P} C^\infty_{sfc}(M, E_2) \to \{0\}
\end{aligned}$$

are exact. The corresponding statement for the support systems c and sc is more complicated and is given by the following theorem.

Theorem 3.6.21. *Let G be the causal propagator of the Green-hyperbolic operator $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$. Then*

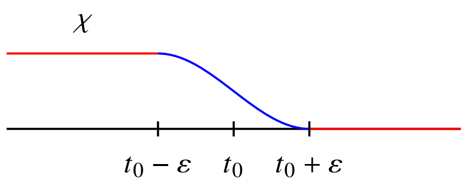
$$\{0\} \to C_c^\infty(M, E_1) \xrightarrow{P} C_c^\infty(M, E_2) \xrightarrow{G} C_{sc}^\infty(M, E_1) \xrightarrow{P} C_{sc}^\infty(M, E_2) \to \{0\} \tag{3.9}$$

is an exact sequence.

Proof. a) Exactness at $C_c^\infty(M, E_1)$:
We know from Theorem 3.6.7 (i) and (ii) that P considered as an operator $C_{pc}^\infty(M, E_1) \to C_{pc}^\infty(M, E_2)$ is bijective. Since $C_c^\infty(M, E_1) \subset C_{pc}^\infty(M, E_1)$ we conclude that P is injective on this smaller space too.
b) Exactness at $C_c^\infty(M, E_2)$:
Since $G_\pm \circ P = \mathrm{id}|_{C_c^\infty}$ we see that $G \circ P = 0$ on $C_c^\infty(M, E_1)$. Hence we conclude that $\mathrm{im}(P|_{C_c^\infty(M,E_1)}) \subset \ker(G|_{C_c^\infty(M,E_2)})$. Conversely, let $f \in \ker(G|_{C_c^\infty(M,E_2)})$. We define $u := G_- f = G_+ f \in C^\infty(M, E_1)$. Since f has compact support we see that $u = G_+ f \in C_{spc}^\infty$ and $u = G_- f \in C_{sfc}^\infty$. Hence u has strictly past and strictly future compact support. This means that u has compact support [5]. Therefore $Pu = PG_+ f = f$ which implies that $f \in \mathrm{im}(P|_{C_c^\infty(M,E_1)})$. This shows $\ker(G|_{C_c^\infty(M,E_2)}) \subset \mathrm{im}(P|_{C_c^\infty(M,E_1)}$.
c) Exactness at $C_{sc}^\infty(M, E_1)$:
First we see that $\mathrm{im}(G|_{C_c^\infty(M,E_2)}) \subset \ker(P|_{C_{sc}^\infty(M,E_1)})$ since

$$P \circ G = P \circ G_+ - P \circ G_- = \mathrm{id} - \mathrm{id} = 0.$$

Conversely, let $f \in \ker(P|_{C_{sc}^\infty(M,E_1)})$. Let $t : M \to \mathbb{R}$ be a Cauchy time function. Let t_0 and $\epsilon > 0$ be such that $[t_0 - \epsilon, t_0 + \epsilon] \subset t(M)$. Choose a function $\chi \in C^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $(-\infty, t_0 - \epsilon)$ and $\chi \equiv 0$ on $(t_0 + \epsilon, \infty)$.



Then $\mathrm{supp}(\chi \circ t) \in fc$ and $\mathrm{supp}((1 - \chi) \circ t) \in pc$. Set

$$f_1 := (\chi \circ t) \cdot f \in C_{sfc}^\infty(M, E_1)$$

and

$$f_2 := ((1 - \chi) \circ t) \cdot f \in C_{spc}^\infty(M, E_1).$$

[5]To see this let $\mathrm{supp}(u) \subset J_+(K_1)$ and $\mathrm{supp}(u) \subset J_-(K_2)$. Then $\mathrm{supp}(u) \subset J_+(K_1 \cup K_2) \cap J_-(K_1 \cup K_2)$ which is a compact set.

Then $f = f_1 + f_2$ and $0 = Pf = Pf_1 + Pf_2$, hence $Pf_1 = -Pf_2$. The support of $u := Pf_1 = -Pf_2$ is both strictly future compact and strictly past compact, hence compact, $u \in C_c^\infty(M, E_2)$. Moreover,

$$Gu = G_+ P f_1 - G_- P f_1 = G_+ P f_1 + G_- P f_2 = f_1 + f_2 = f.$$

d) Exactness at $C_{sc}^\infty(M, E_2)$:
We have to show that P is surjective on $C_{sc}^\infty(M, E_2)$. Let f be in $C_{sc}^\infty(M, E_2)$. Again, decompose f as $f = f_1 + f_2$ with $f_1 \in C_{sfc}^\infty(M, E_2)$ and $f_2 \in C_{spc}^\infty(M, E_2)$. Set $u := G_- f_1 + G_+ f_2 \in C_{sc}^\infty(M, E_1)$. Then

$$Pu = PG_- f_1 + PG_+ f_2 = f_1 + f_2 = f. \qquad \square$$

We extend any differential operator $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ as usual to distributional sections by taking the dual map of $P^* : C_c^\infty(M, E_2^*) \to C_c^\infty(M, E_1^*)$ thus giving rise to a continuous linear map $P : \mathcal{D}'(M, E_1) \to \mathcal{D}'(M, E_2)$.

Lemma 3.6.22. *The Green's operators* $\overline{G}_+ : C_{pc}^\infty(M, E_2) \to C_{pc}^\infty(M, E_1)$ *and* $\overline{G}_- : C_{fc}^\infty(M, E_2) \to C_{fc}^\infty(M, E_1)$ *extend uniquely to continuous linear operators*

$$\widehat{G}_+ : \mathcal{D}'_{pc}(M, E_2) \to \mathcal{D}'_{pc}(M, E_1) \text{ and } \widehat{G}_- : \mathcal{D}'_{fc}(M, E_2) \to \mathcal{D}'_{fc}(M, E_1),$$

respectively. Moreover

(i) $\widehat{G}_+ P f = f$ *holds for all* $f \in \mathcal{D}'_{pc}(M, E_1)$;

(ii) $P\widehat{G}_+ f = f$ *holds for all* $f \in \mathcal{D}'_{pc}(M, E_2)$;

(iii) $\mathrm{supp}(\widehat{G}_+ f) \subset J_+(\mathrm{supp} f)$ *holds for all* $f \in \mathcal{D}'_{pc}(M, E_2)$;

and similiarly for $\widehat{G}_-$.

Proof. Recall from Lemma 3.5.14 and Example 3.5.13 that $\mathcal{D}'_{pc}(M, E_i)$ can be identified with the dual space of $C_{sfc}^\infty(M, E_i^*)$. Let G_-^* be the retarded Green's operator of P^*. We extend to $\tilde{G}_-^* : C_{sfc}^\infty(M, E_1^*) \to C_{sfc}^\infty(M, E_2^*)$. Now let $\widehat{G}_+$ be the dual map of $\tilde{G}_-^*$, namely for $u \in \mathcal{D}'_{pc}(M, E_2)$ and $\varphi \in C_{sfc}^\infty(M, E_1)$ set

$$(\widehat{G}_+ u)[\varphi] := u[\tilde{G}_-^* \varphi].$$

This defines a continuous linear map $\mathcal{D}'_{pc}(M, E_2) \to \mathcal{D}'_{pc}(M, E_1)$. By Lemma 3.6.20, $\widehat{G}_+$ is an extension of $\overline{G}_+$. The extension is unique because $C_c^\infty(M, E_2)$ is dense in $\mathcal{D}'_{pc}(M, E_2)$ by Lemma 3.5.16.
Dualizing (i) and (ii) for P^* and G_-^* in Corollary 3.6.9 we get (i) and (ii) as asserted. As to (iii) let $f \in \mathcal{D}'_{pc}(M, E_2)$ and let $\varphi \in C_c^\infty(M, E_2^*)$ be a test section such that $J_+(\mathrm{supp} f) \cap \mathrm{supp}(\varphi) = \emptyset$.

Then $\mathrm{supp} f \cap J_-(\mathrm{supp}(\varphi)) = \emptyset$ and therefore

$$(\widehat{G}_+ f)[\varphi] = f[\overline{G}_-^* \varphi] = 0.$$

Thus $\mathrm{supp}(\widehat{G}_+ f) \subset J_+(\mathrm{supp} f)$. $\square$

Summarizing Theorem 3.6.7, Corollary 3.6.9 and Lemma 3.6.22 we get the following diagram of continuous extensions of the Green's operator G_+ of P:

$$\begin{tikzcd}
C_c^\infty(M,E_2) \arrow[r, hook] \arrow[d, equal] \arrow[rrr, bend left, "G_+"] & C_{spc}^\infty(M,E_2) \arrow[r, "\tilde{G}_+"] \arrow[d, hook] & C_{spc}^\infty(M,E_1) \arrow[r, hook] \arrow[d, hook] & C^\infty(M,E_1) \arrow[d, equal] \\
C_c^\infty(M,E_2) \arrow[r, hook] \arrow[d, hook] & C_{pc}^\infty(M,E_2) \arrow[r, "\overline{G}_+"] \arrow[d, hook] & C_{pc}^\infty(M,E_1) \arrow[r, hook] \arrow[d, hook] & C^\infty(M,E_1) \arrow[d, hook] \\
\mathcal{D}'_c(M,E_2) \arrow[r, hook] & \mathcal{D}'_{pc}(M,E_2) \arrow[r, "\widehat{G}_+"] & \mathcal{D}'_{pc}(M,E_1) \arrow[r, hook] & \mathcal{D}'(M,E_1)
\end{tikzcd}$$

Using the restriction of $\widehat{G}_+$ to an operator $\mathcal{D}'_c(M,E_2) \to \mathcal{D}'_{spc}(M,E_1) \hookrightarrow \mathcal{D}'_{sc}(M,E_1)$ and $\widehat{G}_- :$ $\mathcal{D}'_c(M,E_2) \to \mathcal{D}'_{sc}(M,E_1)$ we obtain an extension of the causal propagator $G : C_c^\infty(M,E_2) \to$ $C_{sc}^\infty(M,E_1)$ to distributions:

$$\widehat{G} := \widehat{G}_+ - \widehat{G}_- : \mathcal{D}'_c(M,E_2) \to \mathcal{D}'_{sc}(M,E_1).$$

Now we get the analog to Theorem 3.6.21 with essentially the same proof.

Theorem 3.6.23. *The sequence*

$$\{0\} \to \mathcal{D}'_c(M,E_1) \xrightarrow{P} \mathcal{D}'_c(M,E_2) \xrightarrow{\widehat{G}} \mathcal{D}'_{sc}(M,E_1) \xrightarrow{P} \mathcal{D}'_{sc}(M,E_2) \to \{0\} \qquad (3.10)$$

is exact.

3.7 Symmetric hyperbolic systems

Now we consider an important class of operators of first order on Lorentzian manifolds, the symmetric hyperbolic systems. Let M be a timeoriented Lorentzian manifold. Let $E \to M$ be a real or complex vector bundle with a (possibly indefinite) nondegenerate sesquilinear fiber metric $\langle \cdot, \cdot \rangle$.

Definition 3.7.1. A linear differential operator $P : C^\infty(M, E) \to C^\infty(M, E)$ of first order is called a **symmetric hyperbolic system over** M if the following holds for every $x \in M$:

(i) The principal symbol $\sigma(P, \xi) : E_x \to E_x$ is symmetric or Hermitian with respect to $\langle\cdot,\cdot\rangle$ for every $\xi \in T_x^*M$;

(ii) For every future-directed timelike covector $\tau \in T_x^*M$, the bilinear form $\langle\sigma(P, \tau)\cdot,\cdot\rangle$ on E_x is positive definite.

The first condition relates the principal symbol of P to the fiber metric on E, the second relates it to the Lorentzian metric on M. The Lorentzian metric enters only via its conformal class because this suffices to specify the causal types of (co)vectors.

Example 3.7.2. Let $M = \mathbb{R}^{n+1}$ and denote generic elements of M by $x = (x^0, x^1, \dots, x^n)$. We provide M with the Minkowski metric $g = -(dx^0)^2 + (dx^1)^2 + \dots + (dx^n)^2$. The coordinate function $t = x^0/c : M \to \mathbb{R}$ is a Cauchy time function; here c is a positive constant to be thought of as the speed of light.
Let E be the trivial real or complex vector bundle of rank N over M and let $\langle\cdot,\cdot\rangle$ denote the standard Euclidean scalar product on the fibers of E, canonically identified with $\mathbb{K}^N$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Any linear differential operator $P : C^\infty(M, E) \to C^\infty(M, E)$ of first order is of the form

$$P = A_0(x)\frac{\partial}{\partial t} + \sum_{j=1}^{n} A_j(x)\frac{\partial}{\partial x^j} + B(x)$$

where the coefficients A_j and B are $N \times N$-matrices depending smoothly on x. Condition (i) in Definition 3.7.1 means that all matrices $A_j(x)$ are symmetric if $\mathbb{K} = \mathbb{R}$ and Hermitian if $\mathbb{K} = \mathbb{C}$. Condition (ii) with $\tau = dt$ means that $A_0(x)$ is in addition positive definite. Thus P is a symmetric hyperbolic system in the usual PDE sense, see e.g. [1, Def. 2.11]. But (ii) says more than that; it means that $A_0(x)$ dominates $A_1(x), \dots, A_n(x)$ in the following sense: The covector $\tau = dt + \sum_{j=1}^n \alpha_j dx^j$ is timelike if and only if $\sum_{j=1}^n \alpha_j^2 < c^{-2}$. Thus the matrix

$$\sigma(P, \tau) = A_0(x) + \sum_{j=1}^{n} \alpha_j A_j(x)$$

must be positive definite whenever $\sum_{j=1}^n \alpha_j^2 < c^{-2}$.

Example 3.7.3. Let N be an 3-dimensional oriented Riemannian manifold. Then the *Maxwell equations* are given by

$$\begin{aligned} \frac{\partial E}{\partial t} - \operatorname{rot} B &= J, & \frac{\partial B}{\partial t} + \operatorname{rot} E &= 0, \\ \operatorname{div} E &= \varrho, & \operatorname{div} B &= 0, \end{aligned}$$

where E, B and J are time-dependend vector fields on N and ϱ is a function on $\mathbb{R} \times N =: M$. Here J and ϱ are usually given and E and B are to be solved for.
We organize two of the four Maxwell equations into a differential operator P acting on sections in the vector bundle $\pi^*(TN \oplus TN)$ equipped with the Riemannian metric induced by N. Here $\pi : M \to N$ is the projection onto the second factor. We put

$$P \begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} & -\mathrm{rot} \\ \mathrm{rot} & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} .$$

First we calculate the principal symbol to check Condition (i) in Definition 3.7.1. For the timelike covector dt we find

$$\sigma(P, dt) = \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{pmatrix}.$$

For $\xi \in T^*N$ we first recall that for any first-order differential operator P the principal symbol is characterized by

$$P(f \cdot E) = f \cdot PE + \sigma(P, df)E.$$

In case of $P = \mathrm{rot}$ we hence have

$$f \cdot \mathrm{rot}\, E + \sigma(\mathrm{rot}, df)E = \mathrm{rot}(f \cdot E) = f \cdot \mathrm{rot}\, E + \mathrm{grad}\, f \times E$$

which leads to

$$\sigma(\mathrm{rot}, \xi) = \xi^\sharp \times .$$

For $\xi \in T_p^*N$ we choose a positively oriented orthonormal basis e_1, e_2, e_3 of T_pN such that $\xi^\sharp = x \cdot e_1$. Then the linear map $\xi^\sharp \times \cdot$ is given by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{pmatrix}$. Therefore for a spacelike covector $\xi \in T^*N$ the principal symbol is given by

$$\begin{pmatrix} & & & 0 & 0 & 0 \\ & 0 & & 0 & 0 & x \\ & & & 0 & -x & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & -x & & 0 & \\ 0 & x & 0 & & & \end{pmatrix}.$$

Thus the principal symbol is symmetric both for the covector dt and for all covectors perpendicular to dt. Linear combinations of such covectors yield all covectors on M, hence the principal symbol is symmetric for any covector.
Next we want to determine in which cases $\sigma(P, dt + \xi)$ is positive definite in order to check Condition (ii). We compute

$$
\left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & -x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x & 0 & 1 & 0 \\ 0 & x & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 + xu_6 \\ u_3 - xu_5 \\ u_4 \\ u_5 - xu_3 \\ u_6 + xu_2 \end{pmatrix} \right\rangle
$$

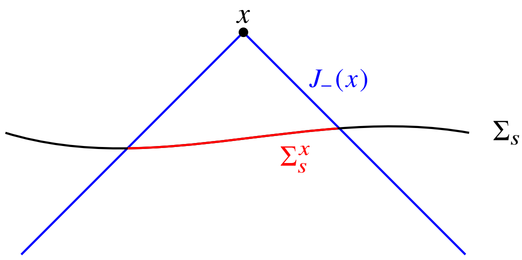
$$
\begin{aligned}
&= u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + 2xu_2u_6 - 2xu_3u_5 \\
&\geq |u|^2 - |x| \left(u_2^2 + u_6^2\right) - |x| \left(u_3^2 + u_5^2\right) \\
&\geq (1 - |x|)\, |u|^2 .
\end{aligned}
$$

Thus the principal symbol $\sigma(P, dt + \xi)$ is positive definite for $|\xi| = |x| < 1$. In the standard Lorentzian metric $\langle \cdot, \cdot \rangle = -dt^2 + g_N$ on M this means that

$$
\langle dt + \xi, dt + \xi \rangle = -1 + x^2 < 0,
$$

i.e., that the covector $dt + \xi$ is timelike. Hence the principal symbol is positive definite for future-directed timelike covectors $dt + \xi$ as required by Condition (ii).
We conclude that the two Maxwell equations which involve a time derivate form a symmetric hyperbolic system.

The following energy estimate will be crucial for controlling the support of solutions to symmetric hyperbolic systems. It will establish finiteness of the propagation speed and the uniqueness of solutions to the Cauchy problem.
Let M be globally hyperbolic and let $t : M \to \mathbb{R}$ be a Cauchy time function. Then the Lorentzian metric on M is given by $g = -N^2 dt^2 + g_t$ where each g_s is the induced Riemannian metric on $\Sigma_s := t^{-1}(s)$. We define $\Sigma_s^x := J_-(x) \cap \Sigma_s$ for $x \in M$.



The scalar product $\langle \cdot, \cdot \rangle_0 := N \langle \sigma(P, dt) \cdot, \cdot \rangle$ is positive definite. Let dA_s be the volume density of Σ_s. We denote the norm corresponding to $\langle \cdot, \cdot \rangle_0$ by $|\cdot|_0$.

Theorem 3.7.4 (Energy estimate). *Let M be globally hyperbolic, let P be a symmetric hyperbolic system over M and let $t : M \to \mathbb{R}$ be a Cauchy time function. For each $x \in M$ and each $t_0 \in t(M)$ there exists a constant $C > 0$ such that*

$$\int_{\Sigma^x_{t_1}} |u|_0^2 \, \mathrm{dA}_{t_1} \le \left[C \int_{t_0}^{t_1} \int_{\Sigma^x_s} |Pu|_0^2 \, \mathrm{dA}_s \, ds + \int_{\Sigma^x_{t_0}} |u|_0^2 \, \mathrm{dA}_{t_0} \right] e^{C(t_1 - t_0)}$$

holds for each $u \in C^\infty(M, E)$ and for all $t_1 \ge t_0$.

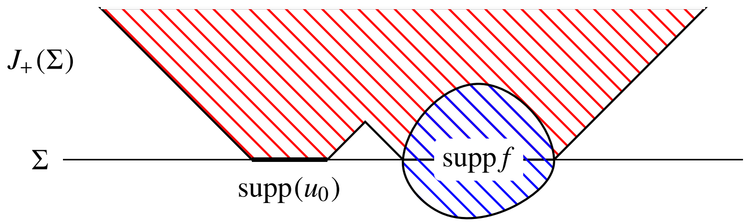
Before we prove the energy estimate, we deduce that a "wave" governed by a symmetric hyperbolic system can propagate with the speed of light at most. As a consequence we obtain uniqueness for the Cauchy problem.

Corollary 3.7.5 (Finite propagation speed). *Let M be globally hyperbolic, let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface and let P be a symmetric hyperbolic system over M. Let $u \in C^\infty(M, E)$ and put $u_0 := u|_\Sigma$ and $f := Pu$. Then*

$$\mathrm{supp}(u) \cap J_\pm(\Sigma) \subset J_\pm((\mathrm{supp} f \cap J_\pm(\Sigma)) \cup \mathrm{supp}\, u_0). \tag{3.11}$$

In particular,

$$\mathrm{supp}(u) \subset J(\mathrm{supp} f \cup \mathrm{supp}(u_0)).$$



Proof. We choose the Cauchy time function such that $\Sigma = t^{-1}(0)$. Let $x \in J_+(\Sigma)$. Assume $x \in M \setminus J_+((\mathrm{supp} f \cap J_+(\Sigma)) \cup \mathrm{supp}(u_0))$. This means that there is no future-directed causal curve starting in $\mathrm{supp} f \cup \mathrm{supp}\, u_0$, entirely contained in $J_+(\Sigma)$, which terminates at x. In other words, there is no past-directed causal curve starting at x, entirely contained in $J_+(\Sigma)$, which terminates in $\mathrm{supp} f \cup \mathrm{supp}\, u_0$. Hence $J_-(x) \cap J_+(\Sigma)$ does not intersect $\mathrm{supp} f \cup \mathrm{supp}(u_0)$. By Theorem 3.7.4, u vanishes on $J_-(x) \cap J_+(\Sigma)$, in particular $u(x) = 0$. This proves (3.11) for J_+. The case $x \in J_-(\Sigma)$ can be reduced to the previous case by time reversal. For the support of u we deduce

$$\mathrm{supp}\, u \subset J_+((\mathrm{supp} f \cap J_+(\Sigma)) \cup \mathrm{supp}\, u_0) \cup J_-((\mathrm{supp} f \cap J_-(\Sigma)) \cup \mathrm{supp}\, u_0)$$

$$\begin{aligned} &\subset J_+(\operatorname{supp} f \cup \operatorname{supp} u_0) \cup J_-(\operatorname{supp} f \cup \operatorname{supp} u_0) \\ &= J(\operatorname{supp} f \cup \operatorname{supp} u_0). \end{aligned}$$ □

Corollary 3.7.6 (Uniqueness for the Cauchy problem). *Let M be globally hyperbolic, let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface and let P be a symmetric hyperbolic system over M. Given $f \in C^\infty(M, E)$ and $u_0 \in C^\infty(\Sigma, E)$ there is at most one solution $u \in C^\infty(M, E)$ to the Cauchy problem*

$$\begin{cases} Pu = f, \\ u|_\Sigma = u_0. \end{cases} \tag{3.12}$$

Proof. By linearity, we only need to consider the case $f = 0$ and $u_0 = 0$. In this case, Corollary 3.7.5 shows $\operatorname{supp} u \subset J(\emptyset) = \emptyset$, hence $u = 0$. □

Proof of Theorem 3.7.4. Denote the dimension of M by $n + 1$. Without loss of generality, we assume that M is oriented; if M is nonorientable replace the $(n + 1)$- and n-forms occurring below by densities or, alternatively, work on the orientation covering of M.
Let vol be the volume form of M. We define the n-form ω on M by

$$\omega := \sum_{j=0}^{n} \Re(\langle \sigma(P, b_j^*)u, u\rangle)\, b_j \lrcorner \operatorname{vol}.$$

Here $b_0, \dots, b_n$ denotes a local tangent frame, $b_0^*, \dots, b_n^*$ the dual basis, and $\lrcorner$ denotes the insertion of a tangent vector into the first slot of a form. It is easily checked that ω does not depend on the choice of $b_0, \dots, b_n$. For the sake of brevity, we write

$$f := Pu. \tag{3.13}$$

We choose a metric connection ∇ on E. The symbol ∇ will also be used for the Levi-Civita connection on TM. Since the first-order operator $\sum_{j=0}^n \sigma(P, b_j^*)\nabla_{b_j}$ has the same principal symbol as P, it differs from P only by a zero-order term. Thus there exists $B \in C^\infty(M, \operatorname{Hom}(E, E))$ such that

$$P = \sum_{j=0}^{n} \sigma(P, b_j^*)\nabla_{b_j} - B. \tag{3.14}$$

To simplify the computation of the exterior differential of ω, we assume that the local tangent frame is synchronous at the point under consideration, i.e., $\nabla b_j = 0$ at the (fixed but arbitrary) point. In particular, the Lie brackets $[b_j, b_k]$ vanish at that point. Then we get at that point

$$\begin{aligned} d\omega(b_0,\dots,b_n) &= \sum_{k=0}^n (-1)^k \partial_{b_k}(\omega(b_0,\dots,\widehat{b_k},\dots,b_n)) \\ &= \sum_{k=0}^n (-1)^k \partial_{b_k}\Big(\sum_{j=0}^n \Re(\langle\sigma(P,b_j^*)u,u\rangle)\ \mathrm{vol}(b_j,b_0,\dots,\widehat{b_k},\dots,b_n)\Big) \\ &= \Re\sum_{j=0}^n \partial_{b_j}(\langle\sigma(P,b_j^*)u,u\rangle)\ \mathrm{vol}(b_0,\dots,b_n) \end{aligned}$$

and thus

$$d\omega = \Re\sum_{j=0}^n \partial_{b_j}(\langle\sigma(P,b_j^*)u,u\rangle)\ \mathrm{vol}.$$

We put $\widetilde{B} := \sum_{j=0}^n \nabla_{b_j}\sigma(P,b_j^*) \in C^\infty(M,\mathrm{Hom}(E,E))$. Using the symmetry of the principal symbol, (3.13), and (3.14) we get

$$\begin{aligned} \sum_{j=0}^n \partial_{b_j}(\langle\sigma(P,b_j^*)u,u\rangle) &= \langle\widetilde{B}u,u\rangle + \sum_{j=0}^n[\langle\sigma(P,b_j^*)\nabla_{b_j}u,u\rangle + \langle\sigma(P,b_j^*)u,\nabla_{b_j}u\rangle] \\ &= \langle\widetilde{B}u,u\rangle + \langle(P+B)u,u\rangle + \langle u,(P+B)u\rangle \\ &= \langle(\widetilde{B}+B)u,u\rangle + \langle u,Bu\rangle + \langle f,u\rangle + \langle u,f\rangle \end{aligned}$$

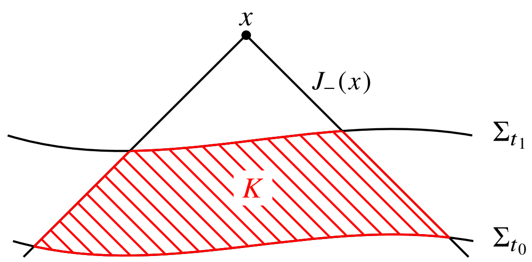
and hence

$$d\omega = \Re(\langle(\widetilde{B}+2B)u,u\rangle + 2\langle f,u\rangle)\ \mathrm{vol}.$$

Thus we have for any compact $K\subset M$

$$\begin{aligned} \int_K d\omega &= \int_K \Re(\langle(\widetilde{B}+2B)u,u\rangle + 2\langle f,u\rangle)\ \mathrm{vol} \\ &\le \int_K (C_1|u|_0^2 + C_2|f|_0|u|_0)\ \mathrm{vol} \\ &\le C_3\int_K (|u|_0^2 + |f|_0^2)\ \mathrm{vol} \end{aligned}$$

with constants C_1, C_2, C_3 depending on P and K but not on u and f. We apply this to $K = J_-(x)\cap t^{-1}([t_0,t_1])$ where $[t_0,t_1]$ is a compact subinterval of the image of t.



By the Fubini theorem,

$$\int_K d\omega \leq C_4 \int_{t_0}^{t_1} \int_{\Sigma_s^x} (|u|_0^2 + |f|_0^2)\, \mathrm{dA}_s\, ds\,. \tag{3.15}$$

The boundary $\partial J_-(x)$ is a Lipschitz hypersurface (see [13, pp. 413–415]). The Stokes' theorem for manifolds with Lipschitz boundary [10, p. 209] yields

$$\int_K d\omega = \int_{\partial K} \omega = \int_{\Sigma_{t_1}^x} \omega - \int_{\Sigma_{t_0}^x} \omega + \int_Y \omega \tag{3.16}$$

where $Y = (\partial J_-(x)) \cap t^{-1}([t_0, t_1])$. Choosing $b_0 = \sqrt{\beta} dt$ and $b_1, \ldots, b_n$ tangent to Σ_s, we see that

$$\int_{\Sigma_s^x} \omega = \int_{\Sigma_s^x} \langle \sigma_P(\sqrt{\beta} dt) u, u\rangle\, \mathrm{dA}_s = \int_{\Sigma_s^x} |u|_0^2\, \mathrm{dA}_s. \tag{3.17}$$

The boundary $\partial J_-(x)$ is ruled by the past-directed lightlike geodesics emanating from x. Thus at each differentiable point $y \in \partial J_-(x)$ the tangent space $T_y \partial J_-(x)$ contains a lightlike vector but no timelike vectors. We choose a positively oriented generalized orthonormal tangent basis $b_0, b_1, \ldots, b_n$ of $T_y M$ in such a way that b_0 is future-directed timelike and $b_0 + b_1, b_2, \ldots, b_n$ is a oriented basis of $T_y \partial J_-(x)$. Then

$$\begin{aligned}
\omega(b_0 + b_1, b_2, \ldots, b_n) &= \sum_{j=0}^{n} \Re(\langle \sigma_P(b_j^*) u, u\rangle) \operatorname{vol}(b_j, b_0 + b_1, b_2, \ldots, b_n) \\
&= \Re\langle \sigma_P(b_0^*) u, u\rangle - \Re\langle \sigma_P(b_1^*) u, u\rangle \\
&= \Re\langle \sigma_P(b_0^* - b_1^*) u, u\rangle.
\end{aligned}$$

Since $\langle \sigma_P(\tau)\cdot, \cdot\rangle$ is positive definite for each future-directed timelike covector, it is, by continuity, still positive semidefinite for each future-directed causal covector. Now $b_0^* - b_1^*$ is future-directed lightlike. Therefore

$$\omega(b_0 + b_1, b_2, \ldots, b_n) = \langle \sigma_P(b_0^* - b_1^*) u, u\rangle \geq 0.$$

This implies

$$\int_Y \omega \geq 0. \tag{3.18}$$

Combining (3.15), (3.16), (3.17), and (3.18) we find

$$\int_{\Sigma_{t_1}^x} |u|_0^2\, \mathrm{dA}_{t_1} - \int_{\Sigma_{t_0}^x} |u|_0^2\, \mathrm{dA}_{t_0} \leq C_4 \int_{t_0}^{t_1} \int_{\Sigma_s^x} (|u|_0^2 + |f|_0^2)\, \mathrm{dA}_s\, ds\,.$$

In other words, the function $h(s) = \int_{\Sigma_s^x} |u|_0^2\, \mathrm{dA}_s$ satisfies the integral inequality

$$h(t_1) \leq \alpha(t_1) + C_4 \int_{t_0}^{t_1} h(s)\, ds$$

for all $t_1 \geq t_0$ where $\alpha(t_1) = C_4 \int_{t_0}^{t_1} \int_{\Sigma_s^x} |f|_0^2 \, \mathrm{dA}_s \, ds + h(t_0)$. Grönwall's lemma 1.5.1 gives

$$h(t_1) \leq \alpha(t_1) e^{C_4(t_1 - t_0)}$$

which is the claim. □

We now want to prove existence of solutions to the Cauchy problem.

Theorem 3.7.7 (Existence for the Cauchy problem). *Let M be a globally hyperbolic manifold, $E \to M$ a vector bundle with non-degenerate metric, $P \in \mathscr{D}\!\mathit{iff}_1(E,E)$ a symmetric hyperbolic system. Let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface. For any $u_0 \in C^\infty(\Sigma, E)$ and $f \in C^\infty(M,E)$ there exists a unique solution $u \in C^\infty(M,E)$ of the Cauchy problem*

$$\begin{cases} Pu = f, \\ u|_\Sigma = u_0. \end{cases} \tag{3.19}$$

Proof. Corollary 3.7.6 gives uniqueness of the solution. We now prove existence.
A) We first assume that M is spatially compact, i.e., Σ is compact.
a) We fix a diffeomorphism $M \approx \mathbb{R} \times \Sigma$ such that the projection $t : M \approx \mathbb{R} \times \Sigma \to \mathbb{R}$ is a Cauchy time function with $t^{-1}(0) = \Sigma$. This is possible by Theorem 1.2.53.
Since P is a symmetric hyperbolic system, the principal symbol with respect to the timelike covector dt, $S := \sigma(P, dt)$ is a positive definite symmetric operator. This yields a new positive definite metric $\langle \cdot, \cdot \rangle_0$ from the possibly indefinite $\langle \cdot, \cdot \rangle$ on E by setting $\langle \cdot, \cdot \rangle_0 = \langle S \cdot, \cdot \rangle$. We choose a metric connection ∇ for $\langle \cdot, \cdot \rangle_0$ and we write

$$P = \sigma(P, dt) \nabla_t - L = S \nabla_t - L \tag{3.20}$$

where L differentiates only in directions tangential to Σ.
We put $B := L + L^t$ where the formal adjoint is taken with respect to the indefinite metric $\langle \cdot, \cdot \rangle$. The operator B, considered as a first-order differential operator, has vanishing principal symbol. Namely, for $\xi \in T^*\Sigma$:

$$\begin{aligned} \sigma_1(L + L^t, \xi) &= \sigma_1(L, \xi) - \sigma_1(L, \xi)^t \\ &= \sigma_1(P, \xi) - \sigma_1(P, \xi)^t \\ &= 0. \end{aligned}$$

Therefore the operator B is of order zero, i.e. $B \in \mathscr{D}\!\mathit{iff}_0(E,E) = C^\infty(M, \mathrm{End}(E))$. Using (3.20) the inhomogeneous equation we have to solve can be written as

$$S \nabla_t u = Lu + f. \tag{3.21}$$

The time-depended operator

$$\Delta := \Delta_{(t)} := \left(\nabla|_{\Sigma_t}\right)^* \left(\nabla|_{\Sigma_t}\right) + 1$$

on the compact Cauchy hypersurfaces Σ_t yields the Sobolev spaces $H^k(\Sigma_t, E)$, c.f. Section 1.4. Let $\varepsilon > 0$. We define mollifier

$$J_\varepsilon := J_\varepsilon^{(t)} := \exp(-\varepsilon\Delta_{(t)}).$$

Now $J_\varepsilon : H^k(\Sigma_t, E) \to H^l(\Sigma_t, E)$ is bounded for any choice of k and l, since

$$\begin{aligned}
\|J_\varepsilon u\|_l &= \|\Delta^{\frac{l}{2}} J_\varepsilon u\|_0 \\
&= \|\Delta^{\frac{l-k}{2}} J_\varepsilon \Delta^{\frac{k}{2}} u\|_0 \\
&\leq c \cdot \|\Delta^{\frac{k}{2}} u\|_0 \\
&= c \cdot \|u\|_k \, .
\end{aligned}$$

The L^2-L^2-operator norm of $\Delta^{\frac{l-k}{2}} J_\varepsilon$ is bounded because the function $\lambda \mapsto \lambda^{\frac{l-k}{2}} \exp(-\varepsilon\lambda)$ decreases exponentially to 0 and hence is bounded on the spectrum of Δ which is contained in $[1, \infty)$.
In particular, the mollifier maps any Sobolev section to a smooth section, $J_\varepsilon : H^k(\Sigma_t, E) \to C^\infty(\Sigma_t, E)$, J_ε is a smoothing operator. In case $k = l$ the above calculation shows that the operator norm $\|J_\varepsilon\|_{H^k, H^k} \leq 1$ for all k. Since the family of functions $\lambda \mapsto \exp(-\varepsilon\lambda)$ converges monotonically to $\lambda \mapsto 1$ on $[1, \infty)$ as $\varepsilon \searrow 0$ the family of operators J_ε converges strongly to id_{H^k} in the space of bounded operators on $H^k(\Sigma_t, E)$.
b) For $\varepsilon > 0$ we now solve

$$\nabla_t u^{(\varepsilon)} = J_\varepsilon S^{-1} L J_\varepsilon u^{(\varepsilon)} + S^{-1} f \tag{3.22}$$

with $u^{(\varepsilon)}\big|_{t=0} = u_0$ in the space $H^k(\Sigma_t, E)$. This is possible since $J_\varepsilon S^{-1} L J_\varepsilon$ acts as a *bounded* operator on $H^k(\Sigma_t, E)$ so that (3.22) is an ODE in the Hilbert space $H^k(\Sigma_t, E)$.
A priori, the solution of this ODE depends on ε and k. But since the Sobolev spaces are embedded into each other with decreasing k, the uniqueness of solution shows that they are actually all the same and the solution does not depend on k. This already shows that the solution is smooth in spatial directions.
Of course, the solution does depend on ε. Our aim is now to obtain a limiting function $u^{(\varepsilon)} \to u$ for $\varepsilon \to 0$ and show that u solves the Cauchy problem.
c) Consider at $u^{(\varepsilon)}$ as a map $\mathbb{R} \to H^k(\Sigma, E)$ where $k \in \mathbb{N}$ is fixed. We will derive estimates for the growth of the $u^{(\varepsilon)}$ in time, and the important fact is that the bounds c_j do not depend on u, u_0, f and ε. They do depend on t but in a continuous fashion (and are hence bounded on compact subintervals).

$$\begin{aligned}
\partial_t \|u^{(\varepsilon)}\|_{2k}^2 &= \partial_t (\Delta^k u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 \\
&\leq 2\Re(\nabla_t \Delta^k u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 + c_1 \cdot \|\Delta^k u^{(\varepsilon)}\|_0^2 \, .
\end{aligned} \tag{3.23}$$

Note here that in order to differentiate $(u,v)_0 = \int_\Sigma \langle u,v\rangle_0 d\Sigma_t$ we also have to differentiate the volume element $d\Sigma_t$. This yields $\frac{\frac{d}{dt}d\Sigma_t}{d\Sigma_t}d\Sigma_t$ a logarithmic change of the volume element which we can estimate by a time-dependend bound c_1.
We next want to exchange ∇_t with Δ^k. A priori, the commutator $[\nabla_t, \Delta^k]$ is a differential operator of order $2k+1$. But it turns out that it is actually of order at most $2k$ as can be seen by computing the principal symbol:

$$\sigma_{2k+1}([\nabla_t, \Delta^k], \xi) = [\sigma_1(\nabla_t, \xi), \sigma_{2k}(\Delta^k, \xi)] = [\sigma_1(\nabla_t, \xi), |\xi|^{2k}] = 0\,.$$

Therefore we can bound the commutator term in the $\|\cdot\|_{2k}$-norm and we continue the estimation (3.23).

$$\begin{aligned}\partial_t \|u^{(\varepsilon)}\|_{2k}^2 &\le c_2 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + 2\Re(\Delta^k \nabla_t u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 \\ &= c_2 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + 2\Re(\Delta^k S^{-1} f, \Delta^k u^{(\varepsilon)})_0 + 2\Re(\Delta^k J_\varepsilon S^{-1} L J_\varepsilon u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 \end{aligned} \tag{3.24}$$

We estimate the second summand in (3.24):

$$\begin{aligned}2\Re(\Delta^k S^{-1} f, \Delta^k u^{(\varepsilon)})_0 &\le \|\Delta^k S^{-1} f\|_0^2 + \|\Delta^k u^{(\varepsilon)}\|_0^2 \\ &= \|S^{-1} f\|_{2k}^2 + \|u^{(\varepsilon)}\|_{2k}^2 \\ &\le c_3 \cdot \|f\|_{2k}^2 + \|u^{(\varepsilon)}\|_{2k}^2 . \end{aligned} \tag{3.25}$$

For the third summand we observe that Δ^k commutes with J_ε and that J_ε is selfadjoint. Hence

$$\begin{aligned}2\Re(\Delta^k J_\varepsilon S^{-1} L J_\varepsilon u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 &= 2\Re(J_\varepsilon \Delta^k S^{-1} L J_\varepsilon u^{(\varepsilon)}, \Delta^k u^{(\varepsilon)})_0 \\ &= 2\Re(\Delta^k S^{-1} L J_\varepsilon u^{(\varepsilon)}, J_\varepsilon \Delta^k u^{(\varepsilon)})_0 \\ &= 2\Re(\Delta^k S^{-1} L J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)})_0 . \end{aligned} \tag{3.26}$$

Inserting (3.25) and (3.26) into (3.24) yields

$$\partial_t \|u^{(\varepsilon)}\|_{2k}^2 \le c_4 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + c_3 \cdot \|f\|_{2k}^2 + 2\Re(\Delta^k S^{-1} L J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)})_0 .$$

Again, we find that $\mathrm{ord}[\Delta^k, S^{-1}L] \le 2k$ and hence

$$\partial_t \|u^{(\varepsilon)}\|_{2k}^2 \le c_4 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + c_3 \cdot \|f\|_{2k}^2 + c_5 \cdot \|J_\varepsilon u^{(\varepsilon)}\|_{2k} \cdot \|J_\varepsilon u^{(\varepsilon)}\|_{2k} + 2\Re(S^{-1} L \Delta^k J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)})_0 \,.$$

Since $\|J_\varepsilon u^{(\varepsilon)}\|_{2k} \le \|u^{(\varepsilon)}\|_{2k}$ we find

$$\partial_t \|u^{(\varepsilon)}\|_{2k}^2 \le c_6 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + c_3 \cdot \|f\|_{2k}^2 + 2\Re(L \Delta^k J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)}) \,.$$

Note that the term S^{-1} has converted the definite scalar product $(\cdot,\cdot)_0$ into $(\cdot,\cdot)$. Moreover

$$\begin{aligned}2\Re(L \Delta^k J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)}) &= (L \Delta^k J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)})_0 + (\Delta^k J_\varepsilon u^{(\varepsilon)}, L \Delta^k J_\varepsilon u^{(\varepsilon)})_0 \\ &= ((L + L^t) \Delta^k J_\varepsilon u^{(\varepsilon)}, \Delta^k J_\varepsilon u^{(\varepsilon)})_0 \\ &\le c_7 \cdot \|\Delta^k J_\varepsilon u^{(\varepsilon)}\|_0^2 \\ &= c_7 \cdot \|J_\varepsilon u^{(\varepsilon)}\|_{2k}^2 \\ &\le c_7 \cdot \|u^{(\varepsilon)}\|_{2k}^2 \end{aligned}$$

and hence
$$\partial_t \|u^{(\varepsilon)}\|_{2k}^2 \leq c_8 \cdot \|u^{(\varepsilon)}\|_{2k}^2 + c_3 \cdot \|f\|_{2k}^2.$$
This implies
$$\|u^{(\varepsilon)}|_{\Sigma_t}\|_{2k}^2 \leq \|u_0\|_{2k}^2 + \int_0^t c_3(s)\|f\|_{2k}^2 ds + \int_0^t c_8(s)\|u^{(\varepsilon)}\|_{2k}^2 ds.$$
Grönwall's lemma 1.5.1 with $\alpha(t) = \|u_0\|_{2k}^2 + \int_0^t c_3(s)\|f\|_{2k}^2 ds$ and $\beta(t) = c_8(t)$ now yields
$$\|u^{(\varepsilon)}|_{\Sigma_t}\|_{2k}^k \leq \left(\|u_0\|_{2k}^2 + \int_0^t c_3(s)\|f\|_{2k}^2 ds\right) \cdot \exp \int_0^t c_8(s) ds\,. \tag{3.27}$$
Note that this bound is independent of ε. For $t < 0$ one obtains an analogue estimate by integrating over $[t, 0]$.
d) We have seen that, for $t \in \mathbb{R}$ fixed, the set $\{u^{(\varepsilon)}|_{\Sigma_t} \mid \varepsilon > 0\}$ is bounded in $H^k(\Sigma_t, E)$. By the Rellich-Kondrachov theorem 1.4.5 $\{u^{(\varepsilon)}|_{\Sigma_t} \mid \varepsilon > 0\}$ is relatively compact in $H^{k-1}(\Sigma_t, E)$ for all $k \in \mathbb{R}$.
Taking $\|J_\varepsilon\|_{H^k,H^k} \leq 1$ into account and that S^{-1} is of order 0 and L is of order 1, we get the estimate
$$\begin{aligned}\|\nabla_t u^{(\varepsilon)}\|_k &\leq \|J_\varepsilon S^{-1} L J_\varepsilon u^{(\varepsilon)}\|_k + \|S^{-1} f\|_k \\ &\leq \|S^{-1} L J_\varepsilon u^{(\varepsilon)}\|_k + \|S^{-1} f\|_k \\ &\leq C \cdot \|J_\varepsilon u^{(\varepsilon)}\|_{k+1} + \|S^{-1} f\|_k \\ &\leq C \cdot \|u^{(\varepsilon)}\|_{k+1} + \|S^{-1} f\|_k \\ &\leq C'\end{aligned}$$
where C' does not depend on ε by (3.27). Thus the map $t \mapsto u^{(\varepsilon)}|_{\Sigma_t}$ is equicontiuous.
For fixed $T > 0$ and fixed k the Arzelà-Ascoli theorem (1.5.2) implies that $\{u^{(\varepsilon)} \mid \varepsilon > 0\} \subset C^0([-T, T], H^k(\Sigma, E))$ is relatively compact. Thus we obtain a subsequence $u^{(\varepsilon_j)}$ of the family $u^{(\varepsilon)}$ with $u^{(\varepsilon_j)} \to u$ for $u \in C^0([-T, T], H^k(\Sigma, E))$, $\varepsilon_j \searrow 0$. By a diagonal subsequence argument we can w.l.o.g. assume $u^{(\varepsilon_j)} \to u \in C^0([-T, T], H^k(\Sigma, E))$ for all $k \in \mathbb{R}$ and all $T > 0$. Therefore the convergence $u^{(\varepsilon_j)} \to u$ is locally unifom in $C^0(\mathbb{R}, H^k(\Sigma, E))$ for every $k \in \mathbb{R}$.
We now want to show that this u is the desired solution of the Cauchy problem. First we see that for $t = 0$
$$u^{(\varepsilon_j)}|_{\Sigma_0} = u_0$$
and therefore
$$u|_{\Sigma_0} = u_0.$$
Showing that $Pu = f$ is more complicated since we also have to control the convergence of the time derivatives of the $u^{(\varepsilon_j)}$ to the time derivatives of u.
We defined the $u^{(\varepsilon)}$ to be solutions of the ODE (3.22). Identifying the Cauchy hypersurfaces Σ_t with Σ via parallel transport along the integral curves of ∇t this ODE translates into
$$\frac{\partial u^{(\varepsilon)}}{\partial t} = J_\varepsilon S^{-1} L J_\varepsilon u^{(\varepsilon)} + S^{-1} f.$$

In order to get rid of the time derivatives we integrate (3.22) and we obtain

$$u^{(\varepsilon_j)}|_{\Sigma_t} - u_0 = \int_0^t (J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} u^{(\varepsilon_j)}|_{\Sigma_s} + S^{-1} f|_{\Sigma_s}) ds \tag{3.28}$$

Now we let $\varepsilon_j \searrow 0$. For the left hand side of (3.28) we find $u^{(\varepsilon_j)}|_{\Sigma_t} - u_0 \to u|_{\Sigma_t} - u_0$. For the right hand side of (3.28) we consider the first summand under the integral which is the one depending on ε_j. We split this summand

$$J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} u^{(\varepsilon_j)} = J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} (u^{(\varepsilon_j)} - u) + J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} u \tag{3.29}$$

and now look seperately at the k-th Sobolev norms of the two parts. For the first summand we find

$$\begin{aligned}
\|J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} (u^{(\varepsilon_j)} - u)\|_k &\le \|S^{-1} L J_{\varepsilon_j} (u^{(\varepsilon_j)} - u)\|_k \\
&\le C \cdot \|J_{\varepsilon_j} (u^{(\varepsilon_j)} - u)\|_{k+1} \\
&\le C \cdot \|(u^{(\varepsilon_j)} - u)\|_{k+1} \to 0 \text{ as } \varepsilon_j \to 0.
\end{aligned}$$

The second summand is split again:

$$J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} u = J_\varepsilon S^{-1} L (J_\varepsilon u - u) + J_\varepsilon S^{-1} L u \tag{3.30}$$

We estimate the k-th Sobolev norm of the first summand of the right hand side of (3.30)

$$\begin{aligned}
\|J_\varepsilon S^{-1} L (J_\varepsilon u - u)\|_k &\le \|S^{-1} L (J_\varepsilon u - u)\|_k \\
&\le C \cdot \|J_\varepsilon u - u\|_{k+1} \\
&= C \cdot \|J_\varepsilon \Delta^{\frac{k+1}{2}} u - \Delta^{\frac{k+1}{2}} u\|_0 \to 0\,.
\end{aligned}$$

For the second summand of the right hand side of (3.30) we directly see $J_\varepsilon S^{-1} L u \to S^{-1} L u$. To summarize, we found that the first summand under the integral on the right hand side of (3.28) converges to $S^{-1} L u$ locally uniformly in t.
Hence for the whole integral on the right hand side of (3.28) we found

$$\int_0^t (J_{\varepsilon_j} S^{-1} L J_{\varepsilon_j} u^{(\varepsilon_j)}|_{\Sigma_s} + S^{-1} f|_{\Sigma_s}) ds \to \int_0^t (S^{-1} L u + S^{-1} f)|_{\Sigma_s} ds\,.$$

Therefore

$$u|_{\Sigma_t} - u_0 = \int_0^t (S^{-1} L u|_{\Sigma_s} + S^{-1} f|_{\Sigma_s}) ds, .$$

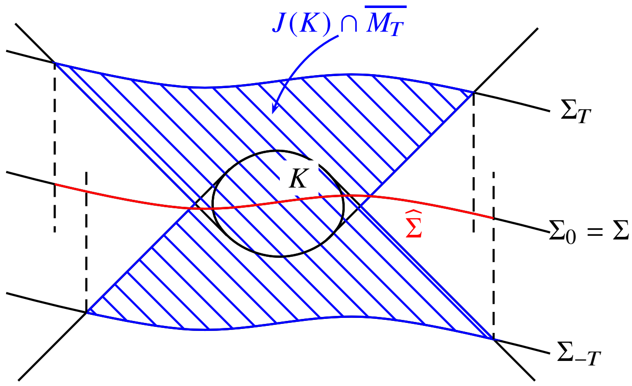
Differentiation yields

$$\frac{\partial u}{\partial t} = S^{-1} L u + S^{-1} f, .$$

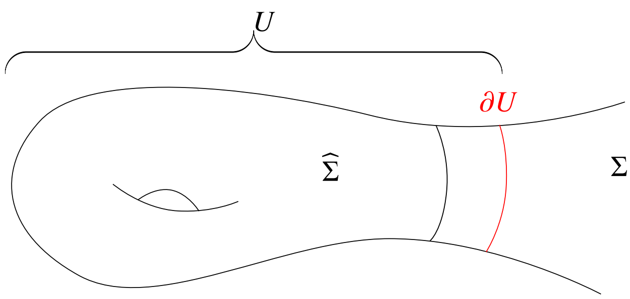
Now we drop the identification $\Sigma_t \to \Sigma$ and hence the ordinary differentiation ∂_t turns into the covariant derivative ∇_t again. Thus we have shown

$$\nabla_t u = S^{-1} L u + S^{-1} f \tag{3.31}$$

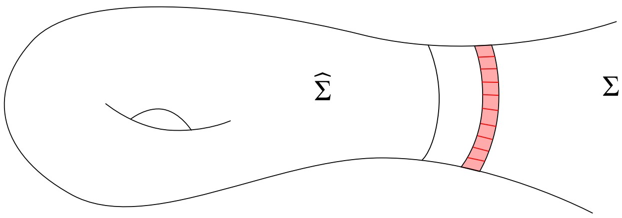
In other words, we have $S\nabla_t u = Lu + f$ which means $Pu = f$.
e) So far we know continuity in time direction and smoothness in spatial direction. Next we want to prove smoothness in time direction. We have $u \in C^0(\mathbb{R}, H^k(\Sigma, E))$ for all $k \in \mathbb{R}$. By (3.31)) we see that $\nabla_t u \in C^0(\mathbb{R}, H^{k-1}(\Sigma, E))$ for all $k \in \mathbb{R}$. Thus $u \in C^1(\mathbb{R}, H^k(\Sigma, E))$ for all $k \in \mathbb{R}$.
To obtain the second time derivative of u we differentiate (3.31) with respect to t. Since on the right hand side we then have at most one time derivative we conclude $\nabla_t\nabla_t u \in C^0$ and therefore $u \in C^2(\mathbb{R}, H^k(\Sigma, E))$. Repeating this argument we obtain $u \in C^\ell(\mathbb{R}, H^k(\Sigma, E))$ for all ℓ and k and hence $u \in C^\infty(\mathbb{R}, H^k(\Sigma, E))$ for all k. The Sobolev embedding theorem then yields $u \in C^\infty(\mathbb{R}, C^\infty(\Sigma, E))$. This implies $u \in C^\infty(M, E)$.
B) We now drop the assumption that M be spatially compact but we still assume that the Cauchy data have compact support, $u_0 \in C_c^\infty(\Sigma, E)$ and $f \in C_c^\infty(M, E)$.
Set $K := \mathrm{supp}(u_0) \cup \mathrm{supp}(f)$. Fix $T > 0$ and set $M_T := (-T, T) \times \Sigma \subset \mathbb{R} \times \Sigma = M$. Choose T big enough so that $K \subset M_T$. Note that M_T is globally hyperbolic itself. Consider the compact set $J(K) \cap \overline{M_T}$ and denote by $\widehat{\Sigma}$ the projection of $J(K) \cap \overline{M_T}$ on Σ, which is compact too.



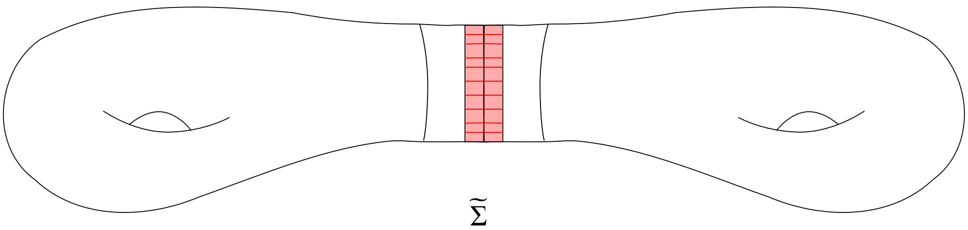
We choose a relatively compact open set $U \subset \Sigma$ with $\widehat{\Sigma} \subset U$ and smooth boundary ∂U.



Now we change the metric g_t of Σ near the boundary ∂U such that it becomes a product metric in a neighboorhood of ∂U. We do this smoothly in $t \in [-T, T]$. We want everything to stay untouched on $\widehat{\Sigma}$.



Then we double this part of Σ and obtain $\widetilde{\Sigma}$.

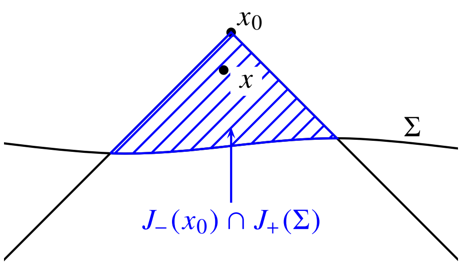


This yields $\widetilde{M}_T$ which is now spatially compact. The supports of f and u_0 are contained in $(-T, T) \times \widehat{\Sigma}$. Therefore we may consider f and u_0 as sections defined on $\widetilde{M}_T$.
By part *A)* of the proof we obtain a solution u_T on $\widetilde{M}_T$. Finite speed of propagation (Corollary 3.7.5) yields $\operatorname{supp}(u_T) \subset J(K) \cap \widetilde{M}_T \subset (-T, T) \times \widehat{\Sigma}$. Thus we can regard u_T as a solution on M_T. For $T' > T$ we analogously obtain a solution $u_{T'}$ on $M_{T'}$. Since the solution is uniquely determined by the initial conditions we find that $u_{T'}|_{M_T} = u_T$. Hence we obtain a solution $u \in C^\infty(M, E)$ on M with $u|_{M_T} = u_T$.
C) Now we also drop the assumption that $\operatorname{supp}(u_0)$ and $\operatorname{supp}(f)$ are compact.
Let $K_1 \subset K_2 \subset K_3 \subset \ldots \subset M$ be an exhaustion by compact subsets such that every compact subset of M is contained in K_j for sufficiently large j. We choose cutoff functions $\chi_j \in C_c^\infty(M)$ with $\chi_j \equiv 1$ on K_j. By *B)* there exists a solution of

$$\begin{aligned} Pu_j &= \chi_j f, \\ u_j|_\Sigma &= \chi_j u_0 \,. \end{aligned}$$

Next we want to show that this sequence of solutions converges to a solution for the general problem.

Fix $x \in M$ W.l.o.g. we may assume $x \in J_+(\Sigma)$. Choose $x_0 \in I_+(x)$. Then $I_-(x_0)$ is an open neighborhood of x.
Since $J_-(x_0) \cap J_+(\Sigma)$ is compact there exists a j_0 such that $J_-(x_0) \cap J_+(\Sigma) \subset K_j$ for all $j \geq j_0$.



Corollary 3.7.5 tells us that u_j is uniquely determined by

$$\begin{aligned} &\chi_j u_0 \text{ on } \Sigma \cap J_-(x_0), \\ &\chi_j f \text{ on } J_+(\Sigma) \cap J_-(x_0)\,. \end{aligned}$$

But since $\chi_j \equiv 1$ for $j \geq j_0$, the section u_j is determined by

$$\begin{aligned} &u_0 \text{ on } \Sigma \cap J_-(x_0), \\ &f \text{ on } J_+(\Sigma) \cap J_-(x_0), \end{aligned}$$

and hence independent of j for $j \geq j_0$.
Therefore $u(x) := \lim_{j\to\infty} u_j(x)$ exists and we can do this for every point x. For $x \in I_-(x_0)$ we have

$$Pu(x) = Pu_{j_0}(x) = \chi_{j_0}(x) f(x) = f(x)$$

and

$$u|_\Sigma = u_{j_0}|_\Sigma = \chi_{j_0} u_0|_\Sigma = u_0\,.$$

Thus u is the desired solution of the Cauchy problem. □

We conclude the discussion of the Cauchy problem for symmetric hyperbolic systems by showing stability. This means that the solutions depend continuously on the data. Note that if u_0 and f have compact supports, then the solution u of the Cauchy problem (3.12) has spatially compact support by Corollary 3.7.6.

Proposition 3.7.8 (Stability of the Cauchy problem). *Let P be a symmetric hyperbolic system over the globally hyperbolic manifold M. Let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface.*

Then the map $C_c^\infty(M,E) \times C_c^\infty(\Sigma,E) \to C_{sc}^\infty(M,E)$ mapping (f,u_0) to the solution u of the Cauchy problem (3.12) is continuous.

Proof. The map $\mathcal{P} : C^\infty(M,E) \to C^\infty(M,E) \times C^\infty(\Sigma,E)$, $u \mapsto (Pu, u|_\Sigma)$, is linear and continuous. Fix a compact subset $A \subset M$. Then $C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E)$ is a closed subset of $C^\infty(M,E) \times C^\infty(\Sigma,E)$ and thus $V_A := \mathcal{P}^{-1}(C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E))$ is a closed subset of $C^\infty(M,E)$. In particular, $C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E)$ and V_A are Fréchet spaces. By Corollary 3.7.6 and Theorem 3.7.7, $\mathcal{P}$ maps V_A bijectively onto $C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E)$. The open mapping theorem for Fréchet spaces tells us that $(\mathcal{P}|_{V_A})^{-1} : C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E) \to V_A$ is continuous. Now $V_A \subset C^\infty(M,E)$ and $C_{J(A)}^\infty(M,E) \subset C^\infty(M,E)$ carry the relative topologies and $V_A \subset C_{J(A)}^\infty(M,E)$ by Corollary 3.7.5. Thus the embeddings $V_A \hookrightarrow C_{J(A)}^\infty(M,E) \hookrightarrow C_{sc}^\infty(M,E)$ are continuous. Hence the solution operator for the Cauchy problem yields a continuous map $C_A^\infty(M,E) \times C_{A\cap\Sigma}^\infty(\Sigma,E) \to C_{sc}^\infty(M,E)$ for every compact $A \subset M$. Therefore it is continuous as a map $C_c^\infty(M,E) \times C_c^\infty(\Sigma,E) \to C_{sc}^\infty(M,E)$. □

Remark 3.7.9. Corollary 3.7.6, Theorem 3.7.7 and Proposition 3.7.8 are often summarized by saying that the Cauchy problem (3.12) is *well posed*.

Finally, we show that symmetric hyperbolic systems over globally hyperbolic manifolds are Green hyperbolic.

Theorem 3.7.10. *Symmetric hyperbolic systems over globally hyperbolic manifolds are Green hyperbolic.*

Proof. Let P be a symmetric hyperbolic system over the globally hyperbolic manifold M. We construct an advanced Green's operator G_+ for P. Let $u \in C_c^\infty(M,E)$. Then $K := \mathrm{supp}(u) \subset M$ is compact. We choose Σ to be a smooth spacelike Cauchy hypersurface such that $K \subset I_+(\Sigma)$.
Let G_+u be the solution of the Cauchy problem $PG_+u = u$ with initial condition $G_+u|_\Sigma = 0$.
We have to show that this definition does not depend on the particular choice of Σ.
First note that by finite speed of propagation $\mathrm{supp}(G_+u) \cap I_+(\Sigma) \subset J_+(\mathrm{supp}(G_+u|_\Sigma) \cup \mathrm{supp}(u)) = J_+(\emptyset \cup K) = J_+(K)$ and $\mathrm{supp}(G_+u) \cap I_-(\Sigma) \subset J_-(\emptyset \cup (\mathrm{supp}(u) \cap J_-(\Sigma))) = \emptyset$. (This already shows condition (iii) in Definition 3.6.1 for an advanced Green's operator.)
Now let Σ' be another smooth spacelike Cauchy hypersurface with $K \subset I_+(\Sigma')$. Then $J_+(K) \subset J_+(I_+(\Sigma')) = I_+(\Sigma')$ and therefore $\mathrm{supp}(G_+u) \subset I_+(\Sigma')$.
Hence we know that $G_+u|_{\Sigma'} = 0$ and thus G_+u is also a solution of the Cauchy problem $PG_+u = u$ with initial condition $G_+u|_{\Sigma'} = 0$. Hence choosing another Cauchy hypersurface gives the same solution G_+u and the definition does not depend on the particular choice of Σ.
We want to show that G_+ is an advanced Green's operator of P.
By construction $P \circ G_+ = \mathrm{id}_{C_c^\infty(M,E)}$ which is condition (ii).

It remains to check condition (i): If $u = Pv$ for some $v \in C_c^\infty(M, E)$, then $u = v$ is the unique solution to the Cauchy problem $Pv = u$ with $v|_\Sigma = 0$ for a smooth spacelike Cauchy hypersurface Σ with $\operatorname{supp}(v) \subset I_+(\Sigma)$. Then we also have $\operatorname{supp}(u) \subset I_+(\Sigma)$ so that we may use this Cauchy hypersurface in the definition of G_+. Therefore $v = G_+ u = G_+ P v$ and for every $v \in C_c^\infty(M, E)$ and hence $G_+ \circ P = \mathrm{id}_{C_c^\infty(M,E)}$.
Hence G_+ is an advanced Green's operator. A retarded Green's operator is constructed similarly by choosing Σ such that $K \subset I^-(\Sigma)$.
Finally, $-P^*$ is again a symmetric hyperbolic system and therefore has Green's operators. Thus P^* has Green's operators and P is Green hyperbolic. □

Remark 3.7.11. It is possible to derive the well-posedness of the Cauchy problem for normally hyperbolic operators from that for symmetric hyperbolic systems. To see this, let $E \to M$ be a hermitian vector bundle and let $Q \in \mathscr{Diff}_2(E, E)$ be normally hyperbolic. Write $M = \mathbb{R} \times \Sigma$ such that $g = -N^2 dt^2 + g_t$. We choose a connection ∇ on E and write Q as

$$Q = \frac{1}{N^2}\nabla_t \nabla_t - \operatorname{tr}(\nabla^\Sigma_\cdot \nabla^\Sigma_\cdot) + \nabla^\Sigma_b + b_0 \cdot \nabla_t + c$$

where $\nabla^\Sigma_\cdot$ denotes the restriction of ∇ to Σ for any fixed t, b is an $\mathrm{End}(E)$-valued vector field tangential to Σ, b_0 and c are endomorphism fields.
We want to solve $Qv = f$ with initial data prescribed at $t = 0$. We add two redundent equations to obtain the system

$$\nabla_t(\nabla^\Sigma_\cdot v) - \nabla^\Sigma_\cdot \nabla_t v + \nabla^\Sigma_{\pi^\Sigma(\cdot)} v + \pi^t(\cdot)\nabla_t v - R(\partial_t, \cdot)v = 0 \tag{3.32}$$

$$\frac{1}{N^2}\nabla_t \nabla_t v - \operatorname{tr}(\nabla^\Sigma_\cdot \nabla^\Sigma_\cdot) + \nabla^\Sigma_b v + b_0 \cdot \nabla_t v + cv = f \tag{3.33}$$

$$\nabla_t v - \nabla_t v = 0 \tag{3.34}$$

where $\pi^\Sigma : C^\infty(\Sigma, T\Sigma) \to C^\infty(\Sigma, T\Sigma)$ and $\pi^t : C^\infty(\Sigma, T\Sigma) \to C^\infty(\Sigma, \mathbb{R})$ are defined by $\nabla_X \partial_t = \pi^\Sigma(X) + \pi^t(X)\partial_t$ for $X \in T\Sigma$. Note that while (3.32) is an equation in $T^*\Sigma \otimes E$, (3.33) and (3.34) are equations in E.
Equation (3.32) holds for any sufficiently smooth section v: For a time dependend vector field X tangential to Σ we see

$$\nabla_t \nabla^\Sigma_X v - \nabla^\Sigma_X \nabla_t v - \nabla_{[\partial_t, X]} v - R(\partial_t, X) = 0\,.$$

Inserting $[\partial_t, X] = \nabla_t X - \nabla_X \partial_t$ (the Levi-Civita connection is torsion-free) yields

$$\nabla_t \nabla^\Sigma_X v - \nabla_{\nabla_t X} v - \nabla^\Sigma_X \nabla_t v - \nabla_{\nabla_X \partial_t} v - R(\partial_t, X) = 0\,.$$

For the first two summands we see $\nabla_t \nabla^\Sigma_X v - \nabla_{\nabla_t X} v = X \lrcorner (\nabla_t(\nabla^\Sigma_\cdot v))$, which is tensorial in X. The splitting $\nabla_X \partial_t = \pi^\Sigma(X) + \pi^t(X)\partial_t$ then yields (3.32).
We now consider the differential operator $P \in \mathscr{Diff}_1(T^*\Sigma \otimes E \oplus E \oplus E, T^*\Sigma \otimes E \oplus E \oplus E)$ given by

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{N^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \nabla_t + \begin{pmatrix} 0 & -1 & 0 \\ -\mathrm{tr} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla^\Sigma + \begin{pmatrix} \pi^\Sigma(\cdot) & \pi^t(\cdot) & -R(\partial_t, \cdot) \\ b & b_0 & c \\ 0 & -1 & 0 \end{pmatrix}.$$

Equations (3.32), (3.33) and (3.34) are equivalent to

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{N^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \nabla_t + \begin{pmatrix} 0 & -1 & 0 \\ -\mathrm{tr} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nabla^\Sigma + \begin{pmatrix} \pi^\Sigma(\cdot) & \pi^t(\cdot) & -R(\partial_t, \cdot) \\ b & b_0 & c \\ 0 & -1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \nabla^\Sigma v \\ \nabla_t v \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}.$$

The operator P is a symmetric hyperbolic system. To see this let $\xi \in T^*\Sigma$. Then $\sigma(P, dt + \xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{N^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\xi\otimes & 0 \\ -\xi\lrcorner & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is symmetric.

Moreover,

$$\begin{aligned}
\left\langle \begin{pmatrix} 1 & -\xi\otimes & 0 \\ -\xi\lrcorner & \frac{1}{N^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u_0 \\ v \end{pmatrix}, \begin{pmatrix} u \\ u_0 \\ v \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} u - \xi \otimes u \\ -\xi\lrcorner u + \frac{1}{N^2} u_0 \\ v \end{pmatrix}, \begin{pmatrix} u \\ u_0 \\ v \end{pmatrix} \right\rangle \\
&= |u|^2 - \langle \xi \otimes u_0, u\rangle - \langle \xi\lrcorner u, u_0\rangle + \frac{1}{N^2}|u_0|^2 + |v|^2 \\
&= |u|^2 + |v|^2 + \frac{1}{N^2}|u_0|^2 - 2\langle \xi \otimes u_0, u\rangle \\
&\geq (1 - N|\xi|)(|u|^2 + \frac{|u_0|^2}{N^2} + |v|^2).
\end{aligned}$$

The last estimate holds because $2|\langle \xi \otimes u_0, u\rangle| \leq 2N|\xi| \cdot \frac{|u_0|}{N}|u| \leq N|\xi|(\frac{|u_0|^2}{N^2} + |u|^2)$.
For $1 - N|\xi| > 0$, i.e. for $dt + \xi$ timelike, the principal symbol $\sigma(P, dt + \xi)$ is positive definite. We conclude that P is a symmetric hyperbolic system.

We saw that if v is a solution of $Qv = f$, then $V = \begin{pmatrix} \nabla^\Sigma v \\ \nabla_t v \\ v \end{pmatrix}$ is a solution of $PV = F$ where $F = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$.
Conversely, does a solution of $PV = F$ also yield a solution of $Qv = f$?
This cannot be true always because the space of initial data for P is too large. Indeed, we have to impose a restriction of the initial data. Let $V = \begin{pmatrix} u \\ u_0 \\ v \end{pmatrix}$ be a solution of $PV = F$ with the assumption that at $t = 0$ we have $u = \nabla^\Sigma v$.
Now $PV = F$ is equivalent to

$$\nabla_t u - \nabla^\Sigma_\cdot u_0 + \pi^\Sigma(\cdot)\lrcorner u + \pi^t(\cdot) u_0 - R(\partial_t, \cdot) v = 0, \tag{3.35}$$

$$\frac{1}{N^2}\nabla_t u_0 - \mathrm{tr}(\nabla^\Sigma_\cdot u) + b\lrcorner u + b_0 u_0 + cv = f, \tag{3.36}$$

$$\nabla_t v - u_0 = 0.$$

Hence we have $u_0 = \nabla_t v$ on M. If we can show that $u = \nabla^\Sigma v$ holds on all of M then (3.36) implies that v solves $Qv = f$.
By assumption $u = \nabla^\Sigma v$ holds at $t = 0$. We differentiate with respect to t. By (3.35) and $u_0 = \nabla_t v$ we find

$$\begin{aligned}\nabla_t u + \pi^\Sigma(\cdot) \lrcorner u &= \nabla^\Sigma u_0 - \pi^t(\cdot)u_0 + R(\partial_t, \cdot)v \\ &= \nabla^\Sigma \nabla_t v - \pi^t(\cdot)\nabla_t v + R(\partial_t, \cdot)v \\ &= \nabla_t(\nabla^\Sigma_\cdot v). \end{aligned} \tag{3.37}$$

On the other hand, for any smooth vector field X tangential to Σ at all times we have

$$\begin{aligned}\nabla^\Sigma_X \nabla_t v - \pi^t(X)\nabla_t v + R(\partial_t, X)v &= \nabla_t \nabla^\Sigma_X v + \nabla_{[X,\partial_t]} v - \pi^t(X)\nabla_t v \\ &= \nabla_t \nabla^\Sigma_X v - \nabla_{\nabla_{\partial_t} X} v + \nabla_{\nabla_X \partial_t} v - \pi^t(X)\nabla_t v \\ &= X \lrcorner (\nabla_t \nabla^\Sigma_\cdot v) + \nabla^\Sigma_{\pi^\Sigma(X)} v\,. \end{aligned}$$

Combined with (3.37) this yields

$$\nabla_t u + \pi^\Sigma(\cdot) \lrcorner u = \nabla_t \nabla^\Sigma_\cdot v + \pi^\Sigma(\cdot) \lrcorner \nabla^\Sigma_\cdot v\,.$$

Hence $u - \nabla^\Sigma_\cdot v$ satisfies the first-order ODE

$$(\nabla_t + \pi^\Sigma(\cdot) \lrcorner)(u - \nabla^\Sigma_\cdot v) = 0$$

along the integral curves of ∂_t and it vanishes at $t = 0$. Hence $u = \nabla^\Sigma_\cdot v$ on all of M.
We have seen that the initial data v and u_0 (the time derivative) can be prescribed arbitrarily as initial data as expected for the Cauchy problem for a normally hyperbolic operator.

3.8 An application: essential selfadjointness on Riemannian manifolds

A (generally unbounded) symmetric operator in a Hilbert space with dense domain is called *essentially selfadjoint* if it has a unique selfadjoint extension. This is a very desirable property because one then can apply a lot of functional analysis to this selfadjoint extension such as spectral and functional calculus.
We will now use symmetric hyperbolic systems to deduce the selfadjointness of certain operators on Riemannian manifolds, following ideas of P. R. Chernoff in [9]. Throughout this section let (N, g) be a complete Riemannian manifold and $E \to N$ a hermitian vector bundle. We consider a first-order differential operator $L \in \mathscr{Diff}_1(E, E)$ with the properties

1.) The operator L is formally skewadjoint, i.e. $L^t = -L$;

2.) There exists $c > 0$ such that $|\sigma(L, \xi)|_{\mathrm{op}} \leq c \cdot |\xi|$ holds for all $\xi \in T^*M$.

Here $|\sigma(L,\xi)|_{\mathrm{op}}$ denotes the operator norm of the linear map $\sigma(L,\xi) : E_p \to E_p$ where $\xi \in T_p^*M$. Since L is of first order $\sigma(L,\xi)$ depends linearly on ξ so that for each $p \in N$ there exists $c_p > 0$ such that $|\sigma(L,\xi)|_{\mathrm{op}} \le c_p \cdot |\xi|$ holds for all $\xi \in T_p^*M$. Condition 2.) says that the constant can be chosen independently of the base point. If N is compact this is automatic.
Now we equip $M = \mathbb{R} \times N$ with the Lorentzian metric $g_M = -c^2dt^2 + g_N$. Since (N, g_N) is complete (M, g_M) is globally hyperbolic. On M we consider the operator

$$P = \frac{\partial}{\partial t} - L \in \mathscr{D}\!\mathit{iff}_1(E,E)$$

where E is also considered as a hermitian vector bundle on M via the pull-back along the projection $\pi : M \to N$. We check that P is a symmetric hyperbolic system:
The principal symbol of L is symmetric because $\sigma(L,\xi)^t = -\sigma(L^t,\xi) = \sigma(L,\xi)$. Thus for $\xi \in T^*N$ we find that $\sigma(P, dt+\xi) = \mathrm{id} - \sigma(L,\xi)$ is symmetric. Moreover, we see that

$$\langle \sigma(P, dt+\xi)u, u\rangle = |u|^2 - \langle \sigma(L,\xi)u, u\rangle \ .$$

Now

$$|\langle \sigma(L,\xi)u, u\rangle| \le |\sigma(L,\xi)u| \cdot |u| \le |\sigma(L,\xi)|_{\mathrm{op}} \cdot |u|^2 \le c \cdot |\xi| \cdot |u|^2$$

and therefore

$$\langle \sigma(P, dt+\xi)u, u\rangle \ge (1 - c\cdot|\xi|)|u|^2.$$

Hence $\langle \sigma(P, dt+\xi)\cdot,\cdot\rangle$ is positive definite in case $1 - c|\xi| > 0$. This is equivalent to $0 > -1 + c^2|\xi|^2 = c^2 \cdot g_M(dt+\xi)$ and therefore to $dt+\xi$ being timelike. Hence P is a symmetric hyperbolic system.
Given $u_0 \in C_c^\infty(N,E)$ there exists a unique solution $u \in C_{sc}^\infty(M,E)$ of $Pu = 0$ and $u(0) = u_0$ by Theorem 3.7.7. We define $V_t : C_c^\infty(N,E) \to C_c^\infty(N,E)$ by $V_t\, u_0 := u(t)$.

Lemma 3.8.1. *$(V_t)_{t\in\mathbb{R}}$ is a unitary one-parameter group with*

(i) $\frac{d}{dt} V_t\, u_0 = L\, V_t\, u_0$ for all $u_0 \in C_c^\infty(N,E)$;

(ii) $L\, V_t = V_t\, L$ on $C_c^\infty(N,E)$.

Proof. *(a)* Condition (i), i.e. $\frac{d}{dt} V_t u_0 = L V_t u_0$, says nothing but $PV_t u = 0$ and therefore holds by definition.
(b) Fix $s \in \mathbb{R}$ and let $\widetilde{u}$ be a solution of $P\widetilde{u} = 0$ and $\widetilde{u}(0) = V_s u_0 = u(s)$. Now $t \mapsto \widetilde{u}(t)$ and $t \mapsto u(s+t)$ both lie in the kernel of P and have the same values $u(s)$ at $t = 0$. By uniqueness they coincide. Thus

$$V_{t+s}u_0 = u(s+t) = \widetilde{u}(t) = V_t V_s u_0$$

and therefore $V_{t+s} = V_t V_s$. Hence $(V_t)_{t\in\mathbb{R}}$ is a one-parameter group.

(c) We compute

$$\begin{aligned}\frac{d}{dt}(V_t u_0, V_t u_0) &= (\frac{d}{dt}V_t u_0, V_t u_0) + (V_t u_0, \frac{d}{dt}V_t u_0) \\ &= (LV_t u_0, V_t u_0) + (V_t u_0, LV_t u_0) \\ &= ((L + L^t)V_t u_0, V_t u_0) \\ &= 0\end{aligned}$$

and hence

$$(V_t u_0, V_t u_0) = (V_0 u_0, V_0 u_0) = (u_0, u_0).$$

Therefore V_t is unitary.
(d) For $u_0 \in C_c^\infty(N, E)$ we have that $u(t) := LV_t u_0$ is a solution of

$$\begin{aligned}Pu &= (\frac{d}{dt} - L)LV_t u_0 \\ &= L(\frac{d}{dt} - L)V_t u_0 \\ &= LPV_t u_0 \\ &= 0.\end{aligned}$$

For $t = 0$ we find $u(0) = LV_0 u_0 = Lu_0$ and thus $u(t) = V_t L u_0$. We conclude $LV_t = V_t L$ on $C_c^\infty(N, E)$ which is Condition (ii). □

The following functional-analytic lemma lies at the heart of the argument.

Lemma 3.8.2. *Let H be a complex Hilbert space and $\mathcal{D} \subset H$ a dense subspace. Let T be a symmetric operator in H with domain $\mathcal{D}$. Let $(V_t)_t$ be a unitary one-parameter group. Assume*

(i) $T\mathcal{D} \subset \mathcal{D}$;

(ii) $V_t \mathcal{D} \subset \mathcal{D}$ *for all* $t \in \mathbb{R}$;

(iii) $T\, V_t = V_t\, T$ *on* $\mathcal{D}$;

(iv) $\frac{d}{dt} V_t\, u_0 = i\, T\, V_t\, u_0$ *for all* $u_0 \in \mathcal{D}$.

Then T^n is essentially selfadjoint in H on $\mathcal{D}$ for all $n \in \mathbb{N}$.

Proof. Set $A := T^n$. Then A is a symmetric operator in H with domain $\mathcal{D}$. We want to show that $A^*\psi = \pm i\psi$ only has the trivial solution $\psi = 0$.
So let $A^*\psi = i\psi$. Fix $u \in \mathcal{D}$ and define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := (V_t u, \psi)$. We want to show that $f \equiv 0$. First we see $|f(t)| \leq ||V_t u|| \cdot ||\psi|| = ||u|| \cdot ||\psi||$ where we have used that V_t is unitary. Therefore f is bounded.

Moreover, using (iv),

$$\begin{aligned} f^{(n)}(t) &= (\frac{d^n}{dt^n} V_t u, \psi) \\ &= ((iT)^n V_t u, \psi) \\ &= (i^n V_t u, A^* \psi) \\ &= (i^n V_t u, i\psi) \\ &= -i^{n+1} f(t). \end{aligned}$$

The solution space for the ODE $f^{(n)} = -i^{n+1} f$ has the basis $t \mapsto \exp(\alpha_k t)$ where $\{\alpha_1, \ldots, \alpha_n\} = \{\alpha \in \mathbb{R} \mid \alpha^n = (-i)^{n+1}\}$. We write $f(t) = \sum_{k=1}^n a_k \exp(\alpha_k t)$ with $a_k \in \mathbb{C}$.
Since $(\frac{\alpha_k}{i})^n = -i$ we find $\Re(\alpha_k) \neq 0$ for all k. Since $|\alpha_k| = 1$ there is at most one $\alpha_{k'}$, $\alpha_k \neq \alpha_{k'}$, for any α_k with $\Re(\alpha_k) = \Re(\alpha_{k'})$, namely $\alpha_{k'} = \overline{\alpha}_k$.
Now suppose there is a k with $a_k \neq 0$ and $\Re(\alpha_k) > 0$. Choose k such that $\Re(\alpha_k)$ maximal.
Case 1: k is unique.
On the one hand,

$$\lim_{t\to\infty} (\exp(-\alpha_k t) f(t)) = 0$$

because f is bounded and $\Re(\alpha_k) > 0$.
On the other hand,

$$\lim_{t\to\infty} (\exp(-\alpha_k t) f(t)) = \lim_{t\to\infty} (\sum_{j=1}^n a_j \exp((\alpha_j - \alpha_k)t)) = a_k.$$

Therefore $a_k = 0$ which yields the contradiction.
Case 2: There is another $\alpha_{k'}$ with $\alpha_k \neq \alpha_{k'}$ and $\Re(\alpha_k) = \Re(\alpha_{k'})$. Then $\alpha_{k'} = \overline{\alpha}_k$. As in the first case we find

$$\lim_{t\to\infty} (\exp(-\alpha_k t) f(t)) = 0.$$

On the other hand,

$$\begin{aligned} \lim_{t\to\infty} (\exp(-\alpha_k t) f(t)) &= \lim_{t\to\infty} (\sum_{j=1}^n a_j (\exp((\alpha_j - \alpha_k)t))) \\ &= \lim_{t\to\infty} (a_k + a_{k'} \exp(-2\Im(\alpha_k) \cdot it)), . \end{aligned}$$

Choosing a sequence $t_m \to \infty$ such that $\exp(-2\Im\alpha_k \cdot it_m) = 1$ implies that $a_k + a_{k'} = 0$. Choosing a sequence $t_m \to \infty$ such that $\exp(-2\Im\alpha_k \cdot it_m) = -1$ gives that $a_k - a_{k'} = 0$. Hence $a_k = a_{k'} = 0$ which yields again a contradiction.
Thus for all $k$with $\Re(\alpha_k) > 0$ we have $a_k = 0$. Similarly, by looking at the limit $t \to -\infty$ we deduce that $a_k = 0$ for all k with $\Re(\alpha_k) < 0$. Therefore $f \equiv 0$.
In particular, $0 = f(0) = (V_0 u, \psi) = (u, \psi)$ for all $u \in \mathcal{D}$. Since $\mathcal{D}$ is dense in H this implies $\psi = 0$.
In the same way one checks that $A^*\psi = -i\psi$ has only the trivial solution. □

Theorem 3.8.3 (Chernoff). *Let N be a complete Riemannian manifold and let $E \to N$ be a hermitian vector bundle. Let $T \in \mathscr{D}\!\mathit{iff}_1(E,E)$ be formally selfadjoint. Moreover, suppose there is a constant $c > 0$ such that $|\sigma(T,\xi)|_{\mathrm{op}} \leq c \cdot |\xi|$ holds for all $\xi \in T^*N$.*
Then T^n is essentially selfadjoint in $L^2(N,E)$ on $C_c^\infty(N,E)$ for every $n \in \mathbb{N}$.

Proof. Set $L := iT$, $H := L^2(N,E)$, and $\mathcal{D} := C_c^\infty(N,E)$. Then Lemma 3.8.1 yields a unitary one-parameter group V_t with

- $V_t\mathcal{D} = V_t$,
- $\frac{d}{dt}V_t u_0 = Lv_t u_0$ for all $u_0 \in \mathcal{D}$
- $V_t T = -V_t iL = -iV_t L = -iLV_t = TV_t$.

Hence all assumptions in Lemma 3.8.2 are fulfilled. Lemma 3.8.2 now implies that T^n is essentially selfadjoint for all $n \in \mathbb{N}$. □

Example 3.8.4. Let $E = \oplus_{p=0}^{\dim N} \Lambda^p T^*N$ and $T = d + \delta$. Then T is formally selfadjoint and we have $|\sigma(T,\xi)|_{\mathrm{op}} = |\xi \wedge \cdot + \xi \lrcorner \cdot|_{op} \leq 2|\xi|$. Then Theorem 3.8.3 implies that $T^n = (d+\delta)^n$ is essentially selfadjoint.
In particular, the Hodge-Laplacian $T^2 = d\delta + \delta d = \Delta$ is essentially selfadjoint. Since the Hodge-Laplacian preserves the degree of forms $\Delta|_{C_c^\infty(N,\Lambda^p T^*N)}$ is essentially selfadjoint for any $p \in \{0,\dots,\dim(N)\}$.

Example 3.8.5. Let T be a formally selfadjoint operator of Dirac-type. We compute

$$\begin{aligned}
|\sigma(T,\xi)^t\sigma(T,\xi)|_{\mathrm{op}} &= |\sigma(T,\xi)\sigma(T,\xi)|_{\mathrm{op}} \\
&= |\sigma(T^2,\xi)|_{\mathrm{op}} \\
&= |\xi|^2
\end{aligned}$$

where we used that T^2 is Laplace-type. Hence

$$|\sigma(T,\xi)u|^2 = \left\langle \sigma(T,\xi)^t\sigma(T,\xi)u, u\right\rangle \leq |\xi|^2 \cdot |u|^2$$

which implies $|\sigma(T,\xi)|_{\mathrm{op}} \leq |\xi|$. Theorem 3.8.3 then shows that T^n is essentially selfadjoint for all $n \in \mathbb{N}$.

Example 3.8.6. In the previous two examples the operator T is elliptic. Here is a non-elliptic example. Let $\dim(N) = 3$ and $E = TN \oplus TN$. Let $T = i\begin{pmatrix} 0 & -\mathrm{rot} \\ \mathrm{rot} & 0 \end{pmatrix}$. Theorem 3.8.3 yields that T^n is essentially selfadjoint for all $n \in \mathbb{N}$.

We have seen that formally selfadjoint Dirac-type operators are always essentially selfadjoint. This is not true for formally selfadjoint Laplace-type operators in general. Adding a potential going to $-\infty$ sufficiently fast can destroy essential selfadjointness. To prevent this a lower bound is assumed in the next theorem.

Theorem 3.8.7 (Chernoff). *Let N be a complete Riemannian manifold and let $E \to N$ be a hermitian vector bundle. Let $P \in \mathscr{Diff}_2(E,E)$ be a formally selfadjoint operator of Laplace type. If there is an $\alpha \in \mathbb{R}$ such that for all $u \in C_c^\infty(N,E)$*

$$(Pu,u) \geq \alpha ||u||^2$$

then P^n is essentially selfadjoint in $L^2(N,E)$ on $C_c^\infty(N,E)$ for every $n \in \mathbb{N}$.

Proof. W.l.o.g. let $\alpha = 1$, otherwise replace P by $P + (1-\alpha)\,\mathrm{id}$. Let H_1 be the completion of $C_c^\infty(N,E)$ with respect to $(u,v)_1 = (Pu,v)$. Set $H := H_1 \oplus L^2(N,E)$ and $\mathcal{D} := C_c^\infty(N,E) \oplus C_c^\infty(N,E)$. We consider

$$A = \begin{pmatrix} 0 & \mathrm{id} \\ -P & 0 \end{pmatrix}.$$

Claim: A is skewadjoint with respect to the scalar product in H.
Indeed, we have

$$\begin{aligned}\left(A\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_H &= \left(\begin{pmatrix}u_2\\-Pu_1\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_H \\ &= (Pu_2,v_1) - (Pu_1,v_2)\end{aligned}$$

and

$$\begin{aligned}\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},A\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_H &= \overline{\left(A\begin{pmatrix}v_1\\v_2\end{pmatrix},\begin{pmatrix}u_1\\u_2\end{pmatrix}\right)_H} \\ &= \overline{(Pv_2,u_1)} - \overline{(Pv_1,u_2)} \\ &= (u_1,Pv_2) - (u_2,Pv_1) \\ &= (Pu_1,v_2) - (Pu_2,v_1)\,.\end{aligned}$$

Hence

$$\left(\begin{pmatrix}u_1\\u_2\end{pmatrix},A\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_H = -\left(A\begin{pmatrix}u_1\\u_2\end{pmatrix},\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)_H$$

which proves skewadjointness. The equation

$$\frac{d}{dt}\begin{pmatrix}u_1\\u_2\end{pmatrix} = A\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

is equivalent to the equations

$$\frac{d}{dt}u_1 = u_2$$
$$\frac{d}{dt}u_2 = -Pu_1,$$

and hence to

$$\frac{d^2}{dt^2}u_1 = -Pu_1.$$

Since $Q = \frac{d^2}{dt^2} + P$ is normally hyperbolic, the well-posedness of the Cauchy problem shows existence of a one-parameter group $(V_t)_t$ that solves the equation. Since A is skew we find that V_t is unitary. Hence the conditions in Lemma 3.8.2 are fulfilled and therefore $(iA)^n$ is essentially selfadjoint in H on $\mathcal{D}$. Now

$$(iA)^{2n} = (-1)^n(A^2)^n = (-1)^n \begin{pmatrix} -P & 0 \\ 0 & -P \end{pmatrix}^n = \begin{pmatrix} P^n & 0 \\ 0 & P^n \end{pmatrix}.$$

Therefore P^n is essentially selfadjoint in $L^2(N,E)$ on $C_c^\infty(N,E)$. □

3.9 Exercises

3.9.1. Consider the "timelike strip" $M = \{(t,x) \in \mathbb{R}^2 \mid -1 < x < 1\}$ in the 2-dimensional Minkowski space, equipped with the Minkowski metric $g = -dt^2 + dx^2$.

a) Show that M is a causally compatible subset of the Minkowski plane.

b) Is (M,g) globally hyperbolic?

c) Show that advanced and retarted fundamental solutions for the d'Alembert operator on M exist but are not unique.

3.9.2. Consider the "spacelike strip" $M = \{(t,x) \in \mathbb{R}^2 \mid -1 < t < 1\}$ in the 2-dimensional Minkowski space, equipped with the Minkowski metric $g = -dt^2 + dx^2$.

a) Show that M is a causally compatible subset of the Minkowski plane.

b) Is (M,g) globally hyperbolic?

c) Show that advanced and retarted fundamental solutions for the d'Alembert operator on M exist and are unique.

3.9.3. For any spatially compactly supported smooth function u on the n-dimensional Minkowski space consider the *spatial Fourier transform*

$$\hat{u}(t,\xi_2,\ldots,\xi_n) = \int_{\mathbb{R}^{n-1}} e^{-i\sum_{j=2}^n x^j\xi_j} \cdot u(t,x^2,\ldots,x^n)\, dx^2 \cdots dx^n .$$

Now let $u_0, u_1 \in C_c^\infty(\mathbb{R}^{n-1})$ and $f \in C_c^\infty(\mathbb{R}^n)$. Translate the Cauchy problem for u

$$\Box u = f, \quad u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = u_1(x),$$

into a problem for $\hat{u}$ and conlude that the Cauchy problem has a solution.

3.9.4. Let $M = \{(t,x) \in \mathbb{R}^2 \mid -1 < x < 1\}$ equipped with the Minkowski metric $g = -dt^2 + dx^2$. Let $S = \{0\} \times (-1,1)$.
Show that uniqueness fails in the Cauchy problem for the d'Alembert operator with initial values along S.

3.9.5. Let $(N.h)$ be a complete Riemannian manifold and let $M = \mathbb{R} \times N$ equipped with the metric $g = -dt^2 + h$. Let Q be a Laplace-type operator on N and $P = \frac{\partial^2}{\partial t^2} + Q$ the corresponding normally hyperbolic operator on M. Let $S = \{0\} \times N \cong N$ and assume that u_0 and u_1 are eigensections of Q for the eigenvalues λ_0 and λ_1, respectively. Note that the eigenvalues are real but not necessarily non-negative.
Determine the solution u of the Cauchy problem $Pu = 0$, $u(0,x) = u_0(x)$, and $\frac{\partial u}{\partial t}(0,x) = u_1(x)$.

3.9.6. *(Inhomogeneous equation with distributional right-hand side)*
Let M be globally hyperbolic and let $F_+(x)$ be the advanced fundamental solutions of the normally hyperbolic operator P acting on sections of $E \to M$. Let $v \in \mathcal{D}''(M,E)$ a distributional section of order 0 and with past-compact support. We define u by

$$u[\varphi] := v[x \mapsto F_+(x)[\varphi]]$$

for any $\varphi \in \mathcal{D}'(M,E^*)$.

a) Show that u is a well-defined distributional section of E and give an upper bound for the order of u.

b) Show $Pu = v$.

c) Show $\mathrm{supp}(u) \subset J_+(\mathrm{supp}(v))$.

Hint: You may use that for each compact subset $K \subset M$ there exists a constant $C > 0$ such that $|F_+(x)[\varphi]| \le C \cdot \|\varphi\|_{C^{n+1}}$ for all $x \in K$ and $\varphi \in \mathcal{D}'(M,E^*)$ with $\mathrm{supp}(\varphi) \subset K$.

3.9.7. *(Radiation of a charged particle)*
Let M be globally hyperbolic and let $E \to M$ be a Hermitian vector bundle. Let $c : [0,\infty) \to M$ be a smooth future-directed timelike curve (the world line of the particle), parametrized by proper time. Let q be a locally integrable section of E along c, i.e., $q \in L^1_{\mathrm{loc}}([0,\infty), c^*E)$ (the charge of the particle, possibly changing with time). We put for each $\varphi \in \mathcal{D}'(M,E^*)$:

$$v[\varphi] := \int_0^\infty \varphi(c(\tau)) \cdot q(\tau)\, d\tau \, .$$

a) Show that for any compact subset $K \subset M$ there is a $T > 0$ such that $c(\tau) \in M \setminus K$ for all $\tau \geq T$.

b) Show that v is a distributional section of order 0.

c) Show that $\mathrm{supp}(v)$ is contained in the closure of the trace of c and is past compact.

Hence the results of the Exercise 3.9.6 can be applied to this v. The solution u then describes the radiation emitted by the charged particle.

3.9.8. We consider the following example for the setup in Exercise 3.9.7: let M be the $1+1$-dimensional Minkowski space and let $P = \Box$ be the d'Alembert operator. Determine the solution u of $\Box u = v$ with past-compact support where $q \equiv 1$ and

a) $c(\tau) = (\tau, 0)$ (source at rest);

b) $c(\tau) = (\tau \cosh(\theta_0), \tau \sinh(\theta_0))$ (source at constant speed) where $\theta_0 \in \mathbb{R}$ is fixed;

c) $c(\tau) = (\sinh(\tau), \cosh(\tau) - 1)$ (accelerated source).

3.9.9. Let $M = S^1 \times \mathbb{R}$ with the metric $g = -g_{S^1} + dx^2$ where g_{S^1} is the standard metric of S^1 and x is the standard coordinate on $\mathbb{R}$. Let $\theta \in S^1$ be fixed and put $S = \{\theta\} \times \mathbb{R}$.
Show that the Cauchy problem for the d'Alembert operator on M with initial values along S does not always have solutions.

3.9.10. *(Cauchy data with noncompact support)*
For a normally hyperbolic operator P on a globally hyperbolic manifold M with smooth spacelike Cauchy hypersurface S we have seen that there is a unique solution to the Cauchy problem

$$\begin{cases} Pu = f & \text{on } M, \\ u = u_0 & \text{on } S, \\ \nabla_\nu u = u_1 & \text{on } S, \end{cases}$$

where u_0, u_1, and f are smooth and have compact support.
Show that there is still a unique solution if we drop the compactness assumption on the supports of u_0 and u_1.
Hint: Use a partition of unity on S with the property that for every $x \in M$ the set $J^\pm(x) \cap S$ meets the supports of only finitely many of the cut-off functions.

3.9.11. *(Cauchy problem for Dirac-type operators)*
Let M be a globally hyperbolic manifold and $E \to M$ a Hermitian vector bundle. Let $D \in \mathscr{D}\!\mathit{iff}_1(E, E)$ be of Dirac type.

a) Show that the principal symbol $\sigma_1(D, \xi)$ is invertible unless ξ is causal.

b) Let $f \in C_c^\infty(M, E)$ and $u_0 \in C_c^\infty(S, E)$ where $S \subset M$ is a smooth spacelike Cauchy hypersurface. Show that there is a unique solution $u \in C^\infty(M, E)$ of the Cauchy problem

$$\begin{cases} Du = f & \text{on } M, \\ u = u_0 & \text{on } S. \end{cases}$$

Hint: Show uniqueness first.

3.9.12. Let $M = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid t < 0\}$ with Minkowski metric $g = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2$ and standard time orientation. Is the support system pc in M essentially countable?

3.9.13. Let G_+ be the advanced Green's operator for the d'Alembert operator $\Box$ on $M = \mathbb{R} \times N$ with the metric $-dt^2 + g_N$. Here (N, g_N) is any complete Riemannian manifold. Show that for any $u \in C_{pc}^\infty(M)$ which depends only on t we have

$$(G_+ u)(t, y) = -\int_{-\infty}^{t} \int_{-\infty}^{s} u(\tau)\, d\tau\, ds\,.$$

3.9.14. Show that the advanced Green's operator for the d'Alembert operator $\Box$ on $1 + 1$-dimensional Minkowski space is given by

$$(G_+ u)(t, x) = -\frac{1}{2} \int_{-\infty}^{t} \left(\int_{x+\tau-t}^{x+t-\tau} u(\tau, \xi)\, d\xi \right) d\tau\,.$$

3.9.15. Let M be globally hyperbolic and let $E \to M$ be a vector bundle. Let $P \in \mathscr{Diff}(E, E)$ be Green hyperbolic with advanced and retarded Green's operators $G_\pm$. Let $Q \in \mathscr{Diff}(E, E)$ be another differential operator. Show that if Q commutes with P then Q also commutes with G_+ and G_-.
Hint: Consider $\tilde{G}_\pm = G_\pm + [G_\pm, Q]$.

3.9.16. *(Electrodynamics I)*
Let M be globally hyperbolic and let G_+ be the advanced Green's operator for $P = d\delta + \delta d$ acting on 1-forms. Let J be a 1-form on M (the 4-*current density*) with past-compact support. We assume it satisfies $\delta J = 0$ (the *continuity equation*).

a) Show that $A := G_+ J$ satisfies the *Lorentz gauge condition* $\delta A = 0$.

b) Show that $F := dA$ (the *electromagnetic field*) satisfies the *Maxwell equations* $dF = 0$ and $\delta F = J$.

3.9.17. *(Electrodynamics II)*
On 4-dimensional Minkowski space with the metric $g = -dt^2 + \sum_{k=1}^3 (dx^k)^2$ write $J = \rho dt + \sum_{k=1}^3 j_k\, dx^k$ and $F = \sum_{k=1}^3 E_k\, dx^k \wedge dt + \sum_\sigma B_{\sigma(1)}\, dx^{\sigma(2)} \wedge dx^{\sigma(3)}$ where the last sum is taken over all even permutations σ of $\{1, 2, 3\}$.

Express the continuity equation and the Maxwell equations in terms of the function ρ (the *charge density*) and the vector fields $\vec{J} = \sum_{k=1}^{3} j_k \frac{\partial}{\partial x^k}$ (the *current density*), $\vec{E} = \sum_{k=1}^{3} E_k \frac{\partial}{\partial x^k}$ (the *electric field*), and $\vec{B} = \sum_{k=1}^{3} B_k \frac{\partial}{\partial x^k}$ (the *magnetic field*).

3.9.18. Show that the Green's operator of $\Box \circ \Box$ on $1+1$-dimensional Minkowski space has a continuous integral kernel, compare Exercise 3 on Sheet 12.

3.9.19. Let $P \in \mathscr{Diff}(E_1, E_2)$ be a Green-hyperbolic operator on a globally hyperbolic manifold M. Show that the following sequence is exact:

$$0 \to C^\infty_{tc}(M; E_1) \xrightarrow{P} C^\infty_{tc}(M; E_2) \xrightarrow{G} C^\infty(M; E_1) \xrightarrow{P} C^\infty(M; E_2) \to 0\,.$$

3.9.20. *(Maxwell equations)*
Let N be a 3-dimensional Riemannian manifold, let J, E and B be smooth time-dependent vector fields on N and ϱ a smooth function on $M = \mathbb{R} \times N$. We assume that the Maxwell equations

$$\frac{\partial E}{\partial t} - \mathsf{rot}\, B = J, \qquad \frac{\partial B}{\partial t} + \mathsf{rot}\, E = 0,$$

are satisfied as well as the *continuity equation*

$$\mathsf{div}\, J - \frac{\partial \varrho}{\partial t} = 0\,.$$

a) Show that if the other two Maxwell equations

$$\mathsf{div}\, E = \varrho, \qquad \mathsf{div}\, B = 0,$$

hold for some $t = t_0$ then they hold for all t.

b) Let $P = \begin{pmatrix} \frac{\partial}{\partial t} & -\mathsf{rot} \\ \mathsf{rot} & \frac{\partial}{\partial t} \end{pmatrix}$ be the corresponding symmetric hyperbolic system. Show that there is no differential operator D acting on sections of $\pi^*(TN \oplus TN) \to M$ such that $P \circ D$ or $D \circ P$ is normally hyperbolic. Here, of course, $\pi : M \to N$ is the projection onto the N-factor.

3.9.21. Let X be a smooth vector field on a time-oriented Lorentzian manifold M.

a) What is the condition on X for the operator ∂_X acting on functions to be symmetric hyperbolic?

b) Give a direct proof of finite propagation speed in this example.

3.9.22. *(Euler momentum equation)*
Let N be a Riemannian manifold and let u_0 be a smooth time-dependent vector field on N. The *Euler momentum equation* for an imcompressible fluid linearized at u_0 is given by

$$\frac{\partial v}{\partial t} + \nabla_{u_0} v + \nabla_v u_0 = 0$$

where v is the time-dependent vector field on N to be found.

a) Show that there are Lorentzian metrics on $M = \mathbb{R} \times N$ with respect to which this equation is symmetric hyperbolic.

b) Show that these Lorentzian metrics can be chosen to be globally hyperbolic if N is compact.

3.9.23. *(Dust)*
Let (M, g) be a 4-dimensional time-oriented Lorentzian manifold, let u be a smooth future-directed timelike vector field with $g(u, u) \equiv -1$ (the 4-*velocity* of the dust) and let ϱ be a positive smooth function on M (the *mass density*). The linearization of the *dust equations* are given by

$$\nabla_v u + \nabla_u v = 0, \quad \partial_v \varrho + \partial_u \theta + \varrho \operatorname{div} v + \theta \operatorname{div} u = 0$$

where the unknowns are v, a vector field with $g(u, v) \equiv 0$, and θ, a function. In this case, the vector bundle for the system is $E = u^\perp \oplus \mathbb{R}$ where the fibers of $u^\perp$ are the orthogonal complements of u and $\mathbb{R}$ is the trivial line bundle. The rank of E is 4 and E carries a natural positive definite fiber metric.

a) Show that this system is not symmetric hyperbolic.

b) Show that it cannot even be made symmetric hyperbolic by changing the fiber metric of E to any other Riemannian metric on E.

Hint: Consider the Jordan normal form of the principal symbol.

Bibliography

[1] S. Alinhac: *Hyperbolic partial differential equations.* Springer-Verlag, New York, 2009

[2] C. Bär: Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes. *Commun. Math. Phys.* **333** (2015), 1585–1615

[3] C. Bär, N. Ginoux, and F. Pfäffle: *Wave equations on Lorentzian manifolds and quantization.* European Mathematical Society, Zürich, 2007, http://arxiv.org/abs/0806.1036

[4] H. Baum: *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Teubner Verlagsgesellschaft, Leipzig, 1981

[5] A. N. Bernal and M. Sánchez: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Commun. Math. Phys.* **257** (2005), 43–50

[6] A. L. Besse: *Einstein Manifolds*. Springer-Verlag, Berlin-Heidelberg, 1987

[7] A. N. Bernal and M. Sánchez: Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions. *Lett. Math. Phys.* **77** (2006), 183–197

[8] A. N. Bernal and M. Sánchez: Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal'. *Class. Quantum Grav.* **24** (2007), 745–749

[9] P. R. Chernoff: Essential self-adjointness of powers of generators of hyperbolic equations. *J. Funct. Anal.* **12** (1973), 401–414

[10] L. C. Evans and R. F. Gariepy: *Measure theory and fine properties of functions*, CRC Press, Boca Raton, FL, 1992

[11] F. G. Friedlander: *The wave equation on a curved space-time*. Cambridge University Press, Cambridge, 1975

[12] A. Jeffrey: *Handbook of Mathematical Formulas and Integrals*. Academic Press, San Diego, 1995

[13] B. O'Neill: *Semi-Riemannian Geometry* Academic Press, San Diego, 1983

[14] M. Reed and B. Simon: *Methods of Modern Mathematical Physics I: Functional Analysis* (revised and enlarged edition). Academic Press, San Diego, 1980

[15] J.M. Shatah and M. Struwe: *Geometric wave equations.* American Mathematical Society, Providence, 1998

[16] C. Sogge: *Lectures on non-linear wave equations.* International Press, Boston, 2008

[17] F. Trèves: *Topological vector spaces, distributions and kernels*, unabridged republication of the 1967 original. Dover Publications, Mineola, NY, 2006

Index

Symbols

R

S

T

W